Conditional Measure

March 2019

We first give a reminder of some basic concepts needed to state and prove the main theorem.

Conditional Probability

Let $(X, \mathcal{B}, \mu)$ be a probability space and $\mathcal{A} \subset \mathcal{B}$ be a sub σ-algebra. The conditional expectation is a continuous linear function:

$$E[*|\mathcal{A}] : L^1(X, \mathcal{B}, \mu) \to L^1(X, \mathcal{A}, \mu)$$

such that for every $\mathcal{A}$ measurable set A:

$$\int_A E[f|\mathcal{A}] d\mu = \int_A f d\mu$$

Countably generated σ algebras

A σ-algebra $\mathcal{A}$ is called countably generated if it is generated by a countable set, that is, $\mathcal{A} = \sigma(\{A_i\}_{i=1}^{\infty})$. For every $x \in X$ we define the $\mathcal{A}$-atom of x to be:

$$[x]_{\mathcal{A}} = \bigcap_{x \in A_i \subset \mathcal{A}} A_i = \bigcap_{x \in A \subset \mathcal{A}} A$$

Notice that this definition does not depend on choice of generating set. Atoms can be defined to any σ-algebra but being countably generated promises measurability of atoms.

Example 1:
Every Borel σ-algebra in a separable metric space is countably generated by rational size balls around dense countable set.

Example 2:
Let $(X, \mathcal{B}, \mu, T)$ be an invertible Ergodic probability space such that points have zero measure. Then the sub σ-algebra $\mathcal{E}$ of T invariant sets is **not** countably generated. To see this, assume by contradiction it is generated by a countable collection $\{E_i\}$. For every i, $E_i \in \mathcal{E}$ so it is T invaraint. Since T is Ergodic $\mu(E_i) \in \{0, 1\}$. We take intersection of all E_i if $\mu(E_i) = 1$ and $X \setminus E_i$ if $\mu(E_i) = 0$

to get a full measure set which is an atom $[x]_{\mathcal{E}}$. Notice that the orbit O_x of x is T invariant hence $O_x \in \mathcal{E}$ and it contains x so $[x]_{\mathcal{E}} \subset O_x$. If $y \in O_x$ then since E_i are T invariant, $y \in E_i$ iff $x \in E_i$ thus $y \in [x]_{\mathcal{E}}$ and $O_x = [x]_{\mathcal{E}}$. But an orbit is at most countable so $\mu([x]_{\mathcal{E}}) \leq \sum_0^\infty \mu(x) = 0$ which stands in contradiction to $[x]_{\mathcal{E}}$ having full measure.

Riesz representation theorem

If (X, d) is a compact Hausdorff space and φ is a functional in the dual space to $C(X)$ which is positive and with $||\varphi|| = 1$ then exists a uniqe Borel probability measure μ on X such that for every $f \in C(X)$:

$$\varphi(f) = \int_X f d\mu$$

Conditional measure

We can now state and prove the main result:

Theorem 1 (Conditional Measure)**.** *Let $(X, \mathcal{B}, \mu)$ be a probability space with $(X, \mathcal{B})$ being a standard Borel space (Locally compact, second countable, metric) and let $\mathcal{A} \subset \mathcal{B}$ a sub-σ-algebra. Then there exists a subset $X' \subset X$ of full measure and Borel probability measures $\mu_x^{\mathcal{A}}$ for every $x \in X'$ such that:*

1. *For every $f \in L^1(X, \mathcal{B}, \mu)$ we have $E(f|\mathcal{A})(x) = \int f(y) d\mu_x^{\mathcal{A}}(y)$ for almost every x. In particular, the right-hand side is $\mathcal{A}$-measurable as a function of x.*

2. *If $\mathcal{A}$ and $\mathcal{A}'$ are equivalent σ-algebras modulo μ, then we have $\mu_x^{\mathcal{A}} = \mu_x^{\mathcal{A}'}$ for almost every x.*

3. *If $\mathcal{A}$ is countably generated, then $\mu_x^{\mathcal{A}}([x]_{\mathcal{A}}) = 1$ for every $x \in X'$ and for $x, y \in X'$ we have that $[x]_{\mathcal{A}} = [y]_{\mathcal{A}}$ implies $\mu_x^{\mathcal{A}} = \mu_y^{\mathcal{A}}$.*

4. *The set X' and the map $\tau(x) = \mu_x^{\mathcal{A}'}$ are $\mathcal{A}$-measurable on X'.*

Moreover, the family of conditional measures $\mu_x^{\mathcal{A}}$ is almost everywhere uniquely determined by its relationship to the conditional expectation described above. Notice that the full measure set X' is not uniquely determined.

Proof: Since X is a Borel space we may take its compactization and assume it is compact. The space $C(X)$ of continuous functions on X is seprable. We choose a countable dense subset $\{f_0 = 1, f_1, f_2 ...\}$ which gives rise to a $\mathbb{Q}$-vector space.
Set $g_0 = f_0$ and $g_i = E[f_i|\mathcal{A}]$. For every i, j, k and every $\alpha, \beta \in \mathbb{Q}$ we can find a null set $N_{i,j,k,\alpha,\beta}$ such that:

- If $\alpha \leq f_i \leq \beta$ for all $x \in X$ then $\alpha \leq g_i(x) \leq \beta$ for all x outside the null set.

- If $\alpha f_i + \beta f_j = f_k$ then $\alpha g_i + \beta g_j = g_k$

This is true since otherwise we could find a $\mathcal{A}$-measurable set that g_i and f_i do not almost agree on. By taking countable union we get a full measure set X' that the two conditions hold on for every i, j, k, α, β. So for all $x \in X'$ we have a continuous positive linear functional $\mathcal{L}_x(f_i) = g_i(x)$ from $C(X)$ to $\mathbb{R}$ of norm $||\mathcal{L}_x|| \leq 1$. Since $\mathcal{L}_x(f_0) = 1$ we get $||\mathcal{L}_x|| = 1$ and so by Riesz representation theorem exists a measure $\mu_x^{\mathcal{A}}$ characterized by $\mu_x^{\mathcal{A}}(f) = E[f|\mathcal{A}](x)$. In particular the function $x \to \mu_x^{\mathcal{A}}(f)$ is measurable for every $f \in C(X)$. The result can then be extended to all integrable functions using standard monotone convergence arguments, which concludes (1). This result immediately implies (4) since a countable base to the weak* topology is given by preimages of continuous functions integrals.

Let $\mathcal{A}_1, \mathcal{A}_2$ be two equivalent σ-algebras mod μ and let $f \in C(X)$. We take common refinement $\mathcal{A}_0$ such that both $\mathcal{A}_i \subset \mathcal{A}_0$. Notice that $\mathcal{A}_i$ and $\mathcal{A}_0$ are equivalent as well. $E[f|\mathcal{A}_i]$ is $\mathcal{A}_0$-measurable and for any $f \in C(X)$ and $A_0 \in \mathcal{A}_0$ the characterizing property holds:

$$\int_{A_0} E[f|\mathcal{A}_i] = \int_{A_i} E[f|\mathcal{A}_i] = \int_{A_i} f = \int_{A_0} f$$

Where $A_i \in \mathcal{A}_i$ such that $\mu(A_0 \triangle A_i) = 0$. So $E[f|\mathcal{A}_1], E[f|\mathcal{A}_2]$ are two versions of $E[f|\mathcal{A}_0]$ and thus are equivalent a.e. We may now repeat the previous technique used to prove existence, leaving out countable null sets, and construct two measure $\mu_x^{\mathcal{A}_1}, \mu_x^{\mathcal{A}_2}$ which are equal almost everywhere.

Suppose $\mathcal{A} = \sigma(\{A_1, A_2, ...\})$ is countably generated. For every i since $\mathbb{1}_{A_i}$ is $\mathcal{A}$-measurable we get $\mathbb{1}_{A_i}(x) = E[\mathbb{1}_{A_i}|\mathcal{A}] = \mu_x^{\mathcal{A}}(A_i)$, thus for almost every x, if $x \in A_i$ we get $\mu_x^{\mathcal{A}}(A_i) = 1$. We may take a countable intersection of full measure set to make sure this holds for all $i$togather and then $\mu_x^{\mathcal{A}}([x]_{\mathcal{A}}) = \mu_x^{\mathcal{A}}(\bigcap_{x \in A_i} A_i) = 1$ for almost every $x \in X$. Let $x, y \in X'$ such that $[x]_{\mathcal{A}} = [y]_{\mathcal{A}}$. For every $f \in C(X)$ the function $\tau(x) = \mu_x^{\mathcal{A}}(f)$ is measurable. then for every f, $x \in \tau^{-1}(\mu_x^{\mathcal{A}}(f))$ hence $[x]_{\mathcal{A}} \subset \tau^{-1}(\mu_x^{\mathcal{A}}(f))$ and since $y \in [x]_{\mathcal{A}}$ we get that $\mu_x^{\mathcal{A}}(f) = \mu_y^{\mathcal{A}}(f)$. This holds for every $f \in C(X)$ which implies $\mu_x^{\mathcal{A}} = \mu_y^{\mathcal{A}}$.

Countably equivalent σ-algebras

We say two countably generated σ-algebras $\mathcal{A}_1, \mathcal{A}_2$ on a space X are countably equivalent if every atom of $\mathcal{A}_1$ can be covered by at most countably many atoms of $\mathcal{A}_2$ and vice versa. When this is the case we get the following lemma:

Lemma: Let $\mathcal{A}_1, \mathcal{A}_2$ be two countably equivalent σ-algebras. Then for a.e. $x \in X$:

$$\mu_x^{\mathcal{A}_1 \vee \mathcal{A}_2} = \frac{\mu_x^{\mathcal{A}_1}|_{[x]_{\mathcal{A}_1 \vee \mathcal{A}_2}}}{\mu_x^{\mathcal{A}_1}([x]_{\mathcal{A}_1 \vee \mathcal{A}_2})} = \frac{\mu_x^{\mathcal{A}_2}|_{[x]_{\mathcal{A}_1 \vee \mathcal{A}_2}}}{\mu_x^{\mathcal{A}_2}([x]_{\mathcal{A}_1 \vee \mathcal{A}_2})}$$

Proof: Since $\mathcal{A}_1$ is countably equivalent to $\mathcal{A}_2$ if and only if $\mathcal{A}_1$ is countably equivalent to $\mathcal{A}_1 \vee \mathcal{A}_2$ it is enough to show that for two countably equivalent $\mathcal{A} \subset \mathcal{A}'$:

$$\mu_x^{\mathcal{A}'} = \frac{\mu_x^{\mathcal{A}}|_{[x]_{\mathcal{A}'}}}{\mu_x^{\mathcal{A}}([x]_{\mathcal{A}'})}$$

We first show that the RHS is well defined and $\mathcal{A}'$ measurable and then that it satisfies the characterizing property of the conditional measure.

Since $\mathcal{A}'$ is countably generated we can take $\mathcal{A}'_n = \sigma(\{A_1, A_2, ..., A_n\}) \subset \mathcal{A}'$. Consider $E[\mathbb{1}_{[x]_{\mathcal{A}'_n}}|\mathcal{A}] = \mu_x^{\mathcal{A}}([x]_{\mathcal{A}'_n})$. Since $[x]_{\mathcal{A}'_n} \searrow [x]_{\mathcal{A}'} = \bigcap_n [x]_{\mathcal{A}'_n}$ we get $\lim_{n\to\infty} \mu_x^{\mathcal{A}}([x]_{\mathcal{A}'_n}) = \mu_x^{\mathcal{A}}([x]_{\mathcal{A}'})$. Since $\mathcal{A} \subset \mathcal{A}'$ the function is $\mathcal{A}'$ measurable.

To show that the RHS is well defined we have to varify that the denominator is non-zero almost everywhere, that is $\mu(Y) = 0$ for $Y = \{x : \mu_x^{\mathcal{A}}([x]_{\mathcal{A}'}) = 0\}$. We first use total expectation $\mu(Y) = \int \mu_x^{\mathcal{A}}(Y) d\mu(x)$. Since $\mu_x^{\mathcal{A}}([x]_{\mathcal{A}}) = 1$ and from countable equivalence:

$$\mu_x^{\mathcal{A}}(Y) = \mu_x^{\mathcal{A}}(Y \cap [x]_{\mathcal{A}}) = \mu_x^{\mathcal{A}}(\bigcup_{i\in I} [x_i]_{\mathcal{A}'} \cap Y) = \sum_{i \in I} \mu_x^{\mathcal{A}}([x_i]_{\mathcal{A}'} \cap Y)$$

So it is suffice to show that each term in the RHS summation is zero. If $[x_i]_{\mathcal{A}'} \cap Y = \emptyset$ this is obvious. Else exists $y \in [x_i]_{\mathcal{A}'} \cap Y$. So $[x_i]_{\mathcal{A}'} = [y]_{\mathcal{A}'}$ and since $y \in Y$ we get $0 = \mu_y^{\mathcal{A}}([y]_{\mathcal{A}'}) = \mu_y^{\mathcal{A}}([x_i]_{\mathcal{A}'})$. Since $y \in [x]_{\mathcal{A}}$ as well we get $\mu_x^{\mathcal{A}} = \mu_y^{\mathcal{A}}$ thus $\mu_x^{\mathcal{A}}([x_i]_{\mathcal{A}'}) = 0$.

Finaly, we show that the RHS satisfies the defining property of $\mu_x^{\mathcal{A}'}$, that is $E[f|\mathcal{A}'](x) = \frac{\mu_x^{\mathcal{A}}|_{[x]_{\mathcal{A}'}}}{\mu_x^{\mathcal{A}}([x]_{\mathcal{A}'})}(f)$. Let $A \in \mathcal{A}'$ and f integrable:

$$\int_A \int \frac{f(y)}{\mu_x^{\mathcal{A}}([x]_{\mathcal{A}'})} d\mu_x^{\mathcal{A}}|_{[x]_{\mathcal{A}'}}(y) d\mu(x) = \int_A \frac{1}{\mu_x^{\mathcal{A}}([x]_{\mathcal{A}'})} \int f(y) d\mu_x^{\mathcal{A}}|_{[x]_{\mathcal{A}'}}(y) d\mu(x) =$$

$$\int_A \frac{1}{\mu_x^{\mathcal{A}}([x]_{\mathcal{A}'})} E[\mathbb{1}_{[x]_{\mathcal{A}'}} f|\mathcal{A}] d\mu(x) = \int_A \frac{1}{\mu_x^{\mathcal{A}}([x]_{\mathcal{A}'})} E[\mathbb{1}_{[x]_{\mathcal{A}'}} E[f|\mathcal{A}']|\mathcal{A}] d\mu(x) =$$

$$\int_A \frac{\mu_x^{\mathcal{A}}([x]_{\mathcal{A}'})}{\mu_x^{\mathcal{A}}([x]_{\mathcal{A}'})} E[f|\mathcal{A}'] d\mu(x) = \int_A f d\mu(x)$$