KHINTCHINE DICHOTOMY FOR SELF-SIMILAR MEASURES, Chapter 2

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A Borel probability measure σ on the real line $\mathbb{R}$ is called **self-similar** if

$$\sigma = \sum_{i=1}^{m} \lambda_i \, \varphi_{i*}\sigma, \tag{1}$$

for some $m \geq 1$, some probability vector $(\lambda_1, \cdots, \lambda_m) \in \mathbb{R}^m_{>0}$, and some invertible affine maps $\varphi_1, \ldots, \varphi_m : \mathbb{R} \to \mathbb{R}$ without a common fixed point.

I.e for any $A \subset \mathbb{R}$ we have that

$$\sigma(A) = \sum_{i=1}^{m} \lambda_i \, \sigma\left(\varphi_i^{-1}(A)\right). \tag{2}$$

Write $\varphi_i(x) = \rho_i x + b_i$. We say σ is **contractive** if for all i $|\rho_i| < 1$.

0.1 Chapter 2

Theorem Lemma 2.1 (Finite moment and Hölder-regularity of σ).
There exists $\gamma > 0$ such that

1. $\displaystyle\int_{\mathbb{R}} |s|^{\gamma} \, d\sigma(s) < \infty,$
2. *There exists $C > 0$ such the for all $r > 0$ we have* $\sup_{x_0 \in \mathbb{R}} \sigma\big(B(x_0, r)\big) < C r^{\gamma}$.

Proof for the case σ is contractive.

(1) is clear because if it is contractive it is also compactly supported.

For (2) we start with the following.

Definitions:

- $\mathcal{A} := \{1, ..., m\}$.
- $A^* := \cup_{k=1}^{\infty} \mathcal{A}^k$ the collection of all finite non-empty words over $\mathcal{A}$.
- $u \in A^*$ is a word. I.e. $u = i_1 i_2 ... i_k$ for some k, where $i_j \in \{1, ...m\}$.
- if $u = i_1 ... i_k$ define
  - $\widetilde{u}$ is the word obtained from u by dropping the last letter. i.e. $\widetilde{u} := i_1 ... i_{k-1}$.

  - $\varphi_u := \varphi_{i_1} \circ ... \circ \varphi_{i_k}$
  - $\rho_u := \rho_{i_1}\rho_{i_2}...\rho_{i_k}$
  - $\lambda_u := \lambda_{i_1}\lambda_{i_2}...\lambda_{i_k}$
  - $K = supp(\sigma)$.

- For any $0 < \eta \leq 1$ define
$$\Gamma_\eta := \{u \,|\, \rho_u < \eta \leq \rho_{\widetilde{u}}\,\}$$
$$\Gamma_\eta(E) := \{u \in \Gamma_\eta \,|\, \varphi_u(K) \cap E \neq \varnothing\}.$$

Theorem Feng and Lau 2009, Lemma 2.1, well known result. *For any $E \subset \mathbb{R}$ Borel and any $0 < \eta \leq 1$ we have*

$$\sigma(E) = \sum_{u \in \Gamma_\eta(E)} \lambda_u \, \sigma\left(\varphi_u^{-1}(E)\right)$$

Theorem Feng and Lau 2009, Proposition 2.2. *Assume that $K = supp(\sigma)$ is not a singleton. Then there exist constants $C_1, C_2 > 0$ and $0 < s_1 < s_2$, and there exists $\delta > 0$ such that for any $x_0 \in K$, $0 < r \leq \delta$ we have that*

$$C_1 r^{s_1} \leq \sigma\big(B(x_0, r)\big) \leq C_2 r^{s_2}.$$

Proof of Feng and Lau Prop 2.2.

Since K is not a singleton, there exist $0 < \eta \leq 1$ and two words $\omega_1, \omega_2 \in \Gamma_\eta$ such that

$$\varphi_{\omega_1}(K) \cap \varphi_{\omega_2}(K) = \varnothing.$$

Therefore there exists $\delta > 0$ such that for any $x \in \mathbb{R}$, the ball $B(x, \delta)$ intersects at most one of $\varphi_{\omega_1}(K)$ and $\varphi_{\omega_2}(K)$.

Then for *any* $x \in \mathbb{R}$ and $0 < r < \delta$, we have, at most, only one of the followings:

1. $B(x, r) \cap \varphi_{\omega_1}(K) = \varnothing$
2. $B(x, r) \cap \varphi_{\omega_2}(K) = \varnothing$.

Define for $0 < r \leq \delta$,
$$\phi(r) = \sup_{x \in \mathbb{R}} \sigma\big(B(x, r)\big) \leq 1,$$
and assume that (1) holds, i.e. $B(x, r) \cap \varphi_{\omega_1}(K) = \varnothing$.

Then by Lemma 2.1 we have

$$\sigma\big(B(x,r)\big) = \sum_{u \in \Gamma_\eta(B(x,r))} \lambda_u \, \sigma\big(\varphi_u^{-1}(B(x,r))\big) \leq \sum_{\substack{u \in \Gamma_\eta \\ u \neq \omega_1}} \lambda_u \, \sigma\big(\varphi_u^{-1}(B(x,r))\big)$$
$$\leq \sum_{\substack{u \in \Gamma_\eta \\ u \neq \omega_1}} \lambda_u \sup_{x \in \mathbb{R}} \sigma\big(B(x, r \cdot (\min\{\rho_u \,|\, u \in \Gamma_\eta\}^{-1})\big) = \sum_{\substack{u \in \Gamma_\eta \\ u \neq \omega_1}} \lambda_u \, \phi(r/c) = \; (1 - \rho_{\omega_1})\phi(r/c),$$

where
$$c = \min\{\rho_u \,|\, u \in \Gamma_\eta\}.$$

Similarly, if the latter (2) case occurs, we have

$$\sigma\big(B(x,r)\big) \leq (1 - \rho_{\omega_2})\varphi(r/c).$$

Hence we always have

$$\sigma\big(B(x,r)\big) \le t_0\, \phi(r/c),$$

where $t_0 = \max\{1-\rho_{\omega_1},\, 1-\rho_{\omega_2}\} < 1$.

Notice t_0 is independent of x.

So it follows that

$$\phi(r) \le t_0\, \phi(r/c), \quad \forall\ 0 < r \le \delta. \tag{3}$$

In particular, there exists $n \in \mathbb{N}$ such that

$$c^n\delta < r \le c^{n-1}\delta. \tag{4}$$

Then

$$\begin{aligned} \sigma\big(B(x,r)\big) &\le \sigma\big(B(x,c^{n-1}\delta)\big) \le \phi(c^{n-1}\delta) \le\ t_0^{n-1}\phi(\delta) \\ &=\ (c^n\delta)^{\frac{\log t_0^{n-1}}{\log(c^n\delta)}}\ \phi(\delta) \le\ \phi(\delta)\, r^{\frac{\log t_0^{n-1}}{\log(c^n\delta)}}\ \le C_2 r^{s_2}, \end{aligned}$$

with $C_2 := \phi(\delta)$ and

$$s_2 := \inf_{n\in\mathbb{N}} \frac{\log t_0^{n-1}}{\log(c^n\delta)}.$$

This holds for any $x \in \mathbb{R}$.

Proof of 2 using Feng and Lau

Want to show $\exists\gamma\ \exists C\ \forall r > 0$ such that $\sup_{x_0\in\mathbb{R}} \sigma\big(B(x_0,r)\big) < Cr^{\gamma}$.

For any $r > 0$ we have

$$\sup_{x_0\in\mathbb{R}} \sigma\big(B(x_0,r)\big) \le \sup_{x_0\in K} \sigma\big(B(x_0,2r)\big).$$

Now assume $0 < r \le \delta/2$ for δ from Feng-Lau. Then by the hypothesis for any $x_0 \in K$ we have that

$$\sigma\big(B(x_0,2r)\big) \le C_2(2r)^{s_2}$$

Therefore,

$$\sup_{x_0\in\mathbb{R}} \sigma\big(B(x_0,r)\big) \le \sup_{x_0\in K} \sigma\big(B(x_0,2r)\big) \le (2^{s_2}C_2)\, r^{s_2}.$$

For $r > \gamma/2$, we use the trivial bound

$$\sigma\big(B(x_0,r)\big) \le 1.$$

Since $r > \gamma/2$, a number β which satisfy that $\beta r^{s_2} \le 1$ is bounded above. So we may increase the constant $\widetilde{C}$ so that

$$\sigma\big(B(x_0,r)\big) \le \widetilde{C} r^{s_2}, \quad \forall r > \gamma/2.$$

Hence for $\gamma = s_2, C = max\{C_2, \widetilde{C}\}$ we have

$$\sup_{x_0\in\mathbb{R}} \sigma\big(B(x_0,r)\big) \le Cr^{s_2}, \quad \forall r > 0.$$

Notation and Conventions.
$G = \mathrm{SL}_2(\mathbb{R})$, $\Lambda \subseteq G$ is a lattice, and $X = G/\Lambda$.

The injectivity radius of X at a point x is

$$\boldsymbol{inj(x)} := \sup\left\{s > 0 : \text{the map } B_s(e) \subset G \to X,\ g \mapsto gx \text{ is injective}\right\}.$$

Driving measures λ and μ.
$\mathrm{Aff}(\mathbb{R})^+$ denote the group of orientation preserving affine transformations of the real line.

Denote by

$$P = \{a(t)u(s) : t > 0,\ s \in \mathbb{R}\} \subseteq G$$

the subgroup of upper triangular matrices with positive diagonal entries, where

$$a(t) = \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix}, \qquad u(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

For every $g \in P$, we let $r_g \in \mathbb{R}_{>0}$ and $b_g \in \mathbb{R}$ be the unique numbers such that

$$g = a(r_g)^{-1}u(b_g) = \begin{pmatrix} r_g^{-1/2} & r_g^{-1/2}b_g \\ 0 & r_g^{1/2} \end{pmatrix}.$$

We identify P with $\mathrm{Aff}(\mathbb{R})^+$ by

$$g \in P \mapsto \varphi(x) = r_g x + b_g.$$

This is an anti-isomorphism between the two groups.

Fix a probability measure λ on $\mathrm{Aff}(\mathbb{R})^+$ with support $\operatorname{supp}\lambda$, and denote by μ the corresponding probability measure on P via the above anti-isomorphism. Throughout this paper, λ and μ determine each other in this way.

For $n \in \mathbb{N}$, we write

$$\lambda^{*n} = \lambda * \cdots * \lambda$$

to denote the n-fold convolution of λ with itself, and define μ^{*n} similarly.

I.e. for any $A \subseteq \mathrm{Aff}(\mathbb{R})^+$ we have

$$\lambda^{*n}(A) = \int_{(\mathrm{Aff}(\mathbb{R})^+)^n} \mathbf{1}_A(\varphi_1\varphi_2\cdots\varphi_n)\, d\lambda(\varphi_1)\cdots d\lambda(\varphi_n).$$

We assume that λ, and equivalently μ, has a finite exponential moment, i.e. there exists $\varepsilon > 0$ such that

$$\int_P \left(|r_g|^\varepsilon + |r_g^{-1}|^\varepsilon + |b_g|^\varepsilon\right)\, d\mu(g) < \infty.$$

We assume that $\operatorname{supp}\lambda$ does not have a global fixed point in $\mathbb{R}$.

Self-similar measure σ. Throughout this paper, we let σ denote a probability measure on $\mathbb{R}$ that is λ-stationary, which means

$$\sigma = \int_{\mathrm{Aff}(\mathbb{R})} \varphi_* \sigma \, d\lambda(\varphi).$$

By a theorem of Bougerol–Picard, the existence of such σ is equivalent to the condition:

$$\int_P \log r_g \, d\mu(g) < 0, \tag{2.1}$$

i.e. the random walk on $\mathbb{R}$ driven by λ is contractive on average. Moreover, provided existence, the measure σ is uniquely determined by λ, see Bougerol-Picard, Corollary 2.7.

In case λ is contractive, σ is a standard self-similar measure.

Regularity of self-similar measures.
Given an integer $n \in \mathbb{N}$, denote by $\sigma^{(n)}$ the image measure of μ^{*n} under the map

$$g \in P \mapsto b_g \in \mathbb{R}.$$

Equivalently, $\sigma^{(n)} = \lambda^{*n} * \delta_0$, where δ_0 denotes the Dirac measure at $0 \in \mathbb{R}$.

We show that the measures $\sigma^{(n)}$ have a uniformly finite positive moment, and uniform positive dimension above an exponentially small scale. For this, we first observe that $\sigma^{(n)}$ converges toward σ at exponential rate.

We denote by $\mathrm{Lip}(\mathbb{R})$ the space of bounded Lipschitz functions on $\mathbb{R}$ with the norm

$$\|f\|_{\mathrm{Lip}} = \|f\|_\infty + \sup_{s \neq t} \frac{|f(s) - f(t)|}{|s-t|}.$$

Lemma 2.2. There exists $\varepsilon > 0$ such that for all $n \geq 0$, all $f \in \mathrm{Lip}(\mathbb{R})$, we have

$$\left|\sigma^{(n)}(f) - \sigma(f)\right| \ll e^{-\varepsilon n} \|f\|_{\mathrm{Lip}}.$$

Lemma 2.3 (Moment and Hölder-regularity of $\sigma^{(n)}$). There exists $\gamma > 0$ such that

1. $\sup_{n \geq 1} \int_{\mathbb{R}} |s|^\gamma \, d\sigma^{(n)}(s) < \infty$,
2. $\forall n \geq 1, \ \forall r > e^{-n}, \ \sup_{s \in \mathbb{R}} \sigma^{(n)}\big(s + [-r, r]\big) \ll r^\gamma$.

Finally, we derive from Lemma 2.3 that $\sigma^{(n)}$ satisfies a non-concentration estimate with respect to polynomials of degree 2.

Lemma 2.4 (Regularity of $\sigma^{(n)}$ for quadratic polynomials). There exists $\gamma > 0$ such that for every $n \geq 1$, $r > e^{-n}$ and $(a, b, c) \in \mathbb{R}^3$ with $\max(|a|, |b|, |c|) \geq 1$, we have

$$\sigma^{(n)}\big\{s : |as^2 + bs + c| \leq r\big\} \ll r^\gamma.$$

Proposition 2.5 (Effective recurrence on X). There exist constants $c, c' > 0$ depending only on μ such that for every $x \in X$, $n \in \mathbb{N}$, and $\alpha > 0$, we have

$$\mu^{*n} * \delta_x\{\mathrm{inj} < \alpha\} \ll \big(\mathrm{inj}(x)^{-c} e^{-c'n} + 1\big)\alpha^c.$$

For walks on homogeneous spaces, results of this type originate from the work of Eskin–Margulis–Mozes on the quantitative Oppenheim conjecture. They are now understood in the context of semisimple random walks, and more generally expanding random walks.