Seminar material

1. Summery – today we will define and explore unitary representations and their connection to ergodicity and apply it to $SL_2(\mathbb{R})$ while exploiting its specific algebraic properties
2. *Mautner theorem (11.3) - ("$SL_2(\mathbb{R})$ preserves or not preserves elements as a group")*
   a. Remainder we defined

      $$SL_2(\mathbb{R}) = \{A|\det A = 1\}, \Gamma \leq SL_2(\mathbb{R})\ discrete$$

   b. *unitary representation – for a metraizeble G and $\mathcal{H}$ Hilbert space an action*

      $$G \times \mathcal{H} \to \mathcal{H}$$

      Is a unitary representation if it's elementwise unitary

      $$\forall g, x, y\ \langle gx, gy\rangle = \langle x, y\rangle$$

      And for every $v \in \mathcal{H}$ the function gv is continius
   c. a ball is

      $$B_\delta^G = \{g|d(g,e) < \delta\}, B_\delta^G(h) = \{g|d(g,h) < \delta\}$$

   d. claim - let X be a metric locally compact space with a borel measure μ and a metrizable group G that acts continouslly on X and preservese the measure then $G \times L_\mu^2(X) \to L_\mu^2(X)$ with $(g * f)(x) = f\left(g^{-1}(x)\right)$ is an unitary representation
      i. unitarity

         $$\langle gf, gh\rangle = \int_X f\left(g^{-1}(x)\right)h\left(g^{-1}(x)\right)d\mu =_{"y=g(x)"} \int_X f(x)h(x)d\mu = \langle f, h\rangle$$

      ii. continuity – let $\varepsilon > 0$. Since $C_c(x)$ is dense in L_μ^2 there exist

         $$f_{\frac{\varepsilon}{3}} \in C_c(X), \mathrm{K} = \sup(f_0), \left|f - f_{\frac{\varepsilon}{3}}\right|_{L_\mu^2} < \frac{\varepsilon}{3}$$

         Since $f_{\frac{\varepsilon}{3}}$ is continuous on a compact set is uniformly continuous therfore

         $$\exists\ \delta_{\frac{\varepsilon}{3}}\ s.t\ d(x,y) < \delta_{\frac{\varepsilon}{3}} \to |f(x) - f(y)| < \frac{\varepsilon}{3}$$

         Define

         $$R_{\delta_{\frac{\varepsilon}{3}}} = \inf\left\{|g|_G \middle| \exists x \in K\ s.t\ d(g^{-1}x, x) > \delta_{\frac{\varepsilon}{3}}\right\}$$

         It's positive because otherwise

         $$d(g_n^{-1}x_n, x_n) > \delta\ \exists x_{n_k} \to x\ , g_{n_k} \to e\ g_{n_k}^{-1}, x_{n_k} \nrightarrow ex$$

         Therefore

         $$\forall g \in B_{R_{\delta_{\frac{\varepsilon}{3}}}}^G\ \ |f - g^{-1}f| \leq \left|f - f_{\frac{\varepsilon}{3}}\right| + \left|f_{\frac{\varepsilon}{3}} - g^{-1}f_{\frac{\varepsilon}{3}}\right| + \left|g^{-1}f_{\frac{\varepsilon}{3}} - g^{-1}f\right| \leq$$

         $$\int_X \left|f(x) - f\left(g^{-1}(x)\right)\right|^2 d\mu \leq \varepsilon^2\mu(B(e, R_\delta)K) \to_{\varepsilon\to 0} 0$$

   e. claim (proof as excersise) let $\Gamma \leq G$ adiscrete subgroup and $g_2 = hg_1h^{-1}$ so the maps

      $$R_{g_1}, R_{g_2}: G/\Gamma \to G/\Gamma$$

      are *conjugate with R_h and if G/Γ has a finite measure then R_h is measure preserving and there is an isomorphism between*

      $$\left(X, \mathcal{B}_X, m_X, R_{g_1}\right) \to \left(X, \mathcal{B}_X, m_X, R_{g_2}\right)$$

f. Every element of $SL_2(\mathbb{R})$ is *conjugate to an element of the following sets*

$$\pm A = \left\{ \pm \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} \middle| t \in \mathbb{R} \right\}, \pm U^{-} = \left\{ \pm \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \middle| t \in \mathbb{R} \right\}, SO_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}$$

  i. *If the matrix is non diagonal has a Jordan form with ± 1 as eigenevalue so conjugate to $\pm U^{-}$ if diagonal with real eigenevalues conjugate to $\pm A$ and if complex eigenevalue they have to be on the unit circle and therefore conjugate to $SO_2(\mathbb{R})$*

g. *There are 3 types of actions on $SL_2(\mathbb{R})$*

  i. *Hyperbolic action that conjugate to $\pm A$*

  ii. *Parabolic action that conjugate to $\pm U^{-}$*

  iii. *Elliptic action that conjugate to $SO_2(\mathbb{R})$ - in those the orbit of each element is close so the Haar measure limited to any orbit is ergodic*

h. *Mautner theorem let $\mathcal{H}$ be a Hilbert space with unitary representation of $SL_2(\mathbb{R})$ and g non elliptic then if $gv_0 = v_0 \to Gv_0 = v_0$*

i. *Let $X = SL_2(\mathbb{R})/\Gamma$ with a finite measure and $g \in SL_2(\mathbb{R})$ non Elliptic so R_g is ergodic in $(X, \mathcal{B}_X, m_X)$ (otherwise there is a preserved indicator but he can't be preserved by all G)*

j. *Lemma $G \times \mathcal{H} \to \mathcal{H}$ unitary representation v_0 preseved by a sub group L then he is preserved by*

$$\left\{ h \middle| \forall \delta > 0 \; B_\delta^G(h) \cap LB_\delta^G L \neq \emptyset \right\}$$

In particular, the horocycles.

k. *Proof of Mautner – since conjugate hgh^{-1}, hv_0 preservers the theorem we only need to check two cases*

  i. *Hyperbolic action $g = a_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}$ have stable and unstable horocycle of*

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$$

  From the lemma they are preserving v_0 from gauss theorm their generate $SL_2(\mathbb{R})$

  ii. *Parbolic action $g = u_s^{-} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ so*

$$\varepsilon = \frac{1}{2^k s} \quad n = 2^k, m = -2^{k-1} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}^m$$

$$= \begin{pmatrix} 1 + ns\varepsilon & (1 + ns\varepsilon)ms + ns \\ \varepsilon & 1 + ms\varepsilon \end{pmatrix} = \begin{pmatrix} 2 & (1+1)\frac{-\frac{1}{2}}{\varepsilon} + \frac{1}{\varepsilon} \\ \frac{1}{2^k s} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ \frac{1}{2^k s} & \frac{1}{2} \end{pmatrix} \to_{k \to \infty} \begin{pmatrix} 2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

  Preserve v_0 from the lemma and since he is hyperbolic we already know he is preserved by the whole G

l. *Proof of lemma*

Let $g_n \to e, l_n g_n l_n' \to h$ so because h is continuous (from Cauchy–Schwarz)

$$\langle l_n g_n l_n'(v_0), v_0\rangle \to_{n\to\infty} \langle h(v_0), v_0\rangle \le |h(v_0)||v_0| = |v_0|^2$$

But because $l_n, l_n' \in L$ we have

$$\langle l_n g_n l_n'(v_0), v_0\rangle = \langle g_n l_n'(v_0), l_n^{-1} v_0\rangle = \langle g_n(v_0), v_0\rangle \to_{n\to\infty} \langle v_0, v_0\rangle = |v_0|^2$$

And from Cauchy–Schwarz there is wqulity only if $h(v_0) = v_0$

3. *Howe–Moore Theorem (11.4) ("$SL_2(\mathbb{R})$ is mixing")*
   a. *Let G a locally compact group $a_n \in G$ then*

   $$S(a_n) = \left\{g \middle| e \in \overline{\{a_n^{-1} g a_n | n \in \mathbb{N}\}}\right\}$$

   b. *(used later write in board edge) G locally compact $\mathcal{H}$ Hilbert on $\mathbb{R}$ or $\mathbb{C}$ with unitary representation $a_n \in G$ and exist $v \in \mathcal{H}$ s.t $a_n(v) \to_* v_0$ so every element in $g \in S(a_n)$ holds $g(v_0) = v_0$*
      i. *Let $\lim_{k\to\infty} a_{n_k}^{-1} g a_{n_k} = e$ so*

      $$\forall w \in \mathcal{H}\ 0 \le |\langle g v_0 - v_0, w\rangle| = |\langle v_0, g^{-1}w\rangle - \langle v_0, w\rangle| =$$

      $$\lim_{n\to\infty} \left|\langle a_{n_k}(v), g^{-1}w\rangle - \langle a_{n_k}(v), w\rangle\right| = \lim_{n\to\infty} \left|\langle a_{n_k}^{-1} g a_{n_k}(v) - v, a_{n_k}^{-1} w\rangle\right|$$

      $$\le \lim_{n\to\infty} \left|a_{n_k}^{-1} g a_{n_k} - e\right| |w| = 0$$

   c. *Conculation also work to the group generated from $\overline{S(a_n)}$*
   d. *A square matrix is unipotent if $\exists k\ (M - I)^k = 0$ in particular non eliptic*
   e. *(used later write in board edge) if $SL_2(\mathbb{R}) \ni g_n \to \infty$ i.e. $\forall K$ compact $|K \cap a_n| < \infty$ then exist a non-identity unipotent $u \in S(g_n)$*
      i. *Let*

      $$g_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, g_n^{-1} = \begin{pmatrix} d_n & -b_n \\ -c_n & a_n \end{pmatrix}$$

      So

      $$g_n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g_n^{-1} = \begin{pmatrix} -a_n c_n & a_n^2 \\ -c_n^2 & a_n c_n \end{pmatrix}$$

      $$g_n \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} g_n^{-1} = \begin{pmatrix} b_n d_n & -b_n^2 \\ d_n^2 & -b_n d_n \end{pmatrix}$$

      At least one of them tends to infinity without loss of generality $g_n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g_n^{-1} \to \infty$

      because the exponent function has derivate I at 0 there is a sphere

      $$|A| = \varepsilon \to |\exp(A) - \exp(0)| \ge \frac{\varepsilon}{2}$$

      So, by scaling $g_n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g_n^{-1}$ to that neighborhood we get

      $$\exp\left(\frac{\varepsilon}{\left|g_n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g_n^{-1}\right|} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \to_{n\to\infty} \exp(0) = I$$

      $$\left| g_n \frac{\varepsilon}{\left|g_n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g_n^{-1}\right|} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g_n^{-1} \right| = \varepsilon$$

      $$\to \forall n \left| \exp\left(g_n \frac{\varepsilon}{\left|g_n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g_n^{-1}\right|} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g_n^{-1} \right) - I \right| \ge \frac{\varepsilon}{2}$$

Since the sphere is compact there a sub srieres with

$\exp\left(g_{n_k}\frac{\varepsilon}{\left|g_n\begin{pmatrix}0&1\\0&0\end{pmatrix}g_n^{-1}\right|}\begin{pmatrix}0&1\\0&0\end{pmatrix}g_{n_k}^{-1}\right)\to u$ it can't be I from the distance but is the limit presevres unipotentcity because it's equvalnt to have the charastic polinomal $(\lambda-1)^2$ and he is a continuous property of matrixes. And every elemnt in unipotent because it's equal $\exp\left(g_{n_k}\frac{\varepsilon}{\left|g_n\begin{pmatrix}0&1\\0&0\end{pmatrix}g_n^{-1}\right|}\begin{pmatrix}0&1\\0&0\end{pmatrix}g_{n_k}^{-1}\right)=g_{n_k}\exp\left(\frac{\varepsilon}{\left|g_n\begin{pmatrix}0&1\\0&0\end{pmatrix}g_n^{-1}\right|}\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)g_{n_k}^{-1}=$ $g_{n_k}\begin{pmatrix}1&\frac{\varepsilon}{\left|g_n\begin{pmatrix}0&1\\0&0\end{pmatrix}g_n^{-1}\right|}\\0&1\end{pmatrix}g_{n_k}^{-1}$

f. An action is mixing if the measure is probabilistic for every A, B borel and $g_n\to\infty$ it holds $\mu(g_nA\cap B)\to\mu(A)\mu(B)$

g. Exercise equvalt to
   i. For every two function in L^2_μ it holds $\int_X f_1 * gf_2 \to_{g\to\infty} \int_X f_1\int_X f_2$
   ii. If $\int_X f_1=0$ then $\int_X f_1 * gf_2\to_{g\to\infty} 0$

h. For a lattice $SL_2(\mathbb{R})/\Gamma$ the product action of $SL_2(\mathbb{R})$ is mixing
   i. Let $g_n\to\infty$ and $f_2\in L^2_\mu(X)$ (μ derived from Haar) from unitarity $\|g_nf_2\|_2=\|f_2\|_2$ from Alaoglu theorm there is a limit $g_{n_k}f_2\to_* f_0$ and we proved $\exists u\in S(g_{n_k})$ unipotent so non elliptic and he preserves f_0 from Mautner theorem everybody in $SL_2(\mathbb{R})$ preservs f_0 witch makes him constant from $f_1=1$ we get

$$f_0=\int_X f_2\to\int_X f_1 * gf_2\to_{n\to\infty}\int_X f_1 * f_0=\int_X f_1\int_X f_2$$

i. $G\times\mathcal{H}\to\mathcal{H}$ unitary representation without fixed points (for all G) where $\mathcal{H}$ on $\mathbb{R}$ or $\mathbb{C}$ then

$$\forall v,w\in\mathcal{H}\ \langle g_nv,w\rangle\to_{g_n\to\infty}0$$

   i. From Alaoglu theorm and the what we proved before there is u unipotent with

$$g_{n_k}v\to_* v_0,\ g_{n_k}^{-1}ug_{n_k}\to I\to$$

$$\langle v_0,v_0\rangle=\lim_{n\to\infty}\langle g_nv,v_0\rangle=\lim_{n\to\infty}\langle g_n^{-1}ug_nv,g_n^{-1}uv_0\rangle=_{coshy\ svarch}\lim_{n\to\infty}\langle v,g_n^{-1}uv_0\rangle$$

$$=\lim_{n\to\infty}\langle g_nv,uv_0\rangle=\langle v_0,uv_0\rangle\le|v_0||uv_0|=\langle v_0,v_0\rangle$$

The Cousy-Schwartz has equality only when

$$uv_0=v_0$$

Because u isn't elliptic it means that $Gv_0=v_0$ in contradiction to the set app

j. Howe–Moore theorem - X metric σ compact with continuous (not neceserlly unitery) action of $SL_2(\mathbb{R})$ and let μ be ergodic relative to $SL_2(\mathbb{R})$ then the action is mixing
   i. Because C_c is dense in L^2_μ it's sufficient to prove

$$\forall f_1,f_2\in C_c\ \int_X f_1(x)f_2(g_nx)d\mu\to_{n\to\infty}\int_X f_1(x)d\mu\int_X f_2(x)d\mu$$

Define the measures

$$\mu_n(S)=m_X(\{x\in X|(x,g_nx)\in S\})$$

So

$$\int_X f_1(x)f_2(g_nx)d\mu=\int_{X\times X} f_1(x)f_2(y)d\mu_n$$

$$\int_X f_1(x)d\mu \int_X f_2(x)d\mu = \int_{X\times X} f_1(x)f_2(y)d\mu \times \mu$$

Witch means we only need to prove $\mu_n \to \mu \times \mu$*. For each measure the action*

$$h(x,y) = (hx, g_n h g_n^{-1} y)$$

preseves him. In addition

$$\mu_n(X \times S) = \mu_n(S \times X) = \mu(S)$$

From Alaoglu there is a limit $\mu_{n_k} \to \nu \neq \mu$*. Because it's a limit*

$$\nu(X \times S) = \nu(S \times X) = \mu(S)$$

Let $u \in S(g_{n_k})$ *unipotent so*

$$h_{n_k} \to I, g_{n_k} h_{n_k} g_{n_k}^{-1} \to u$$

$$\int_X f(x,y)d\nu = \lim_{k\to\infty} \int_X f(x,y)d\mu_{n_k} = \lim_{k\to\infty} \int_X f\left(h_{n_k}x, g_{n_k}h_{n_k}g_{n_k}^{-1}y\right)d\mu_{n_k} = \int_X f(x,uy)d\nu$$

i.e the measure ν *is preserved by the acion* (I,u) *from Mautner theorem it's preserved by all* (I,g)

let $S, X\backslash S \in \mathcal{B}$ *so*

$$\mu_S(A) = \nu(S \times A), \mu_{X\backslash S}(A) = \nu(X\backslash S \times A)$$

Are measure on X *that preserved under* (I,g)*and their sum is ergodic. Therefore there are scalings of the original measure therefore*

$$\forall S, A \in \mathcal{B}\ \nu(S \times A) = \mu_S(A) = \frac{\mu_S(X)}{\mu(X)} * \mu(A) = \frac{\nu(S \times X)}{\mu(X)} * \mu(A) = \mu(S)\mu(A)$$

So for every borel rectangle $\nu = \mu^2$ *and because they determine the mesure* $\nu = \mu^2$*.*