Infinite Games with Perfect Information

Definitions and Notations

Def. A win-lose game Γ is a collection of objects:

$$(x_0, X_{I}, X_{II}, X, f, S, S_{I}, S_{II})$$

- $x_0 \in X$
- $X_I \cap X_{II} = \emptyset$, $X_I \cup X_{II} = X$
- $f : X \setminus \{x_0\} \to X$ onto

  $\forall x \in X \ \exists n \geq 0$ s.t $f^n(x) = x_0$
- S is the set of all sequences satisfying:

  $S(0) = x_0 \qquad \forall i \geq 0 : s(i) = f(s(i+1))$
- $S_I \cap S_{II} = \emptyset$, $S_I \cup S_{II} = S$

* a <u>play</u> of Γ is an element of S. Player I wins if it belongs to S_I.

Def. If $\Gamma = (x_0, X_I, X_{II}, X, f, S, S_I, S_{II})$ is a game, the set

$\Sigma_I(\Gamma)$ – the set of strategies for player I

$$= \{ \sigma \in X^{X_I} : \sigma(x) \in f^{-1}(x) \}$$

$\Sigma_{II}(\Gamma)$ defined respectively

* we'll denote the elements of Σ_{II} by τ for the convinience

* $S = \langle \sigma, \tau \rangle$ is the unique play which results if the two players pick the strategies σ, τ.

satisfying:
- $S(0) = X_0$
- $S(n) \in X_I \longrightarrow S(n+1) = \sigma(S(n))$
- $S_n \in X_{II} \longrightarrow S(n+1) = \tau(S(n))$

Def. Γ a win-lose game,

$\Sigma_I^w(\Gamma)$ - the sets of all winning startegies.

$$= \{\sigma \in \Sigma_I(\Gamma) : \forall \tau \in \Sigma_{II}(\Gamma), \langle \sigma, \tau \rangle \in S_I\}$$

$[\Sigma_{II}^w(\Gamma)$ respectively$]$

Def. A win-lose game Γ is strictly determined if $\Sigma_I^w(\Gamma) \neq \emptyset$ or $\Sigma_{II}^w(\Gamma) \neq \emptyset$

The Indeterminacy Theorem

We construct an example of an infinite game with perfect information which is not strictly determined.

$$X = \bigcup_{n \in \mathbb{N}} \{0,1\}^n \qquad S = \{0,1\}^{\mathbb{N}}$$

$X_I = \bigcup_{n \in \mathbb{N}_{even}} \{0,1\}^n$, $X_{II} = \bigcup_{n \in \mathbb{N}_{odd}} \{0,1\}^n$, X_0 - the sequence of length 0.

facts

1. $|\Sigma_I(\Gamma)| = |\Sigma_{II}(\Gamma)| = 2^{\aleph_0}$

<u>Proof:</u> every function from X_I to $\{0,1\}$ determines and is determined by a strategy $\sigma \in \Sigma_I(\Gamma)$.

$$\Longrightarrow |\Sigma_I(\Gamma)| = |\{0,1\}|^{|X_I|} = 2^{\aleph_0}$$

2. $\forall \tau_0 \in \Sigma_{II}(\Gamma)$, $|S_{\tau_0}| = |\{\langle\sigma, \tau_0\rangle \mid \sigma \in \Sigma_I(\Gamma)\}| = 2^{\aleph_0}$

$\forall \sigma_0 \in \Sigma_I(\Gamma)$, $|S_{\sigma_0}| = |\{\langle\sigma_0, \tau\rangle \mid \tau \in \Sigma_{II}(\Gamma)\}| = 2^{\aleph_0}$

<u>Proof</u>: By the map $\varphi : \{0,1\}^{\mathbb{N}} \to S_{\tau_0}$, $\{a_n\}_{n=0}^{\infty} \mapsto s$

where $s(0) = x_0$, $s_{2n+1} = a_n$, $s(2n) = \tau_0(s(2n-1))$

$s \in S_{\tau_0}$ because for each such play, there is $\sigma \in \Sigma_I(\Gamma)$ s.t $s = \langle\sigma, \tau_0\rangle$

Moreover, the map is injective.

$$\Longrightarrow 2^{\aleph_0} = |\{0,1\}^{\mathbb{N}}| \leq |S_{\tau_0}| \leq |S| = 2^{\aleph_0}$$

Using Axiome of Choice

Let α be the least ordinal s.t there are $2^{\aleph_0}$ ordinals less then α.

By **(1)** we can index the strategies in $\Sigma_I(\Gamma)$ as σ_β, $\beta \in \{\gamma \mid \gamma < \alpha\}$

Similarly, $\tau_\beta \in \Sigma_{II}(\Gamma)$.

Choose $s_0 \in S_{\tau_0}$, and chose $t_0 \in S_{\sigma_0}$, $t_0 \neq s_0$

Proceed inductivly: if s_γ, t_γ have been chosen for all $\gamma < \beta < \alpha$

$$|\{t_\gamma \mid \gamma < \beta\}| < 2^{\aleph_0} \longrightarrow S_{\tau_\beta} \setminus \{t_\gamma \mid \gamma < \beta\} \neq \emptyset$$

Choose one element and call it s_β.

Similary, $S_{\sigma_\beta} \setminus \{s_\gamma \mid \gamma \leq \beta\} \neq \emptyset$. Call one of its elements t_β

Define $A = \{s_\gamma \mid \gamma < \alpha\}$, $B = \{t_\gamma \mid \gamma < \alpha\}$ $A, B \subseteq S$

We show that $A \cap B = \emptyset$, i.e $\forall_{\gamma, \beta}$ $s_\gamma \neq t_\beta$:

<u>CASE I</u> if $\gamma \leq \beta \rightarrow t_\beta \in S_{\sigma_\beta} \setminus \{s_\delta \mid \delta \leq \beta\} \longrightarrow t_\beta \notin \{s_\delta \mid \delta \leq \beta\}$

<u>CASE II</u> if $\gamma > \beta \rightarrow s_\gamma \in S_{\tau_\gamma} \setminus \{t_\delta \mid \delta < \gamma\} \longrightarrow s_\gamma \notin \{t_\delta \mid \delta < \gamma\}$

We now partition S into two sets S_I, S_{II} s.t $A \subset S_I$, $B \subset S_{II}$

Finally, we show that $\Sigma_I^w(\Gamma) = \Sigma_{II}^w(\Gamma) = \emptyset$

Let $\sigma \in \Sigma_I(\Gamma)$. so it has an index, say β.

By the construction, there is a play t_β s.t $t_\beta \in S_{\sigma_\beta}$.

Hence, there is $\tau \in \Sigma_{II}(\Gamma)$ s.t $\langle \sigma_\beta, \tau \rangle = t_\beta \in B \subset S_{II}$

$$\longrightarrow \sigma_\beta \notin \Sigma_I^w(\Gamma) \longrightarrow \Sigma_I^w(\Gamma) = \emptyset$$

Symmetrically, $\Sigma_{II}^w(\Gamma) = \emptyset$

Thus Γ is not strictly determined.

<u>Reminder</u> - Transfinite Induction: / Recursion

1. Define X_0
2. Given X_β for all $\beta \leq \alpha$, define $X_{\alpha+1}$
3. Given X_γ for all $\gamma < \delta$ define X_δ

$\alpha + 1 = \alpha \cup \{\alpha\}$

$\uparrow$

successor ordinals

limit ordinals

$\downarrow$

$\bigcup X$, X is the set of previous ordinals

It is also a Schimdt-Game :

can be also α-digit game $\alpha^*\alpha$-game . where $\alpha=2$

Recall — $(\mathcal{F},G)$- games

- $\Omega' \subset \Omega$, M , $S \subseteq M$ is the winning set
- $\alpha : \Omega \longrightarrow P(M)$
- φ- functions : $\varphi : \Omega \to P(\Omega)$

  s.t $\forall C \in \varphi(B)$ $\alpha(C) \subseteq \alpha(B)$
- $\mathcal{F}, G$ are φ-functions , move 1 : $B_1 \in \Omega'$

  move 2 $W_1 \in \mathcal{F}(B_1)$

  $\vdots$

  $B_i \in G(W_{i-1})$

  $W_i \in \mathcal{F}(B_i)$
- $\alpha(B_1) \supset \alpha(W_1) \supset \alpha(B_2) \supset \alpha(W_2) \supset \dots\dots$

if $\bigcap_{i=1}^{\infty} \alpha(B_i) = \bigcap_{i=1}^{\infty} \alpha(W_i) \subset S$ the White is the winner .

In our example:

- $\Omega = \bigcup_{n=1}^{\infty} \{0,1\}^n$
- $M = \{0,1\}^{\mathbb{N}}$
- $\alpha : \bigcup_{n=1}^{\infty} \{0,1\}^n \longrightarrow P(\{0,1\}^{\mathbb{N}})$

$$\alpha(\langle a_1, a_2, \dots, a_n \rangle) = \{\langle x_1, x_2, \dots \rangle \in \{0,1\}^{\mathbb{N}} : x_1 = a_1, x_2 = a_2, \dots, x_n = a_n\}$$

- $F : \bigcup_{n=1}^{\infty} \{0,1\}^n \to P\left(\bigcup_{n=1}^{\infty} \{0,1\}^n\right)$ $\boxed{F = G}$

$$F(\langle a_1, a_2, \dots a_n \rangle) = \{\langle x_1, x_2, \dots x_{n+1} \rangle \in \{0,1\}^{n+1} : x_1 = a_1, x_2 = a_2, \dots, x_n = a_n\}$$

indeed, $\alpha(\langle a_1, \dots, a_n, x_{n+1} \rangle) \subseteq \alpha(\langle a_1, \dots, a_n \rangle)$

The Topology of infinite game

[Von Neumann has prooved that finite games with perfect information are strictly determined.]

$x \in X$ $\quad U(x) = \{ s \mid s \in S \text{ and, for some } i, \ s(i) = x \}$

* a neighborhood of s is any $U(x)$ containing s

Thm : the neighborhoods of points of s determine a Hausdorff topology for S. In this topology S is totally disconnected

$\boxed{\text{open game} - \text{when } S_I \text{ is open}}$

Def .. A win-lose game $\Gamma' = (X_0', X_I', X_{II}', X', f', S', S_I', S_{II}')$ is a sub-game of Γ if: $X_0' = X_0$, $X_I' \subset X_I$, $X_{II}' \subset X_{II}$, $f' = f|_{X'}$

$S_I' = S_I \cap S'$, $S_{II}' = S_{II} \cap S'$

Thm 4 : if Γ' is a sub-game of Γ then S' is closed, non-empty subset of S.

Proof : Since Γ' is a game, S' cannot be empty.

Let $s \in S \setminus S'$. For some n, $s(n) \notin X'$ from the definitions.

Now if $t \in U(s(n))$ then $t(n) = s(n) \notin X'$. $\Longrightarrow$ $U(s(n)) \subseteq S \setminus S'$

$\Longrightarrow$ $S \setminus S'$ contains a neighborhood of s. $\Rightarrow$ $S \setminus S'$ open.

Thm 5: Γ is a game, $\emptyset \neq F \subseteq S$ is closed, then there is a unique subgame of Γ whose space is F.

(with its relative topology)

Notation: If $\emptyset \neq F \subseteq S$ is closed, then the corresponding game is Γ_F.

If F is a neighborhood $U(x)$ we'll denote Γ_x.

Def. (1) $\Gamma \cap \Gamma' = (x_0, X_I, X_{II}, f, S, S_I \cap S_I', S \setminus (S_I \cap S_I'))$

(2) $\Gamma \cup \Gamma' = (x_0, X_I, X_{II}, X, f, S, S_I \cup S_I', S \setminus S_I \cup S_I'))$

Define only for games with the same x_0, X_I, X_{II}, X, f

(3) $-\Gamma = (x_0, X_I, X_{II}, X, f, S, S_{II}, S_I)$ "The negative"

* Example for not strictly determined subgame.

Def: A win-lose game Γ is <u>absolutely determined</u> if all subgames of Γ and $-\Gamma$ are strictly determined.

Thm 6: The union of an open game and absolutely determined game, is strictly determined.

Cor.: An open or closed game is strictly determined

Proof of the cor. Apply <u>Thm 6</u> to the case where the absolutely determined game is $\Gamma = (X_0, X_I, X_{II}, f, S, \emptyset, S)$ ($S_I = \emptyset$).

For the closed game, take $(X_0, X_{II}, X_I, X, f, S, S_{II}, S_I)$ which is open

correspondences between $\Sigma_I(-\Gamma)$ and $\Sigma_{II}(\Gamma)$,
$\Sigma_{II}(-\Gamma)$ and $\Sigma_I(\Gamma)$.

Proof of Thm 6: Let $G \subseteq S$ be open, and let Γ be absolutely determined.

We show that $\Gamma' = (X_0, X_I, X_{II}, X, f, S_I \cup G, S_{II} \setminus G)$ is strictly determined.

Define
- $\Gamma^* = (X_0, X_I, X_{II}, X, f, S, G, S \setminus G)$ – The open game
- $W_I^* = \{x \mid \Sigma_I^W(\Gamma_x^*) \neq \emptyset\}$ – "Winning positions" for player I
- $F = S \setminus \bigcup_{x \in W_I^*} U(x)$ – closed set

If $U(x) \subset G$ then every $\sigma \in \Sigma_S(\Gamma_x)$ will win for player I, so $x \in W_I^*$.

G is open, so it's the union of all neighborhoods contained in it.

Hence, $G \subset \bigcup_{x \in W_I^*} U(x)$ [for every $s \in G$, there is $U(x)$ s.t $s \in U(x) \subset G$]

$F \cap G = \emptyset$

If $x_0 \in W_I^*$ then any $\sigma \in \Sigma_I^W(\Gamma_{x_0}^*)$ will win Γ'.

If $x_0 \notin W_I^*$, then there is $s \in S$ s.t $s(i) \notin W_I^*$ $\forall i$. (Ex)

$\Rightarrow s \in F$. So $F \neq \emptyset$ and we can look at the subgame Γ_F.

Γ is absolutely determined so it's enough to show:

Because subgames are strictly determined

(A) If $\Sigma_I^W(\Gamma_F) \neq \emptyset \Longrightarrow \Sigma_I^W(\Gamma') \neq \emptyset$

(B) If $\Sigma_{II}^W(\Gamma_F) \neq \emptyset \Longrightarrow \Sigma_{II}^W(\Gamma') \neq \emptyset$.

Proof of A: $\forall x \in W_I^*$ choose $\sigma_x \in \Sigma_I^W(\Gamma_x^*)$.

choose $\sigma_1 \in \Sigma_I^W(\Gamma_F)$ and let σ_0 be any extension of σ_1

Ex.: There is $\sigma \in \Sigma_I(\Gamma)$ s.t for each $\tau \in \Sigma_{II}(\Gamma)$ either

(*) $\langle \sigma, \tau \rangle = \langle \sigma_0, \tau \rangle \in F$ or

(**) for some $x \in W_I^*$ and $\tau_x \in \Sigma_{II}(\Gamma_x)$: $\langle \sigma, \tau \rangle = \langle \sigma_x, \tau_x \rangle$

- If (*), there is $\tau_1 \in \Sigma_{II}(\Gamma_F)$ s.t

$$\langle \sigma, \tau \rangle = \langle \sigma_0, \tau \rangle = \langle \sigma_1, \tau_1 \rangle .$$

Since $\sigma_1 \in \Sigma_I^W(\Gamma_F)$, $\langle \sigma, \tau \rangle \in S_I \cap F \subset S_I$

- If (**), then since $\sigma_x \in \Sigma_I^W(\Gamma_x^*)$, $\langle \sigma, \tau \rangle \in G$.

$\Rightarrow \langle \sigma, \tau \rangle \in S_I \cup G \Rightarrow \sigma \in \Sigma_I^W(\Gamma')$

Proof of B: Choose $\tau_F \in \Sigma_{II}^W(\Gamma_F)$. We'll define a winning strategy of player II in Γ':

If $x \in X_{II}^F$ choose $\tau(x) = \tau_F(x)$. so $\tau(x) \in f^{-1}(x) \setminus W_I^*$.

If $x \in X_{II} \setminus X_{II}^F$ choose any element of $f^{-1}(x)$.

let $\sigma \in \Sigma_I(\Gamma')$.

- If $\langle \sigma, \tau \rangle \in F$ (By lemma 4.1), by the fact that $\tau_F \in \Sigma_{II}^W(\Gamma_F)$,

$\langle \sigma, \tau \rangle \in F \setminus S_I = \left(S \setminus \bigcup_{x \in W_I^*} U(x)\right) \setminus S_I \subset S_{II} \setminus G$. and we're done.

- If $\langle \sigma, \tau \rangle \notin F$ then $\langle \sigma, \tau \rangle \in \bigcup_{x \in W_I^*} U(x)$. i.e there's a least integer $n \geq 0$ s.t $s(n) \in W_I^*$.

$s(n-1) = f(s(n)) \notin X_I$ $\boxed{\text{Ex: } \Sigma_I^W(\Gamma_x^*) \neq \emptyset \longrightarrow \Sigma_I^W(\Gamma_{f(x)}^*) \neq \emptyset}$

But if $s(n-1) \in X_{II} \setminus W_I^*$ then there would be $y \in f^{-1}(s(n-1)) \setminus W_I^*$. (Ex.)
Lemm 4.5

$\Rightarrow$ by the defenition of τ, $s(n) = \tau(s(n-1)) \in f^{-1}(s(n-1)) \setminus W_I^*$ in contradiction.

<u>Cor 10.</u> An open or closed game is absolutely determined

<u>Cor 13</u> An Intersection and Union of open or closed game with absolutely determined game is strictly determined

(Martin 1975) - A Borel game is determined.

[If S_I is Borel set, then Γ is determined]