Homogeneous Dynamics and Applications

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Introduction

Our settings are:

- G is a second countable locally compact semigroup with underlying σ-algebra Σ and probability measure μ
- $(B, \mathscr{B}, \beta, T)$ is the associated Bernoulli shift, meaning $B = G^{\mathbb{N}_+}$ is the space of series with letters from G, $\mathscr{B} = \Sigma^{\otimes \mathbb{N}_+}$ is the product σ-algebra, $\beta = \mu^{\otimes \mathbb{N}_+}$ is the product measure and T is a shift given by $T(b_1, b_2, b_3, ...) = (b_2, b_3, b_4, ...)$
- X is a compact metrizable topological space with Borel probability measure ν.
- G acts continuously on X and ν is μ stationary:

  $$\mu * \nu = \int_G g_* \nu d\mu(g) = \nu$$

  Which means that for an integrable function φ:

  $$\mu * \nu(\varphi) = \int_G \int_X \varphi(gx) d\nu(x) d\mu(g) = \int_X \varphi(x) d\nu(x) = \nu(\varphi)$$

- We also make use of convolution power of μ, that is $\mu^{*n} = \mu * ... * \mu$ (n times). more explictly, μ^{*n} is a probability measure over G given by:

  $$\mu^{*n}(\varphi) = \int_G ... \int_G \varphi(g_1...g_n) d\mu(g_1)...d\mu(g_n)$$

We first construct a family of probability measures ν_b on X, where $b \in B$, such that $lim_{n\to\infty}(b_1...b_n)_* \nu = \nu_b$. Then we show some properties of ν_b. Prior to proving the main results we state Doob martingale theorem which is a main tool in this work. A detailed proof is given at the end of this document.

Doob martingale theorem

Let $(\Omega, \mathscr{B}, P)$ be a probability space. Let $\mathscr{B}_n \subset \mathscr{B}$ be an increasing sequence of sub σ-algebras and ψ_n be a martingale with respect to $\mathscr{B}_n$, that is:

- ψ_n is $\mathscr{B}_n$ measurable
- $E[|\psi_n|] < \infty$
- $E[\psi_n|\mathscr{B}_{n-1}] = \psi_{n-1}$

If $\sup_{n\geq 1} E[|\psi_n|] < \infty$ then exists a P integrable function ψ_∞ such that:

$$\lim_{n\to\infty} \psi_n \overset{P-a.e}{=} \psi_\infty$$

Main Results

ν_b construction

First we define the filteration $\mathscr{B}_n = \sigma(b_1, ...b_n)$ the sub σ-algebra generated by the first n coordinate functions on B. We also define for a bounded Borel function φ on X the function $f_n^\varphi : B \to \mathbb{R}$:

$$f_n^\varphi := \int_X \varphi(b_1...b_n x) d\nu(x) = (b_1...b_n)_* \nu(\varphi)$$

Notice that f_n^φ is a function of $(b_1, ...b_n)$ so it is $\mathscr{B}_n$ measurable and that φ is bounded so $sup_n E[|f_n^\varphi|] < \infty$. Finnaly, to see that $E[f_{n+1}|\mathscr{B}_n](b) = f_n(b)$, we choose arbitrary $A \in \mathscr{B}_n$:

$$\int_A f_{n+1} = \int_A (\int_X \varphi(b_1...b_{n+1}x) d\nu(x)) d\beta(b) =$$

Define $\tilde{\varphi}(x) = \varphi(b_1...b_n x)$

$$\int_A (\int_X \tilde{\varphi}(b_{n+1}x) d\nu(x)) d\beta(b) =$$

Using Fubini, the fact that μ is probability measure over B and that $A \in \mathscr{B}_n$ so $b_{n+1}(A) = G$:

$$\int_{\bar{A}} \int_X \int_G (b_{n+1})_* \tilde{\varphi}(x) d\mu(b_{n+1}) d\nu(x) d\mu^{\otimes n}(b_1, ...b_n) =$$

And since ν is μ invariant:

$$\int_{\bar{A}} \int_X \tilde{\varphi}(x) d\nu(x) d\mu^{\otimes n}(b_1, ...b_n) = \int_A f_n$$

We conclude that f_n^φ is a martingale with respect to $\mathscr{B}_n$.
By applying Doob martingle theorem to f_n^φ we promise existence to $f^\varphi =$

$\lim_{n\to\infty} f_n^\varphi$ for β-a.e $b \in B$. Since X is compact and metrizable we can find a dense subset of bounded functions $D \subset C^0(X)$. We apply Doob to the functions in D and by approximation deduce that for β-a.e b:

$$\nu_b = lim_{n\to\infty}(b_1...b_n)_*\nu$$

We now prove three basic properties of ν_b

ν as avarage of ν_b

We show that $\nu = \int_B \nu_b d\beta(b)$. ν is μ-stationary, so for any $n \in \mathbb{N}$ and bounded φ one has:

$$\int_B (b_1...b_n)_*\nu(\varphi) = \int_B f_n^\varphi(b)d\beta(b) = \int_B \int_X \varphi(b_1...b_n x)d\nu(x)d\beta(b) =$$

Using similar reasoning to the previous section and the fact that ν is μ-stationary we get:

$$\int_{G^{\otimes n-1}} \int_X \int_G \varphi(b_1...b_n x)d\nu(x)d\mu(b_n)d\mu^{\otimes n-1}(b_1,...,b_{n-1}) =$$

$$\int_{G^{\otimes n-1}} \int_X \varphi(b_1...b_{n-1})(x)d\nu(x)d\mu^{\otimes n-1}(b_1,...,b_{n-1}) = (b_1...b_{n-1})_*\nu(\varphi)$$

We deduce that for any $n \in \mathbb{N}$:

$$\int_B f_n^\varphi(b)d\beta(b) = \int_X \varphi d\nu$$

By taking limit on both sides and using dominated convergence theorem we achieve the desired result hence the existence of equivalence in weak-$*$ topology.

ν_b under Bernoulli shift

We show that $\nu_b = (b_1)_*\nu_{Tb}$.
If $\mu_n \to \mu$ then for every continous φ since G acts continuously on X:

$$\lim_{n\to\infty} g_*\mu_n(\varphi) = \lim_{n\to\infty} \int \varphi(gx)d\mu_n(x) = \lim_{n\to\infty} \int \tilde{\varphi}(x)d\mu_n(x) =$$

$$\int \tilde{\varphi}(x)d\mu(x) = \int \varphi(gx)d\mu(x) = g_*\mu(\varphi) = g_* \lim_{n\to\infty} \mu_n(\varphi)$$

Using this the result follows directly from the definition of ν_b:

$$\nu_b = \lim_{n\to\infty}(b_1...b_n)_*\nu = \lim_{n\to\infty} b_{1_*}(b_2....b_n)_*\nu =$$

$$b_{1_*} \lim_{n\to\infty}(b_2...b_n)_*\nu = b_{1_*}\nu_{Tb}$$

Invariance of ν_b

We show that $\forall m \in \mathbb{N}$ for $\beta \otimes \mu^{*m}$-a.e $(b,g) \in B \times G$ one has

$$\lim_{n\to\infty} (b_1 ... b_n g)_* \nu = \nu_b$$

Let $\varphi \in C^0(X)$ and define $\Phi : G \to \mathbb{R}$ by:

$$\Phi(h) = \int_X \varphi(hx) d\nu(x)$$

Also, define $f_n^g : B \to \mathbb{R}$ by $f_n^g(b) = \Phi(b_1 ... b_n g)$. As before, since $C^0(X)$ is seprable, it suffices to show for μ^{*m}-a.e $g \in G$ and for β-a.e $b \in B$ that $\lim_{n\to\infty} |f_n^g(b) - f_n(b)| = 0$.
To see that, consider:

$$I_n = \int_G \int_B |f_n^g(b) - f_n(b)|^2 d\beta(b) d\mu^{*m}(g)$$

Notice that $\int_B f_n^g(b) d\beta(b) = \int_B \Phi(b_1 ... b_n g) d\beta(b) = \int_G \Phi(hg) d\mu^{*n}(h)$ and $\int_B f_n(b) = \int_B \Phi(b_1 ... b_n) d\beta(b) = \int_G \Phi(h) d\mu^{b*n}(h)$, so:

$$I_n = \int_G \int_G |\Phi(hg) - \Phi(h)|^2 d\mu^{*m}(g) d\mu^{*n}(h) =$$

$$\int_G \int_G (\Phi(hg)^2 - 2\Phi(hg)\Phi(h) + \Phi(h)^2) d\mu^{*m}(g) d\mu^{*n}(h)$$

We define $J_n = \int_G \Phi(h)^2 d\mu^{*n}(h)$ and see that:

$$\int_G \int_G \Phi(hg)^2 d\mu^{*m}(g) d\mu^{*n}(h) = \int_G \Phi(h) d\mu^{*(m+m)}(h) = J_{m+n}$$

$$\int_G \int_G \Phi(h)^2 d\mu^{*m}(g) d\mu^{*n}(h) = \int_G J_n d\mu^{*m}(g) \overset{1}{=} J_n$$

$$\int_G \int_G \Phi(hg)\Phi(h) = \int_G [\Phi(h) \int_G \Phi(hg) d\mu^{*m}(g)] d\mu^{*n}(h) \overset{2}{=} \int_G \Phi(h)^2 d\mu^{*n}(h) = J_n$$

Where 1 holds since μ^{*m} is a probability measure and 2 holds since ν is μ-stationary so:

$$\int_G \Phi(hg) d\mu^{*m}(g) = \int_G \int_X \varphi(hgx) d\nu(x) d\mu^{*m}(g) = \int_G \int_X \tilde{\varphi}(gx) d\nu(x) d\mu^{*m}(g) =$$

$$\int_X \tilde{\varphi}(x) d\nu(x) =_X \varphi(hx) d\nu(x) = \Phi(h)$$

We conclude that $I_n = J_{n+m} - J_n$. $||J_n||$ is bounded by $||\varphi||_\infty$ which implies $\sum^\infty I_n \leq \infty$, so $\lim_{n\to\infty} I_n = 0$. The wanted result is then immidate.

Doob's theorem proof

We prove the first part of the theorem under the assumption that the martingale is uniformly L^2 bounded. This makes the theorem slightly weaker but it is sufficient for our needs.
We start by proving the following lemma:

Lemma:
Let ψ_n be a martingale with respect to $\mathscr{B}_n$ and $\epsilon > 0$. Then:

$$P(\sup_{1\leq k\leq n} |\psi_k| \geq \epsilon) \leq \epsilon^{-1} E[|\psi_n|]$$

Proof:
We denote:

$$A_k = \{|\psi_1| < \epsilon, |\psi_2| < \epsilon, ..., |\psi_k| \geq \epsilon,\}$$

So it is enough to bound $P(A)$ where $A = \cup_{k=1}^n A_k$. Using Chebyshev's inequality and martingales properties:

$$P(A) = \sum_{k=1}^n P(A_k) \leq$$

$$\sum_{k=1}^n P(|\psi_k| \mathbb{1}_{A_k} \geq \epsilon) \leq \sum_{k=1}^n \epsilon^{-1} E[|\psi_k| \mathbb{1}_{A_k}]$$

Notice that $\mathbb{1}_{A_k} \in \mathscr{B}_k$ so $\psi_k \mathbb{1}_{A_k} = E[\psi_n \mathbb{1}_{A_k} | \mathscr{B}_k]$ so by total expectation $E[\psi_k \mathbb{1}_{A_k}] = E[\psi_n \mathbb{1}_{A_k}]$:

$$= \epsilon^{-1} \sum_{k=1}^n E[|\psi_n| \mathbb{1}_{A_k}] \leq \epsilon^{-1} E[|\psi_n|]$$

■

We assume $\sup_{n\geq 1} E[\psi_n^2] < \infty$. for $m < n$:

$$E[\psi_n \psi_m] = E[E[\psi_m \psi_n | \mathscr{B}_{n-1}]] = E[\psi_m E[\psi_n | \mathscr{B}_{n-1}]] =$$

$$E[\psi_m \psi_{n-1}] = ... = E[\psi_m \psi_m] = E[\psi_m^2]$$

So:

$$E[(\psi_n - \psi_m)^2] = E[\psi_n^2] - E[\psi_m^2]$$

This means that the sequence $E[\psi_n^2]$ is non decreasing and therefore converging, so ψ_n is Cauchy sequence in L^2 norm hence converging in L^2 norm to some function ψ_∞. The convergence is L^1 as well.
Now, define the martingale $f_m^n = \psi_n - \psi_m$. using the lemma we get:

$$P(\sup_{1\leq k\leq n} |\psi_{k+m} - \psi_m| \geq \epsilon) = P(\sup_{1\leq k\leq n} |f_m^{k+m}| \geq \epsilon) \leq \frac{1}{\epsilon} E[|f_m^{n+m}|] = \frac{1}{\epsilon} E[|\psi_{n+m} - \psi_m|]$$

Taking n to ∞ gets us:

$$P(\sup_{m \leq k} |\psi_k - \psi_m| \geq \epsilon) \leq \frac{1}{\epsilon} E[|\psi_\infty - \psi_m|]$$

Which tends to 0 as m goes to ∞