Definitions and Notations

Def. A win-lose game $\Gamma$ is a collection of objects:

$$(x_0, x_I, x_{II}, x, f, S, S_I, S_{II})$$

- $x_0 \in X$
- $x_I \cap x_{II} = \emptyset$, $x_I \cup x_{II} = X$
- $f : X \setminus \{x_0\} \to X$ onto
  $$\forall x \in X \exists n \geq 0 \text{ s.t. } f^n(x) = x_0$$
- $S$ is the set of all sequences satisfying:
  $$S(0) = x_0 \quad \forall i > 0 : s(i) = f(s(i+1))$$
- $S_I \cap S_{II} = \emptyset$, $S_I \cup S_{II} = S$

* a play of $\Gamma$ is an element of $S$. Player I wins if it belongs to $S_I$.

Def. If $\Gamma = (x_0, x_I, x_{II}, x, f, S, S_I, S_{II})$ is a game, the set

$$\Sigma_I(\Gamma)$$ - the set of strategies for player I

$$= \{ \sigma \in X^{x_I} : \forall x \in f^{-1}(x) \}$$

$\Sigma_I(\Gamma)$ defined respectively

* we'll denote the elements of $\Sigma_{II}$ by $\tau$ for the comments
\( S = (S_1, T_2) \) is the unique play which results if the two players pick the strategies \( S_1, T_2 \).

Satisfying:
- \( S(0) = X_0 \)
- \( S(n) \in X_1 \rightarrow S(n+1) = S(S(n)) \)
- \( S(n) \notin X_2 \rightarrow S(n+1) = T(S(n)) \)

**Def.** If a win-lose game, \( \Sigma_{w}(\Gamma) \) the sets of all winning strategies.

\[ \Sigma_{w}(\Gamma) = \{ S \in \Sigma_{w}(\Gamma) : \forall T \in \Sigma_{w}(\Gamma), <S, T> \in S1 \} \]

\[ \Sigma_{w}(\Gamma) \text{ respectively} \]

**Def.** A win-lose game \( \Gamma \) is strictly determined

if \( \Sigma_{w}(\Gamma) \neq \emptyset \) or \( \Sigma_{w}(\Gamma) \neq \emptyset \)

**The Indeterminacy Theorem**

We construct an example of an infinite game with perfect information which is not strictly determined.

\[ X = \bigcup_{n \in \mathbb{N}} X_n \]

\[ S = \bigcup_{n \in \mathbb{N}} S_n \]

\[ X_1 = \bigcup_{n \in \mathbb{N}} X_{1,n} \]

\[ X_2 = \bigcup_{n \in \mathbb{N}} X_{2,n} \]

\( X_0 \) - the sequence of length 0.
1. \(|\Sigma_{\text{r}}(\Gamma)| = |\Sigma_{\text{p}}(\Gamma)| = 2^{\aleph_0}\)

**Proof:** every function from \(X_1\) to \(\{0,1\}\) determines and is determined by a strategy \(\sigma \in \Sigma_{\text{r}}(\Gamma)\).

\[\Rightarrow |\Sigma_{\text{r}}(\Gamma)| = |\{0,1\}^{X_1}| = 2^{\aleph_0}\]

2. \(\forall \tau \in \Sigma_{\text{r}}(\Gamma), \quad |S_{\tau_0}| = \{|\langle \sigma_0, \tau \rangle | \tau \in \Sigma_{\text{r}}(\Gamma)\}| = 2^{\aleph_0}\)

\(\forall \sigma_0 \in \Sigma_{\text{r}}(\Gamma), \quad |S_{\sigma_0}| = \{|\langle \sigma_0, \tau \rangle | \tau \in \Sigma_{\text{r}}(\Gamma)\}| = 2^{\aleph_0}\).

**Proof:** By the map \(\Psi : \{0,1\}^\mathbb{N} \rightarrow S_{\sigma_0}\), \(\{a_n\}_{n=0}^\infty \rightarrow s\)

where \(S(0) = \sigma_0\), \(S(2n+1) = a_n\), \(S(2n) = \tau_0(S(2n-1))\)

\(S \in S_{\sigma_0}\), because for each such play, there is \(\tau_0 \in \Sigma_{\text{r}}(\Gamma)\) s.t. \(S = \langle \sigma_0, \tau_0 \rangle\).

Moreover, the map is injective.

\[\Rightarrow 2^{\aleph_0} = |\{0,1\}^\mathbb{N}| \leq |S_{\sigma_0}| \leq |\mathbb{N}| = 2^{\aleph_0}\]

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**Using Axioms of Choice**

Let \(\alpha\) be the least ordinal s.t. there are \(2^{\aleph_0}\) ordinals less than \(\alpha\).

By (1) we can index the strategies in \(\Sigma_{\text{r}}(\Gamma)\) as \(\sigma_\beta\), \(\beta \in \{\mathfrak{t} | \mathfrak{t} < \alpha\}\).

Similarly, \(\tau_\beta \in \Sigma_{\text{p}}(\Gamma)\).

Choose \(\sigma_0 \in S_{\tau_0}\), and choose \(\tau_0 \in S_{\sigma_0}\), \(\tau_0 \neq \sigma_0\).

Proceed inductively: if \(S_\beta, \tau_\beta\) have been chosen for all \(\mathfrak{t} < \beta < \alpha\)

\(|\{\tau_\beta | \mathfrak{t} < \beta\}| < 2^{\aleph_0} \rightarrow S_{\tau_\beta} \setminus \{\tau_\beta | \mathfrak{t} < \beta\} \neq \emptyset\)

Choose one element and call it \(S_\beta\).
Similarly, \( S_\rho \setminus \{ s_\rho \mid \rho \leq \beta \} \neq \emptyset \), call one of its elements \( \tau \).

Define \( A = \{ s_\tau \mid \tau \leq \omega \} \), \( B = \{ t_\tau \mid \tau < \omega \} \), \( A, B \subseteq S \).

We show that \( A \cap B = \emptyset \), i.e. \( \forall \gamma, \beta \ s_\gamma \neq t_\beta \).

**Case I** if \( \gamma < \beta \) \( \rightarrow t_\beta \in S_\rho \setminus \{ s_\rho \mid \rho \leq \beta \} \rightarrow t_\beta \notin S_\rho \setminus \{ s_\rho \mid \rho \leq \beta \} \)

**Case II** if \( \gamma > \beta \) \( \rightarrow s_\gamma \notin S_\rho \setminus \{ s_\rho \mid \rho \leq \beta \} \rightarrow s_\gamma \notin S_\rho \setminus \{ s_\rho \mid \rho \leq \beta \} \)

We now partition \( S \) into two sets \( S_A, S_B \) s.t. \( AC S_A, BC S_B \).

Finally, we show that \( \Sigma^w_1 (\gamma) = \Sigma^w_\Pi (\gamma) = \emptyset \).

Let \( \omega \in \Sigma^w_1 (\gamma) \), so it has an index, say \( \beta \).

By the construction, there is a play \( t_\beta \) s.t. \( t_\beta \in S_\rho \).

Hence, there is \( \tau \in \Sigma^w_\Pi (\gamma) \) s.t. \( \sigma_\gamma, t_\gamma = t_\beta \in BC S_\Pi \)

\[ \sigma_\gamma \notin \Sigma^w_1 (\gamma) \rightarrow \Sigma^w_1 (\gamma) = \emptyset \]

Symmetrically, \( \Sigma^w_\Pi (\gamma) = \emptyset \)

Thus \( \gamma \) is not strictly determined.

Reminder - Transfinite Induction: /Recursion

1. Define \( X_\alpha \)
2. Given \( X_\beta \) for all \( \beta < \alpha \), define \( X_{\alpha+1} \)
3. Given \( X_\beta \) for all \( \beta < \delta \) define \( X_\delta \)

\( \alpha + \beta = \alpha (\cup \beta) \)

\( \alpha \downarrow \)

Successor ordinals

Limit ordinals

\( UX, X \) is the set of previous ordinals
It is also a Schmidt game:

Recall - $(F, G)$-games

- $\mathcal{N} \subset \mathcal{M}$, $S \subseteq \mathcal{M}$ is the winning set
- $\alpha : \mathcal{N} \to \mathcal{P}(\mathcal{M})$
- $\psi$-functions: $\psi : \mathcal{N} \to \mathcal{P}(\mathcal{N})$

$s.t. \forall C \in \mathcal{P}(B) \quad \alpha(C) \in \alpha(B)$

- $F, G$ are $\psi$-functions, move 1: $B_1 \in \mathcal{N}$
- move $\alpha : \mathcal{N} \to \mathcal{P}(\mathcal{N})$
- $F, G$ are $\psi$-functions, move 2: $W_i \in F(B_i)$
- $B_i \in G(W_{i-1})$
- $W_{i+1} \in F(B_i)$

$\alpha(B_1) \supset \alpha(W_1) \supset \alpha(B_2) \supset \alpha(W_2) \supset \ldots$

if $\bigcap_{i=1}^{\infty} \alpha(B_i) = \bigcap_{i=1}^{\infty} \alpha(W_i) \subset S$ the White is the winner.

In our example:

- $\mathcal{N} = \bigcup_{n=1}^{\infty} 10, 11^n$
- $\mathcal{M} = 10, 11^n$
- $\alpha : \bigcup_{n=1}^{\infty} 10, 11^n \to \mathcal{P}(10, 11^n)$

$\alpha(\langle a_1, a_2, \ldots, a_n \rangle) = \{ x_1, x_2, \ldots, x_n \in 10, 11^n : x_1 = a_1, x_2 = a_2, \ldots, x_n = a_n \}$

- $F : \bigcup_{n=1}^{\infty} 10, 11^n \to \mathcal{P}(\bigcup_{n=1}^{\infty} 10, 11^n)$

$F(\langle a_1, a_2, \ldots, a_n \rangle) = \{ x_1, x_2, \ldots, x_n \in 10, 11^{n+1} : x_1 = a_1, x_2 = a_2, \ldots, x_n = a_n \}$

indeed, $\alpha(\langle a_1, \ldots, a_n \rangle) \subseteq \alpha(\langle a_1, \ldots, a_n \rangle)$
The Topology of infinite game

Von Neumann has proved that finite games with perfect information are strictly determined.

\[ x \in X \quad U(x) = \{ s \mid s \in S \text{ and, for some } i, s(i) = x \} \]

* a neighborhood of \( s \) is any \( U(x) \) containing \( s \)

**Theorem:** the neighborhoods of points of \( s \) determine a Hausdorff topology for \( S \). In this topology \( S \) is totally disconnected.

\[
\text{open game - when } S_x \text{ is open}
\]

**Definition:** A win-lose game \( \mathcal{G}' = (x', x'_1, x'_2, f', x' \setminus f', S, S^1_1, S^1_2) \) is a sub-game of \( \mathcal{G} \) if:

\[
x'_0 = x_0 \quad x'_1 \subseteq x_1 \quad x'_2 \subseteq x_2 \quad f' = f \setminus x'
\]

\[
S^1_1 = S^1_1 \cap S' \quad S^1_2 = S^2_2 \cap S'
\]

**Theorem:** if \( \mathcal{G}' \) is a sub-game of \( \mathcal{G} \) then \( S' \) is closed, non-empty subset of \( S \).

**Proof:** Since \( \mathcal{G}' \) is a game, \( S' \) cannot be empty.

Let \( s \in S \setminus S' \). For some \( n \), \( s(n) \notin x' \) from the definitions.

Now if \( t \in U(s(n)) \) then \( t(s) = s(n) \notin x' \). \( \Rightarrow U(s(n)) \subseteq S \setminus S' \)

\( \Rightarrow S \setminus S' \) contains a neighborhood of \( S \). \( \Rightarrow S \setminus S' \) open.
**Thm 5:** If $F$ is a game, $\emptyset \neq F \subseteq S$ is closed, then there is a unique subgame of $F$ whose space is $F$.

(With its relative topology)

**Notation:** If $\emptyset \neq F \subseteq S$ is closed, then the corresponding game is $F_x$.

If $F$ is a neighborhood $U(x)$ we'll denote $F_x$.

**Def:**
1. $F \cap F' = (x_0, x, x, f, s, S, S \cap S', s \setminus s')$
2. $F \cup F' = (x_0, x, x, f, s, S, S \cup S', s \setminus s')$

Define only for games with the same $x_0, x, x, f$.

(3) $-F = (x_0, x, x, f, s, S, S, S')$ "The negative"

Example for not strictly determined subgame.

**Def:** A win-lose game $F$ is absolutely determined if all subgames of $F$ and $-F$ are strictly determined.

**Thm 6:** The union of an open game and absolutely determined game, is strictly determined.

**Cor.:** An open or closed game is strictly determined.
Proof of the cor. Apply Thm 6 to the case where the absolutely determined game is \( \Gamma = (x_0, x_1, x_2, r, s, \emptyset, \emptyset ) (S_I = \emptyset) \).

For the closed game, take \((x, x_1, x_2, x_3, r, s_1, s_2, s_I, \overline{s_I})\) which is open correspondences between \( \Sigma_1(\Gamma) \) and \( \Sigma_2(\Gamma) \),
\( \Sigma_3(\Gamma) \) and \( \Sigma_2(\Gamma) \).

Proof of Thm 6: Let \( G \subseteq S \) be open, and let \( \Gamma \) be absolutely determined.

We show that \( \Gamma' = (x_0, x_1, x_2, r, s, \emptyset U G, S_1 \setminus G) \) is strictly determined.

Define \( \Gamma^* = (x_0, x_1, x_2, r, s, G, S_1 \setminus G) \) - The open game

\( W^*_i = \{ x | \Sigma^w_i (\Gamma^*_x) \neq \emptyset \} \) - "Winning positions" for player \( I \)

\( F = S \setminus \cup_{x \in W^*_1} U(x) \) - closed set

If \( U(x) \subseteq G \) then every \( s \in \Sigma s_1 (x) \) will win for player \( I \), so \( x \in W^*_1 \).

\( G \) is open, so it's the union of all neighborhoods contained in it.

Hence, \( G \subseteq \bigcup_{x \in W^*_1} U(x) \) [for every \( s \in G \), there is \( U(x) \) s.t. \( s \subseteq U(x) \subseteq G \)]

\( F \cap G = \emptyset \)

If \( x \in W^*_1 \) then any \( s \in \Sigma^w I (x) \) will win \( \Gamma' \).

If \( x \in W^*_1 \), then there is \( s \in S \) s.t. \( s(x) \in W^*_1 \) \( \forall i \) \( r(x) \).

\( \Rightarrow s \in F \) so \( F \neq \emptyset \) and we can look at the subgame \( \Gamma_f \).

\( \Gamma \) is absolutely determined so it's enough to show:

Because subgames are strictly determined

\( (A) \) If \( \Sigma^w I (\Gamma_f) = \emptyset \) \( \Rightarrow \Sigma^w I (\Gamma') \neq \emptyset \)

\( (B) \) If \( \Sigma^w (\Gamma_f) \neq \emptyset \) \( \Rightarrow \Sigma^w (\Gamma') \neq \emptyset \).
Proof of A: \{X_{T} \in W_{T}^* \text{ choose } \sum_{I}^{w}(P_{x}) \}.

Choose \sum_{I}^{w}(P_{f}) and let \sigma_{0} be any extension of \sigma_{1}.

Existence: There is \sigma \in \Sigma_{I}^{w}(P_{1}) s.t for each \tau \in \Sigma_{II}^{w}(P_{1}) either

(*) \langle \sigma_{1}, \tau \rangle = \langle \sigma_{0}, \tau \rangle \in F \text{ or}

(**) for some \ x \in W_{1}^* and \tau \in \Sigma_{II}^{w}(P_{x}) : \langle \sigma_{1}, \tau \rangle = \langle \sigma_{x}, \tau \rangle

\bullet \text{If (*)}, there is \tau_{1} \in \Sigma_{II}^{w}(P_{f}) \text{ s.t.} \langle \sigma_{1}, \tau \rangle = \langle \sigma_{0}, \tau \rangle = \langle \tau_{1}, \tau \rangle.

Since \tau_{1} \in \Sigma_{II}^{w}(P_{f}) \text{, } \langle \sigma_{1}, \tau \rangle \in \Sigma_{II} \land F \subset S_{II} \text{ we have}

\bullet \text{If (**)}, then since \sigma \in \Sigma_{II}^{w}(P_{x}^*), \langle \sigma_{1}, \tau \rangle \text{ belong to } G.

\Rightarrow \langle \sigma_{1}, \tau \rangle \in S_{II} \cup G \Rightarrow \sigma \in \Sigma_{II}^{w}(P_{1})

Proof of B: Choose \tau_{f} \in \Sigma_{II}^{w}(P_{f}). We define a winning strategy for player II in \eta_{1}.

If \ x \in X_{II} \text{ choose } \tau_{(x)} = \tau_{f}(x). \text{ So } \tau_{(x)} \notin f^{-1}(W_{1}^*)

If \ x \notin X_{II} \text{ choose any element of } f^{-1}(x).

Let \sigma \in \Sigma_{II}^{w}(P_{1}).

\bullet \text{If } \langle \sigma, \tau \rangle \in F \text{ (By lemma u.s.), by the fact that } \tau_{f} \in \Sigma_{II}^{w}(P_{f}) \text{,}

\langle \sigma, \tau \rangle \in F \setminus S_{II} = \left( \left\{ x \in X_{II}^* \mid \Sigma_{II}^{w}(P_{x}) \neq \emptyset \right\} \right) \cup G. \text{ and we are done.}

\bullet \text{If } \langle \sigma, \tau \rangle \notin F \text{ then } \langle \sigma, \tau \rangle \in \sigma_{II} \cup \left( \Sigma_{II}^{w}(P_{x}) \right) \text{ i.e. there's a least integer } n \geq 0

\text{ s.t } s(n) \in W_{I}^* \text{ s.t } \Sigma_{II}^{w}(P_{s(n)}) \neq \emptyset \rightarrow \Sigma_{I}^{w}(P_{s(n)}) \neq \emptyset

\text{Ex: } \Sigma_{II}^{w}(P_{x}) \neq \emptyset \rightarrow \Sigma_{II}^{w}(P_{x}) \neq \emptyset

\text{But if } s(n) \notin X_{II} \text{, then there would be } y \in f^{-1}(s(n-1)) \setminus W_{I}^*. \text{ (Ex.)}

\text{Lemma u.s.}

\Rightarrow \text{by the definition of } \tau_{1}, \text{ } s(n) \rightarrow \tau_{s(n)} \notin f^{-1}(s(n-1)) \setminus W_{I}^*. \text{ (Ex.)}
Cor 10. An open or closed game is absolutely determined.

Cor 13. An intersection and union of open or closed game with absolutely determined game is strictly determined.

(Martin 1975) - A Borel game is determined.

[If $S_1$ is Borel set, then $P$ is determined]