High entropy method

G $\mathbb{R}$-points of a linear $\mathbb{R}$-group

$A \subset G$ $\mathbb{R}$-points of a $\mathbb{R}$-split torus of rank $r \geq 2$

$\Gamma < G$ discrete subgroup, $X = G/\Gamma$

μ Borel A-inv A-ergodic probability measure on X

Weight decomposition of $\mathfrak{g} = \mathrm{Lie}(G)$

$$\mathfrak{g} = \bigoplus_{\lambda \in \Phi} \mathfrak{g}^{\lambda}$$

$\Phi \subset \{\text{charater of } A\}$ is the set of weights

$$\mathfrak{g}^{\lambda} = \{x \in \mathfrak{g} \mid \forall a \in A,\ \mathrm{Ad}(a)x = \lambda(a)x\}$$

Observation $[\mathfrak{g}^{\lambda}, \mathfrak{g}^{\eta}] \subset \mathfrak{g}^{\lambda\eta}$ $\quad \forall \lambda\ \eta \in \Phi$

Proof. $\forall a \in A$, $\forall X, Y \in \mathfrak{g}$. $\quad \mathrm{Ad}(a)[X, Y] = [\mathrm{Ad}(a)X, \mathrm{Ad}(a)Y]$

□

Notation for $\Theta \subset \Phi$, write

$$\mathfrak{g}^{\Theta} = \bigoplus_{\lambda \in \Theta} \mathfrak{g}^{\lambda}$$

For example $\mathrm{Lie}(G_a^-) = \mathfrak{g}^{\{\lambda \in \Phi \mid |\lambda(a)| < 1\}}$ ($= \Phi \cap$ an open half space in $\mathfrak{g}^*$)

Definition $\lambda \sim \eta$ if $\lambda^n = \eta^m$ for some $n, m \geq 1$,

and $[\lambda] = \{\eta \in \Phi \mid \eta \sim \lambda\}$

$\mathfrak{g}^{[\lambda]}$ is called a Coarse Lyapunov subalgebra.

Theorem Let $\lambda, \eta \in \Phi$ with $[\lambda] \neq [\eta] \neq [\lambda^{-1}]$

μ is invariant under $[H_x^{[\lambda]}, H_x^{[\eta]}]$ for a.e $x \in X$

where $H_x^{[\lambda]}$ is the smallest A-normalised connected closed subgroup containing $\mathrm{Supp}(\mu_x^{G^{[\lambda]}})$.

For subgroup $U \subset G_a^-$ for some $a \in A$, for $x \in X$

$$\mathrm{St}_x^U = \mathrm{Stab}_U \mu_x^U = \{u \in U \mid u.\mu_x^U = \mu_x^U\}.$$

Lemma For a.e. $x \in X$, $U \subset G_a^-$ normalised by A

(1) $St_x^u \subset Stab_G(\mu)$

(2) $\forall a \in A$ $\quad a\, St_x^u\, a^{-1} = St_{a.x}^u \quad \overset{(3)}{\Longleftrightarrow} \quad Ad(a)\, st_x^u = st_{a.x}^u$

(3) St_x^u is closed and connected.

(4) St_x^u is normalised by A.

Notation $st_x^u = Lie(St_x^u)$

(2)+(3)+(4)+ ergodicity $\Rightarrow$ St_x^u, st_x^u same for a.e. x.

Proof (1) ✓

(2) $\Leftarrow \quad \mu_{a.x}^u = a^{-1} . \mu_x^u . a$

(3) closed. $(u_n \to u, \quad u_n . \mu_x^u \xrightarrow{*} u . \mu_x^u)$

Let $f(x) = d\left((St_x^u)^0 , (St_x^u) \setminus (St_x^u)^0 \right)$

by (2) $\quad f(a^n x) \longrightarrow 0$

$U \subset G_a^-$

By Poincare recurrence $\mu(\{x \in X \mid f(x) > 0\}) = 0$.

(4) $\Leftrightarrow \quad Ad(a)\, st_x^u = st_x^u \quad \forall a \in A$.

By ergodicity + (2), $\dim st_x^u = k$ is the same.

$A \overset{\rho}{\curvearrowright} \Lambda^k \mathfrak{g} \qquad \rho(a) = \Lambda^k Ad(a)$

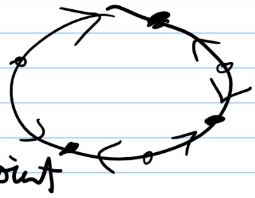
This is algebraic. By the fact (last week), $\rho(A)$ is diagonalisable

$\forall a \in A$, $\quad \rho(a) \curvearrowright \mathbb{P}(\Lambda^k \mathfrak{g})$

$\forall \theta \in \mathbb{P}(\Lambda^k \mathfrak{g})$

either $\rho(a)\theta = \theta$

or $\rho(a^n)\theta \to$ a fixed point



Apply to $\theta = \Lambda^k st_x^u \in \mathbb{P}(\Lambda^k \mathfrak{g})$

θ not fixedpoint contradicts Poincaré recurrence.

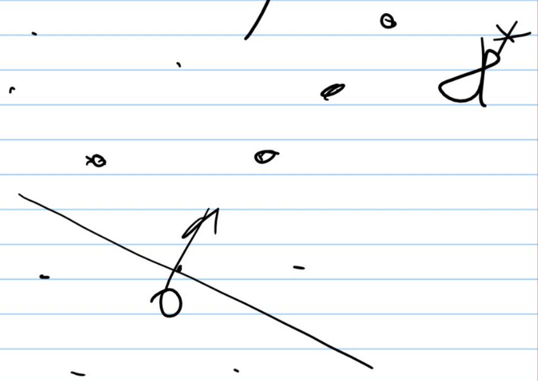
Hence $Ad(a)\, st_x^u = st_x^u$. ∎

Rmk $\mathfrak{h} \subset \mathfrak{g}$ is normalised by A.

then $\left(\forall x \in \mathfrak{h},\quad x = \sum_{\lambda \in \Phi} x_\lambda \quad x_\lambda \in \mathfrak{g}^\lambda \right.$

$\left. \text{then } x_\lambda \in \mathfrak{h} \quad \forall \lambda \in \Phi . \right)$

$\mathfrak{h}$ is A-norm. $\Leftrightarrow \mathfrak{h} = \bigoplus_{\lambda \in \Phi} (\mathfrak{h} \cap \mathfrak{g}^\lambda)$



<u>Proof of theorem</u>

Step 0 $\forall g \in \operatorname{supp} \mu_x^{[\lambda]} = \mu_x^{G^{[\lambda]}}$ $\forall h \in \operatorname{supp} \mu_x^{[\eta]}$

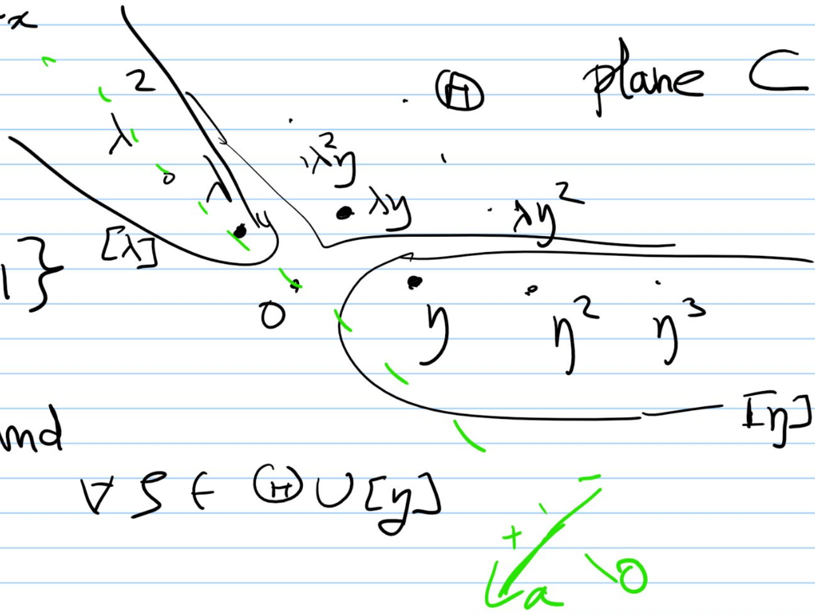
then $(g,h) \in St_x^U$

where

$U = G^{\Theta}$

$\Theta = \{ \lambda^n \eta^m \mid n \geq 1, m \geq 1 \}$

Θ plane $\subset \mathfrak{g}^*$



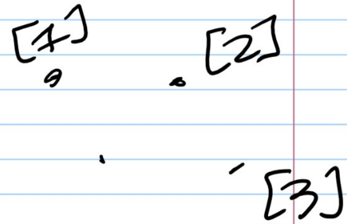
$\exists a \in A,\quad \lambda(a) = 1$ and

$\xi(a) < 1 \quad \forall \xi \in \Theta \cup [\eta]$

$G^{\Theta \cup [\eta]} \subset G_a^-$

and $G^{[\lambda]}$ is centralised by a

and $[\mathfrak{g}^{[\lambda]}, \mathfrak{g}^{\Theta \cup [\eta]}] \subset \mathfrak{g}^{\Theta \cup [\eta]}$



hence $G^{\Theta \cup [\eta]}$ is normalised by $G^{[\lambda]}$.

by product theorem $\forall$ a.e. x

$\mu_x^{G^{[\lambda]} \ltimes G^{\Theta \cup [\eta]}} \propto c_*(\mu_x^{G^{[\lambda]}} \times \mu_x^{G^{\Theta \cup [\eta]}})$

$c : G \times G \to G$
$(x,y) \mapsto xy$

similarly

$\mu_x^{G^{\Theta \cup [\eta]}} \propto \mu_x^{G^{[\eta]} \ltimes G^{\Theta}} \propto c_*(\mu_x^{G^{[\eta]}} \times \mu_x^{G^{\Theta}})$

So

$$\mu_x^{\Theta\cup[\eta]\cup[\lambda]} \propto \iota_*\left(\mu_x^{[\lambda]} \times \mu_x^{[\eta]} \times \mu_x^{U}\right)$$

$$\propto \iota_*\left(\mu_x^{[\eta]} \times \mu_x^{[\lambda]} \times \mu_x^{U}\right)$$

Then by Fubini.

μ_x^U is inv by $(g,h) = ghg^{-1}h^{-1}$ for $\mu_x^{[\lambda]}$ a.e. g and $\mu_x^{[\eta]}$ a.e. h.

<u>Step 1</u>. If $g \in \mathrm{supp}(\mu_x^{[\lambda]})$, $h \in \mathrm{supp}(\mu_x^{[\eta]})$

and $\log g = u = u_1 + u_2 + \cdots$ (with $u_1 \in \mathfrak{g}^{\lambda}$, $u_2 \in \mathfrak{g}^{\lambda^2}$)

$\log h = v = v_1 + v_2 + \cdots$ (with $v_1 \in \mathfrak{g}^{\eta}$, $v_2 \in \mathfrak{g}^{\eta^2}$)

then $[u_1, v_1] \in \mathfrak{st}_x^U$.

> <u>Rmk</u>. $\{(g,h) \mid ghg^{-1}h^{-1}.\mu = \mu\}$ is closed

Proof of Step 1. $\log(ghg^{-1}h^{-1}) \in \mathfrak{st}_x^U$

$\|$ — Baker-Campbell-Hausdorff

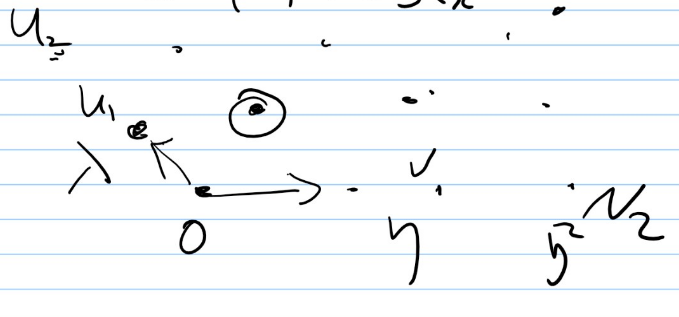
$[u,v]$ + terms of more brakets in u, v

$= [u_1, v_1]$ + terms in $\mathfrak{g}^{\Theta' \setminus \{\lambda\eta\}}$

($[u_1,v_1] \in \mathfrak{g}^{\lambda\eta}$)

$\Theta' = \Theta \cup [\lambda] \cup [\eta]$

$\mathfrak{st}_x^U$ is A-normalised $\Longrightarrow$ $[u_1, v_1] \in \mathfrak{st}_x^U$. ∎



<u>Step 2</u> $\forall n, m \geq 1$. $[u_n, v_m] \in \mathfrak{st}_x^U$

Why do we care? | $u_n = p_{\chi^n}(\log g)$

$v_m = p_{\eta^m}(\log h)$

$\mathfrak{h}^{[\chi]} = \mathrm{Lie}(H_x^{[\chi]})$

$= \mathrm{Lie}$ (smallest closed connected A-normalized subg. containing supp $\mu_x^{[\chi]}$)

$=$ Lie algebra generated by $\bigcup_{n\geq 1} p_{\chi^n}(\mathrm{supp}\,\mu_x^{[\chi]})$.

$p_{\chi^n} : \mathfrak{g} \to \mathfrak{g}^{\chi^n}$ projection // $\mathfrak{g}^{\Phi \setminus \{\chi^n\}}$

Step 3 $[\mathfrak{h}^{[\chi]}, \mathfrak{h}^{[\eta]}] \subset \mathfrak{st}^u$.

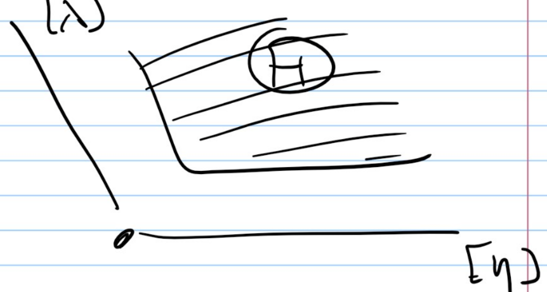
⇓ BCH formula

Step 4 $[H^{[\chi]}, H^{[\eta]}] \subset St^u$

Proof of step 2 Induction on the size of Θ (If $\Theta = \emptyset$ trivial)

Induction on (n,m) (lexicographical order)

$(1,1)$ was step 1.



Next show $[u_2, v_1] \in \mathfrak{st}^u$

$\exp([u_1, v_1]) \in St^u$ by step 1.

implies $\in \mathrm{supp}\,\mu_x^u$

$\in \mathrm{supp}\,\mu_x^{[\chi\eta]}$ (tower rule)

by induction hypothesis (outer)

$[[u_1, v_1], u_n] \in \mathfrak{st}^u$ $\forall n \geq 1$.

$p_{\chi^2\eta}(\log(ghg^{-1}h^{-1})) \in \mathfrak{st}^u$

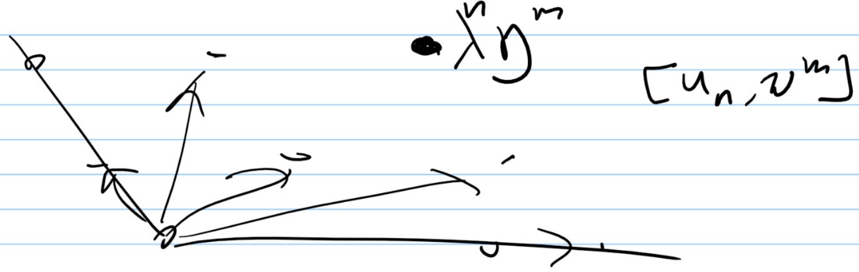
↑ A-norm.

$= [u_2, v_1] +$ terms of at 2 bracket $\in \mathfrak{g}^{\chi^2\eta}$

? $[[u_1, v_1], u_1]$

↑ $\mathfrak{st}^u$

Hence $[u_2, v_1] \in \mathfrak{st}^u$



Proof step3 By step2 and Linearity

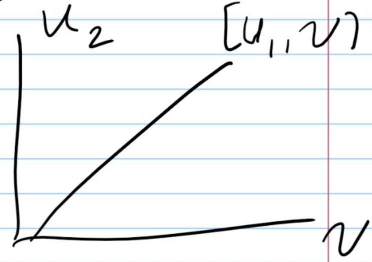
$\forall u \in \mathrm{Span}\left(\bigcup_{n \geq 1} P_{\lambda^n}(\mathrm{supp}\, \mu_x^{[\lambda]}) \right)$

$\forall v \in \mathrm{Span}\left(\bigcup_{m \geq 1} P_{\eta^m}(\mathrm{supp}\, \mu_x^{[\eta]}) \right)$

$[u, v] \in st^u$

claim if $u_1, u_2 \in$

then $[[u_1, u_2], v] \in st^u$



Jacobi $\;=\; [\underbrace{[u_1, v]}_{st_x^u}, u_2] + [u_1, [u_2, v]]$

$\in \mathrm{supp}(\mu_x^?)$ $\quad \in st^u$ by the above. ✓

Next $\forall u_1, u_2, v_1, v_2 \in \ldots$

$[[u_1, u_2], [v_1, v_2]] \in st^u$

Consider

$\{ (u, v) \in \mathfrak{g}^{[\lambda]} \times \mathfrak{g}^{[\eta]} \mid [u, v] \in st^u \}$.

✓ step3

✓ theorem.

Theorem Assume $\mathfrak{g}$ is simple $/\mathbb{R}$

- μ as above.

If $\exists a \in A$. $h_\mu(a) > h_{m_x}(a) - \varepsilon(a)$

$\varepsilon(a) = d(0, \log \mathrm{Spec}(\mathrm{Ad}(a)) \setminus \{0\}) > 0$.

then μ is invariant und G^0.

Lemma $V \subsetneq U \subset G_a$ A-norm.

If $\operatorname{supp} \mu_x^U \subset V$ for a.e x

then $h_\mu(a,U)$ $h_{m_x}(a,V) \leq h_{m_x}(a,U) - \varepsilon(a)$

$h_\mu(a,V) \leq$ ← Brian's lecture

Proof. By tower rule + assumption $\Rightarrow \mu_x^U \propto \mu_x^V$ ⇒ $h_\mu(a,U) = h_\mu(a,V)$ ✓

Recall $h_{m_x}(a,V) = -\log \det \operatorname{Ad}(a)|_{\operatorname{Lie} V}$

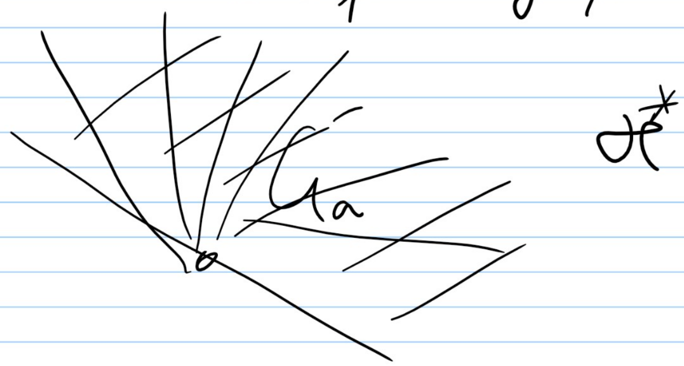
Lemma

$$h_\mu(a) = h_\mu(a, G_a) = \sum h_\mu(a, G^{[\lambda]})$$

↑ Brian's lecture

↑ $\{[\lambda] : \lambda(a) < 1\}$

consequence of product theorem. ✓



Proof of theorem

Claim $H_x^{[\lambda]} = G^{[\lambda]}$ $\forall \lambda$. $\lambda(a) < 1$.

because otherwise $h_\mu(a) \leq h_{m_x}(a) - \varepsilon(a)$. ✓

Also $h_\mu(a^{-1}) = h_\mu(a) > h_{m_x}(a^{-1}) - \varepsilon(a^{-1})$

So $H_x^{[\lambda]} = G^{[\lambda]}$ $\forall \lambda$ $\lambda(a) > 1$.

In view of the first theorem, everything reduces to the following.

Lemma Assume $\mathfrak{g}$ simple / $\mathbb{R}$. then

$[\mathfrak{g}^{[\lambda]}, \mathfrak{g}^{[\eta]}]$, $[\lambda] \neq [\eta] \neq [\lambda^{-1}]$.

$\lambda(a) \neq 1 \neq \eta(a)$.

generates $\mathfrak{g}$.