Properties of leafwise measures

$T \curvearrowright X$

μ is a measure

Thm 6.3 (iii)

① $\mu^T_{tx} = \mu^T_x \circ t = (R_t)_* \mu^T_x$ $\quad t \in T \quad x \in X$

(change of basepoint inside T-leaf).

② For $U \subset T$, μ is U-inv.

Ex. 6.27 (Weikun's lecture)

$\iff \mu^U_x$ is m_U. $\quad x \in X$

③ $x \longmapsto [\mu^T_x]$ is Borel,

Thm. 6.29

8.4

where $[\mu^T_x]$ is proportionality class of μ^T_x, and $\left\{ \begin{array}{l} \text{Radon measures on } U \\ \text{satisfying a growth cond'n} \end{array} \right\} \Big/ \text{proportionality eqn}$

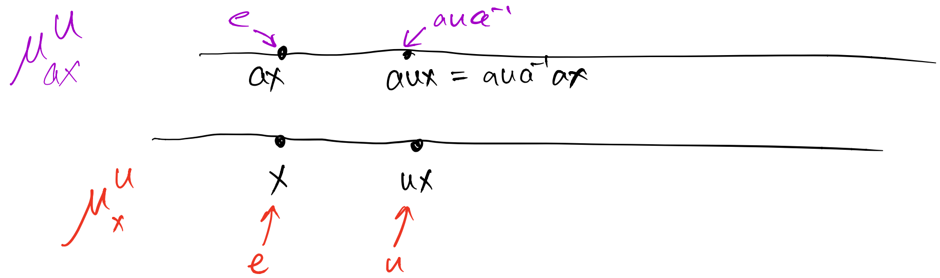
is compact metrizable for weak-$*$ convergence.

④ If $a_* \mu = \mu$ and $a U a^{-1} = U$, denote

$i_a(u) = u^a = a u a^{-1}$, then $\mu^U_{ax} = (i_a)_* \mu^U_x$

Weikun's notes lemma 3.9

7.16



(5) If $H = L\cdot M$ ($(\ell, m) \mapsto \ell m$ is a bijection, L, M are closed subgroups)

8.10 then $\Big[\underbrace{(\mu^H_x)^L_h}_{} \Big] = [\mu^L_{hx}]$ $\quad x\in X \quad h \in H$

measure on H, L acts by left-translation

leafwise measure on a coset Lh

(6) $X = G/\Gamma$, $\quad a \in G$ of Class $\mathcal{A}$

Thm 7.6

$$G^- = \{ g \in G : a^n g a^{-n} \xrightarrow[n\to\infty]{} e \}$$

$$G^+ = \{ g \in G : a^n g a^{-n} \xrightarrow[n\to -\infty]{} e \}$$

Similar, more detailed properties for $U \subset G^-$ normalized by a.

Suppose $a \curvearrowright (G/\Gamma, \mu)$ is ergodic.

Then $h_\mu(a) > 0 \iff \mu_x^{G^-}$ is infinite

$\iff D_\mu(a, G^-)(x) = \lim_{n\to\infty} \dfrac{\log \mu_x^{G^-}(a^{-n} B a^n)}{n} > 0$ ($B \subset G^-$ a ball around e)

<u>Example</u> $G = SL_3(\mathbb{R})$; $a = \mathrm{diag}(e^{-2}, e, e)$

$$H = \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} = T \ltimes U_-$$

$$T = \begin{pmatrix} 1 & & \\ & 1 & * \\ & & 1 \end{pmatrix} \cong \mathbb{R} \qquad G^- = U_- = \begin{pmatrix} 1 & * & * \\ & 1 & \\ & & 1 \end{pmatrix} \cong \mathbb{R}^2$$

both normalized by a: $ata^{-1} = t$, $a u_{xy} a^{-1} = u_{e^{-3}(x,y)}$

T centralized by a, U_- uniformly contracted by a

(7) Baby product structure μ a-inv.

Prop 8.5

$H = T \ltimes U^-$, T centralized by a, U^- contracted by a

Then $\exists X' \subset X$ of full measure s.t. for $h \in H$, $x \in X$

with x, hx both in X',

(The u_i-push is irrelevant)

$$[\mu_x^T] = [(\mu_{hx}^T)t], \text{ where } h = tu_1 = u_2 t.$$

$t \in T$ $u_i \in U_-$.

$$tu_1 = \underbrace{tu_1t^{-1}}_{u_2} t$$

In particular ($h = u$, $t = e$)

$$[\mu_x^T] = [\mu_{ux}^T] \quad \text{if } x \in X' \text{ and } ux \in X'.$$

(8) Product structure. Same as above

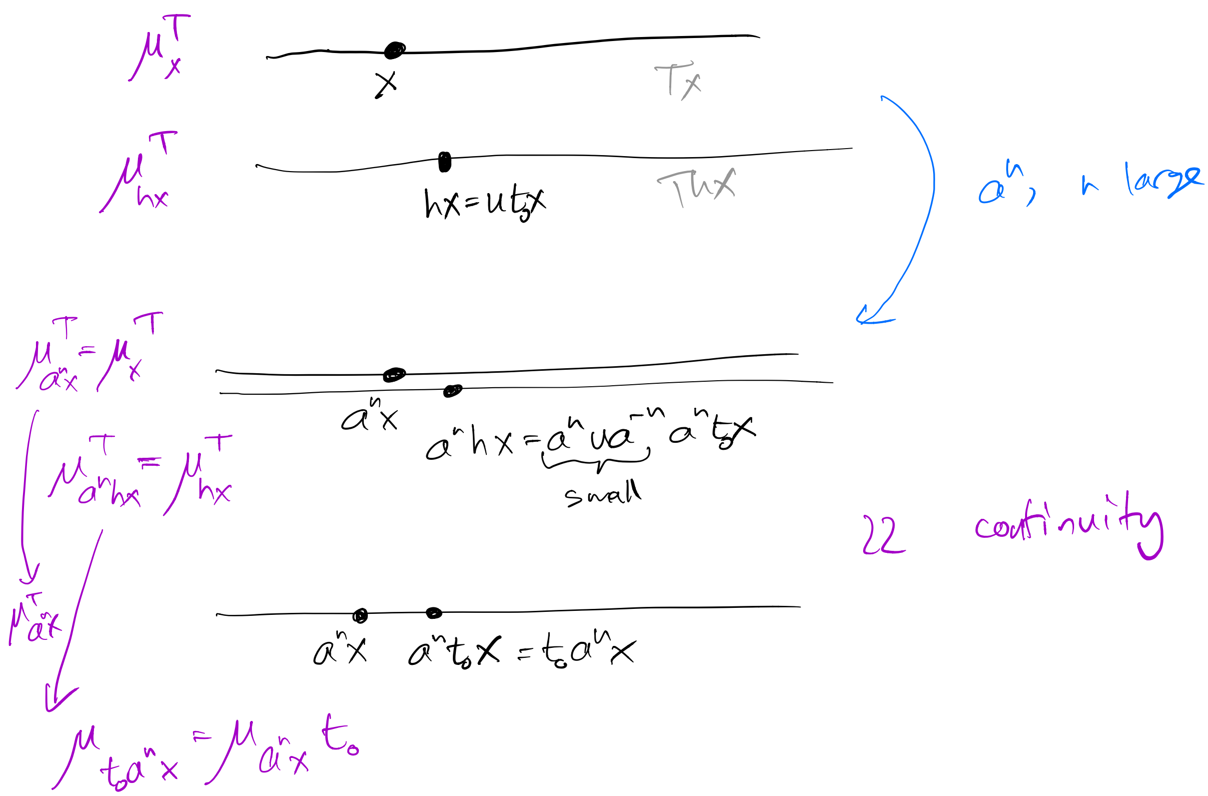
Cor. 8.8 $\exists X' \subset X$ of full measure s.t. for $x \in X'$,

$$[\mu_x^H] = [i_*(\mu_x^T \times \mu_x^{U-})], \quad i(t,u) = tu.$$

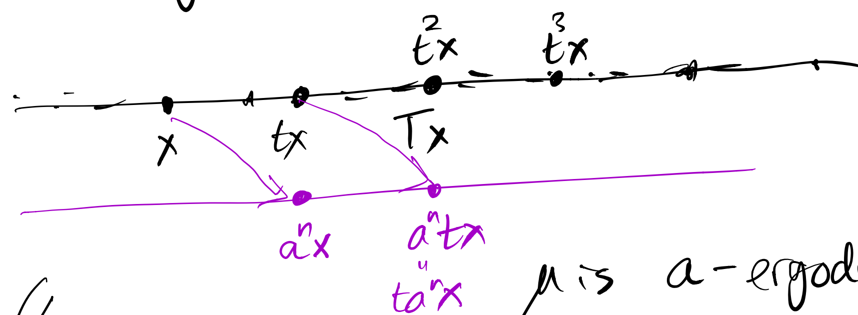
In particular, if $x \in X'$, $tx \in X'$ then

Cor. 8.13 $$[\mu_x^{U-}] = [\mu_{tx}^{U-}].$$

Picture for (7):



Katok-Spatzier argument '96



μ is a-ergodic.

a centralizes T.

μ_x^T is non-atomic.

$\Downarrow$

μ_x^T is t-inv.