Khintchine dichotomy for self-similar measures, Section 5

When a group G acts on X, $\mu \in \text{prob}(G)$ and $\nu \in \text{prob}(X)$ we define the measure $\mu * \nu$ by

$$\mu * \nu(f) = \int_X \int_G f(gx) d\mu(g) d\nu(x), \qquad f \in C_C(X).$$

In particular when $X = G$ with left translation action we can define $\mu^n = \mu * \cdots * \mu$ (n times).

We continue with the notation $G = \text{SL}(2, \mathbb{R})$, $X = G/\Lambda$ where Λ is a cocompact lattice. For $t > 0$, $s \in \mathbb{R}$ we denote

$$a(t) = \begin{pmatrix} \sqrt{t} & \\ & \frac{1}{\sqrt{t}} \end{pmatrix}, \quad u(s) = \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}.$$

$P = \{a(t)u(s) : t > 0, s \in \mathbb{R}\} = \{ \begin{pmatrix} * & * \\ & * \end{pmatrix}$, positive diagonal$\} < G$.

Notation: $f(x) \ll_{h(x)} g(x)$: There exists $c > 0$, which depends on $h(x)$, such that $f(x) \leq cg(x)$.

Also, $f(x) \ll_{h(x)} 1$: There exists small $\varepsilon > 0$ which depends on $h(x)$, such that $f(x) \leq \varepsilon$.

Definition 4.1: Let $\alpha, \tau > 0$ and $I \subseteq [0, 1)$. A Borel measure ν on X is (α, B_I, τ)-robust if we can decompose $\nu = \nu' + \nu''$ such that:

1. $\nu''(X) \leq \tau$,
2. $\nu'\{\text{inj} < \sup I\} = 0$ (in the cocompact case, this holds even for ν if $\sup I$ is sufficiently small).
3. for every $\rho \in I, y \in X$ we have $\nu'(B_\rho y) \leq \rho^{3\alpha}$

If $\{\rho\}$ we write that ν is (α, B_ρ, τ)-robust.

Robustness - simplified definition: Let $\alpha, \rho > 0$. A Borel measure ν on X is (α, ρ)-robust if for every $y \in X$ we have $\nu(B_\rho y) \leq \rho^{3\alpha}$ (dimension$\geq \alpha$ at scale ρ)

Random walks

Fix $\mu \in \text{Prob}(P)$ f.s. Assume the support of μ is not simultaneously diagonalizable, and μ satisfies $\int_G \log \|ge_1\| d\mu(g) > 0$.

Proposition 4.2 (for large n, $\mu^n * \delta_x$ is robust): Let $0 < \kappa < \frac{1}{10}$. For $\rho \ll_\kappa 1$ and $n \gg_\kappa |\log \rho| + |\log \mathrm{inj}(x)|$, the measure $\mu^n * \delta_x$ is $(1-\kappa, \rho)$-robust.

Prop 5.1 (robust implies e.d.) - random walks version: There exist $0 < \kappa < \frac{1}{10}$, $\rho_0 > 0$ such that $\forall$ $0 < \rho \leq \rho_0$:

Let ν be a $(1-\kappa, \rho)$-robust measure on X with $\nu(X) \leq 1$. Then for any $n \in \mathbb{N}$ such that $e^n \in [\rho^{-1/4}, \rho^{-1/2}]$, for any $f \in C_c^\infty(X)$ with $m_X(f) = 0$, we have

$$|\mu^n * \nu(f)| \leq \rho^\kappa S(f).$$

(Sobolev norm of f).

Theorem C: There exists a constant $c > 0$ such that for $x \in X, n \geq 1$ and $f \in C_c^\infty(X)$ we have

$$\mu^n * \delta_x(f) = m_X(f) + O(\mathrm{inj}(x)^{-1} S(f) e^{-cn}).$$

Suffices to prove for f with $m_X(f) = 0$ that $|\mu^n * \delta_x(f)| \leq O(\cdots)$

Proof of Theorem C: Let κ, ρ_0 be as in Prop 5.1. By prop 4.2, for $\rho \ll 1, , m \gg |\log \rho| + |\log \mathrm{inj}(x)|$, $\nu := \mu^m * \delta_x$ is $(1-\kappa, \rho)$-robust. Choose $n > m$ such that $n - m \in [\frac{1}{4}|\log \rho|, \frac{1}{2}|\log \rho|]$, then by Prop 5.1,

$$|\mu^n * \delta_x(f)| \leq \rho^\kappa S(f).$$

Fractals

The proof of Theorem B uses the same idea but is much more technical.

Let σ be a self-similar probability measure on $\mathbb{R}$. For $t > 0$, we define $\eta_t \in \mathrm{prob}(G)$ as follows: For $f \in C_c(X)$,

$$\eta_t(f) = \int_{\mathbb{R}} f(a(t)u(s)) d\sigma(s).$$

Theorem B: There exists a constant $c = c(\Lambda, \sigma) > 0$ such that for all $t > 1, x \in X, f \in C_c^\infty(X)$ we have

$$\eta_t * \delta_x(f) = \int_{\mathbb{R}} f(a(t)u(s)x)d\sigma(s) = m_X(f) + O(\frac{1}{\text{inj}(x)}S(f)t^{-c}).$$

Self-similar measures via random walks:

σ as above. Let $\phi_1, \ldots, \phi_m : \mathbb{R} \to \mathbb{R}$ be contracting invertible affine maps without a global fixed point, and $(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$ a probability vector such that

$$\sigma = \sum \lambda_i (\phi_i)_\star \sigma. \qquad (\blacktriangle)$$

In fact they show that WLOG we may assume $\phi_i \in \text{Aff}^+(\mathbb{R})$.

We can think of (λ_i) as a probability measure on the group $\text{Aff}^+(\mathbb{R})$:

$$\lambda = \sum_i \lambda_i \delta_{\phi_i}.$$

Then Equation ($\blacktriangle$) is equivalent to: $\lambda * \sigma = \sigma$

We can identify Aff$(\mathbb{R})^+$ with P as follows: for $g \in P$ there are unique $r = r_g > 0$, $b = b_g \in \mathbb{R}$ such that $g = a(r)^{-1}u(b) = \begin{pmatrix} r^{-1/2} & r^{-1/2}b \\ & r^{1/2} \end{pmatrix}$. We identify g with the affine map $s \mapsto rs + b$. This is an anti-isomorphism between those groups. The measure on P, corresponding to λ, is denoted by μ. The following lemma (the idea of Barak and Simmons) relates η_t and μ.

Lemma 5.4: Given $t > 0$, $n \in \mathbb{N}$, we have

$$\eta_t = \int_P \eta_{r_g t} * \delta_g d\mu^n(g).$$

Proof: Since σ is λ^n stationary, for $f \in C_c(G)$,

$$\eta_t(f) = \int_{\mathbb{R}} f(a(t)u(s))d\sigma(s) = \int_{\mathbb{R}} \int_{\text{Aff}^+(\mathbb{R})} f(a(t)u(\phi(s)))d\lambda^n(\phi)d\sigma(s)$$
$$= \int_{\mathbb{R}} \int_P f(a(t)u(r_g s + b_g))d\mu^n(g)d\sigma(s)$$

For any $s \in \mathbb{R}$ and $g \in P$,

$$a(t)u(r_g s + b_g) = a(t)a(r_g)u(s)a(r_g)^{-1}u(b_g)$$
$$= a(r_g t)u(s)g$$

and hence we get

$$= \eta_t(f) = \int_{\mathbb{R}} \int_P f(a(r_g t)u(s)g)d\mu^n(g)d\sigma(s) = \int_P \eta_{r_g t} * \delta_g(f)d\mu^n(g).$$

Prop 5.1 - fractals version: There exist $\kappa, \rho_0 > 0$ such that the following holds for all $0 < \rho \leq \rho_0$. Let ν be a Borel measure on X that is $(1-\kappa, \rho)$-robust and $\nu(X) \leq 1$. Then for any $t \in [\rho^{-1/4}, \rho^{-1/2}]$, for any $f \in C_c^\infty(X)$ with $m_X(f) = 0$, we have

$$|\eta_t * \nu(f)| \leq \rho^\kappa S(f)$$

Proof sketch of Theorem B: First observe that for any $r_0, r_1 > 0$,

$$\eta_{tr_0} = \delta_{a(r_0 r_1^{-1})} * \eta_{tr_1}$$

and hence for any finite Borel measure ν on X, we get

$$|\eta_{tr_0} * \nu(f) - \eta_{tr_1} * \nu(f)| \ll |\log(r_0 r_1^{-1})| \nu(X) S(f).$$

Let $\rho > 0$, consider a parameter $\alpha \in (0,1)$ to be specified later, and set $R = \{(1+\rho^\alpha)^k : k \in \mathbb{Z}\}$. For every $r \in R$, denote by μ_r^n the restriction of μ^n to the set $\{g \in P : r_g \in [r, r(1+\rho^\alpha))\}$. Then by the previous lemma

$$|\eta_t * \delta_x(f)| \le \sum_{r\in R} |\int_P \eta_{tr_g} * \delta_g d\mu_r^n(g) * \delta_x(f)|$$
$$\le \sum_{r\in R} |\int_P \eta_{tr} * \delta_g d\mu_r^n(g) * \delta_x(f)| + O(\rho^\alpha S(f)) = \sum |\eta_{tr} * \mu_r^n * \delta_x(f) + O(...).$$

From the proof gets very technical but here is the key point: by Prop 4.2 $\mu_r^n * \delta_x$ is robust for appropriate constants and hence by Prop 5.1,

$$|\eta_{tr} * \mu_r^n * \delta_x(f)| \le \rho^\kappa S(f).$$

Proof of Prop 5.1

Our next goal is to prove Prop 5.1. G acts on $L^2(X, m_X)$: for $g \in G$ and $f \in L^2$ the action is $(g.f)(x) = f(g^{-1}x)$. This is a unitary representation of G. We will use the following known formula: there exists a constant $\delta_0 > 0$ such that for any $f \in C_c^\infty(X)$ with $m_X(f) = 0$ and any $g \in G$ we have

$$|\langle g.f, f\rangle| \ll \|g\|^{-\delta_0} S(f)^2.$$

For $t > 0$, define the markov operator $P_{\eta_t} : L^2(X, m_X) \to L^2(X, m_X)$ by

$$P_{\eta_t} f(x) = \int_G f(gx) d\eta_t(g) = \eta_t * \delta_x(f).$$

Note that P_{η_t} is well defined on L^2.

Prop 5.2 (spectral gap): $\exists c > 0$ s.t. for any $f \in C_c^\infty(X)$ with $m_X(f) = 0$ we have

$$\forall t > 1, \qquad \|P_{\eta_t} f\|_{L^2} \ll t^{-c} S(f).$$

Proof: Using the above formula we get

$$
\begin{aligned}
\|P_{\eta_t} f\|_{L^2}^2 = \langle P_{\eta_t} f, P_{\eta_t} f\rangle &= \int_G \int_G \langle g^{-1}.f, h^{-1}.f\rangle d\eta_t(g) d\eta_t(h) \\
&= \int_G \int_G \langle hg^{-1}.f, f\rangle d\eta_t(g) d\eta_t(h) \ll S(f)^2 \int \int \|hg^{-1}\|^{-\delta_0} \\
&= S(f)^2 \int_{\mathbb{R}} \int_{\mathbb{R}} \|a(t)u(s_1)u(-s_2)a(t)^{-1}\|^{-\delta_0} d\sigma(s_1) d\sigma(s_2)
\end{aligned}
$$

The group element equals $u(t(s_1 - s_2))$ and hence

$$
\|P_{\eta_t} f\|_{L^2}^2 \ll S(f)^2 \int \int \max\{1, t|s_1 - s_2|\}^{-\delta_0} d\sigma(s_1) d\sigma(s_2)
$$

Let

$$
E = \{(s_1, s_2) : t|s_1 - s_2| \leq t^{1/2}\} \subseteq \mathbb{R}^2.
$$

Alon told us that there exists c such that for every $r > 0$ $\sigma(\text{any ball of radius } r) \ll r^c$,and it follows that $\sigma^{\otimes 2}(E) \ll t^{-c}$. Hence, the last integral

$$
\int \int_{\mathbb{R}^2} = \int \int_E 1 + \int \int_{\mathbb{R}^2/E} t|s_1 - s_2|^{-\delta_0} + \ll t^{-c} + t^{-\delta_0/2}.
$$

Robustness - simplified definition: Let $\alpha, \rho > 0$. A Borel measure ν on X is (α, ρ)-robust if for every $y \in X$ we have $\nu(B_\rho y) \leq \rho^{3\alpha}$.

Proof of prop 5.1: For $0 < \rho \leq \rho_0$ define the measure

$$
\nu_\rho(f) = \frac{1}{m_G(B_\rho)} \int_X \int_{B_\rho} f(gx) dm_G(g) d\nu(x)
$$

The first step (which we skip) is to show that ν_ρ is a.c. w.r.t. m_X and

$$
\frac{d\nu_\rho}{dm_X}(x) = \frac{\nu(B_\rho x)}{m_G(B_\rho)}.
$$

It follows from the robustness:

$$\|\frac{d\nu_\rho}{dm_X}(x)\|_{L^\infty(m_X)} \ll \frac{\rho^{3-3\kappa}}{\rho^3} = \rho^{-3k}.$$

For any $t > 1$,

$$\begin{aligned}|\eta_t * \nu(f)| &= |\int\int f(gx)d\eta_t(g)d\nu(x)| = |\int_X P_{\eta_t} f d\nu| \\ &\leq |\int * d\nu_\rho| + |\int * d(\nu_\rho - \nu)\end{aligned}$$

For the first term:

$$\begin{aligned}|\int * d\nu_\rho| &= |\int P_{\eta_t} f \frac{d\nu_\rho}{dm_X} dm_X| \leq \|P_{\eta_t} f\|_{L^1(m_X)} \|\frac{d\nu_\rho}{dm_X}\|_{L^\infty(m_X)} \\ &\leq \|P_{\eta_t} f\|_{L^2} \|\frac{d\nu_\rho}{dm_X}\|_{L^\infty(m_X)} \ll t^{-c} S(f) \rho^{-3k}.\end{aligned}$$

It can be shown that the second term is $\ll \rho t S(f)$. Taking $t \in [\rho^{-1/4}, \rho^{-1/2}]$ completes the proof.