RANDOM WALKS ON REDUCTIVE GROUPS

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We will be looking at two helpful lemmas from the paper by Eskin-Lindenstrauss.

References:

- $[EL]$ - Zariski Dense Random Walks on Homogeneous Spaces, Alex Eskin & Elon Lindenstrauss
- $[BQ]$ - Mesures Stationnaires et ferm 'es Invariants des Espaces Homog'enes I, Yves Benoist & Jean-Franc ois Quint
- $[BQbook]$ - Random walks on reductive groups, Y. Benoist & J.-F. Quint

1 Notations

Let G be an Ad-simple noncompact Lie Group with finite center, and let $\mathfrak{g}$ denote its Lie algebra.
Ad-simplicity means that $\mathfrak{g}$ is simple.
For $g \in G$ and $\mathbf{v} \in \mathfrak{g}$, we will use the shorthand $(g)_*\mathbf{v}$ for $Ad(g)\mathbf{v}$.
Let μ be a countably supported probablity measure on G.
Let $\mathcal{S}$ denote the support of μ, and let G_S denote the closure of the group generated by $\mathcal{S}$.
Recall that for a lattice Γ in G, a measure ν on G/Γ to be μ-stationary if

$$\mu * \nu = \nu, \quad \text{where } \mu * \nu = \int_G g\nu d\mu(g) \tag{1}$$

We say that μ has finite first moment if

$$\int_G \log \|g\| d\mu(g) < \infty \tag{2}$$

We will assume μ has a finite first moment.

We consider the two sided shift space $\mathcal{S}^{\mathbb{Z}}$. For $x \in \mathcal{S}^{\mathbb{Z}}$, we have $x = (\ldots, x_{-1}, x_0, x_1, \ldots)$. We write $x = (x^-, x^+)$ where $x^- = (\ldots, x_{-1})$ is the "past", and $x^+ = (x_0, x_1, \ldots)$

is the "future". Let $T : \mathcal{S}^{\mathbb{Z}} \to \mathcal{S}^{\mathbb{Z}}$ denote the left shift i.e. $(Tx)_n = x_{n+1}$.
We have the "skew product" map $\hat{T} : \mathcal{S}^{\mathbb{Z}} \times G \to \mathcal{S}^{Z} \times G$ given by:

$$\hat{T}(x, g) = (Tx, x_0 g), \qquad \text{where } x = (\ldots, x_0, \ldots) \tag{3}$$

For $x \in \mathcal{S}^{Z}$, and $n \in \mathbb{N}$ we write:

$$T_x^n = x_{n-1} \ldots x_0, \quad T_{T_x^n}^{-n} = (T_x^n)^{-1} \tag{4}$$

so that for $n \in \mathbb{Z}$

$$\hat{T}^n(x, g) = (T^n x, T_x^n g) \tag{5}$$

Let $P(\mathcal{S})$ denote the permutation group of $\mathcal{S}$, i.e. the set of bijections from $\mathcal{S}$ to $\mathcal{S}$.
Let

$$\mathcal{U}_1^+ = P(\mathcal{S}) \times P(\mathcal{S}) \times P(\mathcal{S}) \ldots \tag{6}$$

The way $u = (\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots) \in \mathcal{U}_1^+$ acts on $\mathcal{S}^{Z}$ is given by

$$u \cdot (\ldots, x_{-n}, \ldots, x_{-1}, x_0, x_1, \ldots) = (\ldots, x_{-n}, \ldots, x_{-1}, \sigma_0(x_0), \sigma_1(x_1), \ldots) \tag{7}$$

We then extend the action of $\mathcal{U}_1^+$ to $\mathcal{S}^{\mathbb{Z}} \times G$ by:

$$u \cdot (x, g) = (ux, g) \tag{8}$$

Let $\Omega = \mathcal{S}^{Z} \times [0, 1]$. Let T^t denote the suspension flow on on Ω, i.e. T^t is obtained as a quotient of the flow $(x, s) \to (x, t + s)$ on $\mathcal{S}^2 \times \mathbb{R}$ by the equivalence relation $(x, s + 1) \sim (Tx, s)$.
Let the measure $\tilde{\mu}$ on Ω be the product of the measure $\mu^{\mathbb{Z}}$ on $\mathcal{S}^{\mathbb{Z}}$ and the Lebesgue measure on $[0, 1]$.
We then define

$$T_x^t = T_x^n, \qquad \text{where } n \text{ is the greatest integer smaller than or equal to } t \tag{9}$$

We define $\hat{\Omega} = \Omega \times G$. We then have a skew-product flow $\hat{T}^t$ on Ω, defined by

$$\hat{T}^t(x, g) = \left(T^t x, T_x^t g\right) \tag{10}$$

We have an action on the trivial bundle $\Omega \times \mathfrak{g}$ given by

$$T^t(x, \mathbf{v}) = \left(T^t x, \left(T_x^t\right)_* \mathbf{v}\right) \tag{11}$$

We fix some norm $\| \cdot \|_0$ on g, and apply the Osceledets multiplicative ergodic theorem to the cocycle $(T^t)_*$.
The multiplicative ergodic theorem can be applied as we have a finite first moment and irreducible action (we assumed Ad-simplicity).

Let λ_i denote the i th Lyapunov exponent of this cocycle.
Let

$$\{0\} = \mathcal{V}_{\leq 0}(x) \subset \mathcal{V}_{\leq 1}(x) \subset \cdots \subset \mathcal{V}_{\leq n}(x) = \mathfrak{g} \tag{12}$$

Denote the backwards flag, and let

$$\{0\} = \mathcal{V}_{\geq n+1}(x) \subset \mathcal{V}_{\geq n}(x) \subset \cdots \subset \mathcal{V}_{\geq 1}(x) = \mathfrak{g} \tag{13}$$

denote the forward flag.
This means that for almost all $x \in \Omega$ and for $\mathbf{v} \in \mathcal{V}_{\leq i}(x)$ such that $\mathbf{v} \notin \mathcal{V}_{\leq i-1}(x)$

$$\lim_{t \to -\infty} \frac{1}{t} \log \frac{\|(T_x^t) \cdot \mathbf{v}\|_0}{\|\mathbf{v}\|_0} = \lambda_i \tag{14}$$

Let

$$\mathcal{V}_i(x) = \mathcal{V}_{\leq i}(x) \cap \mathcal{V}_{\geq i}(x) \tag{15}$$

We have $\mathbf{v} \in \mathcal{V}_j(x)$ if and only if

$$\lim_{|t| \to \infty} \frac{1}{t} \log \frac{\|(T_x^t) \cdot \mathbf{v}\|_0}{\|\mathbf{v}\|_0} = \lambda_i \tag{16}$$

The subspace $\mathcal{V}_1(x)$ is a nilpotent subalgebra of $\mathfrak{g}$, and it has corresponding unipotent subgroup $U(x)$ of G:

$$Lie(U)(x) = \mathcal{V}_1(x) \tag{17}$$

An application of the multiplicative ergodic theorem is, that upon choosing some $\mathbf{v} \in \mathfrak{g}$, then for almost every $x \in \Omega$ one have

$$\lim_{t \to -\infty} \frac{1}{t} \log \frac{\|(T_x^t) \cdot \mathbf{v}\|_0}{\|\mathbf{v}\|_0} = \lambda_1 \tag{18}$$

Using Egoroff, we can get a subset of measure close to 1, so that we have a uniform bound:

$$\left\| \left(T_x^t\right)_* \mathbf{v} \right\| \geq c(\mathbf{v}) e^{(\lambda_1/2)t} \|\mathbf{v}\| \tag{19}$$

The second lemma we'll see lets getting such a bound, non-depending on the vector $\mathbf{v}$.

2 The results

Lemma 1. *There exists an inner product $\langle \cdot, \cdot \rangle_x'$ on $\mathcal{V}_1(x)$ and a cocycle $\theta : \Omega \times \mathbb{R} \to \mathbb{R}$ such that for* $\mathrm{v} \in \mathcal{V}_1(x)$ *and* $t \in \mathbb{R}$ *,*

$$\left\langle \left(T_x^t\right)_* \mathbf{v}, \left(T_x^t\right)_* \mathbf{v} \right\rangle_{T^t x}' = e^{\theta(x,t)} \langle \mathbf{v}, \mathbf{v} \rangle_x' \tag{20}$$

Lemma 2. *For every $\alpha > 0$ and every $\eta > 0$ there exists $t_0 = t_0(\alpha, \eta) > 0$ and for every $q_1 \in \Omega$ and every $\mathbf{w} \in \mathfrak{g}$ there exists a subset $Q = Q(q_1, \mathbf{w}) \subset \mathcal{U}_1^+$ with $|Q(q_1) q_1| \geq (1-\alpha) \left|\mathcal{U}_1^+ q_1\right|$ such that for $u \in Q(q_1)$ and $t > t_0$*

$$\left\| \left(T^t_{uq_1}\right)_* \mathbf{w} \right\| \geq c(\alpha) e^{(\lambda_1/2)t} \|\mathbf{w}\| \tag{21}$$

and for $t > t_0$

$$d\left(\frac{\left(T^t_{uq_1}\right)_* \mathbf{w}}{\left\| \left(T^t_{uq_1}\right)_* \mathbf{w} \right\|}, \mathcal{V}_1\left(T^t u q_1\right) \right) \leq \eta \tag{22}$$

where $d(\cdot,\cdot)$ is the Hausdorff distance on $\mathfrak{g}$ defined by the dynamical norm $\|\cdot\|_{T^t uq_1}$. ($d_{\mathrm{H}}(X,Y) = \max\left\{\sup_{x\in X} \inf_{y \in Y} d(x,y), \sup_{y\in Y} \inf_{x\in X} d(x,y)\right\}$)

The first lemma shows that the norm is non-reliant on the angles (conformality), and the second lemma gives a bound like in (19), but non-relevantly on the vector.
We will deduce the lemmas with the results from the paper by Benoist-Quint.

3 Linear Random Walks

First we recall two results from previous semester, concerning the Furstenberg boundary map and proximal representation.

Let V be a finite dimensional real vector space. Let (V, ρ) be a real representation of G of dimension d. That is, we work in a more general context of any finite representation.
Denote by $\mathrm{Gr}_{d_0}(V)$ the Grassmannian variety of d_0-planes in V.
We endow it with a metric using the operator norm (seeing subspace as orthogonal projections).
The proximal dimension r_{G_S} is the smallest integer such that $\exists \pi \in End(V)$ of rank r_{G_S} so that $\pi = \lim_{n\to\infty} \lambda_n g_n$, $\lambda_n \in \mathbb{R}, g_n \in G_S$. If $r_{G_S} = 1, G_S$ is called proximal.

Assume G_S is strongly irreducible, and mark $r = r_{G_S}$. Recall that if an action is strongly irreducible, then for subspaces $\{V_k\}_{k=1}^l$, if $\bigcup V_k$ is stable then for some i, $V_i = V$ (or $\cup V_k = \{0\}$).
We will omit the suspension flow, and work in the standard Bernoulli two-sided shift space. Recall the marking $\nu_x = \lim_{n\to\infty} (x_1 \cdots x_n)_* \nu$ (we've shown that the limit exists).

Theorem 1. *(Furstenberg boundary map)*
a) There is a borel map $\zeta : \mathcal{S}^{\mathbb{Z}} \to G_r(V)$ such that for $\mu^{\mathbb{Z}}-$ almost any $x \in \mathcal{S}^{\mathbb{Z}}$ for every nonzero limit point $f = \lim \lambda_n x_1 \ldots x_n, im(f) = \zeta(x)$. In particular rank$(f) = r$.

b) Let ν be a μ-stationary Borel probability measure on $\mathbb{P}(V)$. Then, for $\mu^{\mathbb{Z}}$-almost any $x \in \mathcal{S}^{\mathbb{Z}}$, $\zeta(x)$ is the smallest vector subspace V_x of V such that $\nu_x(\mathbb{P}(V_x)) = 1$.

Corollary 1. *(The proximal case)*
Assume G_S is proximal.
Then the μ stationary measure ν on $\mathbb{P}(V)$ is unique.
In this case, we can view the boundary map as $\xi : \mathcal{S}^{\mathbb{Z}} \to \mathbb{P}(V)$, since the planes are of dimension 1.
Then for $\mu^{\mathbb{Z}} - a.e\ x \in \mathcal{S}^{\mathbb{Z}}, \nu_x = \delta_{\xi(x)}$(the Dirac Mass).
We have $\xi(x) = x_0\xi(Tx)$, and $\xi_*\mu^{\mathbb{Z}}$ is therefore the unique μ -stationary measure on $\mathbb{P}(V)$.

So the λ_n are used to normalize the norm each time.
Suppose for example $G = SL(n, \mathbb{R})$.
While normalizing the norm, we change the determinant, and eventually get out of $GL(n, \mathbb{R})$. So the result of the theorem is indeed meaningful.
In the next section, using the Zariski density, we'll see that we can identify the subspaces $\zeta(x)$.

4 Random Walks on Reductive Groups

We need to recall some results on Lie groups, before stating the two Lemmas we will prove.

4.1 Real semisimple Lie groups and representations

Definition 1. *We say that a Borel probability measure on G is Zariski dense if G_S has a Zariski dense image in the adjoint group* $\mathrm{Ad}(G) \subset \mathrm{GL}(\mathrm{g})$.

From now we will assume μ is a Zariski dense probability measure.

Denote the root-system decomposition as:

$$\mathfrak{g} = \mathfrak{z} \oplus (\oplus_{\alpha \in \Sigma} \mathfrak{g}^{\alpha}) \tag{23}$$

Where $\mathfrak{a}$ is a Cartan subspace of $\mathfrak{g}$, $\mathfrak{z}$ its centralizer (the Cartan subalgebra), $\Sigma = \{\alpha \in \mathfrak{a}^* \backslash \{0\} | \mathfrak{g}^{\alpha} \neq \{0\}\}$ the roots of non-zero weights,
$\mathfrak{g}^{\alpha} := \{y \in \mathfrak{g} / \forall x \in \mathfrak{a}, \mathrm{ad}\, x(y) = \alpha(x)y\}$ the roots subspaces.

Denote by $\mathfrak{u} := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}^{\alpha}$ the subalgebra correlating to positive roots.
$\mathfrak{u}$ is called the *minimal parabolic subalgebra* (or the Borel subalgebra) of Σ^+.
Any subalgebra contaning $\mathfrak{u}$ is called *parabolic*.

Recall that an element g of $\mathfrak{g}$ is unipotent, if the adjoint action $Ad_g(x)$ on $\mathfrak{gl}_n(\mathbb{R})$ is unipotent ($Ad_g(x) - 1$ is nilpotent).
At the level of the Lie group, an element of G is unipotent if the differential of the adjoint action $x \mapsto gxg^{-1}$, on $\mathfrak{g}$, is unipotent.
The radical of an algebraic group is the identity component of its maximal normal solvable subgroup. The *unipotent radical* of a subalgebra is the set of unipotent elements in its radical.

Using the exponential map, we define the minimal parabolic subgroup, and the unipotent radical, of G. A *reductive group*, for our needs, will be a group without a (non-trivial) unipotent normal subgroup.
Let P be a minimal parabolic subgroup of G, and U its unipotent radical.
Using Levi-decomposition, we then write $P = ZU$, where Z is a maximal reductive subgroup of P.
(note that we can have here a direct product, and not a semi-direct product, as P is a parabolic subgroup.)
Note that A, the Cartan subgroup, is a subgroup of Z. Denote by A^+ the (fundamental) Weyl chamber of A (w.r.t the roots order, by our choice of P).
We pick a Cartan involution on G that fixes Z. Let K be the maximal compact subgroup of G, of points fixed by the involution.

For every character χ of $\mathfrak{a}$, we set the eigenspace

$$V^\chi := \{v \in V / \forall x \in \mathfrak{a}, \rho(x)v = \chi(x)v\}. \tag{24}$$

The set $\Sigma(\rho)$ of the characters χ is called the set of weights of V. We endow it with partial ordering:

$$\chi_1 \leq \chi_2 \Longleftrightarrow \chi_2 - \chi_1 \text{ is a sum of positive roots.} \tag{25}$$

Since ρ is irreducible, we can apply the Highest Weight Theorem and get a largest weight χ.
We mark $V_0 = V^\chi$, and $d_0 = \dim V_0$.

A result in Lie groups is that the unipotent group corresponds to the highest weight space:

$$V_0 = V^U := \{v \in V | Uv = v\} \tag{26}$$

So we have a correspondence between the Highest weight space and the unipotent group.

Picking the appropriate parabolic subgroup P with the needed unipotent radical U, we'll show a correspondence between cosets of V_0 and the spaces $\mathcal{V}_1(x)$.
We have a mapping:

$$\begin{aligned} G/P &\to \mathbb{G}_{d_0}(V) \\ \eta = gP &\mapsto V_\eta := gV_0 \end{aligned} \tag{27}$$

We will choose a representation so that the associated boundary map $\xi(x)$ will have values in the flag variety G/P. Then, for general strongly irreducible representations, with boundary map $\zeta(x)$, we'll have $\zeta(x) = \xi(x)V_0$. Since we have $\mathrm{Lie}(U)(x) = \mathcal{V}_1(x)$, for the appropriate U we get $\xi(x)V_0 = \mathcal{V}_1(x)$.

Let us denote by V_0' the subspace of V which is the sum of the other weight-subspaces, so that $V = V_0 \oplus V_0'$.

4.2 Iwasawa cocycle and good norms

We construct a Borel section $s : G/P \to G/U$ of the projection $G/U \to G/P$. That is, a borel map so that $s(1) = 1$ and $s(x)P = x$.
Let $M = Z \cap K$. Using Iwasawa decomposition we have $G = KP = KZU$. It is possible to choose a section s such that for every $k \in K$,

$$s(kP) = km(k)U \text{ with } m(k) \in M. \tag{28}$$

The group Z acts by right multiplication on G/U.
We denote by $\sigma : G \times G/P \to Z$ the Borel map given by, for every $g \in G$ and $x \in G/P$

$$gs(x) = s(gx)\sigma(g, x). \tag{29}$$

Lemma 3. *The continuous map $\sigma : G \times G/P \to Z$ is a cocycle. It is called the Iwasawa cocycle.*

Proof. For g, g' in G and $\eta = kP$ in G/P with k in K, let $k' \in K$, and $x, x' \in Z$ so that

$$g'k \in k'x'U \quad \text{and} \quad gk' \in KxU \tag{30}$$

We have $\sigma(g', \eta) = x'$ and $\sigma(g, g'\eta) = x$ and

$$gg'k \in gk'x'U \subset KxUx'U = Kxx'U \tag{31}$$

hence $\sigma(gg', \eta) = xx'$ and σ satisfies the cocycle property. $\square$

We denote by $\theta : \mathcal{S}^{\mathbb{Z}} \to Z$ the map given by, for $\mu^{\mathbb{Z}}$ -a.e. $x \in \mathcal{S}^{\mathbb{Z}}$

$$\theta(x) = \sigma(x_0, \xi(Tx)) \tag{32}$$

We introduce the bounded function $\theta_{\mathbb{R}} : \mathcal{S}^{\mathbb{Z}} \to \mathbb{R}$ given, for $\mu^{\mathbb{Z}}$ -a.e. $x \in \mathcal{S}^{\mathbb{Z}}$, by

$$\theta_{\mathbb{R}}(x) = \log|\chi(\theta(x))| \tag{33}$$

Remark. Though it isn't used in the paper of Eskin-Lindenstrauss, let us point out the following observation, which is used in the original paper of Benoist-Quint.
It can be shown that the multiplicative ergodic theorem can applied to the cocycle σ. Using the Furstenberg formula for the first Lyapunov exponent and its positivity, that we've seen previous semester, we get that

$$0 < \lambda_1^\theta = \int_{\mathcal{S}^{\mathbb{Z}}} \theta_{\mathbb{R}}(x) d\mu^{\mathbb{Z}}(x) \tag{34}$$

for the first lyapunov exponent λ_1^θ of the cocycle θ.

We will choose a K-invariant norm on V, following [BQbook] Lemma 5.33:

Lemma 4. *a) There exists a good norm on V i.e. a K-invariant Euclidean norm such that, for all a in $A, \rho(a)$ is a symmetric endomorphism.*
b) For such a good norm, one has, for all g in G, η in $\mathcal{P}$ and v in V_η

$$\begin{array}{rl} i) & \chi(\kappa(g)) = \log(\|\rho(g)\|) \\ ii) & \chi(\sigma(g,\eta)) = \log \frac{\|\rho(g)v\|}{\|v\|} \end{array} \tag{35}$$

$\kappa(g)$ being the cartan projection of g to A^+.

This norm has a corresponding inner-product ([BQbook], lemma 5.18), which we'll refer to at the end.

4.3 Random walks on real Lie groups

We now wish to apply the results on Linear random walks in the current context. We will show the following:

Corollary 2. *There is a unique borel map $\xi : \mathcal{S}^{\mathbb{Z}} \to G/P$ such that, for $\mu^{\mathbb{Z}}$ -a.e. $x \in \mathcal{S}^{\mathbb{Z}}$,*

$$\xi(x) = x_0 \xi(Tx) \tag{36}$$

The image measure $\xi_* \mu^{\mathbb{Z}}$ is therefore the unique μ -stationary measure on G/P.

Theorem 2. *There are borel maps $\zeta : \mathcal{S}^{\mathbb{Z}} \to \mathrm{Gr}_{d_0}(V), x \to V_x$ and $\zeta' : \mathcal{S}^{\mathbb{Z}} \to \mathrm{Gr}_{d-d_0}(V), x \mapsto V'_x$, such that:*
0) The proximal dimension equals d_0, and $V_x = \xi(x) V_0$.
a) For $\mu^{\mathbb{Z}}$ -a.e. $x \in \mathcal{S}^{\mathbb{Z}}$, any accumulation point m of the sequence $\left(\frac{x_0 \cdots x_n}{\|x_0 \cdots x_n\|}\right)_n$, has as its image Im $(m) = V_x$ and is an isometry on $\ker(m)^\perp$.
b) For $\mu^{\mathbb{Z}}$ -a.e. $x \in \mathcal{S}^{\mathbb{Z}}$, any accumulation point m' of the sequence $\left(\frac{x_n \cdots x_0}{\|x_n \cdots x_0\|}\right)_n$, has $\ker(m') = V'_x$ and is an isometry on ker $(m')^\perp$.

c) For any hyperplane $W \subset V$, we have $\mu^{\mathbb{Z}}\left(\left\{x \in \mathcal{S}^{\mathbb{Z}} : V_x \subset W\right\}\right) = 0$.
d) For any nonzero $v \in V$, we have $\mu^{\mathbb{Z}}\left(\left\{x \in \mathcal{S}^{\mathbb{Z}} : v \in V'_x\right\}\right) = 0$.
e) For any $W \in \mathrm{Gr}_{d_0}(V)$, we have $\mu^{\mathbb{Z}}\left(\left\{x \in \mathcal{S}^{\mathbb{Z}} : W \cap V'_x \neq 0\right\}\right) = 0$.

For showing the corollary, we use Tits representation theorem ([Tits 1971]). Denote by Π, the simple root system.

Lemma 5. *For every α in Π, there exists a proximal irreducible algebraic representation (ρ_α, V_α) of G whose highest weight χ_α is a multiple of the fundamental weight ϖ_α associated to α. These weights$(\chi_\alpha)_{\alpha\in\Pi}$ form a basis of the dual space $\mathfrak{a}^*$. Moreover, the product of the maps given by*

$$G/P \to \prod_{\alpha\in\Pi} \mathbb{P}\left(V_\alpha\right) \tag{37}$$

is an embedding in this product of projective spaces.

We wish to combine this result with the corollary from section 3. We have a unique stationary measure for each $\mathbb{P}\left(V_\alpha\right)$. We need to combine this with the following ([BQbook], lemma 1.24):

Lemma 6. *Let $X, X_1, \ldots, X_k$ be compact metrizable topological spaces, all of them equipped with a continuous action of a second count- able locally compact semigroup G and, let $\pi : X \to X_1 \times \ldots \times X_k$ be a continuous injective G-equivariant map. Suppose, for any $1 \leq i \leq k$ there exists a unique μ -stationary Borel probability measure ν_i on X_i and ν_i is μ -proximal. Then, there exists a unique μ-stationary Borel probability measure on X and it is μ-proximal.*

Now, we will show the theorem.

Proof. We emphasize that following is the part where the Zariski-density is needed.
From Zariski density, the set of loxodromic elements is also Zariski dense, and in particular - there exists a loxodromic element. (Theorem 5.36 [BQbook]).
An element g of G is loxodromic if and only if, for all α in Π, the element $\rho_\alpha(g)$ is proximal in V_α. (Lemma 5.37 [BQbook])
If this happens, g has an attracting fixed point ξ_g^+ on the flag variety $\mathcal{P}$ of G. (Lemma 5.39 [BQbook]).
Going from the flag variety to G, it can be shown that the proximal dimension is d_0. Moreover, one can get $V_x = \xi(x)V_0$.

Also, from the ad-simplicity, it also can be shown that the Zariski density gives that G_S is strongly irreducible.

Now, using the theorem from section 3, we almost have (a) and (c). We need to show that each accumulation point is an isometry.

Using the good norm from the lemma, for $v \in V_x$ we have $\|\rho(g)\| = \frac{\|\rho(g)v\|}{\|v\|}$.
Now since each time we normalize (we use $\frac{x_0 \cdots x_n}{\|x_0 \cdots x_n\|}$), then we get $\|\rho(g)v\| = \|v\|$.

For showing b) and d) we'll switch to a dual representation.
Note that in the previous semester, we have proved the results from previous section just for $GL(V)$. Still, they hold for general real Lie groups.
For simplicity, we'll prove the rest of the theorem for GL(V).
b) For $g \in \mathrm{GL}(V)$ we denote by $g^* \in \mathrm{GL}(V^*)$ the adjoint operator of g. The adjoint subsemigroup $G_{\mathcal{S}}^* \subset GL(V^*)$ is also strongly irreducible and one has

$$r_{G_{\mathcal{S}}} = r_{G_{\mathcal{S}}^*} \tag{38}$$

Hence we apply (a) to the image measure μ^* of μ by the adjoint map. This tells us that, for $\mu^{\mathbb{Z}}$ -almost any x in $\mathcal{S}^{\mathbb{Z}}$ and any λ_n in $\mathbb{R}$, any nonzero limit value of $\lambda_n x_1^* \cdots x_n^*$ is a rank d_0 operator in End (V^*) whose image $\zeta^*(x) \in \mathbb{G}_r(V^*)$ does not depend on the limit value. Let $V_x \subset V$ be the vector subspace

$$V_x := (\zeta^*(x))^{\perp} \tag{39}$$

Any limit value of $\lambda_n x_n \cdots x_1$ is a rank d_0 operator in End (V) whose kernel is V_x.
d) Note that, by construction, for $\mu^{\mathbb{Z}}$ -almost any x in $\mathcal{S}^{\mathbb{Z}}$, one has

$$\zeta^*(Tx) = (x_1^*)^{-1} \zeta^*(x). \tag{40}$$

So the Borel probability measure ν^* on $\mathbb{G}_r(V^*)$, the image of $\mu^{\mathbb{Z}}$ by the map ζ^*, is μ^* -stationary. The result now follows from (c) applied to ν^*.
e) Assertion e) is deduced from d by passing to an irreducible sub- representation of the represention of G on $\wedge^{d_0} V$ generated by the line of highest weight $\wedge^{d_0} V_0$.
□

5 Proof of the main results

5.1 Proof of Lemma 1

We'll prove of Lemma 1. We show the following:

Lemma 7. *For $\mu^{\mathbb{Z}}$ -a.e. $x \in \mathcal{S}^{\mathbb{Z}}$, for every $w \in V_x$, we have*

$$\left\|x_0^{-1} w\right\| = e^{-\theta_{\mathbb{R}}(x)} \|w\| \tag{41}$$

Proof. By the definition of θ, for $\mu^{\mathbb{Z}}$ -a.e. $x \in \mathcal{S}^{\mathbb{Z}}$, we have

$$x_0 s(\xi(Tx)) = s(\xi(x)) \theta(x) \tag{42}$$

Since w is in V_x, we can write $w = s(\xi(x))v$ with $v \in V_0$. ($V_x = \xi(x) V_0$)
We note that this expression makes sense because U acts trivially on V_0.
Now we get the following equations:

$$\left\|x_0^{-1} w\right\| = \left\|x_0^{-1} s(\xi(x)) v\right\| = \left\|\theta(x)^{-1} v\right\| = e^{-\theta_{\mathbb{R}}(x)} \|v\| = e^{-\theta_{\mathbb{R}}(x)} \|w\| \tag{43}$$

Where the third step is since $\chi(\sigma(g,\eta)) = \log \frac{\|\rho(g)v\|}{\|v\|}$, and the last step is because the norm is K-invariant. $\square$

For proving Lemma 1, we identify $\xi(x)V_0 = \mathcal{V}_1(x)$. We take the associated inner-product to this norm (that we've mentioned earlier). Now applying Lemma 7 iteratively, we get Lemma 1 (and also with the suspension flow).

5.2 Proof of Lemma 2

For proving Lemma 2, we'll show the following Corollary (which is very close):

Corollary 3. *a) For any $\alpha > 0$, there are $r_0 \geqslant 1, n_0 \geqslant 1$, such that for any $v \in V \backslash \{0\}$, we have*

$$\mu^{\mathbb{Z}}\left\{x \in \mathcal{S}^{\mathbb{Z}} : \forall n \geqslant n_0, \|x_n \cdots x_0 v\| \geqslant \frac{1}{r_0}\|x_n \cdots x_0\|\,\|v\|\right\} \geqslant 1-\alpha \tag{44}$$

b) For every $\alpha > 0$ and $\eta > 0$, there exists $n_0 \geqslant 1$, such that, for every $v \in V \wedge \{0\}$, and every $W \in \mathrm{Gr}_{d_0}(V)$, we have

$$\mu^{\mathbb{Z}}\left\{x \in \mathcal{S}^{\mathbb{Z}} : \forall n \geqslant n_0, d\left(\mathbb{R}x_n \cdots x_0 v, x_n \cdots x_0 W\right) \leqslant \eta\right\} \geqslant 1-\alpha \tag{45}$$

In order to prove the corollary, we will need the following lemma in linear algebra. Denote

$$O_{d_0}(V) = \left\{\pi \in \mathrm{End}(V) : \mathrm{rank}(\pi) = d_0 \text{ and } \left.\pi\right|_{(\ker\pi)^\perp} \text{ is an isometry}\right\} \tag{46}$$

This is a compact subset of End (V).

Lemma 8. *a) For any $\varepsilon > 0$, there are $r_0 \geqslant 1, \varepsilon' > 0$ such that for any $g \in \mathrm{GL}(V)$ and $\pi \in O_{d_0}(V)$ with $\|g-\pi\| < \varepsilon'$, for any $v \in V\backslash\{0\}$ with $d(\mathbb{R}v, \ker\pi) \geqslant \varepsilon$ we have $\|gv\| \geqslant \frac{1}{r_0}\|v\|$.*
b) For any $\varepsilon > 0$ and $\eta > 0$, there is $\varepsilon' > 0$ such that, for every $g \in \mathrm{GL}(V)$ and $\pi \in O_{d_0}(V)$ with $\|g-\pi\| \leqslant \varepsilon'$ we have, for all $v \in V \wedge \{0\}$ and $W \in \mathrm{Gr}_{d_0}(V)$, if $d(\mathbb{R}v$, ker $\pi) \geqslant \varepsilon$ and $\inf_{w\in W\backslash\{0\}} d(\mathbb{R}w, \ker\pi) \geqslant \varepsilon$, then $d(\mathbb{R}gv, gW) \leqslant \eta$.

Proof:
a). Otherwise, we can find sequences π_n in $O_{d_0}(V), g_n \in \mathrm{GL}(V)$ and $v_n \in V$ with $\|v_n\| = 1$, such that $\|g_n - \pi_n\| \to 0, d\left(\mathbb{R}v_n, \ker\pi_n\right) \geqslant \varepsilon$ and $\|g_n v_n\| \to 0$. By compactness, we can assume by passing to subsequences that the π_n converge to $\pi \in O_{d_0}(V)$ and v_n converge to $v \in V, \|v\| = 1$. Our assertions imply that v is simultaneously in ker π and is of distance at least ε from ker π, a contradiction.

b). The argument is similar to the one used for proving a). $\square$

Proof of Corollary 3:
a) By Theorem 2(b), for any $\alpha > 0$, there is $\epsilon > 0$ such that for any $v \in V \backslash \{0\}$

$$\mu^{\mathbb{Z}} \left\{ x \in \mathcal{S}^{\mathbb{Z}} : d\left(\mathbb{R}v, V_a'\right) \geqslant \epsilon \right\} \geqslant 1 - \alpha/2. \tag{47}$$

On the other hand, by Theorem 2(d) for any $\epsilon' > 0$, there is $n_0 \geqslant 1$ such that

$$\mu^{\mathbb{Z}} \left\{ x \in \mathcal{S}^{\mathbb{Z}} : \forall n \geqslant n_0, d\left(\frac{x_n \cdots x_0}{\|x_n \cdots x_0\|}, O_{d_0}(V) \right) < \epsilon' \right\} \geqslant 1 - \alpha/2. \tag{48}$$

It now suffices to apply Lemma 8(a).
b) By Therorem 2(e), for any $\alpha > 0$, there is $\varepsilon > 0$ such that for $W \in \mathrm{Gr}_{d_0}(V)$,

$$\mu^{\mathbb{Z}} \left\{ x \in \mathcal{S}^{\mathbb{Z}} : \inf_{w \in W \backslash \{0\}} d\left(\mathbb{R}w, V_x'\right) \geqslant \varepsilon \right\} \geqslant 1 - \alpha/2. \tag{49}$$

It suffices to apply, as above, Theorem 2(d) 5.2, and Lemma 8(b).$\square$

What's left is to deduce Lemma 2 from Corollary 3.
Note that as we've seen in (19), picking $v = (1, 1, ..., 1)$, for each $\alpha > 0$ there is a subset of measure $> 1 - \alpha$ so that $\|x_n \cdots x_0\|$ grows according to the first lyapunov exponent. Taking $\lambda_1/2$ we get the uniform bound in the first estimate

$$\left\| \left(T_{uq_1}^t \right), \mathbf{w} \right\| \geq c(\alpha) e^{(\lambda_1/2)t} \|\mathbf{w}\| \tag{50}$$

This time, the bound holds for every vector (that is c is non-reliant on the vector).
Note that by the result from previous semester, the first Lyapunov exponent is positive, so this is indeed meaningful.
In the second estimate, set $W = \mathcal{V}_1(uq_1) = \mathcal{V}_1(q_1)$.
What's left is to convert the subset of points $x \in \mathcal{S}^Z$, of measure $1-\alpha$, in a subset of $\mathcal{U}_1^+$ upon choosing some $x \in \mathcal{S}^Z$. We do so by choosing all permutations $Q(x)$ that keep $Q(x)\,x$ in the previous subset of $\mathcal{S}^{\mathbb{Z}}$.
Lastly, we see that changing to the suspension flow, the results hold the same.