Geometric and arithmetic aspects of approximation vectors - Section 5

Throughout the lecture we will work with $(X, \mathcal{B}_X, \{a_t\}, \mu)$, where

- X is a locally compact second countable Hausdorff space
- $\mathcal{B}_X$ is the Borel σ-algebra.
- $\{a_t\}_{t\in\mathbb{R}}$ is a one-parameter group which acts continuously on X
- μ is a Borel probability measure.

$\mathcal{S} \in \mathcal{B}_X$ is a μ-cross section, i.e.

- $\exists X_0 \subset X$ measurable $\{a_t\}$-invariant with $\mu(X_0) = 1$ and $\mathcal{S} \cap X_0$ is a Borel cross section for X_0: for every $x \in X_0$,
$$\{t \in \mathbb{R} : a_t x \in \mathcal{S}\} \subset \mathbb{R}$$
is discrete and unbounded from below and from above.
- The return time function is Borel.

Example to have in mind: irrational flow in $\mathbb{T}^2$, $\mathcal{S}$ is the x-axis.

Moreover, we will assume that $\mathcal{S} \subset X$ is locally compact second countable as well.

For $E \in \mathcal{B}_X$ and $I \subset \mathbb{R}$
$$E^I = \{a_t x : x \in E, t \in I\}.$$

Recall that we saw that there exists a measure $\mu_{\mathcal{S}}$ on $\mathcal{S}$ such that (among other things), for every Borel $E \subset \mathcal{S}$
$$\mu_{\mathcal{S}}(E) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\mu(E^{(0,\varepsilon)}).$$

$\mu_{\mathcal{S}}$ is called the cross-section measure of μ. We will assume that $\mu_{\mathcal{S}}$ is finite.

Let $E \subset \mathcal{S}$, $x \in X$ and $T > 0$. we denote
$$N(x, T, E) = \#\{t \in [0, T] : a_t x \in E\}.$$

A Borel $E \subset X$ is μ-JM (Jordan measurable) if $\mu(\partial_X E) = 0$.

Definition + lemma: $x \in X$ is called (a_t, μ) generic if $\frac{1}{T}\int_0^T \delta_{a_t x} dt \to \mu$ as $T \to \infty$. $x \in X$ is (a_t, μ) generic if and only if for any μ-JM set $E \subset X$,

$$\frac{1}{T}\int_0^T \chi_E(a_t x) dt \to \mu(E).$$

Let $\mathcal{S}' \subset \mathcal{S}$ be $\mu_{\mathcal{S}}$-JM with $\mu_{\mathcal{S}}(\mathcal{S}') > 0$. $x \in X$ is called $(a_t, \mu_{\mathcal{S}}|_{\mathcal{S}'})$ generic if the sequence of visits of $\{a_t x : t > 0\}$ to $\mathcal{S}'$ equidistributes w.r.t. $\frac{1}{\mu_{\mathcal{S}}(\mathcal{S}')}\mu_{\mathcal{S}}|_{\mathcal{S}'}$. $x \in X$ is $(a_t, \mu_{\mathcal{S}}|_{\mathcal{S}'})$ generic if and only if for any $\mu_{\mathcal{S}}$-JM set $E \subset \mathcal{S}'$,

$$\frac{N(x, T, E)}{N(x, T, \mathcal{S}')} \to_T \frac{\mu_{\mathcal{S}}(E)}{\mu_{\mathcal{S}}(\mathcal{S}')}.$$

Our goal will be to understand the relation between (a_t, μ) and $(a_t, \mu_{\mathcal{S}})$ genericity. Since this is a topological property we will add some topological requirements:

Recall that the first return time of $x \in \mathcal{S}$ is min$\{t > 0 : a_t x \in \mathcal{S}\}$ and that for $\varepsilon > 0$,

$$\mathcal{S}_{\geq \varepsilon} = \{x \in \mathcal{S} : \text{first return time of } x \text{ to } \mathcal{S} \text{ is } \geq \varepsilon\}$$
$$\mathcal{S}_{< \varepsilon} = \mathcal{S} \backslash \mathcal{S}_{\geq \varepsilon}.$$

Definition: $\mathcal{S}$ is μ-reasonable if

- for all sufficiently small ε, the sets $\mathcal{S}_{\geq \varepsilon}$ are $\mu_{\mathcal{S}}$-JM.
- $\exists \mathcal{U} \subset \mathcal{S}$ open in $\mathcal{S}$ such that the map $(0,1) \times \mathcal{U} \to X$, $(t, x) \mapsto a_t x$ is open and $\mu((\mathrm{cl}_X(S) \backslash \mathcal{U})^{(0,1)}) = 0$.

We assume from now on that $\mathcal{S}$ is μ-reasonable.

Lemma 5.5: $\forall$ $\mu_{\mathcal{S}}$-JM set $E \subset \mathcal{S}$ and interval $I \subset [0, 1]$, E^I is μ-JM.

Prop 5.6: $\forall$ (a_t, μ) generic $x \in X$, $\forall \varepsilon > 0$ and $\forall \mu_{\mathcal{S}}$-JM set $E \subset \mathcal{S}_{\geq \varepsilon}$

$$\lim_{T \to \infty} \frac{1}{T} N(x, T, E) = \mu_{\mathcal{S}}(E).$$

Proof: By the last lemma, $E^{(0,\varepsilon)}$ is μ-JM and thus

$$\frac{1}{T}N(x,T,E) = \frac{1}{T}(\frac{1}{\varepsilon}\int_0^T \chi_{E^{(0,\varepsilon)}}(a_t x)dt + O(1)) \to \frac{1}{\varepsilon}\mu(E^{(0,\varepsilon)}) = \mu_{\mathcal{S}}(E)$$

(second equality due to lemma 4.9, last one due to properties of $\mu_{\mathcal{S}}$).

Our goal is to replace the assumption $E \subset \mathcal{S}_{\geq\varepsilon}$ with a weaker one. For $\delta > 0$, let

$$\Delta_{\mathcal{S},\delta} \coloneqq \{x \in \mathcal{S} : \forall \varepsilon > 0, \limsup_{T\to\infty} \frac{1}{T}N(x,T,\mathcal{S}_{<\varepsilon}) > \delta\}$$

$$\Delta_{\mathcal{S}} \coloneqq \bigcup_{\delta>0} \Delta_{\mathcal{S},\delta}$$

(and as always $\Delta_{\mathcal{S}}^{\mathbb{R}} = \{a_t x : x \in \Delta_{\mathcal{S}}, t \in \mathbb{R}\}$).

Prop. 5.7: $\forall$ (a_t,μ) generic $x \in X\backslash\Delta_{\mathcal{S}}^{\mathbb{R}}$ which is and any $\mu_{\mathcal{S}}$-JM set $E \subset \mathcal{S}$,

$$\lim_{T\to\infty} \frac{1}{T}N(x,T,E) = \mu_{\mathcal{S}}(E).$$

Proof: Since $\mathcal{S}$ is reasonable, for any small enough $\varepsilon > 0$, $E \cap \mathcal{S}_{\geq\varepsilon}$ is $\mu_{\mathcal{S}}$-JM. By last prop,

$$\liminf_T \frac{1}{T}N(x,T,E) \geq \lim_T \frac{1}{T}N(x,T,E\cap\mathcal{S}_{\geq\varepsilon}) = \mu_{\mathcal{S}}(E\cap\mathcal{S}_{\geq\varepsilon}).$$

Since the sets $\mathcal{S}_{\geq\varepsilon}$ exhaust $\mathcal{S}$, it follows that

$$\liminf_T \frac{1}{T}N(x,T,E) \geq \mu_{\mathcal{S}}(E).$$

Fix $\delta > 0$. Since $x \notin \Delta_{\mathcal{S}}^{\mathbb{R}}$, $\exists\varepsilon > 0$ such that

$$\limsup_T \frac{1}{T}N(x,T,\mathcal{S}_{<\varepsilon}) \leq \delta$$

(and we may assume ε is small enough). Therefore,

$$\limsup_T \frac{1}{T} N(x,T,E) = \limsup_T \frac{1}{T}(N(x,T,E\cap \mathcal{S}_{\geq\varepsilon}) + \frac{1}{T}N(x,T,\mathcal{S}_{<\varepsilon}))$$
$$\leq \mu_{\mathcal{S}}(E\cap\mathcal{S}_{\geq\varepsilon}) + \delta \leq \mu_{\mathcal{S}}(E) + \delta.$$

Since δ was arbitrary, we are done.

Theorem 5.11: Let $\mathcal{S}' \subset \mathcal{S}$ be $\mu_{\mathcal{S}}$-JM such that $\mu_{\mathcal{S}}(\mathcal{S}') > 0$ and let $x \in X\backslash\Delta_{\mathcal{S}}^{\mathbb{R}}$ be (a_t,μ) generic. Then x is $(a_t, \mu_{\mathcal{S}}|_{\mathcal{S}'})$ generic.

Proof: We need to show that for any $\mu_{\mathcal{S}}$-JM set $E \subset \mathcal{S}'$,

$$\lim_T \frac{N(x,T,E)}{N(x,T,\mathcal{S}')} = \frac{\mu_{\mathcal{S}}(E)}{\mu_{\mathcal{S}}(\mathcal{S}')}$$

and we already know the convergence of both enumerator and denominator by Prop 5.7.

In the last theorem we assume that $x \in X\backslash\Delta_{\mathcal{S}}^{\mathbb{R}}$, but we don't know yet if there are many such x? or maybe its an empty set?

Lemma 5.8: $\mu_{\mathcal{S}}(\Delta_{\mathcal{S}}) = 0$ and $\mu(\Delta_{\mathcal{S}}^{\mathbb{R}}) = 0$.

Proof: By the properties of the measure $\mu_{\mathcal{S}}$, the latter follows from the former. To show the former, it suffices to show that for any fixed $\delta > 0$, $\mu(\Delta_{\mathcal{S},\delta}) = 0$. Take $0 < \varepsilon_1 < \varepsilon_0$ small enough so that

$$\mu_{\mathcal{S}}(\mathcal{S}_{\geq\varepsilon_0}) > 0, \qquad \mu_{\mathcal{S}}(\mathcal{S}_{<\varepsilon_1}) < \delta.$$

Consider the ergodic decomposition of μ:

$$\mu = \int \nu d\Theta(\nu)$$

that is, Θ is a probability measure on $\mathcal{P}(X)$ and for Θ-a.e. ν, ν is an $\{a_t\}$-invariant ergodic measure on X. Then for $\Theta - a.e.\nu$, the sets $\mathcal{S}_{\geq\varepsilon_0}, \mathcal{S}_{<\varepsilon_1}$ are ν-JM and $\mathcal{S}$ is a ν cross section.

Our goal is to prove that for such ν, for ν-a.e. x,

$$\lim_T \frac{1}{T} N(x, T, \mathcal{S}_{<\varepsilon_1}) = \nu_{\mathcal{S}}(\mathcal{S}_{<\varepsilon_1}).$$

Since $\nu_{\mathcal{S}}(\mathcal{S}_{<\varepsilon_1}) < \delta$ (why??), it will imply that $\nu_{\mathcal{S}}(\Delta_{\mathcal{S},\delta}) = 0$ and hence also $\mu_{\mathcal{S}}(\Delta_{\mathcal{S},\delta}) = 0$.

Recall that $\phi : \mathcal{S} \to \mathcal{S}$ is the first return map: for $x \in \mathcal{S}$, if $t > 0$ is the first return time of x to $\mathcal{S}$ then $F_{\mathcal{S}} x = a_t x$. Then $(\mathcal{S}, \frac{1}{\nu_{\mathcal{S}}(\mathcal{S})}\nu_{\mathcal{S}}, \phi)$ is an ergodic dynamical system and if we denote $N_T = N(x, T, \mathcal{S})$ then for $E \subset \mathcal{S}$ and $\nu_{\mathcal{S}}$-a.e. x

$$\frac{N(x,T,E)}{N_T} = \frac{1}{N_T} \sum_{n=0}^{N_T - 1} \chi_E(\phi^n x) \to_T \int \chi_E d\frac{\nu_{\mathcal{S}}}{\nu_{\mathcal{S}}(\mathcal{S})} = \frac{\nu_{\mathcal{S}}(E)}{\nu_{\mathcal{S}}(\mathcal{S})},$$

so we can choose $F \subset \mathcal{S}$ of full $\nu_{\mathcal{S}}$ measure such that this convergence holds for every $y \in F$ and $E = \mathcal{S}_{\geq \varepsilon_0}$, $E = \mathcal{S}_{<\varepsilon_1}$. Thus

$$\lim_{T\to\infty} \frac{\frac{1}{T} N(y, T, \mathcal{S}_{<\varepsilon_1})}{\frac{1}{T} N(y, T, \mathcal{S}_{\geq \varepsilon_0})} = \frac{\nu_{\mathcal{S}}(\mathcal{S}_{<\varepsilon_1})}{\nu_{\mathcal{S}}(\mathcal{S}_{\geq \varepsilon_0})}.$$

Replace F with a smaller set of full $\nu_{\mathcal{S}}$-measure of (a_t, ν) generic points. We already know that the denominator of LHS converges to this of RHS for every $y \in F$ by Prop. 5.6, so we have the enumerator convergence which is what we wanted.

We also want another variant of this theorem with different assumptions.

<u>Definition:</u> Let $M \in \mathbb{N}$ and $E \subset \mathcal{S}$ Borel. We say that E is M-tempered if for $\mu_{\mathcal{S}}$-a.e. x,

$$\#\{t \in [0,1] : a_t x \in E\} < M.$$

E is tempered if it is tempered for some M.

<u>Theorem $5.11(ii)$:</u> Let $\mathcal{S}' \subset \mathcal{S}$ be $\mu_{\mathcal{S}}$-JM and tempered such that $\mu_{\mathcal{S}}(\mathcal{S}') > 0$. Then every (a_t, μ) generic $x \in X$ is $(a_t, \mu_{\mathcal{S}}|_{\mathcal{S}'})$ generic.

If time allows:

Lemma 5.5: For any $\mu_{\mathcal{S}}$-JM set $E \subset \mathcal{S}$ and interval $I \subset [0,1]$, E^I is μ-JM.

Proof sketch: Assume for concreteness that $I = [\tau_1, \tau_2]$ is a closed interval. Since $\mathcal{S}$ is μ-reasonable, $\exists \mathcal{U} \subset \mathcal{S}$ open in $\mathcal{S}$ such that the map $(0,1) \times \mathcal{U} \to X$, $(t,x) \mapsto a_t x$ is open and $\mu((\mathrm{cl}_X(S) \backslash \mathcal{U})^{(0,1)}) = 0$. It is possible to show that

$$\partial_X(E^I) \subset (\mathrm{cl}_X(\mathcal{S}) \backslash \mathcal{U})^I \cup a_{\tau_1}\mathcal{S} \cup a_{\tau_2}\mathcal{S} \cup \partial_{\mathcal{S}}(E)^I.$$

All sets on the RHS are μ-null: the first one by the choice of $\mathcal{U}$.

The fourth one since E is $\mu_{\mathcal{S}}$-JM and by the relation between μ, $\mu_{\mathcal{S}}$.

The second and third one because $\mu_{\mathcal{S}}$ finite implies $\mu(\mathcal{S}) = 0$. ($\mu(S) \leq \mu(S^{(0,\varepsilon)}) \leq \mu_{\mathcal{S}}(S)\varepsilon = \varepsilon$)