Deriving the Diophantine Properties of IFS Fractals

Itamar Cohen-Matalon

Abstract

We introduce the concepts if IFSes - Iterated Function Systems and the fractals they define. In addition we shall introduce the concept of random walks on the space of lattices. Using a theorem regarding equidistribution of the random walk trajectory for such cases (that we shall prove later on in the seminar) and machinery from the previous meetings we will prove Diophantine results for these types of fractals.

1 Definitions (IFSs)

Definition 1. (Similarity Map) A map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is called a similarity map if:

$$\phi(x) = cOx + y \tag{1}$$

Where O is a $d \times d$ matrix orthogonal with respect to the inner product, $c \in \mathbb{R}$, $c > 0$ and $y \in \mathbb{R}^d$. A similarity map is called *contracting* if $c < 1$.

Definition 2. (IFS - Iterated Function System) An IFS is a collection of similarity maps, $\Phi = (\phi_e)_{e \in E}$, where E is called the *alphabet*. We will later consider some probability measure on E, and denote it by μ.

An IFS is called strictly contracting if $\sup\limits_{e \in E} |\phi'_e| < 1$, and finite if $|E| < \infty$.

Definition 3. (Open Set Condition) An IFS Φ is said to satisfy the open set condition if $\exists U \subseteq \mathbb{R}^d$ open such that $(\phi_e(U))_{e \in E}$ is a disjoint collection of subsets of U.

Definition 4. (Irreducible IFS) An IFS Φ is irreducible if there is no affine proper subset $\mathcal{L} \subsetneq \mathbb{R}^d$ such that:

$$\phi_e(\mathcal{L}) = \mathcal{L} \qquad \forall e \in E \tag{2}$$

Definition 5. $B = E^{\mathbb{N}}$, equipped with the measure $\beta = \mu^{\otimes \mathbb{N}}$. For $b \in B$ we shall denote:

$$b_n^1 = (b_n, ..., b_1) \tag{3}$$

Definition 6. (Coding Map) The coding map of Φ shall be defined as $\pi : B \to \mathbb{R}^d$ by:

$$\pi(b) = \lim_{n \to \infty} \phi_{b_n^1}(\alpha_0) \tag{4}$$

Where $\alpha_0 \in \mathbb{R}^d$ is an arbitrary fixed point and:

$$\phi_{b_n^1} = \phi_{b_1} \circ \cdots \circ \phi_{b_n} \tag{5}$$

We shall later see that this limit exists for our setting (which will require a few more assumptions on Φ).

Definition 7. (Limit Set) The image of B under the coding map π is called the *limit set* of Φ. We denote it by $\mathcal{K} = \mathcal{K}(\phi)$

Definition 8. (Compact similarity IFS) Let E be a compact set, and $\Phi = (\phi_e)_{e \in E}$ a family of continuously varying similarities. Then Φ is a compact similarity IFS.

Definition 9. (Contracting on Average) $\mu \in \operatorname{Prob}(E)$ is contracting on average if:

$$\int \log \phi'_e \, \mathrm{d}\mu(e) < 0 \tag{6}$$

If μ is contracting on average then by the ergodic theorem, $\phi'_{b_n^1} \to 0$ exponentially fast for β a.e. $b \in B$, and so the limit in Definition 6 converges almost everywhere, thereby defining a measure preserving map $\pi : (B, \beta) \to (\mathbb{R}^d, \pi_* \beta)$.

Notice that if we only have contraction on average (as opposed to a contracting IFS where all similarities are contracting), π must not be continuous, but only measurable, and so the limit set from Definition 7 is not necessarily compact.

Notice that in our more limited setting, of a finite strictly contracting IFS, π converges everywhere and is continuous, and therefore the limit set $\mathcal{K}$ is compact.

Definition 10. (General Algebraic Self-Similar Measure) Let Φ be a compact, irreducible similarity IFS, and fix $\mu \in \operatorname{Prob}(E)$ contracting on average, such that $\operatorname{supp}(\mu) = E$. Then the Bernoulli measure $\pi_* \beta$ is called a *general algebraic self-similar measure.*

Note 1. From here on, in order to simplify the discussion, we will only consider the case where $d = 1$ and $O = 1$.

Theorem 1. *(SW - Theorem 8.9) If ν is a general algebraic self similar measure on $\mathbb{R}$, then for ν-a.e. $\alpha \in \mathbb{R}$, the forward orbit of the point $\alpha - \lfloor \alpha \rfloor$ under the Gauss map is equidistributed with respect to the Gauss measure (i.e. of generic type).*

To simplify the discussion, we will prove the following, restricted result:

Theorem 2. *Suppose ν is a general algebraic self similar measure on $\mathbb{R}$, originating from a finite strictly contracting IFS. Then for ν-a.e. $\alpha \in \mathbb{R}$, the forward orbit of the point $\alpha - \lfloor \alpha \rfloor$ under the Gauss map is equidistributed with respect to the Gauss measure (i.e. of generic type).*

We will also prove a simpler result to begin with:

Theorem 3. *Suppose ν is a general algebraic self similar measure on $\mathbb{R}$, originating from a finite strictly contracting IFS. Then for $\nu(\mathrm{BA}) = 0$.*

2 Examples

Some examples of fractals $\mathcal{K}$ that are relevant to our discussion:

- $\mathcal{K} = \mathcal{C}$, the Cantor set.
- $\mathcal{K} = \mathcal{C} + x$, a translate of the Cantor set. *(This result is new in SW)*
- $\mathcal{K}$ is the middle-ε Cantor set constructed by starting with the closed interval $[0, 1]$ and removing at each stage the open middle subinterval of relative length ε from each closed interval kept in the previous stage of the construction, for some $\varepsilon \in (0, 1)$. *(This result is new in SW for $\varepsilon \notin \left\{\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, ...\right\}$)*
- $\mathcal{K}$ is the limit set of the IFS: $\phi_1(x) = \frac{x}{3}, \phi_2(x) = \frac{3+x}{4}$ *(This result is new in SW)*

In addition, the framework in the paper expands in what we will see in the seminar, and handles fractals of higher dimension such as $\mathcal{K} = \mathcal{C} \times \mathcal{C} \subseteq \mathbb{R}^2$.

3 Random Walks

We will observe a Markov random walk on a space $(X, \mathcal{B})$. In general, such a random walk determined by the transition probabilities, $P(A \mid x)$ from x into $A \in \mathcal{B}$, or alternatively defined by a Borel map from X to $\operatorname{Prob}(X)$, $x \mapsto P_x$. We will also be interested in space $\left(X^{\mathbb{N}}, \mathcal{B}^{\mathbb{N}}\right)$ the space of infinite sequences with coordinates in X, which are called random paths or trajectories.

Given a sequence $s \in X^{\mathbb{N}}$, we will say that $s = (x_0, x_1, ...)$ equidistributes in X with respect to the measure m if the sampling measures:

$$\frac{1}{N} \sum_{n=0}^{N-1} \delta_{x_i} \tag{7}$$

Converge to m as $N \to \infty$ in the weak-* topology. (δ_{x_i} is the Dirac mass on X centered at x_i).

In our seminar we will be interested in a more specific setting. As follows:

Notation 1. (Random walk setting)

- $G = \mathrm{SL}_2(\mathbb{R})$, $\Lambda = \mathrm{SL}_2(\mathbb{Z})$, $X_1 = G/\Lambda$ and $x_0 \in X_1$ corresponds to the coset of Λ.
- m is the measure on X derived from the Haar measure on G.
- E is a compact set, $e \mapsto g_e$ is a continuous map from E to G and $\mu \in \operatorname{Prob}(E)$ is a measure such that $\operatorname{supp}(\mu) = E$. Sometimes we will choose to consider $E \subseteq G$, and then $e = g_e$.
- Γ^+ (resp. Γ) is the semigroup (resp. group) generated by $\{g_e : e \in E\}$
- We denote $B = E^{\mathbb{N}}$. For $b = (e_1, e_2, ...) \in B$ and $n \in \mathbb{N}$, $g_{b_1^n}$ denotes the product $g_{e_n} \cdots g_{e_1}$. B is equipped with the measure $\beta = \mu^{\otimes \mathbb{N}}$, and $T : B \to B$ is the left shift. Similarly we denote by $\bar{B} = E^{\mathbb{Z}}$ and $\bar{\beta} = \mu^{\otimes \mathbb{Z}}$

The transition probabilities are determined by μ in the following manner: P_x is the pushforward of μ under the map $e \mapsto g_e x$, where g_e is the corresponding element in G. I.e.:

$$P_x(A) = \int_G \mathbb{1}_A(g_e x) \, \mathrm{d}\mu(e) \tag{8}$$

Theorem 4. *(SW - Theorem 1.1) Take $E = \{1, ..., t\}$. For each $i \in E$, fix $d_i > 1$, $h_i \in \mathbb{R}$ and let:*

$$g_i = \begin{bmatrix} d_i & h_i \\ 0 & d_i^{-1} \end{bmatrix} \in G \tag{9}$$

And assume that $h_1 = 0$, and some $h_i \neq 0$. Fix $p_1, ..., p_t > 0$ with $\sum_{i=1}^t p_i = 1$, and let $\mu = \sum_{i=1}^t p_i \delta_i$, where δ_i is the Dirac mass on E centered at i. Then for any $x \in X$, and for β a.e. $b \in B$ the sequence:

$$\left(g_{b_1^n} x\right)_{n \in \mathbb{N}} \tag{10}$$

Equidistribues in X with respect to m.

Notation 2.

$$a_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}, \quad u_\alpha = \begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix} \tag{11}$$

And $A = \{a_t : t \in \mathbb{R}\}, U = \{u_\alpha : \alpha \in \mathbb{R}\}, P = AU$. Notice that $P < G$, and A normalizes U.

A more refined version of the Theorem 4 is:

Theorem 5. *(SW - Theorem 10.1) Let μ be a probability measure with compact support $E \subseteq G$, that satisfies:*

1. $\operatorname{supp}(\mu) \subseteq P$*, i.e. every $g \in \operatorname{supp}(\mu)$ decomposes into: $g = a_{t_g} u_{\alpha_g}$ for some $t_g, \alpha_g \in \mathbb{R}$. We denote: $\theta_1(g) = t_g$ and $\theta_2(g) = \alpha_g$.*

2. The function $\theta_1 : P \to \mathbb{R}$ satisfies:

$$c_1 \overset{def}{=} \int_P \theta_1(g)\ \mathrm{d}\mu(g) > 0 \tag{12}$$

3. The Lie algebra of Γ contains V^+, the Lie algebra of U.

Then for all $x \in X$:

1. Γ^+ is dense in X.

2. For β-a.e. $b \in B$, the random walk trajectory:

$$\left(g_{b_1^n} x\right)_{n \in \mathbb{N}} \tag{13}$$

Equidistribues in X with respect to m.

Theorem 6. *(SW - Theorem 10.4) Fix $x \in X$ and suppose that for β-a.e. $b \in B$ the random walk trajectory $\left(g_{b_1^n} x\right)_{n \in \mathbb{N}}$ is equidistributed in X with respect to m. Let Y be a locally compact topological space and let $f : \bar{B} \to Y$ be a measurable transformation. Then for $\bar{\beta}$-a.e. $b \in \bar{B}$, the sequence:*

$$\left(g_{b_1^n} x, f\left(T^n b\right)\right)_{n \in \mathbb{N}} \tag{14}$$

Equidistributes in $X \times Y$ with respect to $m \otimes f_* \bar{\beta}$.

4 Connection Between IFSes and Random Walks - Original

For our proof, we would like to translate the language if IFSes into the language of random walks, so that we can use Theorems 6 and 4 to prove Theorem 2. Note that if we wanted to prove Theorem 1, we will use Theorem 5 instead of 4.

Therefore, given an IFS $(\phi_e)_{e \in E}$ our objective is to translate the maps ϕ_e into elements $g_e \in G$ that act on X. Specifically, we want them to be in upper triangular form, as that is the form we use in Theorems 4 and 5.

Notice that in the IFS setup, we were interested in $\phi_{b_n^1}$, and in the homogeneous space we are interested in the objects $g_{b_1^n}$. Therefore we would like the relation to be of the form $g_e = \phi_e^{-1}$, so that the coding map agrees with the random walk, and we have $g_{b_1^n} = \phi_{b_n^1}^{-1}$. Therefore, we need to understand how to view the similarity map as an element of G.

Notice that the elements of $G = \mathrm{SL}(2, \mathbb{R})$ can be views as Mobius transformations acting on the real line. Specifically, the upper triangular matrices are exactly the group of similarity maps, as a matrix:

$$u_\alpha a_t = \begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} = \begin{bmatrix} e^t & -e^{-t}\alpha \\ 0 & e^{-t} \end{bmatrix} \tag{15}$$

Defines the similarity map: $\phi(x) = \frac{e^t x - e^{-t}\alpha}{e^{-t}} = e^{2t}x - \alpha$. Therefore, for a similarity map $\phi(x) = cx + b$ we will need to take $\alpha = -b, t = \frac{1}{2}\log c$, so is represented by the element:

$$\phi = u_{-b} a_{\frac{1}{2}\log c} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c^{\frac{1}{2}} & 0 \\ 0 & c^{-\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} c^{\frac{1}{2}} & c^{-\frac{1}{2}}b \\ 0 & c^{-\frac{1}{2}} \end{bmatrix} \tag{16}$$

The corresponding group element for the random walk will be:

$$g = \phi^{-1} = \begin{bmatrix} c^{-\frac{1}{2}} & -c^{-\frac{1}{2}}b \\ 0 & c^{\frac{1}{2}} \end{bmatrix} \tag{17}$$

Example 1. We'll observe the Cantor set $\mathcal{C}$ which is defined by the maps:

$$\phi_1(x) = \frac{x}{3}, \quad \phi_2(x) = \frac{2+x}{3} \tag{18}$$

Here, the corresponding group elements will be:

$$g_1 = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix}, \quad g_2 = \begin{bmatrix} \sqrt{3} & -\sqrt{3}\frac{2}{3} \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & -\frac{2}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \tag{19}$$

5 Common Steps in the Proof

Our objective is to show that for ν-a.e. α, $\alpha \notin \mathrm{BA}$ for 3 and that $\left(\mathcal{G}^n(\alpha)\right)_{n\in\mathbb{N}}$ equidistributes for Theorem 2. In a previous lecture we saw that this translates into the language of the diagonal flow $\{a_t u_\alpha x_0 : t \geq 0\}$, where $\alpha \notin \mathrm{BA}$ if and only if the trajectory is bounded, and is of generic type if and only if the trajectory equidistributes in X with respect to m.

Denote by $\pi_+(b) = \pi(b_1^\infty)$. Therefore, by the definition of ν, if we must prove that for $\bar{\beta}$-a.e. $b \in \bar{B}$, these properties hold for the trajectory $\left\{a_t u_{\pi_+(b)} x_0 : t \geq 0\right\}$.

Observe our setup: We have a finite set $E = \{1, ..., t\}$ indexing contracting similarity maps:

$$\phi_i(x) = c_i x + y_i \tag{20}$$

I.e. for all $i \in E$, $0 < c_i < 1$. We'll denote the corresponding elements in G given by the translation above (17) by $\{g_i : i \in E\}$.

We first intend to apply Theorem 4 to show that for any $x \in X$, for β-a.e. $b \in B$ the associated random walk trajectory is equidistributed in X.

Note that replacing the elements g_i by their pushforward under conjugation in G does not affect the validity of the conclusion, as if $\left(g_{b_1^n} x\right)_{n\in\mathbb{N}}$ is bounded or equidistributed, then so is $\left(g_0 g_{b_1^n} x\right)_{n\in\mathbb{N}} = \left(g_0 g_{b_1^n} g_0^{-1} g_0 x\right)_{n\in\mathbb{N}}$ which is the trajectory of $g_0 x$ with respect to the conjugated elements $\left\{g_i' = g_0 g_i g_0^{-1} : i \in E\right\}$. Taking:

$$g_0 = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \tag{21}$$

We'll get:

$$\begin{aligned} g_1' = g_0 g_1 g_0^{-1} &= \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{c_1}} & -\frac{y_1}{\sqrt{c_1}} \\ 0 & \sqrt{c_1} \end{bmatrix} \begin{bmatrix} 1 & -\gamma \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{c_1}} & -\frac{y_1}{\sqrt{c_1}} + \gamma\sqrt{c_1} \\ 0 & \sqrt{c_1} \end{bmatrix} \begin{bmatrix} 1 & -\gamma \\ 0 & 1 \end{bmatrix} = \\ &= \begin{bmatrix} \frac{1}{\sqrt{c_1}} & -\frac{\gamma}{\sqrt{c_1}} - \frac{y_1}{\sqrt{c_1}} + \gamma\sqrt{c_1} \\ 0 & \sqrt{c_1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{c_1}} & \left(\sqrt{c_1} - \frac{1}{\sqrt{c_1}}\right)\left(\gamma - \frac{y}{c_1 - 1}\right) \\ 0 & \sqrt{c_1} \end{bmatrix} \overset{(*)}{=} \begin{bmatrix} \frac{1}{\sqrt{c_1}} & 0 \\ 0 & \sqrt{c_1} \end{bmatrix} \end{aligned} \tag{22}$$

Where $(*)$ will hold for the value: $\gamma = \frac{y}{c_1 - 1}$, which is well defined as $c_1 < 1$.

Notice that we conjugated by an element $u_\alpha \in U$, so we so not affect the projection to A, and remain in P. Therefore we have that the elements are of the form:

$$g_i' = \begin{bmatrix} \sqrt{c_i}^{-1} & h_i \\ 0 & \sqrt{c_i} \end{bmatrix} = \begin{bmatrix} d_i & h_i \\ 0 & d_i^{-1} \end{bmatrix} \tag{23}$$

Where by (22) we have that $h_1 = 0$. Because ϕ_i are contracting, we have $0 < c_i < 1$, and so $d_i = \sqrt{c_i}^{-1} > 1$. Therefore, g_i' are of the form required in Theorem 4, and so we can apply the theorem, i.e. we have that for every $x \in X$ and β-a.e. $b \in B$ the random walk trajectory $\left(g_{b_1^n} x\right)_{n\in\mathbb{N}}$ equidistributes in X.

6 Proof of Theorem 3

Denote:

- $g_n = g_{b_1^n} = a_{t_n} u_{\alpha_n}$ (as it is an element of P)
- $\beta_n = \pi_+(T^n b)$
- $h_n = u_{-\beta_n} a_{t_n} u_{\pi_+(b)}$

Clearly $h_n, g_n \in P$ and agree on their projections to A. On the other hand, observing their action on $\mathbb{R}$ as Mobius transformations (where $u_\alpha = \left[\begin{smallmatrix} 1 & -\alpha \\ 0 & 1 \end{smallmatrix}\right]$, $a_t = \left[\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right]$ encode the maps $\phi_{u_\alpha}(x) = x - \alpha$, $\phi_{a_t}(x) = e^{2t}x$):

$$h_n^{-1}(\beta_n) = u_{\pi_+(b)}^{-1} a_{t_n}^{-1} u_{-\beta_n}^{-1}(\beta_n) = u_{-\pi_+(b)} a_{t_n}^{-1} u_{\beta_n}(\beta_n) = u_{-\pi_+(b)} a_{t_n}^{-1}(0) = u_{-\pi_+(b)}(0) = \\ = \pi_+(b) \overset{(*)}{=} \phi_{b_n^1}(\beta_n) \overset{(**)}{=} g_{b_1^n}^{-1}(\beta_n) = g_n^{-1}(\beta_n) \quad (24)$$

Where $(*)$ is by the definition of the coding map (Definition 6):

$$\pi_+(b) = \lim_{N\to\infty} \phi_{b_N^1}(\alpha_0) = \lim_{N\to\infty} \phi_{b_n^1}\phi_{b_N^{n+1}}(\alpha_0) = \phi_{b_n^1}\left(\lim_{N\to\infty} \phi_{b_N^{n+1}}(\alpha_0)\right) = \\ = \phi_{b_n^1}\left(\lim_{N\to\infty} \phi_{T^n b_N^1}(\alpha_0)\right) = \phi_{b_n^1}(\pi_+(T^n b)) = \phi_{b_n^1}(\beta_n) \quad (25)$$

And $(**)$ is the relation between the similarity maps and their corresponding group elements (17). Therefore the U element must be equal as well (the A element is equal and so they must have the same "c" if viewed as similarity maps, and because they agree on a point they must have the same "y" as well, and so are equal).

Thus $h_n x_0 = g_n x_0$. Since Φ is strictly contracting, the limit $\mathcal{K}$ is compact, and so the sequence $(\beta_n)_{n\in\mathbb{N}}$ is bounded. Therefore the distance from $g_n x_0 = h_n x_0 = u_{-\beta_n} a_{t_n} u_{\pi_+(b)} x_0$ to $a_{t_n} u_{\pi_+(b)} x_0$ (which is affected by $u_{-\beta_n}$) is bounded by a number independent of n. In addition, the sequence $(a_{t_n})_{n\in\mathbb{N}}$ has bounded gaps in $(a_t)_{t\geq 0}$ (each gap is determined by an a_{t_e} factor of an element from $\{g_e : e \in E\}$). So we have that:

$$(g_n x_0)_{n\in\mathbb{N}} \text{ is bounded} \Leftrightarrow \left(a_{t_n} u_{\pi_+(b)} x_0\right)_{n\in\mathbb{N}} \text{ is bounded} \Leftrightarrow \left(a_t u_{\pi_+(b)} x_0\right)_{t\geq 0} \text{ is bounded} \quad (26)$$

And therefore, as for β-a.e. $b \in B$ the random walk trajectory $\left(g_{b_1^n}\right)_{n\in\mathbb{N}}$ equidistributes in X, it is not bounded, and so $\left(a_t u_{\pi_+(b)} x_0\right)_{t\geq 0}$ is not bounded as well and $\pi_+(b) \notin \mathrm{BA}$.

7 Proof of Theorem 2

We shall now apply Theorem 6. The equidistribution above is the one required by the theorem. We will choose $Y = E \times \mathbb{R}$ and $f : \bar{B} \to Y$ to be $f(b) = (b_0, \pi_+(b))$. Then by the theorem, for $\bar{\beta}$-a.e. $b \in \bar{B}$, the sequence:

$$\left(g_{b_1^n} x_0, f(T^n b)\right)_{n\in\mathbb{N}} \quad (27)$$

Equidistributes in $X \times Y$ with respect to $m \otimes f_*\bar{\beta}$. Note that $f_*\bar{\beta} = \mu \otimes \nu$ where $\nu = f_*\beta$. Consider the map $f_2 : X \times Y \to X \times E$ define by:

$$f_2(x, (e, \alpha)) = (u_\alpha x, e) \quad (28)$$

Since f_2 is continuous, the image of (27) under f_2, i.e. the sequence:

$$(x_n, b_n)_{n\in\mathbb{N}} \quad \text{where} \quad x_n = u_{\pi_+(T^n b)} g_{b_1^n} x_0 \quad (29)$$

is equidistributed in $X \times E$ with respect to the measure $(f_2)_*\left[m \otimes f_*\bar{\beta}\right] = m \otimes \mu$ (equality on the second coordinate is clear, and the first is by left invariance of the Haar measure).

As in (24) here too we have $g_n = h_n = u_{-\pi_+(T^n b)} a_{t_n} u_{\pi_+(b)}$. Substituting in (29) we get:

$$x_n = u_{\pi_+(T^n b)} g_n x_0 = u_{\pi_+(T^n b)} u_{-\pi_+(T^n b)} a_{t_n} u_{\pi_+(b)} x_0 = a_{t_n} u_{\pi_+(b)} x_0 \quad (30)$$

for all $n \in \mathbb{N}$.

For each $e \in E$, let $t_e \in \mathbb{R}$ be such that $\pi_A(g_e) = a_{t_e}$. Since π_A is a homomorpism we have that $t_n = t_{n-1} + t_{b_n}$ for all $n \in \mathbb{N}$. Now let $F : X \to \mathbb{R}$ be a bounded continuous function. Then the function $F' : X \times E \to \mathbb{R}$ defined by the formula:

$$F'(x, e) = \int_{-t_e}^{0} F(a_t x)\ \mathrm{d}t \quad (31)$$

is also a bounded continuous function. We use the convention: $\int_a^b F \,\mathrm{d}t = -\int_b^a F \,\mathrm{d}t$. Since $(x_n, b_n)_{n\in\mathbb{N}}$ from (29) is equidistributed, plugging in (30) we get that:

$$\int F' \,\mathrm{d}(m\otimes\mu) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} F'(x_n, b_i) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} F'\left(a_{t_i}u_{\pi_+(b)}x_0, b_i\right) =$$
$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} F\left(a_t u_{\pi_+(b)}x_0\right) \,\mathrm{d}t \overset{(*)}{=} \left(\int t_e \,\mathrm{d}\mu(e)\right)\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n} F\left(a_t u_{\pi_+(b)}x_0\right) \,\mathrm{d}t \quad (32)$$

where the equality $(*)$ is due to limit multiplication with $\left(\int t_e \,\mathrm{d}\mu(e)\right)\lim_{n\to\infty}\frac{n}{t_n} = 1$ which is due to $\lim_{n\to\infty}\frac{t_n}{n} = \int t_e \,\mathrm{d}\mu(e)$. On the other hand:

$$\int F' \,\mathrm{d}(m\otimes\mu) = \int\left(t_e\int F \,\mathrm{d}m\right) \,\mathrm{d}\mu(e) = \left(\int t_e \,\mathrm{d}\mu(e)\right)\left(\int F \,\mathrm{d}m\right) \quad (33)$$

Since $t_n \to \infty$ and the gaps $t_{n+1} - t_n$ $(n \in \mathbb{N})$ are bounded, it follows that:

$$\frac{1}{T}\int_0^T F\left(a_t u_{\pi_+(b)}x_0\right) \,\mathrm{d}t \to \int F \,\mathrm{d}m \quad (34)$$

i.e. that $\left(a_t u_{\pi_+(b)}\right)_{t\geq 0}$ is equidistributed with respect to m. Notice that we have not used Theorem 5 in the proof here. This theorem should be used instead of 4 when proving the more general result, Theorem 1. It is required as it allows more general measures, coming from contracting on average, and not necessarily finite IFSes.