SELECTED APPLICATIONS OF THE VARIATIONAL PRINCIPLE

1. Recall

(1) The successive minima function: for a matrix A, $h = (h_1, \dots, h_d) : [0, \infty) \to \mathbb{R}^d$ given by
$$h_i(t) = \log \lambda_i(g_t u_A \mathbb{Z}^d),$$
where λ_i is the minimum λ so that $\{r \in g_t u_A \mathbb{Z}^d : \|\mathbf{r}\| \le \lambda\}$ contains i linearly independent vectors.

(2) Dani correspondence: A is singular iff $\liminf_{t\to\infty} -h_1(t) = \infty$.

(3) An $m \times n$ *template* is a continuous piecewise linear map $f : [0, \infty) \to \mathbb{R}^d$ with
   (a) $f_1 \le f_2 \le \cdots \le f_d$
   (b) $-\frac{1}{n} \le f_i' \le \frac{1}{m}$ wherever the derivative exists
   (c) for all $0 \le j \le d$ and every interval I such that $f_j < f_{j+1}$ on I,
$$F_j := \sum_{1 \le i \le j} f_i$$
   is a convex and continuous piecewise linear function on I with slopes in $Z(j)$, where
$$Z(j) = \left\{ \frac{L_+}{m} - \frac{L_-}{n} : L_+ \in [0, m]_{\mathbb{Z}}, L_- \in [0, n]_{\mathbb{Z}}, L_+ + L_- = j \right\}.$$
   Here, $S_{\mathbb{Z}} := S \cap \mathbb{Z}$. Note the convention $f_0 = -\infty$ and $f_{d+1} = +\infty$.

   Note in particular that $F_d = f_1 + \cdots + f_d$ has slope 0 by (c), so every template adds to a constant.

2. Theorem 3.12

Theorem 2.1 (Thm 3.12)**.** *If ψ is such that*
$$q^{n/m}\psi(q) \to 0 \text{ as } q \to \infty,$$
then the set of $m \times n$ singular matrices that are not ψ-approximable has Hausdorff dimension $\Delta_{m,n}$.

Equivalently, if
$$\phi(t) \to \infty \text{ as } t \to \infty,$$
then the set of $m \times n$ singular matrices A satisfying
$$-\log \lambda_1(g_t u_A \mathbb{Z}^d) \le \phi(t)$$

for all sufficiently large $t \geq 0$ has Hausdorff dimension $\delta_{m,n}$. The same is true for the packing dimension.

First, let's understand the justification for the equivalent statement.

Recall: an $m \times n$ matrix A is **singular** if for all $\varepsilon > 0$, there exists $Q_\varepsilon > 0$ such that for all $Q \geq Q_\varepsilon$, there exists $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^m$ such that $0 < \|q\| \leq Q$ and

$$\|A\mathbf{q} + \mathbf{p}\| \leq \varepsilon Q^{-n/m}.$$

And A is <u>not</u> ψ-approximable if for all but finitely many $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n$ with $\mathbf{q} \neq 0$,

$$\|A\mathbf{q} + \mathbf{p}\| > \psi(\|\mathbf{q}\|).$$

Thus, we see that if A is both singular and not ψ-approximable, then for sufficiently large Q_ε, there exists $\mathbf{p}, \mathbf{q}$ as in the definition of singularity of A so that

$$\psi(\|\mathbf{q}\|) < \|A\mathbf{q} + \mathbf{p}\| \leq \varepsilon Q^{-n/m}.$$

Equivalently,

$$\psi(\|q\|)Q^{n/m} < \|A\mathbf{q} + \mathbf{p}\| \leq \varepsilon.$$

Since $0 < \|q\| \leq Q$, the lower bound is in turn bounded below by $q^{n/m}\psi(q)$, which tends to zero by assumption.

Also recall from Daniel's talk:

Theorem 2.2 (Thm 3.1)**.** *For all $(m, n) \neq (1, 1)$, we have that*

$$\dim_H(\mathrm{Sing}(m, n)) = \dim_P(\mathrm{Sing}(m, n)) = \delta_{m,n} = mn\left(1 - \frac{1}{m + n}\right).$$

Thus, for the proof of Theorem 3.12, we need only show that the dimension is bounded below by $\delta_{m,n}$.

We will also need to recall the variational principle. Recall that for every template $\mathbf{f}$, we have seen that

$$D(\mathbf{f}) := \{A : \mathbf{h}_A \asymp_+ \mathbf{f}\}$$

is nonempty, and for a collection $\mathcal{F}$ of templates, we say that $\mathcal{F}$ is **closed under finite perturbations** if

$$\mathbf{g} \asymp_+ \mathbf{f} \in \mathcal{F} \implies \mathbf{g} \in \mathcal{F}.$$

In particular, we saw that if $\mathcal{F}$ is the collection of singular templates, it is closed under finite perturbations.

Theorem 2.3 (Thm 4.3, Variational principle version 1)**.** *Let $\mathcal{F}$ be a (Borel) collection of templates closed under finite perturbations. Then*

$$\dim_H(D(\mathcal{F})) = \sup_{\boldsymbol{f} \in \mathcal{F}} \underline{\delta}(\boldsymbol{f}), \quad \dim_P(D(\mathcal{F})) = \sup_{\boldsymbol{f} \in \mathcal{F}} \overline{\delta}(\boldsymbol{f}),$$

where $\underline{\delta}, \overline{\delta}$ will be defined in this document as their use arises (but in the paper are defined in Definition 4.5).

It then follows immediately from the variational principle that in order to establish a lower bound on the Hausdorff dimension in Theorem 3.12, it is sufficient to construct a sequence of templates with $\underline{\delta}(\mathbf{f})$ converging to $\delta_{m,n}$. This will be achieved using standard templates.

2.1. **Standard templates.** Recall Definition 9.1 of a **standard template** defined by $(t_k, -\varepsilon_k)$ and $(t_{k+1}, -\varepsilon_{k+1})$:

Fix $0 \leq t_k < t_{k+1}$ and $\varepsilon_k, \varepsilon_{k+1} \geq 0$. Let $\Delta t = t_{k+1} - t_k$ and $\Delta\varepsilon_k = \varepsilon_{k+1} - \varepsilon_k$. Assume:

- $\frac{-1}{m}\Delta t \leq \Delta\varepsilon \leq \frac{1}{n}\Delta t$
- $\Delta\varepsilon \geq -\frac{n-1}{2n}\Delta t$ if $m = 1$ and $\Delta\varepsilon \leq \frac{m-1}{2m}\Delta t$ if $n = 1$
- either:
  (1) $(n-1)(\frac{1}{n}\Delta t - \Delta\varepsilon) \geq d\varepsilon_k$ or
  (2) $(m-1)(\frac{1}{m}\Delta t + \Delta\varepsilon) \geq d\varepsilon_{k+1}$.

  Then the **standard template** defined by $(t_k, -\varepsilon_k)$ and $(t_{k+1}, -\varepsilon_{k+1})$ is the partial template $\mathbf{f} : [t_k, t_{k+1}] \to \mathbb{R}^d$ defined as follows:

  (1) Let $g_1, g_2 : [t_k, t_{k+1}] \to \mathbb{R}$ be piecewise linear functions such that
  $$g_i(t_j) = -\varepsilon_j$$
  for $j = k, k+1$ and g_i each have two intervals of linearity: one on which $g_i' = \frac{1}{m}$ and another on which $g_i' = -1/n$. For $i = 1$, it's $-1/n$ first and then $1/m$, and for $i = 2$ it is in the other order. These functions exist because of the first assumption about the t_k, ε_k's. Finally, let $g_3 = \cdots = g_d$ be chosen so that
  $$g_1 + \cdots + g_d = 0.$$
  (2) For each $t \in [t_k, t_{k+1}]$, let $\mathbf{f}(t) = \mathbf{g}(t)$ if $g_2(t) \leq g_3(t)$. Otherwise, let $f_1(t) = g_1(t)$ and let $f_2(t) = \cdots = f_d(t)$ be chosen so that $f_1 + \cdots + f_d = 0$.

We denote this standard template by
$$s[(t_k, -\varepsilon_k), (t_{k+1}, -\varepsilon_{k+1})].$$

Lemma 2.4. *The map* $(\varepsilon_1, \varepsilon_2) \mapsto \Delta(s[(0, -\varepsilon_1), (1, -\varepsilon_2)], 1)$ *is continuous.*

Proof of Theorem 3.12. Without loss of generality, assume that ϕ is increasing and $\phi(t) \to \infty$. Then since $\phi(t) \to \infty$, we can find points $(t_k, -\varepsilon_k)$ satisfying:

(1) $\Delta t_k \leq \frac{1}{2}\phi(t_k)$ for all k
(2) $\varepsilon_k \leq \frac{1}{2}\phi(t_k)$ for all k
(3) $\varepsilon_k \to \infty$ as $k \to \infty$
(4) $\frac{\varepsilon_k}{\Delta t_k} \to 0$ and $\frac{\varepsilon_{k+1}}{\Delta t_k} \to 0$ as $k \to \infty$.

(Basically, we need to grow Δt_k much faster than $\Delta\varepsilon_k$: but that's OK, there is room to do so because $\phi(t) \to \infty$).)

We now consider the standard template $\mathbf{f}$ corresponding to this sequence of points, i.e. stick together the one for each pair of points. Then $f_1(t) = g_1(t)$, where we recall from the construction of the standard template that g_1 first has $g_1' = -1/n$ and then $g_1' = 1/m$.

Note that the first two conditions imply that $f_1(t) \geq -\phi(t_k) \geq -\phi(t)$ for all $t \in [t_k, t_{k+1}]$. To see this, observe that by the construction of f_1, a lower bound is given by

$$-\varepsilon_k - \frac{1}{n}\Delta t_k,$$

simply because this is the maximum amount the line of slope $-1/n$ can decrease to on the interval $[t_k, t_{k+1}]$. (In fact, f_1 does not reach this lower bound, since it must have slope $1/m$ somewhere.) Thus, by the first two assumptions, for any $t \in [t_k, t_{k+1}]$,

$$\begin{aligned} f_1(t) &\geq -\varepsilon_k - \frac{1}{n}\Delta t_k \\ &\geq -\frac{1}{2}\phi(t_k) - \frac{1}{2n}\phi(t_k) \\ &= -\frac{n+1}{2n}\phi(t_k) \\ &\geq -\phi(t), \end{aligned}$$

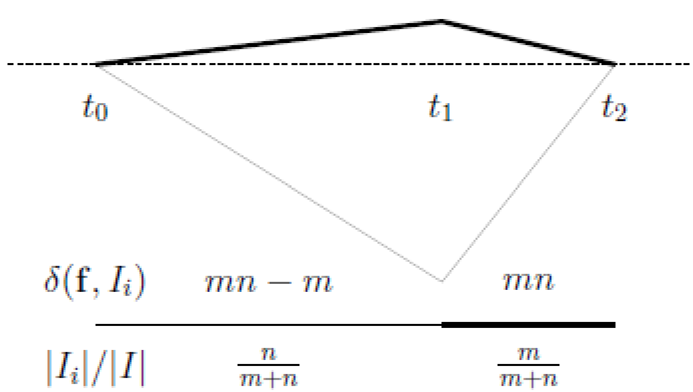
where the last line follows because $-\phi$ is decreasing.

Since $\varepsilon_k \to \infty$, $f_1(t) \to -\infty$ as $t \to \infty$, which implies that $\mathbf{f}$ is singular (i.e. corresponds to a singular matrix), but at rate bounded by $-\phi(t)$ by the above computation.

It remains to verify that $\underline{\delta}(\mathbf{f}) = \delta_{m,n}$. First, that by the fourth condition and Lemma 2.4, (

$$\begin{aligned} \Delta(\mathbf{f}, [t_k, t_{k+1}]) &= \Delta\left(s\left[\left(0, -\frac{\varepsilon_k}{\Delta t_k}\right), \left(0, -\frac{\varepsilon_{k+1}}{\Delta t_k}\right)\right], 1\right) \\ &\to \Delta(s[(0,0),(1,0)],1). \end{aligned}$$

We will understand this via Figure 4 on p.36 of the paper:



Recall that

$$\Delta(\mathbf{f},T) := \frac{1}{T}\int_0^T \delta(f,t)dt,$$
$$\underline{\delta}(\mathbf{f}) = \liminf_{T\to\infty} \Delta(\mathbf{f},T),$$
$$\delta(f,I) = \#\{(i_+,i_-) \in S_+ \times S_- : i_+ < i_-\},$$

where I is an interval of equality for $\mathbf{f}$ (that is, it's $(p,q]_{\mathbb{Z}}$ so that

$$f_p < f_{p+1} = \cdots = f_q < f_{q+1}$$

on I.) Also,

$$S_+ := \bigcup_{(p,q]_{\mathbb{Z}}} (p, p+M_+(p,q)]_{\mathbb{Z}}$$

and

$$S_- := \bigcup_{(p,q]_{\mathbb{Z}}} (p+M_+(p,q), q]_{\mathbb{Z}}$$

where in both cases, the union is taken over all intervals of equality of $\mathbf{f}$. Recall/observe that intervals of equality will always partition $\{1,\dots,d\}$. In turn, we must still recall that $M_\pm(p,q)$ are the unique integers such that

$$M_+ + M_- = q-p \text{ and } f'_{p+1} + \cdots + f'_q = \frac{M_+}{m} - \frac{M_-}{n} \text{ on } I.$$

We see that $\mathbf{f}$ has two intervals of equality on both intervals of linearity: $(0,1]_{\mathbb{Z}}$ and $(1,d]_{\mathbb{Z}}$. On the first interval of linearity, $f'_1 = -1/n$. Since $f_1 + \cdots + f_d$ must have slope in $Z(d) = \{0 = m/m - n/n\}$, we know that $f_2 + \cdots + f_d$ must have slope $1/n$.

Thus, on $(0,1]_{\mathbb{Z}}$, we compute: $M_+(0,1) + M_-(0,1) = 1$ and must satisfy

$$\frac{1}{n} = f'_1 = \frac{M_+(0,1)}{m} - \frac{M_-(0,1)}{n}.$$

Thus,

$$M_+(0,1) = 0, M_-(0,1) = 1.$$

Similarly, on $(1,d]_{\mathbb{Z}}$, $M_+(1,d) + M_-(1,d) = d-1$ and

$$\frac{1}{n} = f'_2 + \cdots + f'_d = \frac{M_+(1,d)}{m} - \frac{M_-(1,d)}{n}.$$

Thus, $M_+(1,d) = m$ and $M_-(1,d) = n-1$.

So, we compute that

$$S_+ = (0,0]_{\mathbb{Z}} \cup (1,1+m]_{\mathbb{Z}} = \{2,\dots,m+1\}$$

and

$$S_- = (0+0,1]_{\mathbb{Z}} \cup (1+m,d]_{\mathbb{Z}} = \{1, m+2, \dots, d\}.$$

Thus,

$$\begin{aligned}\delta(\mathbf{f}, I_1) &= \#\{(i_+, i_-) \in S_+ \times S_- : i_+ < i_-\}\\ &= \#S_+ \cdot (\#S_- - 1)\\ &= (m+1-1)(d-(m+2)+1)\\ &= m(n-1)\\ &= mn - n\end{aligned}$$

A similar computation shows that we get mn on interval I_2.

Moreover, a direct computation shows that I_1 has proportion $\frac{n}{m+n}$ and I_2 has proportion $\frac{m}{m+n}$. Thus, we can compute the average

$$\Delta(\mathbf{f}, [t_k, t_{k+1}]) = \frac{n}{m+n}(mn-m) + \frac{m}{m+n}mn = \delta_{m,n}.$$

□

3. Theorem 3.14

Schmidt conjectured that for all $2 \le k \le m$, there exists an $m \times 1$ matrix A such that

$$\lambda_{k-1}(g_t u_A \mathbb{Z}^d) \to 0 \text{ but } \lambda_{k+1}(g_t u_A \mathbb{Z}^d) \to \infty \text{ as } t \to \infty.$$

This was proved by Moschevitin, who proved that there is such an $m \times 1$ matrix which is not contained in any rational hyperplane (turns out to be obvious for any $(\mathbf{x}, 0)$ with $\mathbf{x} \in \mathbb{R}^{k-1}$ or $\mathbb{R}^{k-2}$ is a BA vector).

This can be extended to the matrix framework. We say that an $m \times n$ matrix is **k-singular** for $2 \le k \le m+n-1$ if

$$\lambda_{k-1}(g_t u_A \mathbb{Z}^d) \to 0 \text{ and } \lambda_{k+1}(g_t u_A \mathbb{Z}^d) \to \infty \text{ as } t \to \infty.$$

Theorem 3.14 improves Moschevitin's result by computing a lower bound on the Hausdorff dimension.

Theorem 3.1 (Thm 3.14)**.** *For all $(m, n) \neq (1, 1)$ and for all $2 \le k \le m+n-1$, the Hausdorff dimension of the set of matrices A that satisfy*

$$\lambda_{k-1}(g_t u_A \mathbb{Z}^d) \to 0 \text{ and } \lambda_{k+1}(g_t u_A \mathbb{Z}^d) \to \infty \text{ as } t \to \infty$$

is at least

$$\max(f_{m,n}(k), f_{m,n}(k-1)),$$

where

$$f_{m,n}(k) := mn - \frac{k(m+n-k)mn}{(m+n)^2} - \left\{\frac{km}{m+n}\right\}\left\{\frac{kn}{m+n}\right\}.$$

Here, $\{x\}$ denotes the fractional part of a real number x.

This is conjectured to be optimal.

Proof. By alternating long $S_j^{\pm}$ intervals with short intervals along which f_k crosses 0 and returns, we can construct a template $\mathbf{f}$ satisfying, for fixed $2 \le k \le d-1$ and $j \in \{k-1, k\}$:

(1) $f_{k-1}(t) \to -\infty$ as $t \to \infty$,
(2) $f_{k+1}(t) \to +\infty$ as $t \to \infty$,
(3) $\frac{1}{t}\mathbf{f}(t) \to 0$ as $t \to \infty$,
(4) $\frac{1}{T}\lambda([0,T] \cap (S_j^+ \cup S_j^-)) \to 1$ as $T \to \infty$, where S_j^+ (respectively S_j^-) is the set of all times $t \geq 0$ such that:
  - $f_1(t) = \cdots = f_j(t) < f_{j+1}(t) = \cdots = f_d(t)$,
  - $(L_+, L_-) = (\lceil \frac{jm}{d} \rceil, \lfloor \frac{jn}{d} \rfloor)$ (respectively $(L_+, L_-) = (\lfloor \frac{jm}{d} \rfloor, \lceil \frac{jn}{d} \rceil)$), where $L_\pm = L_\pm(\mathbf{f}, t, j)$.

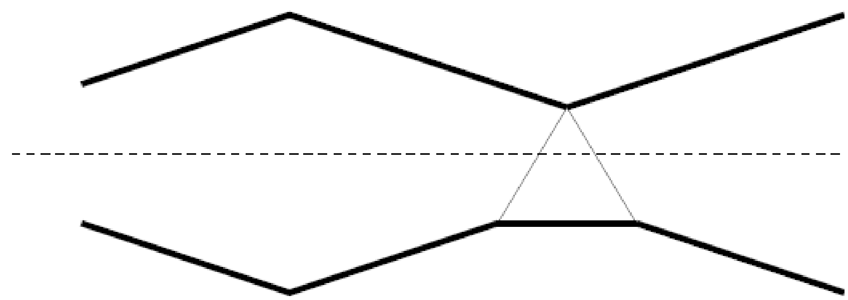


FIGURE 13. A piece of a template f with the desired properties, as described in §23 (Proof of Theorem 3.14). The triangular portion of the figure can be made arbitrarily small in proportion to the rest.

The key idea of the construction of the template is that if $t \in S_j^+$, then $f_1'(t) \geq 0$ and if $t \in S_j^-$, then $f_1'(t) \leq 0$, with equality iff $jm/d \in \mathbb{Z}$.

By the variational principle, to compute the lower bound on the Hausdorff dimension, we need to compute $\underline{\delta}(\mathbf{f})$.

Recall that

$$\underline{\delta}(\mathbf{f}) = \liminf_{T\to\infty} \Delta(\mathbf{f}, T),$$
$$\Delta(\mathbf{f}, T) := \frac{1}{T}\int_0^T \delta(f,t)dt,$$
$$\delta(f, I) = \#\{(i_+, i_-) \in S_+ \times S_- : i_+ < i_-\},$$

where I is an interval of equality for $\mathbf{f}$ (that is, it's $(p,q]_\mathbb{Z}$ so that

$$f_p < f_{p+1} = \cdots = f_q < f_{q+1}$$

on I.) Also,

$$S_+ := \bigcup_{(p,q]_\mathbb{Z}} (p, p + M_+(p,q)]_\mathbb{Z}$$

and

$$S_- := \bigcup_{(p,q]_{\mathbb{Z}}} (p + M_+(p,q), q]_{\mathbb{Z}}$$

where in both cases, the union is taken over all intervals of equality of $\mathbf{f}$.

By the construction of the template, there are long intervals of equality that are either of the form S_j^+ or S_j^-: on each one, the calculation for $\delta(\mathbf{f}, I)$ is identical. So, we simply call these quantities $\delta(\mathbf{f}, S_j^+)$ and $\delta(\mathbf{f}, S_j^-)$.

So we have that

$$\Delta(\mathbf{f}, T)$$
$$= \frac{1}{T}\lambda([0,T] \cap S_j^+)\delta(\mathbf{f}, S_j^+) + \frac{1}{T}\lambda([0,T] \cap S_j^-)\delta(\mathbf{f}, S_j^-) + \frac{1}{T}\lambda(\text{the rest})\delta(\text{the rest})$$

By assumption, as $T \to \infty$,

$$\frac{1}{T}\lambda([0,T] \cap (S_j^+ \cup S_j^-)] = 1,$$

so "the rest", which arises from the triangles in the diagram, is irrelevant.

Computation of $\delta(\mathbf{f}, S_j^+)$**:** To make it align with the conclusion, we actually compute $mn - \delta(\mathbf{f}, S_j^+)$. Observe that this is equal to

$$\#\{(i_+, i_-) \in S_+ \times S_- : i+ > i_-\},$$

where S_+, S_- are as above. The intervals of equality are $(0, j]_{\mathbb{Z}}$ and $(j, d]_{\mathbb{Z}}$. So

$$S_+ = (0, 0 + M_+(0,j)]_{\mathbb{Z}} \cup (j, j + M_+(j,d)]_{\mathbb{Z}}$$

where

$$M_+(p,q) = L_+(q) - L_+(p).$$

The values of the L's are given in the definition of S_j^+!

$$M_+(0,j) = L_+(j) - L_+(0) = \lceil \frac{jm}{d} \rceil := L_+.$$

$$M_+(j,d) = L_+(d) - L_+(j) = m - L_+.$$

So, we conclude that

$$S^+ = (0, L_+]_{\mathbb{Z}} \cup (j, j + m - L_+]_{\mathbb{Z}}.$$

Similarly,

$$\begin{aligned} S_- &= (0 + M_+(0,j), j]_{\mathbb{Z}} \cup (j + M_+(j,d), d]_{\mathbb{Z}} \\ &= (L_+, j]_{\mathbb{Z}} \cup (j + m - L_+, d]_{\mathbb{Z}} \end{aligned}$$

Thus, $\delta(\mathbf{f}, S_j^+)$ is equal to the number of elements in $(j, j + m - L_+]_{\mathbb{Z}}$ multiplied by the number of elements in $(L_+, j]_{\mathbb{Z}}$.

This is

$$[(j + m - L_+) - (j+1) + 1][j - (L_+ + 1) + 1] = (m - L_+)(j - L_+) = L_-(m - L_+).$$

Thus, we have

$$
\begin{aligned}
mn - \delta(\mathbf{f}, S_j^+) &= L_-(m - L_+) \\
&= \left\lfloor \frac{jn}{d} \right\rfloor \left(m - \left\lceil \frac{jm}{d} \right\rceil \right) \\
&= \left(\frac{jn}{d} - \left\{ \frac{jn}{d} \right\} \right) \left(m - \frac{jm}{d} - \left\{ -\frac{jm}{d} \right\} \right)
\end{aligned}
$$

Note that if $x - y \in \mathbb{Z}$, then $\{x\} = \{y\}$. Thus,

$$\left\{ -\frac{jm}{d} \right\} = \left\{ \frac{jn}{d} \right\}.$$

So the above simplifies to

$$mn - \delta(\mathbf{f}, S_j^+) = \frac{j(d-j)mn}{d^2} - \frac{(d-j)m + jn}{d} \left\{ \frac{jn}{d} \right\} + \left\{ \frac{jn}{d} \right\}^2.$$

For $\delta(\mathbf{f}, S_j^-)$: The same ideas allow you to compute that

$$mn - \delta(\mathbf{f}, S_j^-) = \frac{j(d-j)mn}{d^2} + \frac{(d-j)m + jn}{d} \left\{ \frac{jm}{d} \right\} + \left\{ \frac{jm}{d} \right\}^2.$$

Putting it together:

We know that $f_1 + \cdots + f_d$ is a constant, by definition of a template ($Z(d) = \{0\}$, so this function has slope 0). For the idea of the proof, assume that there are no triangles. Then all the functions start at 0, they go up and down as in the figure, and then they meet again at zero.

Define

$$\alpha^+ := \lim_{T \to \infty} \frac{1}{T} \lambda([0,T] \cap S_j^+), \quad \alpha^- = \lim_{T \to \infty} \frac{1}{T} \lambda([0,T] \cap S_j^-).$$

Then the amount that the function f_1 goes up is $\alpha^+ f_1'(S_j^+)$ and the amount it goes down is $\alpha^- f_1(S_j^-)$.

Since it returns to 0,

$$\alpha^+ f_1'(S_j^+) + \alpha^- f_1(S_j^-) = 0.$$

This idea is actually precise, because of the assumption that the triangles become negligible. In particular,

$$\alpha^+ + \alpha^- = 1.$$

To put it together, we need to compute $f_1'(S_j^+)$ and $f_1'(S_j^-)$.

Computing f_1' on the sets:

First, observe that for $t \in S_j^+$,

$$f_1'(S_j^+) := f_1'(t) = \frac{1}{j} \left(\frac{\lceil \frac{jm}{d} \rceil}{m} - \frac{\lfloor \frac{jn}{d} \rfloor}{n} \right),$$

because the term in the brackets is the slope of $\sum_{i=1}^{j} f_i'$, and all these functions are equal by definition of S_j^+.

Observe that $\{x\}$ is somewhat counterintuitive for negative numbers, e.g. $\{-1.4\} = 0.6$. Using this, we see that

$$\begin{aligned}
f_1'(t) &= \frac{1}{j}\left(\frac{\frac{jm}{d} + \left\{-\frac{jm}{d}\right\}}{m} - \frac{\frac{jn}{d} - \left\{\frac{jn}{d}\right\}}{n}\right) \\
&= \frac{1}{jmn}\left(n\left\{-\frac{jm}{d}\right\} + m\left\{\frac{jn}{d}\right\}\right) \\
&= \frac{n+m}{jmn}\left\{\frac{jn}{d}\right\}
\end{aligned}$$

The last line follows because if $x - y \in \mathbb{Z}$, then $\{x\} = \{y\}$.

A similar computation gives that

$$f_1'(S_j^-) = \frac{-1}{j}\frac{m+n}{mn}\left\{\frac{jm}{d}\right\}.$$

Thus, by comparing the two, we see that

$$\alpha^+ = \left\{\frac{jm}{d}\right\}, \quad \alpha^- = \left\{\frac{jn}{d}\right\}.$$

Computing the result:

Because the triangles are negligible in the limit,

$$\begin{aligned}
\underline{\delta}(\mathbf{f}) &= \lim_{T\to\infty}\frac{1}{T}\Delta(\mathbf{f}, T) \\
&= \alpha^+\delta(\mathbf{f}, S_j^+) + \alpha^-\delta(\mathbf{f}, S_j^-)
\end{aligned}$$

Observe from the computations that the $\frac{(d-j)m+jn}{d}$ terms appear with opposite signs, and once we multiply by α^+, α^-, these terms will cancel out. Also recall that $\alpha^+ + \alpha^- = 1$.

So overall the result we get is

$$mn - \frac{j(d-j)mn}{d^2} - \left\{\frac{jm}{d}\right\}\left\{\frac{jn}{d}\right\}^2 - \left\{\frac{jn}{d}\right\}\left\{\frac{jm}{d}\right\}^2.$$

Note that the last two terms combine into

$$\left\{\frac{jm}{d}\right\}\left\{\frac{jn}{d}\right\}\left(\left\{\frac{jm}{d}\right\} + \left\{\frac{jn}{d}\right\}\right) = \left\{\frac{jm}{d}\right\}\left\{\frac{jn}{d}\right\}$$

and hence we see that indeed

$$\underline{\delta}(f) = f_{m,n}(j).$$

$\square$

Then the Hausdorff dimension of the overall set of such templates will be the maximum over $j = k-1, k$ because each set gives one type of behaviour of λ_k.