UNDERSTANDING THE CROSS-SECTION MEASURES IN CASE I

Main Theorems

Let $m_{\mathscr{X}_d}, m_{\mathscr{E}_n}, m_{\widehat{\mathbb{Z}}^n_{\mathrm{prim}}}$ be Haar probability measures on $\mathscr{X}_d,\ \mathscr{E}_n,\ \widehat{\mathbb{Z}}^n_{\mathrm{prim}}$.

Theorem 1. *For any norm $\|\cdot\|$ on $\mathbb{R}^d$ there is a probability measure $\mu = \mu_{\mathrm{best},\|\cdot\|}$ on $\mathscr{X}_d \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n$ such that for Lebesgue almost any $\theta \in \mathbb{R}^d$, the following holds. Let $\mathbf{v}_k \in \mathbb{Z}$ be the sequence of best approximations to θ with respect to the norm $\|\cdot\|$. Then the sequence*

$$([\pi_{\mathbb{R}^d}^{\mathbf{v}_k}(\mathbb{Z}^n)], \mathrm{disp}(\theta, \mathbf{v}_k), \mathbf{v}_k)_{k\in\mathbb{N}} \in \mathscr{X}_d \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n$$

equidistributes with respect to μ. The measure μ has the following properties:

1. *It is a product $\mu = \mu^{(\infty)} \times \mu^{(f)}$ where $\mu^{(\infty)} \in \mathcal{P}(\mathscr{X}_d \times \mathbb{R}^d),\ \mu^{(f)} \in \mathcal{P}(\widehat{\mathbb{Z}}^n)$.*
2. *The measure $\mu^{(f)}$ is $m_{\widehat{\mathbb{Z}}^n_{prim}}$ (and in particular, does not depend on the choice of the norm).*
3. *The projection $\mu^{(\mathscr{X}_d)}$ of $\mu^{(\infty)}$ to $\mathscr{X}_d$ is equivalent to $m_{\mathscr{X}_d}$, but is equal to it only in case $d = 1$.*
4. *The projection $\mu^{(\mathbb{R}^d)}$ of $\mu^{(\infty)}$ to $\mathbb{R}^d$ is boundedly supported, absolutely continuous w.r.t. Lebesgue with a nontrivial density (i.e., is not the restriction of Lebesgue measure to a subset of $\mathbb{R}^d$). If $\|\cdot\|$ is the Euclidean norm, then it is $\mathrm{SO}_d(\mathbb{R})$-invariant.*
5. *For $d > 1$, $\mu^{(\infty)} \neq \mu^{(\mathscr{X}_d)} \times \mu^{(\mathbb{R}^d)}$.*

Furthermore, each of the coordinate sequences

$$\left([\pi_{\mathbb{R}^d}^{\mathbf{v}_k}(\mathbb{Z}^n)]\right) \subset \mathscr{X}_d,\ (\mathrm{disp}(\theta, \mathbf{v}_k)) \subset \mathbb{R}^d,\ (\mathbf{v}_k) \subset \widehat{\mathbb{Z}}^n$$

equidistributes in its respective space, with respect to the pushforward of $\mu^{(\mathscr{X}_d)},\ \mu^{(\mathbb{R}^d)},\ \mu^{(f)}$ respectively.

Theorem 2. *For any norm $\|\cdot\|$ on $\mathbb{R}^d$ and any $\varepsilon > 0$ there is a probability measure $\nu = \nu_{\varepsilon-\mathrm{approx},\|\cdot\|}$ on $\mathscr{X}_d \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n$ such that for Lebesgue almost any $\theta \in \mathbb{R}^d$, the following holds. Let $\mathbf{w}_k \in \mathbb{Z}$ be the sequence of best approximations to θ with respect to the norm $\|\cdot\|$. Then the sequence*

$$([\pi_{\mathbb{R}^d}^{\mathbf{w}_k}(\mathbb{Z}^n)], \mathrm{disp}(\theta, \mathbf{w}_k), \mathbf{w}_k)_{k\in\mathbb{N}} \in \mathscr{X}_d \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n$$

equidistributes with respect to μ. The measure μ has the following properties:

1. *It is a product $\nu = \nu^{(\mathscr{X}_d)} \times \nu^{(\mathbb{R}^d)} \times \nu^{(f)}$ where $\nu^{(\mathscr{X}_d)} \in \mathcal{P}(\mathscr{X}_d),\ \nu^{(\mathbb{R}^d)} \in \mathcal{P}(\mathbb{R}^d),\ \nu^{(f)} \in \mathcal{P}(\widehat{\mathbb{Z}}^n)$.*
2. *The measure $\nu^{(\mathscr{X}_d)}$ is $m_{\mathscr{X}_d}$ and the measure $\nu^{(f)}$ is $m_{\widehat{\mathbb{Z}}^n_{\mathrm{prim}}}$ (in particular, these measures do not depend on the choice of ε or of the norm).*
3. *The measure $\nu^{(\mathbb{R}^d)}$ is the normalized restriction of the Lebesgue measure on $\mathbb{R}^d$, to the ball of radius ε around the origin with respect to the norm $\|\cdot\|$.*

Furthermore, each of the coordinate sequences

$$\left([\pi_{\mathbb{R}^d}^{\mathbf{w}_k}(\mathbb{Z}^n)]\right) \subset \mathscr{X}_d,\ (\mathrm{disp}(\theta, \mathbf{w}_k)) \subset \mathbb{R}^d,\ (\mathbf{w}_k) \subset \widehat{\mathbb{Z}}^n$$

equidistributes in its respective space, with respect to the pushforward of $\nu^{(\mathscr{X}_d)},\ \nu^{(\mathbb{R}^d)},\ \nu^{(f)}$ respectively.

Let $\mathbf{v} = (v_1, \ldots, v_n)^{\mathrm{t}} \in \mathbb{Z}^n_{\mathrm{prim}}$, then the maps $\pi_{\mathscr{X}_d}, \rho_{\mathscr{E}_n}, \rho_{\mathscr{X}_d}$ are defined as follows

$$\rho_{\mathscr{X}_d}(\mathbf{v}) = \left[\pi^{\mathbf{v}}_{\mathbb{R}^d}\left(\mathbb{Z}^n\right)\right],$$
$$\pi_{\mathscr{X}_d}(\Lambda) = \pi_{\mathbb{R}^d}(\Lambda),$$
$$\rho_{\mathscr{E}_n}(\mathbf{v}) \overset{\text{def}}{=} a_t u(v)\mathbb{Z}^n,$$

where

$$v = -\frac{1}{v_n}\left(v_1, \ldots, v_d\right)^{\mathrm{t}},$$
$$u(v) = \begin{pmatrix} I_d & v \\ 0 & 1 \end{pmatrix}, \quad a_t = \operatorname{diag}\left(e^t, \ldots, e^t, e^{-dt}\right),$$

where $t = \log|v_n|$.

The following diagram is commutative:

$$\begin{tikzcd} & \mathscr{E}_n \arrow[d, "\pi_{\mathscr{X}_d}"] \\ \mathbb{Z}^n_{\mathrm{prim}} \arrow[ur, "\rho_{\mathscr{E}_n}"] \arrow[r, "\rho_{\mathscr{X}_d}"'] & \mathscr{X}_d \end{tikzcd}$$

Set

$$H = \left\{ \left(\begin{smallmatrix} A & 0 \\ \mathbf{x}^t & 1 \end{smallmatrix}\right) \in \mathrm{SL}_n(\mathbb{R}) : A \in \mathrm{SL}_d(\mathbb{R}), \mathbf{x} \in \mathbb{R}^d \right\}.$$

Then the lattice $\mathbb{Z}^n$ is contained in $\mathscr{E}_n$, the group H acts transitively on $\mathscr{E}_n$, and $H(\mathbb{Z})$ is the stabilizer of $\mathbb{Z}^n$ for this action. Thus we may identify $\mathscr{E}_n \simeq H/H(\mathbb{Z})$.

Since $H(\mathbb{Z})$ is a lattice in H, there is a unique H-invariant probability measure on $\mathscr{E}_n$ which we denote $m_{\mathscr{E}_n}$.

From the uniqueness of invariant probability measures for transitive actions we obtain

$$\left(\pi_{\mathscr{X}_d}\right)_* m_{\mathscr{E}_n} = m_{\mathscr{X}_d}.$$

We defined before a linear functional $f : \mathbb{R}^d \to \mathbb{R}$ that helps reconstruct Λ from its projection Λ' to $\mathbb{R}^d$ along $\mathbf{v}$:

$$\forall\, \mathbf{w} \in \Lambda \ \exists\, k \in \mathbb{Z} \text{ such that } \mathbf{w} = \pi_{\mathbb{R}^d}(\mathbf{w}) + \left(f\left(\pi_{\mathbb{R}^d}(\mathbf{w})\right) + k\right)\mathbf{v}. \tag{1}$$

For any two functionals f_1, f_2 satisfying (1) we will have $f_1(\mathbf{w}) - f_2(\mathbf{w}) \in \mathbb{Z}$ for any $\mathbf{w} \in \Lambda'$, that is they differ by an element of the dual lattice $(\Lambda')^*$, so f is well defined as a class in the torus $\mathbb{T}_{\Lambda'}$, where

$$\mathbb{T}_{\Lambda'} = \left(\mathbb{R}^d\right)^* / \left(\Lambda'\right)^*.$$

Given $\Lambda' \in \mathscr{X}_d$, a lift functional f, and a vector $\mathbf{v} \in \Lambda_{\mathrm{prim}} \setminus \mathbb{R}^d$ we can recover Λ as follows:

$$\Lambda = \Lambda(\Lambda', f, \mathbf{v}) = \left\{ |v_n|^{-1/d}\mathbf{x} + (f(\mathbf{x}) + k)\mathbf{v} : \mathbf{x} \in \Lambda',\, k \in \mathbb{Z} \right\}. \tag{2}$$

We use $\mathbf{v} = \mathbf{e}_n$, so the image of the map $\Lambda(\cdot, \cdot, \mathbf{v})$ is $\mathscr{E}_n$.

Theorems 1 and 2 both follow from this Theorem.

Theorem 3. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$, let $\varepsilon > 0$. Then there are measures*

$$\mu^{(\mathbf{e}_n)}, \nu^{(\mathbf{e}_n)}$$

on $\mathscr{E}_n \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n$ such that, denoting by $(\mathbf{v}_k)_{k\in\mathbb{N}}, (\mathbf{w}_k)_{k\in\mathbb{N}}$ the sequence of best approximations and ε-approximations of $\theta \in \mathbb{R}^d$, the sequences

$$(\rho_{\mathscr{E}_n}(\mathbf{v}_k),\, \mathrm{disp}(\theta, \mathbf{v}_k),\, \mathbf{v}_k)_{k\in\mathbb{N}} \in \mathscr{E}_n \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n$$
$$(\rho_{\mathscr{E}_n}(\mathbf{w}_k),\, \mathrm{disp}(\theta, \mathbf{w}_k),\, \mathbf{w}_k)_{k\in\mathbb{N}} \in \mathscr{E}_n \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n$$

equidistribute with respect to $\mu^{(\mathbf{e}_n)}$ and $\nu^{(\mathbf{e}_n)}$ respectively for Lebesgue a.e. θ. Furthermore, the properties of these measures, listed in Theorems 1 and 2, remain valid, provided we replace everywhere $\mathscr{X}_d$ with $\mathscr{E}_n$.

Chapter 11 – Properties of the cross-section measures (Case I)

We let $\overline{B}_{r_0} \subset \mathbb{R}^d$ denote the closed ball centered at $0 \in \mathbb{R}^d$, with respect to our chosen norm (note that the norm is suppressed from the notation). Consider the map

$$\varphi : \mathscr{E}_n \times \overline{B}_{r_0} \to \mathcal{S}_{r_0}, \quad \varphi(\Lambda, v) = u(v)\Lambda.$$

Note that the map $v \mapsto u(v)\mathbf{e}_n$ is a bijection between $\overline{B}_{r_0}$ and D_{r_0}. It follows that φ is onto $\mathcal{S}_{r_0}$, and

$$\text{for any } r \in (0, r0), \quad \varphi(\mathscr{E}_n \times \overline{B}_r) = \mathcal{S}_r.$$

Furthermore, for $\Lambda \in \mathcal{S}_{r_0}$,

$$\#\varphi^{-1}(\Lambda) = \#(\Lambda_{\mathrm{prim}} \cap D_{r_0}).$$

Indeed, for any $v \in \Lambda_{\mathrm{prim}} \cap D_{r_0}$,

$$\big(u(\pi_{R^d}(v))^{-1}\Lambda, \pi_{R^d}(v)\big) \in \varphi^{-1}(\Lambda),$$

and this assignment is easily seen to be a bijection. Let

$$\psi : \mathcal{S}^{\#}_{r_0} \to \mathscr{E}_n \times \overline{B}_{r_0},$$
$$\psi(\Lambda) = (u(v_\Lambda)^{-1}\Lambda, v_\Lambda),$$

where

$$v_\Lambda = \pi_{R^d}(v(\Lambda)) \text{ and } \{v(\Lambda)\} = \Lambda \cap D_{r_0}.$$

Proposition 4. *In Case I,*

$$\mu_{\mathcal{S}_{r_0}} = \frac{1}{\zeta(n)} \varphi_*(m_{\mathscr{E}_n} \times m_{\mathbb{R}^d}|_{\overline{B}_{r_0}})$$

(where $\zeta(n) = \sum_{k\in\mathbb{N}} k^{-n}$). In particular, $\mu_{\mathcal{S}_{r_0}}$ is finite and $\mathrm{supp}(\mu_{\mathcal{S}_{r_0}}) = \mathcal{S}_{r_0}$.

Let $\mathscr{X}_n^{\mathbb{A}}$ be the adelic space, let $\pi : \mathscr{X}_n^{\mathbb{A}} \to \mathscr{X}_n$ be the projection, and let $m_{\mathscr{X}_n^{\mathbb{A}}}$ be a $\mathrm{SL}_(\mathbb{A})$-invariant probability measure on $\mathscr{X}_n^{\mathbb{A}}$.

Let $\widetilde{\mathcal{S}}^{\#}_{r_0} = \pi^{-1}(\mathcal{S}^{\#}_{r_0})$. We augment ψ and define a map

$$\widetilde{\psi} : \widetilde{\mathcal{S}}^{\#}_{r_0} \to \mathscr{E}_n \times \overline{B}_{r_0} \times \widehat{\mathbb{Z}}^n \ \text{ by } \ \widetilde{\psi} = (\psi \circ \pi, \psi_f).$$

I am not defining ψ_f here. Just think of it as nice continuous map to $\widehat{\mathbb{Z}}^n$.

We now describe the image of the cross-section measure under $\widetilde{\psi}$. Given an $\{a_t\}$-invariant measure μ on $\mathscr{X}_n^{\mathbb{A}}$, let $\mu_{\widetilde{\mathcal{S}}_{r_0}}$ be the cross-section measure, and define a measure on $\mathscr{E}_n \times \mathbb{R}^d \times \widehat{\mathbb{Z}}^n$ by

$$\nu = \widetilde{\psi}_* \mu_{\widetilde{\mathcal{S}}_{r_0}}.$$

Let $\nu^{(\mathscr{E}_n)}$, $\nu^{(\mathbb{R}^d)}$, $\nu^{(f)}, \nu^{(\infty)}$, $\nu^{(\mathbb{S}^{d-1})}$, $\nu^{(\mathscr{X}_d)}$ denote the projection of ν to $\mathscr{E}_n$, $\mathbb{R}^d$, $\widehat{\mathbb{Z}}^n$, $\mathscr{E}_n \times \mathbb{R}^d$, $\mathbb{S}^{d-1}$, $\mathscr{X}_d$ respectively.

Theorem 5. *In Case I, with $\mu = m_{\mathscr{X}_n^{\mathbb{A}}}$, we have*

$$\nu = \frac{1}{\zeta(n)} \left(m_{\mathscr{E}_n} \times m_{\mathbb{R}^d}|_{\overline{B}_{r_0}} \times m_{\widehat{\mathbb{Z}}^n_{\mathrm{prim}}} \right);$$

in particular, the measures $\nu^{(\mathscr{E}_n)}, \nu^{(\mathbb{R}^d)}, \nu^{(f)}, \nu^{(\mathscr{X}_d)}$ are scalar multiples of the measures $m_{\mathscr{E}_n}, m_{\mathbb{R}^d}|_{\overline{B}_{r_0}}, m_{\widehat{\mathbb{Z}}^n_{\mathrm{prim}}}, m_{\mathscr{X}_d}$ and the measures $\nu^{(\mathbb{R}^d)}$ and $\nu^{(\mathcal{S}^{d-1})}$ are invariant under any linear transformations of $\mathbb{R}^d$ preserving the norm $\|\cdot\|$.

Let

$$\lambda_\varepsilon = \widetilde{\psi}_* \left(\mu_{\widetilde{\mathcal{S}}_{r_0}} \Big|_{\widetilde{\mathcal{S}}_\varepsilon} \right), \quad \lambda_{\mathrm{best}} = \widetilde{\psi}_* \left(\mu_{\widetilde{\mathcal{S}}_{r_0}} \Big|_{\widetilde{\mathcal{B}}} \right)$$

Let $\lambda^{(\mathscr{E}_n)}, \lambda^{(\mathbb{R}^d)}, \lambda^{(f)}, \lambda^{(\infty)}, \lambda^{(\mathbb{S}^{d-1})}, \lambda^{(\mathscr{X}_d)}$ denote the projection of λ to $\mathscr{E}_n$, $\mathbb{R}^d$, $\widehat{\mathbb{Z}}^n$, $\mathscr{E}_n \times \mathbb{R}^d$, $\mathbb{S}^{d-1}$, $\mathscr{X}_d$ respectively.

Proposition 6 (Case I, ε-approximations)**.** *Let $\varepsilon \in (0, r_0)$. Let $\mu = m_{\mathscr{X}_n^{\mathbb{A}}}$. Let B_ε denote the ball of radius ε around the origin in $\mathbb{R}^d$ and let $V_d = m_{\mathbb{R}^d}(B_\varepsilon)$. Then*

$$\lambda_\varepsilon^{(\mathscr{E}_n)} = \frac{\varepsilon^d V_d}{\zeta(n)} m_{\mathscr{E}_n}, \qquad \lambda_\varepsilon^{(\mathbb{R}^d)} = \frac{1}{\zeta(n)}\, m_{\mathbb{R}^d}|_{B_\varepsilon},$$

$$\lambda_\varepsilon^{(f)} = \frac{\varepsilon^d V_d}{\zeta(n)} m_{\widehat{\mathbb{Z}}^n_{\mathrm{prim}}}, \qquad \lambda_\varepsilon^{(\mathscr{X}_d)} = \frac{\varepsilon^d V_d}{\zeta(n)} m_{\mathscr{X}_d}.$$

The measures $\lambda_\varepsilon^{(\mathbb{R}^d)}$, $\lambda_\varepsilon^{(\mathbb{S}^{d-1})}$ are preserved by any linear transformations of $\mathbb{R}^d$ preserving the norm $\|\cdot\|$.

For the best approximations, let

$$\widehat{\mathcal{B}} = \psi(\mathcal{B}) \subset \mathscr{E}_n \times \mathbb{R}^d,$$

so that

$$(\Lambda, v) \in \widehat{\mathcal{B}} \Leftrightarrow \varphi(\Lambda, v) \in \mathcal{B},$$

and denote the indicator function of $\widehat{\mathcal{B}}$ by $\mathbf{1}_{\widehat{\mathcal{B}}}$.

Proposition 7. *Let λ_{best} be the measure (best approximations). Then $\lambda_{\mathrm{best}} = \lambda_{\mathrm{best}}^{(\infty)} \times \lambda_{\mathrm{best}}^{(f)}$, where $\lambda_{\mathrm{best}}^{(f)}$ is a multiple of $m_{\widehat{\mathbb{Z}}^n_{\mathrm{prim}}}$. The measure $\lambda_{\mathrm{best}}^{(\infty)}$ is absolutely continuous with respect to the measure $\nu^{(\infty)}$, and the Radon-Nikodym derivative is given by*

$$\frac{d\lambda_{\mathrm{best}}^{(\infty)}}{d\nu^{(\infty)}}(\Lambda, v) = \mathbf{1}_{\widehat{\mathcal{B}}}(\Lambda, v).$$

Proposition 8. *In Case I, for best approximations, the measures*

$$\lambda_{\mathrm{best}}^{(\mathscr{E}_n)},\ \lambda_{\mathrm{best}}^{(\mathbb{R}^d)},\ \lambda_{\mathrm{best}}^{(\mathbb{S}^{d-1})},\ \lambda_{\mathrm{best}}^{(\mathscr{X}_d)}$$

are absolutely continuous with respect to $m_{\mathscr{E}_n}$, $m_{\mathbb{R}^d}$, $m_{\mathbb{S}^{d-1}}$, $m_{\mathscr{X}_d}$, *and we have the following formulae for the Radon-Nikodym derivatives:*

$$\frac{d\lambda_{\mathrm{best}}^{(\mathscr{E}_n)}}{dm_{\mathscr{E}_n}}(\Lambda) = \frac{1}{\zeta(n)} \int_{B_{r_0}} \mathbf{1}_{\widehat{\mathcal{B}}}(\Lambda, v)\, dm_{\mathbb{R}^d}(v),$$
$$\frac{d\lambda_{\mathrm{best}}^{(\mathbb{R}^d)}}{dm_{\mathbb{R}^d}}(v) = \frac{1}{\zeta(n)} \int_{\mathscr{E}_n} \mathbf{1}_{\widehat{\mathcal{B}}}(\Lambda, v)\, dm_{\mathscr{E}_n}(\Lambda),$$
$$\frac{d\lambda_{\mathrm{best}}^{(\mathscr{X}_d)}}{dm_{\mathscr{X}_d}}(\Lambda') = \frac{1}{\zeta(n)} \int_{B_{r_0}} \int_{\mathbb{T}_{\Lambda'}} \mathbf{1}_{\widehat{\mathcal{B}}}(\Lambda(\Lambda', f, \mathbf{e}_n), v)\, dm_{\mathbb{T}_{\Lambda'}}(f)\, dm_{\mathbb{R}^d}(v),$$

where $\mathbb{T}_{\Lambda'} = \left(\mathbb{R}^d\right)^* / (\Lambda')^*$ *and* $\Lambda(\cdot)$ *as in (2), and for some* $c > 0$,

$$\frac{d\lambda_{\mathrm{best}}^{(\mathbb{S}^{d-1})}}{dm_{\mathbb{S}^{d-1}}}(w) = c \int_0^{r_0} t^{d-1} \int_{\mathscr{E}_n} \mathbf{1}_{\widehat{\mathcal{B}}}(\Lambda, tw)\, dm_{\mathscr{E}_n}(\Lambda)\, dt.$$

Proposition 9. *In Case I, for best approximations, the measures*

$$\lambda_{\mathrm{best}}^{(\infty)}, \lambda_{\mathrm{best}}^{(\mathscr{E}_n)},\ \lambda_{\mathrm{best}}^{(\mathbb{R}^d)},\ \lambda_{\mathrm{best}}^{(\mathbb{S}^{d-1})},\ \lambda_{\mathrm{best}}^{(\mathscr{X}_d)}$$

satisfy:

- *(a)* $\lambda_{\mathrm{best}}^{(\infty)}$ *is not a scalar multiple of* $\lambda_{\mathrm{best}}^{(\mathscr{E}_n)} \times \lambda_{\mathrm{best}}^{(\mathbb{R}^d)}$.
- *(b) The measures* $\lambda_{\mathrm{best}}^{(\mathscr{E}_n)}$, $\lambda_{\mathrm{best}}^{(\mathbb{S}^{d-1})}$, $\lambda_{\mathrm{best}}^{(\mathscr{X}_d)}$ *have full support, and the support of* $\lambda_{\mathrm{best}}^{(\mathbb{R}^d)}$ *contains a neighborhood of the origin.*
- *(c) For* $d > 1$, *there is* $c > 0$ *such that for any* $\Lambda \in \mathscr{E}_n$ *and any* $\Lambda' \in \mathscr{X}_d$,

  $$\frac{d\lambda_{\mathrm{best}}^{(\mathscr{E}_n)}}{dm_{\mathscr{E}_n}}(\Lambda) \leqslant c \cdot \mathrm{sys}(\pi_{\mathbb{R}^d}(\Lambda))^d \ \text{ and } fracd\lambda_{\mathrm{best}}^{(\mathscr{X}_d)} dm_{\mathscr{X}_d}(\Lambda') \leqslant c \cdot \mathrm{sys}(\Lambda')^d,$$

  where $\mathrm{sys}(\Lambda')$ *is the length of the shortest nonzero vector of* $\Lambda' \in \mathscr{X}_d$. *In particular* $\lambda_{\mathrm{best}}^{(\mathscr{E}_n)}$ *and* $\lambda_{\mathrm{best}}^{(\mathscr{X}_d)}$ *are not scalar multiplies of* $m_{\mathscr{E}_n}$ *and* $m_{\mathscr{X}_d}$.
- *(d) For any* $\Lambda \in \mathscr{E}_n$ *with no nonzero horizontal vectors, and any* $w \in \mathbb{S}^{d-1}$, *the function*

  $$t \mapsto \frac{d\lambda_{\mathrm{best}}^{(\mathbb{R}^d)}}{dm_{\mathbb{R}^d}}(tw)$$

  is monotone non-increasing, is not a.e. an indicator function, and $\mathrm{supp}\,\lambda^{(\mathbb{R}^d)}$ *is star-shaped around the origin.*
- *(e) The measures* $\lambda_{\mathrm{best}}^{(\mathbb{R}^d)}, \lambda_{\mathrm{best}}^{(\mathbb{S}^{d-1})}$ *are invariant under any linear transformation of* $\mathbb{R}^d$ *preserving the norm* $\|\cdot\|$. *In particular, for the Euclidean norm,* $\lambda_{\mathrm{best}}^{(\mathbb{R}^d)}$ *and* $\lambda_{\mathrm{best}}^{(\mathbb{S}^{d-1})}$ *are* $\mathrm{SO}_d(\mathbb{R})$*-invariant.*