EXERCISE

In this exercise we are going to construct a set $A \subset \mathbb{R}^2$ of positive Hausdorff dimension whose projections to the x and y axis both have Hausdorff dimension 0.

Definition. Let $A \subset \mathbb{R}^n$ be some compact set, the Hausdorff dimension of A is defined as:

$$dim_H(A) = inf\left\{d \geq 0 : \, inf\left\{\sum r_i^d : \, A \subseteq \bigcup B_{r_i}(x_i) \text{ where } B_{r_i}(x_i) \text{ are closed balls of radii } r_i \right\} = 0\right\}$$

Exercise. Define two subsets of $\mathbb{N}$ as follows:

$$\Lambda_1 = \left(\bigcup_{n \in \mathbb{N}, \, n \text{ is even}} [n!, (n+1)!) \right) \cap \mathbb{N}, \; \Lambda_2 = \left(\bigcup_{n \in \mathbb{N}, \, n \text{ is odd}} [n!, (n+1)!) \right) \cap \mathbb{N}$$

then define the sets:

$$B = \left\{ \sum_{j \in \Lambda_1} \frac{\varepsilon_j}{2^j} : \, \varepsilon_j \in \{0, 1\} \right\}, \; C = \left\{ \sum_{j \in \Lambda_2} \frac{\varepsilon_j}{2^j} : \, \varepsilon_j \in \{0, 1\} \right\}$$

finally define $A = B \times C$.

(1) Prove that $dim_H B = dim_H C = 0$. (hint: find a sequence of coverings by segments of rapidly decreasing lengths.)

(2) Prove that $dim_H A \geq 1$. (hint: use the fact that Lipschitz functions never increase the Hausdorff dimension and find an appropriate function.)