High entropy method, general case

k local field $\mathrm{char}(k)=0$, e.g. $k=\mathbb{R}$ or $\mathbb{Q}_p$

G k-points of a linear algebraic group over k

(i.e. $G < SL_d(k)$ a subgroup
and an algebraic set)

Assume G is $\underline{\text{simple}}$.
$= \{1\}$, G are the only normal algebraic subg.

(Rmk: $\underline{k\text{-simple}}$ is enough)
$\Leftrightarrow$ $\mathfrak{g} = \mathrm{Lie}(G)$ is simple
$\{0\}$ and $\mathfrak{g}$ are the only ideals in $\mathfrak{g}$.

Assume G is connected in usual topology.

$\Gamma < G$ discrete subgroup

$A \subset G$ k-points of a k-split torus.

(i.e. $A \subset \{\text{diagonal matrices}\} \cap G$
A isomorphic to $(k^\times)^r$ as an algebraic group
r is called rank of A.)

Example
$G = SL_3(\mathbb{R})$ A = full diagonal group $rk=2$

$G = SL_{5}(\mathbb{R})$ $A = \left\{ \begin{pmatrix} t & & & & \\ & t^2 & & & \\ & & ts & & \\ & & & s^3 & \\ & & & & t^{-4}s^{-4} \end{pmatrix} t, s \in \mathbb{R}^\times \right\}$ $rk=2$

$\underline{\text{Fact}}$ If V is an algebraic representation of A over k
then A is diagonalisable over k in $GL(V)$.

Assume $\mathrm{rank}(A) \geq 2$.

$\underline{\text{Theorem}}$ [Theorem 9.20] (entropy gap) Assume the above
$\forall a \in A \setminus \{1\}$ $\exists\, \varepsilon = \varepsilon(a) > 0$ s.t.

$\forall$ μ Borel prob on $X = G/\Gamma$, A-invariant and A-ergodic

If $h_\mu(a) > h_{m_X}(a) - \varepsilon$ (Haar on X) then $\mu = m_X$.

In $SL_3(\mathbb{R})$ it was $h_\mu(a) > \frac{1}{2} h_{m_X}(a)$

Rmk. $\varepsilon(a) = \min \underbrace{\mathbb{R}_{>0} \cap \{-\log \text{ eigenvalue of } Ad(a)\}}$.

$h_{m_X}(a) = \sum$ this set

(Rmk A can be replaced by $A^\circ = \exp(\mathfrak{a})$ identity component $\mathfrak{a} = \mathrm{Lie}(A)$)

(Rmk If G not connected then conclusion μ is G°-invariant)

Weights $A < G \overset{Ad}{\curvearrowright} \mathfrak{g}$ adjoint action algebraic representation

By the fact.

$$\mathfrak{g} = \bigoplus_{\lambda \in \{\text{character of } A\} \approx \mathbb{Z}^r} \mathfrak{g}^\lambda$$

$\mathfrak{g}^\lambda = \{x \in \mathfrak{g} \mid \forall a \in A \quad Ad(a)x = \lambda(a)x\}$

weight space

λ is called a weight if $\mathfrak{g}^\lambda \neq \{0\}$.

Example. $G = SL_3(\mathbb{R})$ A = full diagon.

$\mathfrak{g}^{[1]} = \begin{pmatrix} 0 & * & \\ & 0 & \\ & & 0 \end{pmatrix}$ $\lambda\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = a_1 a_2^{-1}$

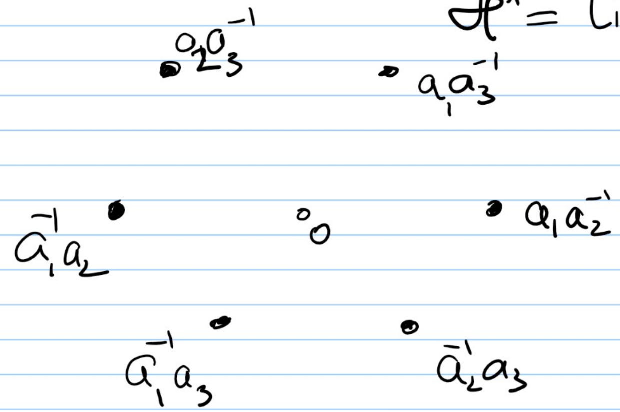
$\mathfrak{g}^{[2]} = \begin{pmatrix} 0 & & * \\ & 0 & \\ & & 0 \end{pmatrix}$ $\lambda(\quad) = a_1 a_3^{-1}$

$\mathfrak{g}^{[3]} = \begin{pmatrix} 0 & & \\ & 0 & * \\ & & 0 \end{pmatrix}$ $\lambda(\quad) = a_2 a_3^{-1}$

$\begin{pmatrix} 0 & & \\ * & 0 & \\ & & 0 \end{pmatrix}$ $\lambda(\quad) = a_1^{-1} a_2$

$\begin{pmatrix} 0 & & \\ & 0 & \\ * & & 0 \end{pmatrix}$ $\qquad \lambda(\qquad) = a_1^{-1}a_3$

$\begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}$ $\qquad \lambda(\qquad) = a_2^{-1}a_3$



$\mathfrak{a}^* = \mathrm{Linear}(\mathfrak{a}, \mathbb{R})$

$\forall \lambda \in \{\text{characters of } A\}$

$\exists! \ \alpha \in \mathfrak{a}^* \quad \forall x \in \mathfrak{a}$

$\lambda(\exp(x)) = e^{\alpha(x)}$

$\forall a \in A$ Recall $G_a^- = \{ g \in G \mid a^n g a^{-n} \xrightarrow[n\to+\infty]{} 1 \}$

<u>G_a^- is unipotent</u> (i.e. $\forall g \in G_a^-$ $\quad g-1$ nilpotent matrix.)

$\mathrm{Lie}(G_a^-) \xrightarrow{\exp} G_a^-$ is bijective.

$$\mathrm{Lie}(G_a^-) = \bigoplus_{\lambda : |\lambda(a)| < 1} \mathfrak{g}^\lambda = \{ x \in \mathfrak{g} \mid \mathrm{Ad}(a)^n x \xrightarrow[n\to+\infty]{} 0 \}$$

proof WLOG assume $a = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & \ddots \\ & & & a_d \end{pmatrix}$ with $|a_1| \geq |a_2| \geq \cdots \geq |a_d|$

then $G_a^- \subset \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \right\}$ $\square$

Def let ξ, η be weights. $\xi \sim \eta$ if $\exists n, m \in \mathbb{N}$
equivalent

$\xi^n = \eta^m$.

Notat. $[\xi]$ = equivalence class of ξ

$$\mathfrak{g}^{[\xi]} = \bigoplus_{\eta \in [\xi]} \mathfrak{g}^\eta$$

$$G^{[\xi]} = \exp(\mathfrak{g}^{[\xi]})$$

Theorem 9.14 Let η, ξ be s.t. $[\eta] \neq [\xi] \neq [\eta^{-1}]$

then $\forall$ a.e. $x \in X$ μ is invariant under $[g, h]$

① with $g \in \operatorname{supp} \mu_x^{G^{[\xi]}}$, $h \in \operatorname{supp} \mu_x^{G^{[\eta]}}$

② $g \in$ (smallest A-normarised Zariski closed subgroup containing $\operatorname{supp} \mu_x^{G^{[\xi]}}$)

$h \in$ ..... $\operatorname{supp} \mu_x^{[\eta]}$

the same for a.e. $x \in X$ by ergodicity.