using Schmidt

$BA_d \subset$

Incompressability:

$\dim\left(\bigcap_{i=1}^{\infty} f_i^{-1}(S)\right) = d$

where $\{f_i\}$ are uniformly Lip.

full measure subsets

for affine maps by: Fishman

for certain K's

Strong Incompressability of S on K:

$\dim\left(\bigcap_{i=1}^{\infty} f_i^{-1}(S) \cap K\right) = \dim(U \cap K)$

$U \cap K \neq \emptyset$ f_i are ______.

this paper

HAW, K is diffuse

new game

BA_d

Schmidt games:

- set of nested balls in $\mathbb{R}^d$: $B_1 \supseteq A_1 \supseteq B_2 \supseteq \dots$

s.t. $\rho(A_i) = \alpha \rho(B_i)$; $\rho(B_{i+1}) = \beta \rho(A_i)$

$0 < \alpha, \beta < 1$ are fixed.

McMullen games:

- Alice takes away balls instead of intersecting: $B_1 \supseteq B_1 \setminus A_1 \supseteq B_2 \supseteq B_2 \setminus A_2 \supseteq \dots$

s.t. $\rho(A_i) \leq \beta \rho(B_i)$, $\rho(B_{i+1}) \geq \beta \rho(A_i)$.

for $\beta < 1/3$ fixed.

makes it hard for both.

- Here, $S \subseteq \mathbb{R}^d$ is <u>β-absolute winning</u> if $\exists$ strategy leading to $\bigcap_{i=1}^{\infty} B_i \cap S \neq \emptyset$.

- BAd is not absolutely winning since Bob can always choose points on a hyperplane.

k-dimensional absolute winning

- $k \in \{0, 1, \dots, d-1\}$
- Alice removes ε-nbhd of k-dim. affine space $\mathcal{L}$ w/ $0 < \varepsilon < \beta \rho_i$ (ε can depend on i!), $A_i := \mathcal{L}^{(\varepsilon)}$, $B_i = B(x_i; \rho_i)$ then Bob chooses x_{i+1}, $\rho_{i+1} \geq \beta \rho_i$ s.t.

$$B_{i+1} := B(x_{i+1}, \rho_{i+1}) \subseteq B_i \setminus A_i.$$

- Winning if $\bigcap B_i \cap S \neq \emptyset$.
- This is clearly a gen. of McMullen games. Winning sets for $k = d-1$ are

called HAW sets.

Proposition 2.3:

a. HAW $\longrightarrow$ α winning $\forall \alpha < \frac{1}{2}$ $\longrightarrow$

straight forward

$\dim(S \cap U) = d$ $\forall k \leq d-1$, k-dim abs. winning $S \subseteq \mathbb{R}^d$ & $\emptyset \neq U \subseteq \mathbb{R}^d$ open.

b. k-dim abs. winning is preserved under countable intersection.

c. Image of k-dim abs. winning under C^1 diffeo of $\mathbb{R}^d$ is again... .

Theorem 2.4: $S \subseteq \mathbb{R}^d$ k-di absolute winning, $U \subseteq \mathbb{R}^d$ open, $f: U \to \mathbb{R}^d$ a C^1 non singular map. Then $f^{-1}(S) \cap U^c$ is k-dim absolute winning. Consequently, k dim abs winning $\Longrightarrow$ strongly C^1 incompressible.

Theorem 2.5: $BA_d \subseteq \mathbb{R}^d$ is hyperplane absolute winning (HAW).

- **Remark:** Thm's 2.4, 2.5 are the main Thms we prove.

Pf of Thm 2.4: (Idea: on $f^{-1}(S) \cap U^c$ Alice is playing w/ some $\beta \in (0, 1/3)$ = Game 1. Game 2 will be played w/ some appropriate const. $\beta' \in (0, 1/3)$. For Game

2 Alice has winning strategy. Bob's moves B_i on Game 1 will be translated using f to moves (legal) on Game 2, denoted B_i'. She'll have legal moves A_i' on game 2 & use them & f^{-1} to get A_i - legal moves on Game 1.)

(Reduction 1:) We can assume $\rho(B_i) \to 0$, otherwise the $\bigcap \mathring{B}_i \neq \emptyset$ & S winning $\Rightarrow S$ dense $\Rightarrow f^{-1}(S) \cup U^c$ dense $\Rightarrow$ $f^{-1}(S) \cap U^c \cap \bigcap B_i \neq \emptyset \Rightarrow$ Alice wins.

(Reduction 2:) We may assume $\exists i$ s.t. $B_i \subseteq U$. Otherwise, $B_i \subseteq U^c$ are nested balls inside the closed U^c

so $\bigcap B_i \cap U^c \neq \emptyset$.

(Controlling distortion of f)

- $J_z := f'(z) \left(\in GL_n(\mathbb{R})\right)$

- $\forall$ closed ball $B \subseteq U$, $x, y \in B$:

$$\|f(x) - f(y)\| \leq \sup_{z \in B} \|J_z\|_{op} \cdot \|x-y\|.$$

- $\forall x_0 \in B$:

$$L_{x_0}(x) := f(x_0) + J_{x_0}(x - x_0)$$

- $\tilde{L}_{y'}$ defined for f^{-1}, $y \in f(B)$.

Claim: (*) $\sup_{y, y_0 \in f(B)} \dfrac{\|\tilde{L}_{y_0}(y) - f^{-1}(y)\|}{\|y - y_0\|} \xrightarrow[\rho(B) \to 0]{} 0$.

where B is a subset of a fixed cpt set.

<u>Indeed</u> let $H_{y_0} := f^{-1}(z) - J_{y_0}^{-1}(z)$.

H_{y_0} is differentiable, $H_{y_0}'(z) = J_z^{-1} - J_{y_0}^{-1}$.

$\Rightarrow f$ is C^1 & non-singular $\Longrightarrow \sup\limits_{z, y_0 \in f(B)} \| H_{y_0}'(z) \|_{op} \xrightarrow[\rho(f(B)) \to 0]{} 0$

where $f(B) \subseteq$ fixed cpt subset. Then:

$$\frac{\| L_{y_0}(y) - f^{-1}(y) \|}{\| y - y_0 \|} = \frac{\| H_{y_0}(y_0) - H_{y_0}(y) \|}{\| y - y_0 \|} \leq \sup_{z \in B} \| d_z H_{y_0} \|_{op}$$

and this proves the claim.

(Defining appropriate Constants:) Let B_1 be the initial ball chosen by Bob in Game 1 & let:

$$C_1 := \sup_{x \in B_1} \|J_x\|_{op} \quad ; \quad C_2 := \sup_{x \in f(B_1)} \|(J_x)^{-1}\|_{op}$$

and let n be large enough s.t.

6 $C(\beta+1)\beta^{n-2} < 1$ where $C = 2C_1 \cdot C_2$.

Let $\beta' = \beta^n < 1/3$.

By the first reduction, making enough dummy moves, Alice can make sure that

$$(*) \leqslant C_2 \, (1 + 1/\beta) \, \beta'$$

and may assume $B \subseteq U$.

$\uparrow$ Bob's ball.

We renumber the new ball B_1.

We will construct by induction legal moves:

$$B_1' \supseteq B_1' \setminus A_1' \supseteq B_2' \supseteq \ldots$$

in Game 2 played w/ target set S.

We will describe a strategy in Game 1 & $1 = j_1 < j_2 < \dots$ s.t. $f(B_{j_i}) \subseteq B_i'$ which will be enough.
$(**)$

We will make sure that A_i, B_i', A_i will satisfy for every i:

1. All chosen sets are valid moves;
2. if $B_{j_i} := B(x_i, \rho_i)$, $B_i' = B(f(x_i), \rho_i')$,

$$\rho_i' = c_1 \rho_i \text{ , then:}$$

$$(**) \quad \beta^n \leq \frac{\rho_i}{\rho_{i-1}} < \beta^{n-1} \qquad (n \text{ as defined})$$

3. $A_i' = \mathcal{L}_i'(\varepsilon_i)$ are from Alice strategy

in Game 2 &:

$$A_{j_i} \supseteq f^{-1}\left(\mathcal{L}_i^{(\eta\rho_i')}\right) ; \quad \eta = (1+{}^{1}/_{\beta})\beta'.$$

Indeed the definition of c_1 and 2 ensure that (**) holds, so it suffices to verify 1–3.

(Verification of Conditions:) By induction. Suppose we verified 1–3 up to step $\ell-1$. Alice plays dummy moves until Bob's ball B_m has radius ρ s.t. ${}^{\rho}/_{\rho_{\ell-1}} < \beta^{n-1}$, then let $j_\ell = m$. Our choice ensures that (***) holds. Let $B_\ell' = B(f(x_\ell), \rho_\ell')$ $(\rho_\ell' = c_1\rho_\ell)$.

Note:

$$(\Delta) \quad \beta' = \beta^n \le \frac{\rho_\ell}{\rho_{\ell-1}} \overset{(**)}{=} \frac{\rho_\ell'}{\rho_{\ell-1}'} < \beta^{n-1} = \frac{\beta'}{\beta}.$$

$$B_{j_\ell} \subseteq B_{j_{\ell-1}} \implies d(x_{\ell-1}, x_\ell) \le \rho_{\ell-1} - \rho_\ell$$

$$\implies d\big(f(x_{\ell-1}), f(x_\ell)\big) \le C_1 d(x_{\ell-1}, x_\ell) \le C_1(\rho_{\ell-1} - \rho_\ell)$$

$$= \rho_{\ell-1}' - \rho_\ell'$$

$\implies B_\ell' \subseteq B_{\ell-1}'$. Next,

induction

$$B_{j_\ell} \cap A_{j_{\ell-1}} = \emptyset, \quad A_{j_{\ell-1}} \supseteq f^{-1}\left(\mathcal{L}_{\ell-1}'^{\,(\eta\rho_{\ell-1}')}\right)$$

$$\Downarrow \qquad \Downarrow + (\Delta)$$

$$d(f(x_\ell), \mathcal{L}_{\ell-1}') \ge \eta\rho_{\ell-1}' = \left(1+\frac{1}{\beta}\right)\beta'\rho$$

$> \beta' \rho'_{\ell-1} + \rho'_\ell$

$\Longrightarrow B'_\ell \cap \mathcal{L}'^{\,(\beta'\rho'_{\ell-1})}_{\ell-1} = \emptyset \Longrightarrow B'_\ell \subseteq B'_{\ell-1} \setminus A'_{\ell-1}$

so we proved that it's a legal move.

(Next step: playing in Game 2:) Let A'_ℓ be the next step in Alice's strategy in Game 2, $A'_\ell = \mathcal{L}'^{\,(\varepsilon'_\ell)}_\ell$ for some $\varepsilon'_\ell \leq \beta'\rho'_\ell$.

Fix $y' \in \mathcal{L}'_\ell$ and let $\mathcal{L}_\ell := L_{y'}(\mathcal{L}'_\ell)$ (it is also k-dimensional).

$\forall y \in B'_\ell$ s.t. $d(y, \mathcal{L}'_\ell) < \eta \rho'_\ell$ denote y_0 to be the orthogonal projection of y

onto $\mathcal{L}_\ell'$. Then:

$$\|f^{-1}(y) - f^{-1}(y_0)\| \underset{\substack{\uparrow \\ \text{def of } C_2}}{\leq} C_2 \|y - y_0\| \leq C_2' \eta \rho_\ell'.$$

We made sure (by re-indexing) that

$(*) \leq C_2 \eta$, so that:

$$\|f^{-1}(y) - L_{y'}(y_0)\| \leq \|f^{-1}(y) - f^{-1}(y_0)\| + \|L_{y'}(y_0) - f^{-1}(y_0)\|$$

$$\leq 2 C_2 \eta \rho_\ell'$$

Define: $\varepsilon_\ell := 2 C_2 \eta \rho_\ell' = C \eta \rho_\ell$

$$\Downarrow$$

$$A_{j\ell} := \mathcal{L}_\ell^{(\varepsilon_\ell)} \supseteq f^{-1}(\mathcal{L}_\ell'^{(\eta \rho_\ell')})$$

satisfies 3.

By ⊙ , $C\eta < \beta \Rightarrow \varepsilon_\ell < \beta \varsigma_\ell$

$\Rightarrow A_{j\ell}$ is a valid move $\Rightarrow \square$.

Next, we are proving Thm 2.5. for that we need to recall the "Simplex Lemma":

<u>Lemma 3.1:</u> $\forall \beta \in (0,1), k \in \mathbb{N}$ let:

$$U_k := \left\{ \frac{\bar{p}}{q} : q \in \mathbb{N}, p \in \mathbb{Z}^d, \beta^{\frac{-d}{d+1}(k-1)} \leq q < \beta^{\frac{-d}{d+1}k} \right\}.$$

let $V_d = \mathrm{Vol}(B_1^d)$. Then for every

(3.6) $\qquad 0 < r < \beta (d!\, V_d)^{-1/d}$

and for every $x \in \mathbb{R}^d$ $\exists$ affine hyperplane $\mathcal{L}$ s.t. :

$$U_k \cap B(x, \beta^{k-1} r) \subseteq \mathcal{L}.$$

Pf of Thm 2.5 (that BA_d is HAW):

Let $\beta < 1/3$. Alice makes dummy moves until ρ satisfies (3.6) for $r \leq \rho$.

Let $c = \beta^2 \rho$. Let B_{j_k} be the subsequence of moves s.t. $\beta^{k-1}\rho \geq \rho_{j_k} > \beta^k \rho$ firstly.

Outside of j_k Alice makes dummy moves.

On j_k, she chooses $A_{j_{k+1}} := L_k^{(\beta^{k+1}\rho)}$

(its a legal move).

Note: $\beta^{k+1}\rho = c\beta^{k-1} \geq c q^{-(1+1/d)}$

whenever q is denominator from U_k.

$\Longrightarrow$ $\forall x \in B_{j_{k+1}}$ is at least $c q^{-(1+1/d)}$ away from $p/q \in U_k$. Since it holds for any k we get $\bigcap B_j \in BA_d$ $\square$

Playing Games on Fractals:

- Given a closed subset $K \subseteq \mathbb{R}^n$, Alice & Bob play the same game but on K (the centers are all chosen from K). $S \subseteq \mathbb{R}^n$ is (α,β)-winning on K if Alice can guarantee $\bigcap_i B_i \in S$. (this is the previous game, just Schmidt game).

Def: For $K \subseteq \mathbb{R}^d$, $x \in K$, $\rho > 0$, $0 < \beta < 1$, $N_K(\beta, x, \rho)$ is the maximal # of disjoint balls centered in K w/ radius $\beta\rho$ and contained in $B(x,\rho)$.

Lemma: (Fishman–Schmidt) Suppose $\exists$ $M, \delta, \rho_0, \beta_0 > 0$ s.t.

$$N_K(\beta, x, \rho) \geq M\beta^{-\delta}$$

$\forall x \in K, \rho < \rho_0, \beta < \beta_0$. Then

$$\dim(S \cap K \cap U) \geq \delta$$

for any open U s.t. $U \cap K \neq \emptyset$ and S winning on K.

Next, k-dim β-absolute games on K:

- same as before, just that Bob chooses

centers from K.

- Note that Alice may remove all or most K after finitely many steps. (this may happen for $\mathbb{R}^n$ when $\beta > 1/3$ hence the restriction on β).
- If K is removed, we declare Bob a winner, to make it not worthwhile for Alice.
- $S \subseteq \mathbb{R}^d$ is k-di absolute winning on K if $\exists\, \beta_0 \leq 1/3$ s.t. Alice can win k-di abs. game on K for some $\beta \leq \beta_0$.

- $d-1$-dim abs. winn. on K is

<u>HAW on K</u>.

Since Alice anyway looses when Bob has no moves, we introduce a definition that prevents it:

<u>Def:</u> A closed $K \subseteq \mathbb{R}^d$ is said to be k-dim. β-diffuse ($\beta \in (0,1)$) if $\exists\, \rho_K > 0$ s.t. $\forall\ 0 < \rho < \rho_K, x \in K$ and any k-dim affine subspace $\mathcal{L}$ $\exists\, x' \in B(x,\rho) \setminus \mathcal{L}^{(\beta\rho)}$.

- K is k-dim diffuse if it is k-dim β_0 diffuse for some $\beta_0 < 1$

– $k = d-1$ $\rightsquigarrow$ hyperplane diffuse.

Examples: – $\mathbb{R}^d$ is hyperplane diffuse for $\beta < 1$

– bounded open subsets of $\mathbb{R}^d$ w/ smooth bdry.

Lemma 4.5: If K is k-dim β-diffuse & S is winning on K then S is β'-$k$dim abs. winning on K whenever $\beta' \leq \frac{\beta}{2+\beta}$.

Thm 4.6: Let $K \subseteq \mathbb{R}^d$ be k-dim β-diffuse set, $S \subseteq \mathbb{R}^d$ winning on K, U open s.t. $U \cap K \neq \emptyset$. Then:

$$\dim(S \cap K \cap U) \geq \frac{-\log(k+2)}{\log\beta - \log(2+\beta)}$$

("Winning implies bound on Hausdorff dim of K", pf: using def of diffuseness $\Downarrow$ bound on covering #)

& the Lemma by Fishman – Schmidt
(w/ covering numbers)).

Proposition 4.7: ("k-dim abs. winning $\Rightarrow$ winning")
Let $K \subseteq \mathbb{R}^d$ be k-dim β-diffuse, $S \subseteq \mathbb{R}^d$ k-dim abs. winning on K. Then S is $\frac{\beta}{2+\beta}$-winning on K.

Proposition 4.8: ("Countable intersection of k-dim β abs. winning on K is β abs. winning on K") Let $K \subseteq \mathbb{R}^d$, $\beta > 0$ and for all $i \in \mathbb{N}$ let S_i be k-dim β'-abs. winning on K for any $\beta' \leq \beta$. Then the countable intersection $\bigcap_i S_i$ is also k-dim β' abs. winning for any $\beta' \leq \beta$.

- **Note:** We already know that k-dim abs. winning passes to intersection, but this result is on K.

(Next, k-dim abs. winning passes to diffuse subsets)

Proposition: If $L \subseteq K$ are k-dim diffuse sets, $S \subseteq \mathbb{R}^d$ is k-dim abs. winning on K, then it is also k-dim abs. winning on L. In particular, every k-dim abs. winning set on $\mathbb{R}^d$ is k-dim absolutely winning on any k-dim diffuse set.

<u>Corollary:</u> Let $S \subseteq \mathbb{R}^d$ be k-dim abs. winning, $U \subseteq \mathbb{R}^d$ open, K - a k-dim diff. set. Then for any C^1 non singular map $f : U \to \mathbb{R}^d$, the set $f^{-1}(S) \cup U^c$ is k-dim absolutely wining on K.

Consequently:

$$\dim\left(\bigcap_{i=1}^{\infty} f_i^{-1}(S) \cap K\right) > 0 .$$