RATNER'S THEOREMS

1. Preliminaries, Formulations and Examples

Orbit Closure Theorem.

Theorem. *Let $X = G/\Gamma$, where G is a connected Lie group and Γ a lattice in G. Let U be a connected subgroup of G generated by one-parameter unipotent subgroups. Then for any $x \in X$ there exist a subgroup H containing U such that $\overline{Ux} = Hx$ and Hx carries a finite H-invariant measure.*

The space $X = G/\Gamma$.

- We can think of G as a (closed) group of matrices.
- Γ being a lattice means that it is discrete and X admits a G invariant probability measure.

Assume G acts transitively on a space X. Fix $x_0 \in X$, and let $\Gamma = \{g \in G : \, gx_0 = x_0\}$. Then

$$g\Gamma \mapsto gx_0$$

defines the orbit map, which gives a bijection between the space and the coset space, and so X is a manifold as the quotient of two Lie groups.

Example. Let $X = \mathcal{L}_n = \{\text{covolume}\, 1\, \text{lattices}\}$. A lattice is $\Lambda = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_n$ with v_i linearly independent and $\det(v_1, \ldots, v_n) = \pm 1$. $G = \mathrm{SL}_n(\mathbb{R})$ acts transitively on $\mathcal{L}_n$ by multiplication on the vectors $\{v_1, \ldots, v_n\}$, and

$$\Gamma = \mathrm{SL}_n(\mathbb{Z}) = \{g \in \mathrm{SL}_n(\mathbb{R}) : \, g\mathbb{Z}^n = \mathbb{Z}^n\}$$

and so we can identify the space of latticed $\mathcal{L}_n$ with the quotient $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$.

Remark. The conclusion Hx is closed (for some closed H) mean that $H/H_x \to X$ where $H_x = \{h \in H : \, hx = x\}$ then $hH_x \mapsto hx$ is a homeomorphism and H_x is a lattice in H.

The group U. Unipotent matrix: all eigenvalues are equal to 1.

Example. In $\mathrm{SL}_2(\mathbb{R})$, consider the groups

$$A = \left\{ \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} : \, t \in \mathbb{R} \right\}, \quad U = \left\{ \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} : \, t \in \mathbb{R} \right\}.$$

U is unipotent, A is not generated by unipotents.

In $G=\mathrm{SL}_3(\mathbb{R})$ we could take for example the following groups U:

$$\left\{\begin{pmatrix}1 & t & \\ & 1 & \\ & & 1\end{pmatrix}\right\},\left\{\begin{pmatrix}1 & t & t^2/2\\ & 1 & t\\ & & 1\end{pmatrix}\right\},\left\{\begin{pmatrix}1 & & s\\ & 1 & t\\ & & 1\end{pmatrix}\right\},\left\{\begin{pmatrix}A & \\ & 1\end{pmatrix} : A\in\mathrm{SL}_2(\mathbb{R})\right\}$$

Corollary of the orbit classification theorem. Let $X=\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ and let U as above. Then any orbit is either closed or dense.

Proof. This is because the only subgroups which contain U are G which would give the dense case and U which would give the closed case, and the upper triangular matrices which do not admit a lattice so by Ratner's Theorem is not possible. □

Example. Here for groups which are not matrices: $G=\mathbb{R}^n$ and $\Gamma=\mathbb{Z}^n$, so $X=\mathbb{T}^n$. Let $U=\{tx : t\in\mathbb{R}\}$ for some fixed $x\in\mathbb{R}^n$. All U-orbits are tori of dimension $\dim_{\mathbb{Q}}\left(Span_{\mathbb{Q}}\{v_i\}\right)$.

Measure Classification Theorem.

Definition. A Borel probability measure μ on X is homogeneous if μ is H- invariant and its support is a closed orbit of a point under H, that is $\mathrm{supp}\mu=Hx$ for some closed group $H\subset G$ and some $x\in X$.

Theorem. *Let $X=G/\Gamma$, and U as above. Then for any U-inv. ergodic measure is homogeneous.*

Genericity Theorem.

Definition. Let $U=\{u_s : s\in\mathbb{R}\}$ a one parameter group, μ a measure on X, $x_0\in X$. We say x_0 is generic for μ if for any $f\in C_C(X)$ we have

$$\frac{1}{T}\int_0^T f(u_s x_0)\,ds\to\int_X f\,d\mu$$

Theorem. *Let $X=G/\Gamma$ where G is a connected Lie group, Γ a lattice, and U is a one-parameter connected unipotent subgroup. Then any $x\in X$ is a generic point for some homogeneous measure on X.*

Overall structure of argument:

(1) Ratner proved the measure classification theorem (long hard part).
(2) Genericity deduced by measure classification.
(3) Orbit closure deduced by genericity.

A couple of Lemmas - getting used to the definitions.

Definition. Let μ be an ergodic Borel probability measure on $X = G/\Gamma$. The stabilizer of μ is

$$\operatorname{stab}(\mu) := \{g \in G : \text{the action of } g \text{ on } X \text{ preserves } \mu\}.$$

Lemma. $\operatorname{stab}(\mu)$ *is a closed subgroup of G and therefore a Lie group.*

Proof. Let g_n be a sequence in G such that all g_n preserve μ and assume $g_n \to g$. We have to show that g preserves μ. By the Riesz representation theorem, g preserves μ if for for any $f \in C_C(X)$ we have

$$\int_X f(gx)\, d\mu = \int_X f(x)\, d\mu.$$

For every point $x \in X$, since $g_n x \to gx$ and f is continuous, we have $f(g_n x) \to f(gx)$. By the Lebesgue dominated convergence theorem

$$\int_X f(g_n x)\, d\mu \to \int_X f(gx)\, d\mu.$$

But for all $n \in \mathbb{N}$, since g_n preserves μ,

$$\int_X f(g_n x)\, d\mu = \int_X f(x)\, d\mu,$$

so the sequence is a constant sequence and

$$\int_X f(gx)\, d\mu = \int_X f(x)\, d\mu.$$

So $\operatorname{stab}(\mu)$ is a closed subgroup of a Lie group. By the closed subgroup theorem it is also a Lie group. $\square$

Recall that μ is homogeneous if it is H- invariant and supported on a closed orbit of H, for some closed subgroup H.

Lemma. *μ is homogeneous if and only if it is supported on one orbit of* $\operatorname{stab}(\mu)$.

Proof. Since $\operatorname{stab}(\mu)$ is closed, if we assume that μ is supported on one orbit of $\operatorname{stab}(\mu)$, then by definition $\mu$is homogeneous with $H = \operatorname{stab}(\mu)$.
Assume now that μ is homogeneous, so it is H-invariant for some closed H, and therefore $H \subset \operatorname{stab}(\mu)$. In addition $S = \operatorname{supp}\mu = Hx$ for some $x \in X$, and so H acts transitively on S. Since H acts transitively on S so does $\operatorname{stab}(\mu)$, and since $\operatorname{stab}(\mu)$ preserves S by definition, this means that S is one orbit under $\operatorname{stab}(\mu)$. $\square$

2. The case $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$

From here on $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and $X = G/\Gamma$ is the space of lattices $\mathcal{L}_2$. Let

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

The group $U = \{u_t : t \in \mathbb{R}\}$ is unipotent and acts on $\mathcal{L}_2$. We wish to classify the U-inv. measures on $\mathcal{L}_2$.

Lemma. *For $L \in \mathcal{L}_2$, the U-orbit of L is closed is and only if L contains a horizontal vector.*

Proof. Note that the action of U preserves the y-components and fixes horizontal vectors. Assume $v \in L$ is horizontal, then $v \in u_t L$ for all t and so $v \in \overline{UL}$. Let a matrix for L be

$$\begin{pmatrix} a & c \\ 0 & d \end{pmatrix}$$

containing the horizontal vector $(a, 0)$. All vectors in $\overline{UL}$ have y-components that are multiples of d and the horizontal vectors in $\overline{UL}$ are the same as those in L. We will show that the U-orbit is closed: Let $L' \in \overline{UL}$, then it can be represented by the matrix

$$\begin{pmatrix} a & c' \\ 0 & d' \end{pmatrix}$$

and since the covolume of the lattice is $|ad|$ it follows that $d' = \pm d$ and without loss of generality we can assume $d' = d$, and since $d \neq 0$ then there is some t so that $c' = c + td$, and so $L' = u_t L \in UL$.
Assume now that L does not contain a horizontal vector, then it is generated by two vectors whose y-coordinates are incommensurable, and in particular L contains vectors with y-coordinates arbitrarily close to 0. Let $v_n \in L$ be primitive vectors with

$$0 < (v_n)_y < \frac{1}{n}.$$

Pick t so that $u_t v_n = \left(1, (v_n)_y\right)$ and find a second vector generating $u_t L$ so that it's x-coordinate is in $[0, 1]$. Its y-coordinate should be close to 1 because the lattice is o covolume 1. Let w_n be the sequence of such second vectors, then they are contained in compact set and therefore have a converging subsequence, and the sequence of pairs $(u_t v_{n_k}, w_{n_k})$ converges to a pair of generators for a lattice with the first coordinate horizontal, therefore $UL \neq \overline{UL}$. □

Any closed U-orbit supports a U-invariant probability measure, and these measures are ergodic. Denote by ν the Haar measure on $\mathcal{L}_2$.

Theorem. *Let μ be an ergodic U-invariant probability measure on $\mathcal{L}_2$. Then either μ is supported on a closed orbit, or $\mu = \nu$.*

Proof. Denote by $\mathcal{L}_2'$ the U-invariant set of lattices which contain a horizontal vector, then either $\mu(\mathcal{L}_2') = 0$ or $\mu(\mathcal{L}_2') = 1$.

Assume first $\mu(\mathcal{L}_2') = 1$, and parameterize orbits in $\mathcal{L}_2'$ by a, which is the length of the primitive horizontal vector. Let $S_n \subset \mathcal{L}_2$ be the set of lattices with a-coordinate in $[n, n+1)$. Since S_n is U-invariant then $\mu(S_n)$ is 1 for some unique n. Further partitioning of this interval will determine a unique value of a, such that μ is supported on that particular orbit.

Assume now $\mu(\mathcal{L}_2') = 0$. Let $f \in C_C(\mathcal{L}_2)$ and let $\varepsilon > 0$. We want to show that

$$\left| \int_{\mathcal{L}_2} f d\mu - \int_{\mathcal{L}_2} f d\nu \right| < \varepsilon.$$

Let

$$a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad v_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

A direct computation shows

$$a_t u_s a_t^{-1} = u_{e^{2t}s}, \; a_t v_s a_t^{-1} = v_{e^{-2t}s},$$

Consider the subgroups U as before, $V = \{v_t : t \in \mathbb{R}\}$ and $A = \{a_t : t \in \mathbb{R}\}$, so conjugation by a_t with $t > 0$ contracts V and expands U.

By uniform continuity of f, there exist neighborhoods of the identity $W_0' \subset A$ and $W_-' \subset V$ such that for any $a \in W_0'$ and $v \in W_-'$ and any $L \in \mathcal{L}_2$

$$|f(vaL) - f(L)| < \frac{\varepsilon}{3} \quad (\forall L \in \mathcal{L}_2). \tag{2.1}$$

Before we continue we need the following definition of flowbox:

Definition. Let $W_+ \subset U$, $W_- \subset V$ and $W_0 \subset A$ be (images of) open intervals containing 0, that is the identity matrix. A subset of G of the form $W_- W_0 W_+ g$ for some $g \in G$ is called a flowbox, and it is an open set containing g which is isometric to $W_- W_0 W_+$.

Definition. Let $\mathcal{L}_2(\varepsilon) \subset \mathcal{L}_2$ denote the set of lattices with shortest non-zero vector of length at least ε.

Theorem (Mahler compactness)**.** *For any $\varepsilon > 0$ the set $\mathcal{L}_2(\varepsilon)$ is compact.*

Let W_-, W_0, W_+ be small enough so that for all $g \in G$ with $\pi(g) \in \mathcal{L}_2(\varepsilon)$, where $\pi : G \to \mathcal{L}_2$ is the natural projection, the restriction of π to the flowbox $W_- W_0 W_+ g$ is injective (such flowbox exists because of compactness of $\mathcal{L}_2(\varepsilon)$). We can also assume $W_0 \subset W_0'$ and $W_- \subset W_-'$. Denote $\delta = \nu(W_- W_0 W_+)$.

Lemma. *Let $\phi_t : X \to X$ be a flow preserving an ergodic probability measure μ, and let $f \in L^1(\mu)$. For any $\varepsilon > 0$ and $\delta > 0$ there exists $T_0 > 0$ and a set E with $\mu(E) < \varepsilon$ such that for every $x \notin E$ and any $T > T_0$ we have*

$$\left| \frac{1}{T} \int_0^T f(\phi_t(x))\, dt - \int_X f d\mu \right| < \delta.$$

Proof. Let E_n be the set of $x \in X$ such that for some $T > n$

$$\left| \frac{1}{T} \int_0^T f(\phi_t(x))\, dt - \int_X f d\mu \right| \geq \delta$$

then by Birkhoff ergodic theorem

$$\frac{1}{T} \int_0^T f(\phi_t(x))\, dt \to \int_X f d\mu$$

almost everywhere and so $\mu(\cap E_n) = 0$. Therefore for some n we have $\mu(E_n) < \varepsilon$, and we take $T_0 = n$ and $E = E_n$. $\square$

Since the flow by U is ergodic for the measure ν, the lemma implies that there exists a set $E \subset \mathcal{L}_2$ with $\nu(E) < \delta$ and $T_1 > 0$ so that or any interval I containing the origin and of length $|I| \geq T_1$ and any lattice $L' \notin E$,

$$\left| \frac{1}{|I|} \int_I f(u_t L')\, dt - \int_{\mathcal{L}_2} f dv \right| < \frac{\varepsilon}{3} \quad (\forall L' \notin E)\,. \tag{2.2}$$

On the other hand, applying Birkhoff's ergodic theorem to the measure μ, we have for μ-a.e $L \in \mathcal{L}_2$ and for some $T_2 > 0$ that for all intervals I containing the origin and of length $|I| \geq T_2$

$$\left| \frac{1}{|I|} \int_I f(u_t L)\, dt - \int_{\mathcal{L}_2} f d\mu \right| < \frac{\varepsilon}{3} \quad (\text{for}\, \mu\, \text{almost every}\, L \in \mathcal{L}_2)\,. \tag{2.3}$$

Lemma. *There exists an absolute constant $\varepsilon > 0$ such that $L \in \mathcal{L}_2$ cannot contain two linearly independent vectors each of length less than ε. In addition, if $L \in \mathcal{L}_2$ does not contain a horizontal vector then there exists $t \geq 0$ so that $a_t^{-1} L \in \mathcal{L}_2(\varepsilon)$.*

Proof. For any two linearly independent vectors v_1, v_2 in L we have

$$\|v_1\|\, \|v_2\| \geq \text{covolume} L = 1,$$

and so we can take $\varepsilon \leq 1$. Suppose L does not contain a horizontal vector and $L \notin \mathcal{L}_2(\varepsilon)$, then L contains a vector v of norm less than ε which is not horizontal. Note that a_t^{-1} stretches the second coordinate of v, so there exists a smallest t_0 so that $\left\|a_{t_0}^{-1} v\right\| = \varepsilon$. For all $t \in [0, t_0)$, the lattice L contains no vectors shorter of ε except $a_t^{-1} v$ and possibly multiple of it, and so $a_{t_0}^{-1} L \in \mathcal{L}_2(\varepsilon)$. $\square$

Since $\mu(\mathcal{L}_2') = 0$, we may assume that L does not contain any horizontal vectors. By repeatedly applying the lemma we obtain t arbitrarily large with $a_t^{-1}L \in \mathcal{L}_2(\varepsilon)$. For such values of t consider

$$Q = Q(L) = a_t W_- W_0 W_+ a_t^{-1} L = \left(a_t W_- a_t^{-1}\right) W_0 \left(a_t W_+ a_t^{-1}\right) L$$

so when t is large, Q is long in the U direction and short in the A and V directions. Note that Q is a copy of a flowbox containing L and $\nu(Q) = \delta$.

Consider the foliation of Q by orbits of U. For $\widetilde{L} \in Q$ let $I\left(\widetilde{L}\right)$ be the connected component containing the origin of the set $\left\{t \in \mathbb{R} : u_t\widetilde{L} \in Q\right\}$. Note that the length of $I\left(\widetilde{L}\right)$ is just the length of W_+ multiplied by e^{2t} and is independent of the choice of $\widetilde{L} \in Q$. For all large t we have $\left|I\left(\widetilde{L}\right)\right| \geq \max\{T_1, T_2\}$. Note that $a_t W_- a_t^{-1} \subset W_- \subset W_-'$ and $W_0 \subset W_0'$ and applying equation 2.1 we have for any $\widetilde{L} \in Q(L)$ with $L \in \mathcal{L}_2 \backslash \mathcal{L}_2'$

$$\left|\frac{1}{\left|I\left(\widetilde{L}\right)\right|}\int_{I\left(\widetilde{L}\right)} f\left(u_t\widetilde{L}\right)dt - \frac{1}{|I(L)|}\int_{I(L)} f(u_tL)\,dt\right| < \frac{\varepsilon}{3} \quad \left(\forall\widetilde{L} \in Q(L),\, L \in \mathcal{L}_2\backslash\mathcal{L}_2'\right). \tag{2.4}$$

Since $\nu(E) < \delta$ and $\nu(Q) = \delta$, there exists $\widetilde{L}' \in Q \cap E^c$, and so we obtain

$$\left|\int_{\mathcal{L}_2} f d\mu - \int_{\mathcal{L}_2} f d\nu\right| < \varepsilon.$$

from equations 2.2, 2.3 and 2.4. □

We are now in the position to prove the orbit closure result:

Theorem. *Let $L \in \mathcal{L}_2$. Then the U-orbit of L is either closed or dense.*

Proof. Suppose UL is not closed, then as we have seen this means that $L \notin \mathcal{L}_2'$. We wish to show that UL passes through every open set $\widetilde{O} \subset \mathcal{L}_2$. Let O be an open subset of a compact subset $C \subset \widetilde{O}$. Let f be a uniformly continuous nonnegative function supported on C and equal to 1 on O, then

$$0 < \nu(O) \leq \int_{\mathcal{L}_2} f d\nu \leq \nu\left(\widetilde{O}\right),$$

and let $\varepsilon < \nu(O)$. As in the proof above, using equations 2.2 and 2.4, we can find an interval I such that

$$\left|\frac{1}{|I|}\int_I f(u_tL)\,dt - \int_{\mathcal{L}_2} f d\nu\right| < \varepsilon.$$

However, by the definition of f this is only possible if $f(u_tL)$ visits $O \subset \widetilde{O}$. Since $\widetilde{O}$ was arbitrary we deduce that UL is dense in $\mathcal{L}_2$. □