GRADUATE STUDIES IN MATHEMATICS 227

Geometric Structures on Manifolds

William M. Goldman

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Providence, Rhode Island

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Dedicated to Carolyn, Emily, Evan, Lizzie, Michael, Amelia, Liana, Leonardo, Jonah and the memory of Morris Cohen and Stanley Goldman

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Preface

This book explores geometric structures on manifolds locally modeled on a classical geometry.

This subject mediates between *topology* and *geometry,* where a fixed topology is given local coordinate systems in the geometry of a homogeneous space of a Lie group. A familiar example puts Euclidean geometry on a manifold; such a *Euclidean structure* is nothing more than a Riemannian metric of zero curvature. In this sense, the topology of the 2-dimensional sphere $\mathbb{S}^2$ is incompatible with the geometry of Euclidean space: *There is no metrically accurate atlas of the world.* In contrast, however, the topology of the 2-dimensional torus $\mathbb{T}^2$ *does* support Euclidean geometry. Indeed, the classification of Euclidean structures on the torus is part of a rich and central area of mathematics (elliptic curves, modular forms). Indeed, Euclidean structures on $\mathbb{T}^2$ are classified by the action of the modular group on the Poincaré upper halfplane.

Topology and geometry communicate via *group theory. Topology* contributes its group, — the fundamental group — and *Geometry* contributes the group of symmetries of the given geometry. Thus our approach starts from the Klein–Lie algebraicization of geometry via Lie groups and homogeneous spaces, and quickly evolves into studying representations of discrete groups in Lie groups.

This book surveys the theory, with a special emphasis on affine and projective geometry. Many important geometries (for example, hyperbolic geometry) have projective models, and these projective models unify the diverse geometries.

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This work is based on examples. I have tried to present examples as a way to suggest the general theory. Because of the dramatic growth of this subject in the last decades, I tried to collect many facets of this subject and present them from a single viewpoint. Since Ehresmann's 1936 initiation of this subject, there have been many "success stories" in the classification of geometric structures on a given topology. I have to tried to present some of these in this book.

Despite the profound interrelations between different geometries, each geometry enjoys special features. A developing map only goes so far, and heavier machinery is often required, drawing on techniques special to the particular geometry. Learning new techniques and adapting to different areas of mathematics has been an exciting part of this journey.

Furthermore some of the material — which I feel should be better known — is unpublished, untranslated, or aimed at a different readership. The literature suffers from many errors (including some of my own) which I have tried to correct and clarify. However, I am certain many errors still persist, and I take full responsibility.

This book is suitable for a graduate textbook and contains many exercises. Some exercises are routine and others are more difficult. Many are used in other parts of the text. Others are meant to introduce ideas and examples before a subsequent detailed discussion.

To preserve the expository flow, several developments have been put in appendices. I have tried to illustrate geometric ideas with pictures and algebraic ideas with tables.

I have tried to keep the prerequisites fairly minimal. Material from beginning graduate courses in topology, differential geometry, and algebra are assumed, although some of the material which is crucial or less standard is summarized. The relationship between Lie groups and Lie algebras is heavily used, but little of the general structure theory/representation theory is assumed.

I began this area of research working with Dennis Sullivan and Bill Thurston at Princeton University in 1976. Their influence is evident throughout this work. Thurston formulated his *geometrization* of 3-manifolds in the context of geometric structures modeled on 3-dimensional Riemannian homogeneous spaces. Since then the study of more general (but not necessarily Riemannian) locally homogeneous geometric structures has become a very active field with interactions to other areas of mathematics and physics.

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Several important topics have been omitted or only briefly mentioned. Flat structures on Riemann surfaces — namely, singular Euclidean structures modeled on translations — are barely mentioned despite their fundamental role in modern Teichmüller theory. Their strata are fascinating and mysterious examples of incomplete complex affine manifolds. Nor are holomorphic affine and projective structures on complex manifolds. The algebraic theory of character varieties and representations of fundamental groups, is not really developed thoroughly. In particular the theory of surface group representations into Lie groups of higher rank, sometimes called *higher Teichmüller theory*, is not extensively discussed, despite its remarkable recent activity. Integral affine structures (important in mirror symmetry) are not discussed. Other very natural topics in this subject have not been discussed in detail, for reasons of space: These include the convex decomposition theorem of Suhyoung Choi, completeness results of Carrière and Klingler for constant curvature Lorentzian manifolds, and affine structures with diagonal holonomy as developed by Smilie and Benoist.

I welcome suggestions, comments, and feedback of (almost) all sorts. Through the AMS bookstore, I plan to maintain a website of errata, comments, graphics, and interactive software in connection with this book.

I hope you enjoy this journey as much as I have!

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Acknowledgments

This book grew out of lecture notes *Projective Geometry on Manifolds,* from a course I gave at the University of Maryland in 1988. Since then I have given minicourses at international conferences, graduate courses and summer schools on various aspects of this material.

Comments and suggestions by Suhyoung Choi, Daniele Alessandrini, Jean-Philippe Burelle, Francisco Arana Herrera, Daryl Cooper, Charles Daly, Jeff Danciger, François Guéritaud, Misha Kapovich, Fanny Kassel, Qiongling Li, John Loftin, Darren Long, Ludovic Marquis, Richard Schwartz, Stephan Tillmann, Tengren Zhang and various anonymous referees have been extremely helpful. The final version of the manuscript greatly benefitted from an extremely careful reading by Olivier Guichard.

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This work has benefitted tremendously from a large community of research mathematicians at various career stages, who have contributed in many ways to its development. I sincerely apologize for any omissions.

My interest in this subject began with my 1977 undergraduate thesis [**145**] from Princeton University. There Dennis Sullivan and Bill Thurston suggested looking at affine and projective structures, which I continued to pursue in graduate school in Berkeley with my doctoral adviser Moe Hirsch. David Fried and John Smillie spent summers in Berkeley and the four of us discussed affine structures extensively. Their influence is clearly evident

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from this book's content. This led to correspondence with Jacques Vey and Jacques Helmstetter in Grenoble. Conversations with my teachers at that time, especially Dan Burns, Shiing-Shen Chern, Bob Gunning, Shoshichi Kobayashi, Joe Wolf, and S.T. Yau were particularly valuable for specific parts of this work.

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The series of conferences *Crystallographic Groups and their Generalizations* (1996, 1999, 2002, 2005, 2008, and 2011), in Kontrijtk, Belgium (and later in Oostende) were also very formative, and I am extremely grateful to Paul Igodt and Karel Dekimpe of the Katholieke University of Louvain in Kortrijt for organizing these highly stimulating events.

I am grateful to the hospitality of the University of Maryland, Mathematical Sciences Research Institute, the Institute for Advanced Study, Brown University, International Center for Theoretical Physics, Institute for Computation and Experimentation in Research in Mathematics, and Institut Henri Poincaré. This work was completed while on sabbatical at the Institute for Advanced Study in 2021–2022, and I benefitted from the superb working conditions there.

I would like to thank Ed Dunne at the American Mathematical Society for suggesting, at the turn of the last century, expanding the original 1988 notes into a book in this series. His advice on the process of publishing over the years was extremely useful. I also want to thank Eko Hironaka for more informative suggestions and encouragement of this project. Without her gentle but firm support—especially near the end of this project—this work would have taken even longer to complete.

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Introduction

Symmetry powerfully unifies the various notions of geometry. Based on ideas of Sophus Lie, Felix Klein's 1872 Erlangen program proposed that geometry is the study of properties of a space X invariant under a group G of transformations of X. For example Euclidean geometry is the geometry of n-dimensional Euclidean space $\mathbb{R}^n$ invariant under its group of rigid motions. This is the group of transformations which transforms an object ξ into an object congruent to ξ. In Euclidean geometry one can speak of points, lines, parallelism of lines, angles between lines, distance between points, area, volume, and many other geometric concepts. All these concepts can be derived from the notion of distance, that is, from the metric structure of Euclidean geometry. Thus any distance-preserving transformation or *isometry* preserves all of these geometric entities.

Notions more primitive than that of distance are the *length* and *speed* of a smooth curve. Namely, the distance between points a, b is the infimum of the length of curves γ joining a and b. The length of γ is the integral of its speed $\|\gamma'(t)\|$. Thus Euclidean geometry admits an infinitesimal description in terms of the *Riemannian metric tensor,* which allows a measurement of the size of the velocity vector $\gamma'(t)$. In this way standard Riemannian geometry generalizes Euclidean geometry by imparting Euclidean geometry to each tangent space.

Other geometries "more general" than Euclidean geometry are obtained by removing the metric concepts, but retaining other geometric notions. *Similarity geometry* is the geometry of Euclidean space where the equivalence relation of congruence is replaced by the broader equivalence relation of similarity. It is the geometry invariant under similarity transformations. Similarity geometry does not involve distance, but rather involves angles,

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lines, and parallelism. *Affine geometry* arises when one speaks only of points, lines and the relation of parallelism. And when one removes the notion of parallelism and only studies lines, points and the relation of incidence between them (for example, three points being *collinear* or three lines being *concurrent*) one arrives at *projective geometry.* However in projective geometry, one must enlarge the space to *projective space,* which is the space upon which all the projective transformations are defined.

Here is a basic example illustrating the differences among the various geometries. A particle moving along a smooth path has a well-defined velocity vector field, representing its *infinitesimal displacement* at any time. This uses only the differentiable structure of $\mathbb{R}^n$. The magnitude of the velocity is the *speed,* which makes sense in Euclidean geometry. Thus "motion at unit speed" (that is, "arc-length-parametrized geodesic") is a meaningful concept there. But in affine geometry, the concept of "speed" or "arc-length" must be abandoned: yet "motion at constant speed" remains meaningful since the property of moving at constant speed along a straight line can be characterized as motion with zero acceleration. This is equivalent to the parallelism of the velocity vector field. In projective geometry this notion of "constant speed along a straight line" (or "parallel velocity") must be further weakened to the concept of "projective parameter" introduced by J. H. C. Whitehead [**346**].

Synthetic projective geometry was developed by the architect Desargues in 1636–1639 out of attempts to understand the geometry of perspective. Two hundred years later non-Euclidean (hyperbolic) geometry was developed independently — and practically simultaneously — by Bolyai in 1833 and Lobachevsky in 1826–1829. These geometries were unified in 1871 by Klein who noticed that Euclidean, affine, hyperbolic, and elliptic geometry were all "present" in projective geometry.

Later in the nineteenth century, mathematical crystallography developed, leading to the theory of *Euclidean crystallographic groups.* Answering Hilbert's eighteenth problem on the finiteness of the number of space groups in any given dimension n, Bieberbach developed a structure theory in 1911–1912. For torsion free groups, the quotient spaces identified with *flat Riemannian manifolds* of dimension n, that is, Riemannian n-manifolds having zero sectional curvature. Such Riemannian structures are locally isometric to Euclidean space E^n. In particular, every point has an open neighborhood isometric to an open subset of E^n. These local isometries define a local Euclidean geometry on the neighborhood. Furthermore on overlapping neighborhoods, the local Euclidean geometries "agree," that is, they are related by restrictions of global isometries $\mathsf{E}^n \to \mathsf{E}^n$. The neighborhoods form *coordinate patches,* the local isometries from the patches to

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E^n are the *coordinate charts,* and the restrictions of isometries of E^n are the corresponding *coordinate changes.* In this way a flat Riemannian manifold is defined by a coordinate atlas for a *Euclidean structure.*

More generally, for any geometry one can define geometric structures on a manifold M modeled on the homogeneous space (G, X). A geometric atlas consists of an open covering of M by patches $U \hookrightarrow M$, together with a system of charts $U \xrightarrow{\psi} X$ such that the coordinate changes are locally restrictions of transformations of X which lie in G.

The plethora of different geometries suggests that, at least at a superficial level, no general inclusive theory of locally homogeneous geometric structures exists. Each geometry has its own features and idiosyncrasies, and special techniques particular to each geometry are used in each case. For example, a surface modeled on $\mathbb{C}\mathsf{P}^1$ has the underlying structure of a Riemann surface, and viewing a $\mathbb{C}\mathsf{P}^1$-structure as a projective structure on a Riemann surface provides a satisfying classification of $\mathbb{C}\mathsf{P}^1$-structures. Namely, as was presumably understood by Poincaré, *the deformation space of* $\mathbb{C}\mathsf{P}^1$*-structures on a closed surface* Σ *with* $\chi(\Sigma) < 0$ *identifies with a holomorphic affine bundle over the Teichmüller space of* Σ. When X is a complex manifold upon which G acts biholomorphically, holomorphic mappings provide a powerful tool in the study, a class of local mappings more flexible than "constant" maps (maps which are "locally in G") but more rigid than general smooth maps. Another example occurs when X admits a G-invariant connection, such as an invariant (pseudo-)Riemannian structure. Then the geodesic flow provides a powerful tool for the study of (G, X)-manifolds.

We emphasize the interplay between different mathematical techniques as an attractive aspect of this general subject. See [**160**] for a recent historical account of this material.

Organization of the text

The book divides into three parts. Part One describes affine and projective geometry and provides some of the main background on these extensions of Euclidean geometry. As noted by Lie and Klein, most classical geometries can be modeled in projective geometry. We introduce projective geometry as an extension of affine geometry, so we begin with a detailed discussion of affine geometry as an extension of Euclidean geometry and projective geometry as an extension of affine geometry. Part Two describes how to put the geometry of a Klein geometry (G, X) on a manifold M, and gives the basic examples and constructions. One goal is to *classify* the (G, X)-structures on a fixed topology in terms of a *deformation space* whose points correspond to equivalence classes of *marked structures,* whereby a marking is an extra piece of information which fixes the topology as the geometry of

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M varies. Part Three describes recent developments in this general theory of locally homogeneous geometric structures.

Part One: Affine and projective geometry

Chapter 1 introduces affine geometry as the geometry of parallelism. Two objects are *parallel* if they are related by a *translation.* Translations form a vector space V, and act *simply transitively* on affine space. That is, for two points $p, q \in \mathsf{A}$ there is a unique translation taking p to q. In this way, points in A identify with the vector space V, but this identification depends on the (arbitrary) choice of a basepoint, or *origin* which identifies with the zero vector in V. One might say that an affine space is a vector space, where the origin is forgotten. More accurately, the special role of the zero vector is suppressed, so that all points are regarded equally.

The action by translations now allows the definition of *acceleration* of a smooth curve. A curve is a *geodesic* if its acceleration is zero, that is, if its velocity is parallel. In affine space itself, unparametrized geodesics are straight lines; a parametrized geodesic is a curve following a straight line at "constant speed." Of course, the "'speed" itself is undefined, but the notion of "constant speed" just means that the acceleration is zero.

This notion of parallelism is a special case of the notion of an *affine connection,* except the existence of *globally defined* translations effecting the notion of parallelism is a special feature to our setting — the setting of *flat connections.* Just as Euclidean geometry is affine geometry with a parallel Riemannian metric, other linear-algebraic notions enhance affine geometry with parallel tensor fields. The most notable (and best understood) are flat Lorentzian (and pseudo-Riemannian) structures.

Chapter 2 develops the geometry of projective space, viewed as the compactification of affine space. *Ideal points* arise as "where parallel lines meet." A more formal definition of an ideal point is an equivalence class of lines, where the equivalence relation is parallelism of lines. Linear families (or *pencils*) of lines form planes, and indeed the set of ideal points in a projective space form a *projective hyperplane,* that is, a projective space of one lower dimension. Projective geometry appears when the ideal points lose their special significance, just as affine geometry appears when the zero vector $\mathbf{0}$ in a vector space loses its special significance.

However, we prefer a more efficient (if less synthetic) approach to projective geometry in terms of linear algebra. Namely, the *projective space associated to a vector space* V is the space $\mathsf{P}(\mathsf{V})$ of 1-dimensional linear subspaces of V (that is, lines in V passing through $\mathbf{0}$). *Homogeneous coordinates* are introduced on projective space as follows. Since a 1-dimensional linear subspace is determined by any nonzero element, its coordinates determine

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a point in projective space. Furthermore the homogeneous coordinates are uniquely defined up to *projective equivalence,* that is, the equivalence relation defined by multiplication by nonzero scalars. Projectivizing linear subspaces of V produces projective subspaces of $\mathsf{P}(\mathsf{V})$, and projectivizing linear automorphisms of V yields *projective automorphisms,* or *collineations* of $\mathsf{P}(\mathsf{V})$.

The equivalence of the geometry of incidence in $\mathsf{P}(\mathsf{V})$ with the algebra of V is remarkable. Homogeneous coordinates provide the "dictionary" between projective geometry and and linear algebra. The collineation group is compactified as a projective space of "projective endomorphisms;" this will be useful for studying limits of sequences of projective transformations. These "singular projective transformations" are important in controlling developing maps of geometric structures, as developed in the second part.

Chapter 3 discusses, first from the classical viewpoint of polarities, the Cayley–Beltrami–Klein model for hyperbolic geometry. Polarities are the geometric version of nondegenerate symmetric or skew-symmetric bilinear forms on vector spaces. They provide a natural context for hyperbolic geometry, which is one of the principal examples of geometry in this study.

The Hilbert metric on a properly convex domain in projective space is introduced and is shown to be equivalent to the categorically defined Kobayashi metric [**220**, **222**]. Later this notion is extended to manifolds with projective structure.

Chapter 3 develops notions of convexity. The Cayley–Beltrami–Klein metric on hyperbolic space is a special case of the Hilbert metric on properly convex domains. These define natural metric structures on certain well-studied projective structures. An application of the Hilbert metric is Vey's semisimplicity theorem [**339**], which is later used to characterize closed hyperbolic projective manifolds as quotients of sharp convex cones. Then another metric (due to Vinberg [**340**]) is introduced, and is used to give a new proof of *Benzécri's Compactness theorem* [**46**] that the collineation group acts properly and cocompactly on the space of convex bodies in projective space — in particular the quotient is a compact (Hausdorff) manifold. This is used to characterize the boundary of convex domains which cover convex projective manifolds. Recently Benzécri's theorem has been used by Cooper, Long, and Tillmann [**100**] in their study of cusps of $\mathbb{R}\mathsf{P}^n$-manifolds.

Part Two: Geometric manifolds

The second part globalizes these geometric notions to manifolds, introducing *locally homogeneous geometric structures* in the sense of Whitehead [**345**] and Ehresmann [**122**] in Chapter 5. We associate to every transformation

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group (G, X) a category of geometric structures on manifolds locally modeled on the geometry of X invariant under the group G. Because of the "rigidity" of the local coordinate changes of open sets in X which arise from transformations in G, these structures on M intimately relate to the fundamental group $\pi_1(M)$.

Chapter 5 presents three different viewpoints to study these structures. First are coordinate atlases for the pseudogroup arising from (G, X). Using the aforementioned rigidity, these are globalized in terms of a *developing map*

$$\widetilde{M} \xrightarrow{\mathsf{dev}} X,$$

defined on the universal covering space $\widetilde{M}$ of the geometric manifold M. The developing map is equivariant with respect to the holonomy homomorphism

$$\pi_1(M) \xrightarrow{h} G$$

which represents the group $\pi_1(M)$ of deck transformations of $\widetilde{M} \to M$ in G. Each of these two viewpoints represents M as a quotient: in the coordinate atlas description, M is the quotient of the disjoint union

$$\mathcal{U} := \coprod_{\alpha \in A} U_\alpha$$

of the coordinate patches U_α; in the second description, M is represented as the quotient of $\widetilde{M}$ by the action of the group $\pi_1(M)$. While a map defined on a connected space $\widetilde{M}$ may seem more tractable than a map defined on the disjoint union $\mathcal{U}$, the space $\widetilde{M}$ can still be quite large.

The third viewpoint replaces $\widetilde{M}$ with M and replaces the developing map by a section of a bundle defined over M. The bundle is a *flat bundle,* (that is, has *discrete structure group* in the sense of Steenrod [**317**]). The corresponding *developing section* is characterized by transversality with respect to the foliation arising from the flat structure. This replaces the coordinate charts (respectively the developing map) being local diffeomorphisms into X.

Chapter 6 discusses examples of geometric structures from these three points of view. Although the main interest in these notes is structures modeled on affine and projective geometry, we describe other interesting structures.

These structures interrelate: Geometries may "contain" or "refine" other geometries. For example, affine geometry *contains* Euclidean geometry — abandon the metric notions but retain the notion of *parallelism.* This corresponds to the inclusion of the Euclidean isometry group (consisting of transformations $x \longmapsto Ax + b$, where A is orthogonal) as a subgroup of the affine automorphism group (consisting of transformations $x \longmapsto Ax + b$

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where A is only assumed to be linear). Other examples include the projective and conformal models for non-Euclidean geometry. In these examples, the model space of the refined geometry is an open subset of the larger model space, and the transformations in the refined geometry are restrictions of transformations in the larger geometry.

This *hierarchy of geometries* plays a crucial role in the theory. This is simply the geometric interpretation of the inclusion relations between closed subgroups of Lie groups. This *algebraicization* of geometries in the 19^{th} century by Lie and Klein satisfactorily organized the proliferation of classical geometries. This viewpoint is the cornerstone in our construction and classification of geometric structures. The classification of geometric manifolds often shows that a manifold modeled on one geometry may actually have a *stronger* geometry. For example, Fried's theorem [**135**] (see 11.4) asserts a closed manifold M with a similarity structure is either Euclidean or a manifold modeled on $\mathbb{R}^n \backslash \{\mathbf{0}\} \cong \mathbb{S}^{n-1} \times \mathbb{R}$ with its invariant (product) Riemannian metric. In particular M admits a finite covering space which is either a flat torus (the Euclidean case) or a *Hopf manifold,* a cyclic quotient of $\mathbb{R}^n \backslash \{\mathbf{0}\}$.

Chapter 7 deals with the general classification of (G, X)-structures from the point of view of developing sections. The main result is an important observation due to Thurston [**323**] that the *deformation space* of marked (G, X)-structures on a fixed topology Σ is itself "locally modeled" on the quotient of the space $\mathsf{Hom}\big(\pi_1(\Sigma), G\big)$ by the group $\mathsf{Inn}(G)$ of inner automorphisms of G. The description of $\mathbb{R}\mathsf{P}^1$-manifolds is described in this framework. The deformation space, however, is a non-Hausdorff 1-manifold, while the subspace consisting of closed affine 1-manifolds identifies with $[0, \infty)$. For affine structures on $\mathbb{T}^2$, the deformation space is not even a (non-Hausdorff) manifold.

Chapter 8 deals with the important notion of *completeness,* for taming the developing map. In general, the developing map may be quite pathological — even for closed (G, X)-manifolds — but under various hypotheses, can be proved to be a covering space onto its image. However, the main techniques borrow from Riemannian geometry, and involves *geodesic completeness* of the Levi–Civita connection (the Hopf–Rinow theorem). A complete affine manifold M is a quotient $\Gamma\backslash\mathsf{A}$, where A is an affine space and $\Gamma < \mathsf{Aff}(\mathsf{A})$ is a discrete subgroup acting properly on A. Equivalently, a developing map $\widetilde{M} \to \mathsf{A}$ is a homeomorphism (an affine isomorphism) of the universal covering space $\widetilde{M}$ onto A.

This requires relating geometric structures to *connections,* since all of the locally homogeneous geometric structures discussed in this book can be approached through this general concept. However, we do *not* discuss

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the general notion of *Cartan connections,* but rather refer to the excellent introduction to this subject by R. Sharpe [**305**]. Some aspects of the general theory of affine connections have been relegated to Appendix B.

Chapter 8 introduces some of the basic examples in our theory. Bieberbach's theorems [**53**, **54**] successfully describe the structure and classification of Euclidean structures on closed manifolds:[1] *Every closed Euclidean manifold* M *is a biquotient* $\Lambda\backslash\mathbb{R}^{/}\Phi$ *where* $\Lambda < \mathbb{R}^n$ *is a lattice and* Φ *is a finite group of automorphisms of* Λ*.* In other words M is finitely covered by *flat torus,* such as $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

One wonders if a similar picture holds when M is only assumed to be *affine,* and this is the *Auslander–Milnor question,* (sometimes called the "Auslander conjecture"): whether the fundamental group $\pi_1(M)$ is virtually polycyclic. In that case, M is finitely covered by a *solvmanifold* $\Gamma\backslash G$ where G is a solvable Lie group and $\Gamma < G$ is a lattice. Here G has a left-invariant complete affine structure, meaning that it acts simply transitively and affinely on affine space. It plays the role of the group of translations for Euclidean manifolds.

This question is open for closed manifolds, but Margulis [**253**] found proper affine actions of the 2-generator free group $\mathbb{F}_2$ on A^3, and their quotients, called *Margulis spacetimes,* are discussed in §15.4.

The first 3-dimensional examples are described, including $\mathsf{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, the Heisenberg nilmanifold $\mathsf{Heis}_{\mathbb{Z}}\backslash\mathsf{Heis}_{\mathbb{R}}$, and hyperbolic torus bundles $\mathsf{Sol}_{\mathbb{Z}}\backslash\mathsf{Sol}_{\mathbb{R}}$. They represent three of the eight *Thurston geometries* in dimension 3.

We classify complete affine structures on the 2-torus $\mathbb{T}^2$ (originally due to Kuiper [**234**]). The Hopf manifolds introduced in §6.2 are fundamental examples of incomplete structures. That affine structures on compact manifolds are generally incomplete is one dramatic difference between affine geometry and traditional Riemannian geometry.

The successful classification of affine (and projective) strucutres on $\mathbb{T}^2$ began with Kuiper [**234**] in the convex case. It was completed by Nagano–Yagi [**279**] and Arrowsmith–Furness [**141**]; Baues [**33**] provides an excellent exposition. They provide many basic examples, some of which generalize to higher dimensions. The classification of affine (and projective) 2-manifolds is somewhat messy but provides a paradigm for the problems discussed in this book. The classification is revisited several times to motivate some of the general theory, including deformation spaces and affine Lie groups.

[1] See Charlap [**84**] for a good exposition of this theory.

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Part Three: Affine and projective structures

Chapter 9 begins the classification of affine structures on surfaces. We prove Benzécri's theorem [**45**] that a closed surface Σ admits an affine structure if and only if its Euler characteristic vanishes. We discuss the famous conjecture of Chern that the Euler characteristic of a closed affine manifold vanishes, giving the proof of Kostant–Sullivan [**225**] in the complete case.

Chapter 10 offers a detailed study of left-invariant affine structures on Lie groups. We will call a Lie group with a left-invariant affine structure an *affine Lie group.* These provide many examples; in particular all the non-radiant affine structures on $\mathbb{T}^2$ are *invariant* affine structures on the *Lie group* $\mathbb{T}^2$. For these structures the holonomy homomorphism and the developing map blend together in an intriguing way.[2] Covariant differentiation of left-invariant vector fields lead to well-studied nonassociative algebras called *algèbres symétriques à gauche* or (*left-symmetric* algebras). Such algebras have the property that their *associators* are s c in the left two variables. Commutator defines the structure of an underlying Lie algebra. Associative algebras correspond to *bi-invariant affine structures,* so the "group objects" in the category of affine manifolds correspond naturally to associative algebras. These structures were introduced by Ernest Vinberg [**340**] in his study of homogeneous convex cones in affine space, and further developed by Jean-Louis Koszul and his school. We take a decidedly geometric approach to these ubiquitous mathematical structures. For example, many closed affine surfaces are affine Lie groups.

Chapter 11 describes the question (apparently first raised by L. Markus [**254**]) of whether, for a closed orientable affine manifold, completeness is equivalent to *parallel volume.* The existence of a parallel volume form is equivalent to unimodularity of the linear holonomy group, that is, whether the holonomy preserves volume. An "infinitesimal analog" of this question for left-invariant affine structures on Lie groups is the conceptual and suggestive result that completeness is equivalent to parallelism of *right-invariant* vector fields, (Exercise 10.3.9 in §10.3.5.)

This tantalizing question has led to much research, subsuming various questions which we discuss. Carrière's proof that compact flat Lorentzian manifolds are complete [**78**] is a special case, and Smillie's nonexistence theorem is another special case, discussed in §11.3. Section 11.2 treats the case when the affine holonomy group Γ is nilpotent. Another example is Fried's sharp classification of closed similarity manifolds [**135**] (proved independently by a much different argument by Vaisman–Reischer [**331**]).

[2]Perhaps this provides a conceptual basis for the unexpected relation between the 1-dimensional property of geodesic completeness and the top-dimensional property of volume-preserving holonomy.

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Chapter 12 expounds the notions of "hyperbolicity" of Vey [**337**] and Kobayashi [**222**]. *Hyperbolic affine manifolds* are quotients of properly convex cones. A closed hyperbolic manifold is a radiant suspension of an $\mathbb{R}\mathsf{P}^n$-manifold, which itself is a quotient of a divisible domain. In particular we describe how a *completely incomplete* closed affine manifold must be affine hyperbolic in this sense. (That is, we tame the developing map of an affine structure with *no* two-ended complete geodesics.) This striking result is similar to the tameness where *all* geodesics are complete — complete manifolds are also quotients. The key ingredient is the *infinitesimal Kobayashi pseudo-metric,* which measures the (in)completeness of a geodoesic with given velocity.

Chapter 13 summarizes some aspects of the now blossoming subject of $\mathbb{R}\mathsf{P}^2$-structures on surfaces, in terms of the explicit coordinates and deformations which extend some of the classic geometric constructions on the deformation space of hyperbolic structures on closed surfaces. We describe the analog of Fenchel–Nielsen coordinates and other coordinate systems, briefly mentioning a more analytic approach due independently to Loftin and Labourie. Then we describe the grafting construction, and the first examples, due to Smillie and Sullivan–Thurston, of a projective structure on $\mathbb{T}^2$ with pathological developing map.

Chapter 14 describes the classic subject of $\mathbb{C}\mathsf{P}^1$-manifolds, which traditionally identify with *projective structures on Riemann surfaces.* Using the Schwarzian derivative, these structures are classified by the points of a holomorphic affine bundle over the Teichmüller space of Σ. This parametrization (presumably known to Poincaré), is remarkable in that is completely *formal,* using standard facts from the theory of Riemann surfaces. One knows precisely the deformation space without any knowledge of the developing map (besides it being a local biholomorphism). This is notable because the developing maps can be pathological; indeed the first examples of pathological developing maps were $\mathbb{C}\mathsf{P}^1$-manifolds on hyperbolic surfaces. The theory of projective structures on Riemann surfaces is a suggestive paradigm for a successful classifaction of highly nontrivial geometric structures.

Chapter 15 surveys known results, and the many open questions, in dimension three. This complements Thurston's book [**324**] and expository articles of Scott [**302**] and Bonahon [**56**], which deal with geometrization and the relations to 3-manifold topology. In particular we describe the classification, due to Serge Dupont [**119, 120**], of projective structures on hyperbolic torus bundles

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Prerequisites

This book is aimed roughly at first-year graduate students and advanced undergraduate students, although some knowledge of advanced material will be useful.

For general treatments of geometry, we refer to the two-volume text of Berger [**49**, **50**] (see also Berry–Pansu–Berry–Saint Raymond [**51**]) and Coxeter [**102**].

We also assume basic familiarity with elementary topology, smooth manifolds, and the rudiments of Lie groups and Lie algebras. Much of this can be found in Lee's book "Introduction to Smooth Manifolds" [**244**], including its appendices. For topology, we require basic familiarity with the notion of metric spaces, covering spaces, and fundamental groups.

Fiber bundes, as discussed in the still excellent treatise of Steenrod [**317**], or the more modern treatment of principal bundles given in Sontz [**315**], will be used.

Some familiarity with the properties of proper maps and proper group actions will also be useful.

Some familiarity with the theory of connections in fiber bundles and vector bundles is useful, for example, Kobayashi–Nomizu [**224**], or Milnor [**269**], do Carmo [**113**] Lee [**244**], O'Neill [**283**].

We put the discussion of Fenchel–Nielsen coordinates on Fricke space in the context of Darboux's theorem in symplectic geometry; we recommend §22 of Lee [**244**] for a good general treatment consistent with our notation.

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Notation, terminology, and general background

Vectors and matrices

We work over a field k, usually the field $\mathbb{R}$ of real numbers, but sometimes the field $\mathbb{C}$ of complex numbers. We shall denote vectors and matrices in bold font. Let V be a vector space over k of dimension n. A basis determines an isomorphism $\mathsf{V} \cong \mathsf{k}^n$. Thus a vector in V corresponds to a column vector:

$$\mathbf{v} \quad \longleftrightarrow \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$$

A *covector* is defined as a linear functional $\mathsf{V} \xrightarrow{\omega} \mathsf{k}$, corresponding to a row vector:

$$\omega \quad \longleftrightarrow \quad \begin{bmatrix} \omega_1 & \dots & \omega_n \end{bmatrix}$$

and the duality pairing between V and V^* is:

$$\begin{aligned} \mathsf{V} \times \mathsf{V}^* &\longrightarrow \mathsf{k} \\ (\mathbf{v}, \omega) &\longmapsto v^i \omega_i \end{aligned}$$

(summation over paired indices). A linear transformation $\mathsf{k}^m \longrightarrow \mathsf{k}^n$ is defined by an $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} A^i{}_j \end{bmatrix}$$

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mapping

$$\mathsf{k}^m \xrightarrow{\mathbf{A}} \mathsf{k}^n$$

$$\mathbf{v} = \begin{bmatrix} v^1 \\ \vdots \\ v^m \end{bmatrix} \longmapsto \begin{bmatrix} A^1{}_j v^j \\ \vdots \\ A^n{}_j v^j \end{bmatrix}$$

where $j = 1, \dots, m$.

Affine vector fields on A correspond to affine maps $\mathsf{A} \to \mathsf{A}$:

$$(A^i{}_j x^j + b^i)\partial_i \quad \longleftrightarrow \quad \hat{\mathbf{A}} := \left[\mathbf{A} \;\middle|\; \mathbf{b}\right]$$

where

$$\mathbf{A} = \begin{bmatrix} A^1{}_1 & \dots & A^1{}_i & \dots & A^1{}_n \\ \vdots & & \vdots & & \vdots \\ A^i{}_1 & \dots & A^i{}_j & \dots & A^i{}_n \\ \vdots & & \vdots & & \vdots \\ A^n{}_1 & \dots & A^n{}_j & \dots & A^n{}_n \end{bmatrix}$$

is the linear part and

$$\mathbf{b} = \begin{bmatrix} b^1 \\ \vdots \\ b^i \\ \vdots \\ b^n \end{bmatrix}$$

is the translational part. In this notation,

$$\left[\mathbf{A} \;\middle|\; \mathbf{b}\right] = \left[\begin{array}{ccc|c} A^1_1 & \dots & A^1_n & b^1 \\ \vdots & & \vdots & \vdots \\ \dots & A^i{}_j & \dots & b^i \\ \vdots & & \vdots & \vdots \\ A^n_1 & \dots & A^n_n & b^n \end{array}\right]$$

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Projective equivalence of vectors. Denote the multiplicative group of nonzero scalars in k by $\mathsf{k}^\times$, and let V be a vector space over k. Then $\mathsf{k}^\times$ acts by scalar multiplication on V. Define nonzero vectors $\mathbf{w}, \mathbf{u} \in \mathsf{V}$ to be *projectively equivalent* if and only if $\exists \lambda \in \mathsf{k}^\times$ such that $\mathbf{w} = \lambda \mathbf{u}$. Projective equivalence classes $[\mathbf{v}]$ of nonzero vectors $\mathbf{v}$ form the *projective space* $\mathsf{P}(\mathsf{V})$ associated to V. Denote the projective equivalence class of a vector

$$\mathbf{v} = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \in \mathsf{V}$$

by

$$[\mathbf{v}] := \left[\!\!\left[\begin{matrix} v^1 \\ \vdots \\ v^n \end{matrix} \right]\!\!\right]$$

and the projective equivalence class of a covector $\omega = [\, \omega^1 \, \dots \, \omega^n \,] \in \mathsf{V}^*$ by

$$[\omega] := \Big[\omega^1 \quad \dots \quad \omega^n\Big].$$

Projective equivalence classes of nonzero *covectors* comprise the projective space $\mathsf{P}(\mathsf{V}^*)$ *dual* to $\mathsf{P}(\mathsf{V})$. Just as points in the projective space $\mathsf{P}(\mathsf{V})$ correspond to projective equivalence classes of vectors, projective hyperplanes in $\mathsf{P}(\mathsf{V})$ correspond to projective equivalence classes of covectors (see §3.1).

General topology

For background in topology see Lee [**244**], Willard [**349**], Hatcher [**186**], and Greenberg–Harper [**172**].

If A and B are topological spaces, and $A \xrightarrow{f} B$ is a continuous map, then f is a *local homeomorphism* if and only if $\forall a \in A$, the restriction $f|_U$ is a homeomorphism $U \to f(U)$ for some open neighborhood $U \ni a$.

If A is a topological space, and $B \subset A$ is a subspace, then we write $B \subset\subset A$ if B is compact (in the subspace topology).

We denote the space of mappings $A \longrightarrow B$ by $\mathsf{Map}(A, B)$, given the compact-open topology and the group of homeomorphisms $X \to X$ by $\mathsf{Homeo}(X)$.

If f_n (for $n = 1, 2, \dots, \infty$) are mappings on a space X, write $f_n \rightrightarrows f_\infty$ if f_n converges *uniformly* to f_∞ on X, with respect to a given *uniform structure* (for exampe a metric) on X.

Suppose (X, d) is a metric space. If $x \in X, r > 0$, define the *(open) ball* with center x and radius r as:

$$\mathsf{B}_r(x) \;:=\; \big\{ y \in X \mid \mathsf{d}(x, y) < r \big\}.$$

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The open balls in a metric space are partially ordered by inclusion. More generally, if $A \subset X$, define

$$\mathsf{B}_r(A) \ := \ \big\{ y \in X \mid \exists a \in A \text{ such that } \mathsf{d}(y,a) < r \big\}.$$

If (X, d) is a metric space, and $S, T \subset\subset X$, then define their *Hausdorff distance*

$$\mathsf{d}(S,T) \ := \ \mathsf{inf}\big\{ r \in \mathbb{R} \mid S \subset \mathsf{B}_r(T) \text{ and } T \subset \mathsf{B}_r(S) \big\}.$$

If X is compact, then the set of closed subsets of X with Hausdorff distance d is a metric space.

Denote the group of isometries of a metric space (X, d) by $\mathsf{Isom}(X, \mathsf{d})$, or just $\mathsf{Isom}(X)$ if the context is clear.

Fundamental group and covering spaces. For this material, we recommend the first chapter of Hatcher [**186**].

If $[a,b] \xrightarrow{\gamma} X$ is a continuous path, write

$$\gamma(a) \overset{\gamma}{\rightsquigarrow} \gamma(b)$$

to indicate that γ runs between its two endpoints $\gamma(a), \gamma(b)$. Two such paths are *relatively homotopic* if they are homotopic by a homotopy fixing their endpoints. In that case we write $\gamma_1 \simeq \gamma_2$.

Fix an (arbitrary) basepoint $p_0 \in X$. A *loop based at* p_0 is a path $p_0 \overset{\gamma}{\rightsquigarrow} p_0$, that is, a continuous map $[0,1] \xrightarrow{\gamma} X$ with

$$\gamma(0) = p_0 = \gamma(1).$$

The *fundamental group* $\pi_1(M; p_0)$ corresponding to p_0 consists of relative homotopy classes $[\gamma]$ of based loops γ.

The group operation is defined by *concatenation* of paths: Suppose

$$[a_i, b_i] \xrightarrow{\gamma_i} X, \ \text{for } i = 1,2$$

are paths, with $b_1 = a_2$ and $\gamma_1(b_1) = \gamma_2(a_2)$. Define $\gamma_1 \star \gamma_2$ to be the continuous path

$$\gamma_1(a_1) \rightsquigarrow \gamma_2(b_2),$$

given by:

$$\begin{aligned} [a_1, b_2] &\xrightarrow{\gamma_1 \star \gamma_2} X \\ t &\longmapsto \begin{cases} \gamma_1(t) & \text{if } a_1 \le t \le b_1 \\ \gamma_2(t) & \text{if } a_2 \le t \le b_2 \end{cases} \end{aligned}$$

If γ_1, γ_2 are loops based at p_0, so is $\gamma_1 \star \gamma_2$, and concatenation defines a binary operation on $\pi_1(X, p_0)$.

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The *constant path* p_0 defines an identity element on $\pi_1(X, p_0)$ since

$$p_0 \star \gamma \simeq \gamma \star p_0 \simeq \gamma.$$

Define the *inverse* of a path $[a, b] \xrightarrow{\gamma} M$

$$\begin{aligned} [a, b] &\xrightarrow{\gamma^{-1}} M \\ t &\longmapsto \gamma(a + b - t). \end{aligned}$$

If γ is a loop based at p_0, then

$$\gamma \star \gamma^{-1} \simeq \gamma^{-1} \star \gamma \simeq p_0,$$

obtaining *inversion* in $\pi_1(M; p_0)$. If $[a_3, b_3] \xrightarrow{\gamma_3} X$ with $\gamma_2(b_2) = \gamma_3(a_2)$, then

$$(\gamma_1 \star \gamma_2) \star \gamma_3 \simeq \gamma_1 \star (\gamma_2 \star \gamma_3),$$

implying associativity. Thus $\pi_1(X, p_0)$ is indeed a group.

Under rather general conditions on X (such as being a topological manifold) define the *universal covering space* (corresponding to p_0)

$$\widetilde{X^{(p_0)}} \xrightarrow{\Pi} X$$

as the collection of relative homotopy classes of paths γ starting at p_0, and ending at another point which we will call $\Pi(\gamma)$. Give $\widetilde{X^{(p_0)}}$ the quotient topology, which is the coarsest topology such that Π is continuous.

Then Π is a local homeomorphism, and indeed a *Galois covering space* (or regular covering space) with covering group $\pi_1(X, p_0)$.

The (left) action on $\widetilde{X^{(p_0)}}$ by deck transformations from $\pi_1(X, p_0)$ is defined as follows. Choose a point $p \in X$, a path $p_0 \overset{\eta}{\rightsquigarrow} p$ and a loop γ based at p_0. The action of $[\gamma]$ on $[\eta]$ is defined by:

$$[\eta] \overset{[\gamma]}{\longmapsto} [\gamma \star \eta].$$

The action is free and proper, preserves $p = \Pi([\eta])$. The quotient map naturally identifies with Π and the quotient space $\widetilde{X^{(p_0)}}/\pi_1(X, p_0)$ naturally identifies with X.

Smooth manifolds

We shall work in the context of smooth manifolds, for which a good general reference is Lee [**244**]. This will enable the use of differential calculus locally, and notions of smooth mappings between manifolds. A *smooth manifold* is a Hausdorff space built from open subsets of $\mathbb{R}^n$, which we call *coordinate patches.* The *coordinate changes* are general smooth locally invertible maps. If M and N are given such structures, a continuous map $M \longrightarrow N$ is *smooth* if in the local coordinate charts it is given by a smooth map.

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Smooth functions $M \longrightarrow \mathbb{R}$ form a commutative associative $\mathbb{R}$-algebra which we denote $\mathsf{C}^\infty(M)$.

This structure enables the *tangent bundle* $\mathsf{T}M$, whose points are the *infinitesimal displacements* of points in M. That is, to every smooth curve $(a, b) \xrightarrow{\gamma} M$, and parameter t with $a \le t \le b$, is a *velocity vector*

$$\gamma'(t) \in \mathsf{T}_{\gamma(t)}M$$

representing the infinitesimal effect of displacing $\gamma(t)$ along γ. Since the local coordinates change by general smooth locally invertible maps, there is no natural way of identifying these infinitesimal displacements at *different* points. Therefore we attach to each point $p \in M$, a "copy" T_pM of the model space $\mathbb{R}^n$, which represents the vector space of *infinitesimal displacements of* p. It is important to note that although the *fibers* T_pM are disjoint, that the union

$$\mathsf{T}M := \bigcup_{p \in M} \mathsf{T}_pM$$

is topologized as a smooth manifold (indeed, a smooth *vector bundle),* and not as the disjoint union (see below).

The velocity vector of a smooth curve is a *tangent vector at p,* which can be defined in two equivalent ways:

- Equivalence classes of smooth curves $\gamma(t)$ with $\gamma(0) = p$, where curves $\gamma_1 \sim \gamma_2$ if and only if
$$\frac{d}{dt}\bigg|_{t=0} f \circ \gamma_1(t) = \frac{d}{dt}\bigg|_{t=0} f \circ \gamma_2(t)$$
for all smooth functions $U \xrightarrow{f} \mathbb{R}$, where $U \subset M$ is an open neighborhood of p.
- Linear operators $\mathsf{C}^\infty(M) \xrightarrow{D} \mathbb{R}$ satisfying
$$D(fg) = D(f)g(p) + f(p)D(g).$$

The tangent space T_pM is a vector space *linearizing* the smooth manifold M at the point $p \in M$.

The space of tangent vectors forms a smooth vector bundle $\mathsf{T}M \xrightarrow{\Pi} M$, with fiber $\Pi^{-1}(p) := \mathsf{T}_pM$. If $U \ni p$ is a coordinate patch, then $\Pi^{-1}(U)$ identifies with $U \times \mathbb{R}^n$, and this defines a smooth coordinate atlas on $\mathsf{T}M$.

Let M, N be smooth manifolds, and $p \in M$. A mapping

$$M \xrightarrow{f} N$$

is *differentiable at p* if every infinitesimal displacement $\mathbf{v} \in \mathsf{T}_pM$ maps to an infinitesimal displacement $\mathsf{D}_pf(\mathbf{v}) \in \mathsf{T}_qN$, where $q = f(p)$. That is, if γ is a smooth curve with $\gamma(0) = p$ and $\gamma'(0) = \mathbf{v}$, then we require that

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$f \circ \gamma$ is a smooth curve through q at $t = 0$; then we call the new velocity $(f \circ \gamma)'(0) \in \mathsf{T}_q N$ the value of the *differential* or *derivative*

$$\begin{aligned} \mathsf{T}_p M &\xrightarrow{(\mathsf{D}f)_p} \mathsf{T}_q N \\ \mathbf{v} &\longmapsto (f \circ \gamma)'(0). \end{aligned}$$

Clearly a smooth mapping is differentiable in the above sense.

If P is a third smoooth manifold, and $N \xrightarrow{g} P$ is a smooth map, the composition $M \xrightarrow{g \circ f} P$ is defined, and is a smooth map. The *Chain Rule* expresses the derivative of the composition as the composition of the derivatives of f and g: and $M \xrightarrow{f} N \xrightarrow{g} P$ are smooth maps, then the differential of a composition

$$M \xrightarrow[f]{} N \xrightarrow[g]{} P, \qquad M \xrightarrow{g \circ f} P$$

induces a commutative diagram

$$\mathsf{T}_x M \xrightarrow[(\mathsf{D}f)_x]{} \mathsf{T}_{f(x)} N \xrightarrow[(\mathsf{D}g)_{f(x)}]{} \mathsf{T}_{(g \circ f)(x)} P, \qquad \mathsf{T}_x M \xrightarrow{\big(\mathsf{D}(g \circ f)\big)_x} \mathsf{T}_{(g \circ f)(x)} P$$

that is, $\mathsf{D}(g \circ f)_x = (\mathsf{D}g)_{f(x)} \circ (\mathsf{D}f)_x$.

If M, N are smooth manifolds, a *diffeomorphism* $M \longrightarrow N$ is an invertible smooth mapping whose inverse is also smooth. In particular a diffeomorphism is a smooth homeomorphism.

Now suppose $M \xrightarrow{f} N$ is a smooth map and $p \in M$ such that the differential

$$\mathsf{T}_p M \xrightarrow{(\mathsf{D}f)_p} \mathsf{T}_{f(p)} N$$

is an isomorphism of vector spaces. The *Inverse Function Theorem* guarantees the existence of an open neighborhood $U \ni p$ such that the restriction $f|_U$ is a diffeomorphism $U \to f(U)$. In particular $f(U) \subset N$ is open. Furthermore U can be chosen so that $(\mathsf{D}f)_q$ is an isomorphism for every $q \in U$. Such a map is called a *local diffeomorphism* (at p). Proposition 4.30 Lee [**244**] characterizes when a local diffeomrorphism is a smooth covering space.

Under the C^∞ topology, diffeomorphisms $M \to M$ form a topological group, denoted by $\mathsf{Diff}(M)$. Indeed $\mathsf{Diff}(M)$ has more structure as a *Fréchet Lie group.* If N is a smooth manifold, then a map $N \to \mathsf{Diff}(M)$ is *smooth* if

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the natural composition $N \times M \to M$ is smooth. A smooth homomorphism $\mathbb{R} \xrightarrow{\Phi} \mathsf{Diff}(M)$ is called a *smooth flow* on M.

Denote the group of diffeomorphisms of a smooth manifold X by $\mathsf{Diff}(X)$, with the C^∞ topology (uniform convergence to all orders, on all $K \subset\subset X$). If f, g are smooth maps between smooth manifolds $X \longrightarrow Y$, then we say that f and g are *isotopic* if and only if there is a smooth path

$$\phi_t \in \mathsf{Diff}(X),\ 0 \le t \le 1,$$

with $\phi_0 = \mathbb{I}_X$ such that $g = \phi_1 \circ f$. Denote this relation by $f \simeq g$.

Vector fields. A *vector field* on M is a section of the tangent bundle $\mathsf{T}M \xrightarrow{\Pi} M$, that is a mapping $M \xrightarrow{\xi} \mathsf{T}M$ such that

$$\Pi \circ \xi = \mathbb{I}_M,$$

or, equivalently, $\xi(p) \in \mathsf{T}_pM$ for all $p \in M$. Denote the space of all vector fields on M by $\mathsf{Vec}(M)$. Just as individual tangent vectors at $p \in M$ define derivations $\mathsf{C}^\infty(M) \longrightarrow \mathbb{R}$ over the evaluation map

$$\begin{aligned} \mathsf{C}^\infty(M) &\longrightarrow \mathbb{R} \\ f &\longmapsto f(p) \end{aligned}$$

vector fields in $\mathsf{Vec}(M)$ define derivations of the algebra $\mathsf{C}^\infty(M)$.

Let $a < b \in \mathbb{R}$. A smooth curve $(a, b) \xrightarrow{\gamma} M$ is an *integral curve* for $\xi \in \mathsf{Vec}(M)$ if and only if

$$\gamma'(t) = \xi\big(\gamma(t)\big) \in \mathsf{T}_{\gamma(t)}M$$

for all $a < t < b$. If Φ is a smooth flow as above, then for each $p \in M$,

$$\begin{aligned} \mathbb{R} &\xrightarrow{\Phi_p} M \\ t &\longmapsto \Phi(t)(p) \end{aligned}$$

is a smooth curve in M with velocity vector field $(\Phi_p)'(t) \in \mathsf{T}_{\Phi_p(t)}M$. In particular

$$\xi(p) := (\Phi_p)'(0) \in \mathsf{T}_pM \text{ (since } \Phi_p(0) = p)$$

and Φ_p being an integral curve of ξ defines a smooth vector field $\xi \in \mathsf{Vec}(M)$.

The *Fundamental Theorem on Flows* is a statement in the converse direction: every vector field $\xi \in \mathsf{Vec}(M)$ is tangent to a *local flow.* That is, through every point there exists a unique maximal integral curve, defined for some open interval (a, b) containing 0. When M is a closed manifold, then the integral curves are defined on all of $\mathbb{R}$ and corresponds to a flow Φ on M. Such a vector field is called *complete.* More generally (Lee [**244**], Theorem 9.16), if ξ is compactly supported, it is complete. See Lee [**244**], §9, for full details; a precise statement of the Fundamental Theorem on Flows is given in Theorem 9.12.

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If f is a *local diffeomorphism,* and $\xi \in \mathsf{Vec}(N)$, then define the *pullback* $f^*\xi \in \mathsf{Vec}(M)$ by:

$$(f^*\xi)_p := \big((\mathsf{D}f)_p\big)^{-1}(\xi_{f(p)}). \tag{0.1}$$

In particular, in the terminology of Lee [**244**], the vector fields ξ and $f^*\xi$ are f-related.

Suppose that $M \xrightarrow{f} N$ is a smooth map and $\xi \in \mathsf{Vec}(M)$ and $\eta \in \mathsf{Vec}(N)$ are *f-related* vector fields, that is,

$$(\mathsf{D}f)_p\big(\xi(p)\big) = \eta\big(f(p)\big), \quad \forall p \in M.$$

The *Naturality of Flows* (Lee [**244**], Theorem 9.13) implies that if $\Phi(t)$ is the local flow defined by $\xi \in \mathsf{Vec}(M)$ and $\Psi(t)$ the local flow on N defined by $\eta \in \mathsf{Vec}(N)$, then

$$f\big(\Phi_t(p)\big) = \Psi_t\big(f(p)\big)$$

whenever these objects are defined.

Tensor fields and differential forms. Given any smooth vector bundle $\mathsf{W} \longrightarrow M$ over a smooth manifold M, its sections $M \to E$ comprise a module $\Gamma(\mathsf{W})$ over the ring $\mathsf{C}^\infty(M)$. For example

$$\mathsf{Vec}(M) = \Gamma(\mathsf{T}M)$$

is a $\mathsf{C}^\infty(M)$-module; $\mathsf{C}^\infty(M)$ acts on $\mathsf{Vec}(M)$ by scalar multiplication of vector fields by functions. Furthermore $\mathsf{Vec}(M)$ is a Lie algebra under Lie bracket. Although Lie multiplication is not $\mathsf{C}^\infty(M)$-blinear, these two structures relate via:

$$[f\,\xi,\, g\,\eta] \;=\; fg[\xi,\eta] + f\,(\xi g)\,\eta - g\,(\eta f)\,\xi.$$

Suppose V, W are vector bundles over M. A bundle map $\mathsf{V} \to \mathsf{W}$ determines a homomorphism of $\mathsf{C}^\infty(M)$-modules $\Gamma(\mathsf{V}) \longrightarrow \Gamma(\mathsf{W})$. Conversely an $\mathbb{R}$-linear mapping $\Gamma(V) \longrightarrow \Gamma(\mathsf{W})$ corresponds to a bundle map if and only if it is linear over $\mathsf{C}^\infty(M)$, that is, a homomorphism of $\mathsf{C}^\infty(M)$-modules. Such bundle maps identify with sections of the vector bundle $\mathsf{Hom}(\mathsf{V},\mathsf{W})$. Compare Lee [**244**], §5.16.

For example, suppose that W is a vector bundle over M and

$$\underbrace{\mathsf{Vec}(M) \times \cdots \times \mathsf{Vec}(M)}_{k} \longrightarrow \Gamma(\mathsf{W})$$

is multilinear over $\mathsf{C}^\infty(M)$. Then F is induced by a vector bundle homomorphism

$$\otimes^k \mathsf{T}M \;:=\; \underbrace{\mathsf{T}M \otimes \cdots \otimes \mathsf{T}M}_{k} \longrightarrow \mathsf{W},$$

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or, equivalently, a section of $\mathsf{Hom}\big(\otimes^k(\mathsf{T}M), \mathsf{W}\big)$. Denote the space of such W-*valued covariant tensor fields* by

$$\mathcal{T}^k(M;\mathsf{E}) \longleftrightarrow \Gamma\Big(\mathsf{Hom}\big(\otimes^k \mathsf{T}M, \mathsf{W}\big)\Big).$$

The case when $k = 1$ is particularly important. Then

$$\mathsf{Hom}\big(\otimes^k \mathsf{T}M, \mathsf{W}\big) \cong \mathsf{T}^*M \otimes \mathsf{W},$$

whose sections are W-*valued* 1-*forms* on M. When $\mathsf{W} = \mathsf{T}M$, these are sections of

$$\mathsf{End}(\mathsf{T}M) := \mathsf{Hom}(\mathsf{T}M, \mathsf{T}M) \longleftrightarrow \mathsf{T}^*M \otimes \mathsf{T}M,$$

which we call *endomorphism fields.* An example of an endomorphism field is the identity map on $\mathsf{T}M$, which we can also think of as a $\mathsf{T}M$-valued 1-form. It is sometimes called the *solder form.*

Exterior differential calculus

Sections of the associated exterior algebra bundle $\Lambda^k(\mathsf{T}M)$ are *exterior differential forms* of degree $k \geq 0$; they comprise the $\mathsf{C}^\infty(M)$-module $\mathcal{A}^k(M)$. The direct sum

$$\mathcal{A}^*(M) := \bigoplus_{k=0}^{n} \mathcal{A}^k(M)$$

is a graded algebra over $\mathsf{C}^\infty(M)$; if $\alpha \in \mathcal{A}^p(M)$ then p is its *degree* and we write $p = |\alpha|$. Explicitly, If $\alpha \in \mathcal{A}^p(M)$, $\beta \in \mathcal{A}^q(M)$, their exterior product $\alpha \wedge \beta \in \mathcal{A}^{p+q}(M)$ is defined by:

$$(\alpha\wedge\beta)(\xi_1,\dots,\xi_{p+q}) \;=\; \frac{1}{p!q!}\sum_{\sigma\in\mathfrak{S}_{p+q}} (-1)^{|\sigma|}\alpha\big(\xi_{\sigma(1)},\dots,\xi_{\sigma(p)}\big)\,\beta\big(\xi_{\sigma(p+1)},\dots,\xi_{\sigma(p+q)}\big)$$

where $\xi_1, \dots, \xi_{p+q} \in \mathsf{Vec}(M)$.

This graded algebra is associative and graded-commutative under exterior (wedge) product, where *graded-commutativity* means

$$\beta \wedge \alpha \;=\; (-1)^{pq}\alpha \wedge \beta.$$

A collections of maps $\mathcal{A}^p(M) \xrightarrow{D_p} \mathcal{A}^{p+k}(M)$ is a *derivation of degree* k if and only if

$$D_{p+q}(\alpha\wedge\beta) \;=\; (D_p\alpha)\wedge\beta) + (-1)^{pk}\alpha\wedge D_q(\beta),$$

and if D, D' are derivations, their *commutator* $[D, D']$ defined by:

$$[D, D'] \;:=\; D\circ D' - (-1)^{|D||D'|}D'\circ D$$

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is a derivation of degree $|D| + |D'|$. We describe three derivations: *exterior differentiation* of degree $+1$, *interior multiplication* of degree -1 and depending on a vector field, and *Lie differentiation* of degree 0 and depending on a vector field.

- *Exterior differentiation* $\mathcal{A}^k(M) \xrightarrow{d} \mathcal{A}^{k+1}(M)$ is a derivation of degree 1:
$$d(\alpha \wedge \beta) \;=\; d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta$$
and satisfies:
$$df(\xi) \;=\; \xi f$$
for $\xi \in \mathsf{Vec}(M)$. Furthermore $d \circ d = 0$, and these properties uniquely characterize d.
- For any vector field $\xi \in \mathsf{Vec}(M)$, *interior multiplication* (or contraction) ι_ξ defined by
$$\iota_\xi(\omega)(\eta_1, \dots, \eta_{k-1}) \;:=\; \omega(\xi, \eta_1, \dots, \eta_{k-1}),$$
for $\omega \in \mathcal{A}^k(M), \eta_1, \dots, \eta_{k-1} \in \mathsf{Vec}(M)$.
- If $\xi \in \mathsf{Vec}(M)$ generates a local flow Φ_t, and ω is a tensor field on M, then the *Lie derivative* $\mathfrak{L}_\xi(\omega) \in \mathsf{Vec}(M)$ is defined as:
$$\mathfrak{L}_\xi(\omega) \;:=\; \frac{\partial}{\partial t}\Big|_{t=0} (\Phi_t)_*(\omega)$$
and defines a derivation of degree 0 on $\mathcal{A}^*(M)$.

Cartan's magic formula (Lee [**244**], Proposition 18.13) relates these these derviations through the graded commutator operation:

$$[d, \iota_\xi] \;:=\; d\iota_\xi + \iota_\xi d \;=\; \mathfrak{L}_\xi \tag{0.2}$$

Furthermore the graded commutator

$$[\mathfrak{L}_\xi,\, \iota_\eta] \;:=\; \mathfrak{L}_\xi\, \iota_\eta - \iota_\eta\, \mathfrak{L}_\xi \;=\; \iota_{[\xi,\eta]} \tag{0.3}$$

(Lee [**244**], Proposition 18.9(e)). In particular, if $\alpha \in \mathcal{A}^1(M), \omega \in \mathcal{A}^2(M)$ and $\xi, \eta, \zeta \in \mathsf{Vec}(M)$,[3] then

$$d\alpha(\xi, \eta) \;=\; \xi\alpha(\eta) - \eta\alpha(\xi) - \alpha([\xi, \eta]) \tag{0.4}$$

$$\begin{aligned} d\omega(\xi, \eta, \zeta) \;=\; &\xi\omega(\eta, \zeta) + \eta\omega(\zeta, \xi) + \zeta\omega(\xi, \eta) - \\ &\omega([\xi, \eta], \zeta) - \omega([\eta, \zeta], \xi) - \omega([\zeta, \xi], \eta) \end{aligned} \tag{0.5}$$

[3]Use the formula for $d\omega$ as found in Lee [**244**], or Kobayashi–Nomizu [**224**] — but note that [**224**] uses the "Alt-convention" for differential forms, Lee [**244**], §12, p.302.

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For any subbundle $E \subset \mathsf{T}M$, the annihilators

$$\mathsf{Ann}^p(E) \;:=\; \Big\{\alpha \in \mathcal{A}^p(M) \;\Big|\; \alpha(v_1,\dots,v_p) = 0,\ \forall v_1,\dots,v_p \in E\Big\}$$

define an ideal in the graded algebra $\mathcal{A}^*(M)$. Integrability is equivalent to this ideal being stable under d, that is, $\mathsf{Ann}^*(E)$ is a *differential ideal* in $\mathcal{A}^*(M)$ (Lee [**244**], Proposition 19.9).

Connections on vector bundles

We briefly summarize some general facts about Koszul connections which we use later. Compare Kobayashi–Nomizu [**224**] for more details.

If W is a vector bundle, then a *connection* on W is an $\mathbb{R}$-bilinear mapping

$$\begin{aligned} \mathsf{Vec}(M) \times \Gamma(\mathsf{W}) &\longrightarrow \Gamma(W) \\ (\xi, w) &\longmapsto \nabla_\xi(w) \end{aligned}$$

satisfying:

$$\begin{aligned} \nabla_{f\xi}(w) &= f\,\nabla_\xi w \\ \nabla_\xi(fw) &= f\,\nabla_\xi w \,+\, (\xi f)\,w \end{aligned}$$

for all $f \in \mathsf{C}^\infty(M)$. We call $\nabla_\xi w$ the *covariant derivative* of w with respect to ξ. Alternatively, tensoriality implies that this bilinear mapping is equivalent to an $\mathbb{R}$-linear mapping

$$\Gamma(\mathsf{W}) \xrightarrow{\nabla} \Gamma\big(\mathsf{T}^*M \otimes \mathsf{W})\big)$$

(called the *covariant differential*) satisfying

$$\nabla(fw) \;=\; f\,\nabla w + df \otimes w$$

for all $f \in \mathsf{C}^\infty(M)$. (Compare [**224**],Propositions 2.10,2.11,2.12.) The difference between two connections on W, as linear maps

$$\Gamma(\mathsf{W}) \to \Gamma(\mathsf{T}^*M \otimes \mathsf{W}) \;=\; \mathcal{A}^1(M, \mathsf{W}),$$

is an $\mathsf{End}(\mathsf{W})$-valued 1-form. Indeed, the vector space $\mathcal{A}^1\big(M, \mathsf{End}(\mathsf{W})\big)$ of $\mathsf{End}(\mathsf{W})$-valued 1-forms acts simply transitively on the space $\mathfrak{A}(\mathsf{W})$ of connections on E. That is, $\mathfrak{A}(\mathsf{W})$ is an affine space with underlying vector space $\mathcal{A}^1\big(M, \mathsf{End}(\mathsf{W})\big)$, in other words, an $\mathcal{A}^1\big(M, \mathsf{End}(\mathsf{W})\big)$*-torsor.*

The *Riemann curvature tensor* (or simply the *curvature*) of ∇ is the $\mathsf{End}(\mathsf{W})$-valued exterior 2-form

$$\mathsf{Riem}_\nabla \;\in\; \mathcal{A}^2\big(M; \mathsf{End}(\mathsf{W})\big) \;=\; \Gamma\Big(\mathsf{Hom}\big(\Lambda^2\mathsf{T}M, \mathsf{End}(\mathsf{W})\big)\Big)$$

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defined by

$$\begin{array}{ccc} \mathsf{Vec}(M) \times \mathsf{Vec}(M) \times \Gamma(\mathsf{W}) & \xrightarrow{\mathsf{Riem}_\nabla} & \Gamma(\mathsf{W}) \\ (\xi, \eta; w) & \longmapsto & \Big(\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi - \nabla_{[\xi,\eta]}\Big) w. \end{array}$$

A pleasant exercise is to show that this expression for Riem_∇ is $\mathsf{C}^\infty(M)$-trilinear, and thus corresponds to an $\mathsf{End}(\mathsf{W})$-valued exterior 2-form

$$\mathsf{Riem}_\nabla \in \mathsf{A}^2\big(M; \mathsf{End}(\mathsf{W})\big).$$

(Compare Kobayashi–Nomizu [**224**], §5, Chapter 6 of Burago–Burago–Ivanov [**68**] for Riemannian connections, and Appendix C of Milnor–Stasheff [**273**].)

The covariant differential operator

$$\mathcal{A}^0(M) \;=\; \Gamma(\mathsf{W}) \xrightarrow{\nabla} \Gamma(\mathsf{T}^*M \otimes \mathsf{W}) \;=\; \mathcal{A}^0(M, \mathsf{W})$$

extends to a mapping, the *covariant exterior differential,*

$$\mathcal{A}^k(M; \mathsf{W}) \xrightarrow{D_\nabla} \mathcal{A}^{k+1}(M; \mathsf{W})$$

such that if $\alpha \in \mathcal{A}^k(M), \beta \in \mathcal{A}^l(M; \mathsf{W})$, (so that $\alpha \wedge \beta \in \mathcal{A}^{l+1}(M; \mathsf{W})$), then

$$D_\nabla(\alpha \wedge \beta) \;=\; d\alpha \wedge \beta + (-1)^k \alpha \wedge D_\nabla \beta.$$

Then the square

$$\mathcal{A}^k(M; \mathsf{W}) \xrightarrow{D_\nabla \,\circ\, D_\nabla} \mathcal{A}^{k+2}(M; \mathsf{W})$$

equals exterior multiplication/composition with Riem_∇. In particular, if ∇ is *flat,* that is, $\mathsf{Riem}_\nabla = 0$, then $(D_\nabla)^2 = 0$ and W-valued de Rham cohomlogy $H^k(M; \mathsf{W})$ is defined.

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Part 1

Affine and projective geometry

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Chapter 1

Affine geometry

This section introduces the geometry of affine spaces. After a rigorous definition of affine spaces and affine maps, we discuss how linear algebraic constructions define geometric structures on affine spaces. Affine geometry is then transplanted to manifolds. The section concludes with a discussion of affine subspaces, vector fields, volume, and the notion of center of gravity.

1.1. Euclidean space

We begin with a short summary of Euclidean geometry in terms of its underlying space and its group of isometries.

Euclidean geometry can be described in many different ways. Here is one simple approach. Denote by E^n the set of points in the vector space $\mathbb{R}^n$ (that is, ordered n-tuples of real numbers) with the distance function

$$\begin{aligned} \mathsf{E}^n \times \mathsf{E}^n &\xrightarrow{\mathsf{d}} \mathbb{R} \\ (p,q) &\longmapsto \|p-q\|. \end{aligned}$$

Exercise 1.1.1. Let $(\mathsf{E}^n, \mathsf{d}) \xrightarrow{g} (\mathsf{E}^n, \mathsf{d})$ be an isometry. Then

$$g(p) \;=\; \mathbf{A}p + \mathbf{b}$$

for an orthogonal matrix $\mathbf{A} \in \mathsf{O}(n)$ and a vector $\mathbf{b} \in \mathbb{R}^n$. (*Hint:* First show that translations and orthogonal linear transformations are isometries. Then it suffices to prove that an isometry which fixes the origin and all tangent vectors at the origin is the identity. This can be done by characterizing straight-line paths — that is, *geodesics* — as curves which locally minimize length. Compare Exercise 1.3.4.)

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That is, the isometry g of Euclidean n-space E^n is a composition of the *linear isometry* defined by $\mathbf{A}$ and the *translation*

$$p \overset{\tau_{\mathbf{b}}}{\longmapsto} p + \mathbf{b}$$

by $\mathbf{b}$.

Two objects X, Y are *parallel* if they are related by the action of a translation, in which case we write $X \parallel Y$.

Exercise 1.1.2. Show that translations form a normal subgroup $\mathsf{Trans}(\mathsf{E}^n)$ isomorphic to $\mathbb{R}^n$ and

$$\mathsf{Isom}(\mathsf{E}^n) \;=\; \mathsf{Trans}(\mathsf{E}^n) \rtimes \mathsf{O}(n).$$

Deduce that $\mathsf{Isom}(\mathsf{E}^n)$ preserves the relation of parallelism.

Another feature of Euclidean geometry is the notion of *angle:*

Exercise 1.1.3. Every isometry of E^n preserves angles. (Hint: use the fact that angles can be defined in terms of inner products on $\mathbb{R}^n$.)

That is, every isometry is angle-preserving, or *conformal.* The equivalence relation of *similarity* is generated by the group $\mathsf{Sim}(\mathsf{E}^n)$ of conformal transformations of E^n.

An element of $\mathsf{Sim}(\mathsf{E}^n)$ which is not an isometry is the *homothety* given by scalar multiplication $p \longmapsto \lambda p$, where $\lambda \in \mathbb{R}^\times$ and $\lambda \neq \pm 1$. (See §1.6.2 for the general definition of homotheties.) Denote the group of scalar multiplications by $\lambda > 0$ by $\mathbb{R}^+$.

Exercise 1.1.4. The group $\mathsf{Sim}(\mathsf{E}^n)$ is generated by $\mathsf{Isom}(\mathsf{E}^n)$ and $\mathbb{R}^+$. Indeed,

$$\mathsf{Sim}(\mathsf{E}^n) \;=\; \mathsf{Sim}_0(\mathsf{E}^n) \ltimes \mathsf{Trans}(\mathsf{E}^n)$$

where

$$\mathsf{Sim}_0(\mathsf{E}^n) := \mathbb{R}^+ \times \mathsf{O}(n)$$

is the group of *linear similarities* of E^n. Explicitly a transformation g of E^n lies in $\mathsf{Sim}(\mathsf{E}^n)$ if it has the form

$$p \longmapsto \lambda \mathbf{A}(p) + \mathbf{b}$$

where $\mathbf{A} \in \mathsf{O}(n)$ and $\lambda \in \mathbb{R}^+$.

Yet another feature of Euclidean geometry is *volume:*

Exercise 1.1.5. Show that an orientation-preserving isometry of Euclidean space is volume-preserving. Show that an orientation-preserving similarity transformation preserves volume if and only if it is an isometry.

Compare §1.4.3 for further discussion of volume in affine geometry.

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1.2. Affine space

What geometric properties of E^n do not involve the metric notions of distance, angle and volume? For example, the notion of *straight line* is invariant under translations and more general linear maps which are not Euclidean isometries. It enjoys a metric characterization as a curve which is locally length-minimizing — that is, every subpath is "the shortest path joining its endpoint." However *geodesics* are more fundamentally characterized as curves of zero acceleration. However the definition of acceleration requires comparing the velocity vectors at *different* points along the curve. This is achieved by the *parallel transport* of the velocity along the curve, and hence involves the notion of *parallelism.* (This is the notion of an *affine connection,* which is way to "connect" the infinitesimal displacements at different locations.)

Here is our first definition of an *affine transformation:*

Definition 1.2.1. An *affine transformation* of $\mathbb{R}^n$ is a mapping of the form

$$\mathbb{R}^n \xrightarrow{g} \mathbb{R}^n$$
$$p \longmapsto \mathbf{A}p + \mathbf{b}$$

where $\mathbf{A} \in \mathsf{GL}(n, \mathbb{R})$ is an $n \times n$ invertible matrix and $\mathbf{b} \in \mathbb{R}^n$ is a vector. $\mathbf{A}$ is called the *linear part* of g, and denoted $\mathsf{L}(g)$ and $\mathbf{b}$ is called the *translational part* of g, and denoted $\mathbf{u}(g)$.

Thus an affine transformation g is:

- a translation if and only if $\mathsf{L}(g) = \mathbb{I}$;
- a Euclidean isometry if and only if $\mathsf{L}(g) \in \mathsf{O}(n)$;
- a Euclidean similarity (conformal transformation) if and only if $\mathsf{L}(g) \in \mathsf{Sim}_0(\mathbb{R}^n) = \mathbb{R}^+ \times \mathsf{O}(n)$;
- a volume-preserving (ore *special*) affine transformation if and only if $\mathsf{L}(g) \in \mathsf{SL}(n, \mathbb{R})$.

1.2.1. The geometry of parallelism. Here is a more formal definition of an affine space. Although less intuitive, it embodies the idea that affine geometry is the geometry of *parallelism.*

Recall that subsets $X, Y \subset \mathsf{E}^n$ are *parallel* (written $X \parallel Y$) if and only if $\tau_{\mathbf{v}}(X) = Y$ for some vector $\mathbf{v} \in \mathbb{R}^n$. (Here $\tau_{\mathbf{v}} \in \mathsf{Trans}(\mathsf{E}^n)$ denotes the translation $p \longmapsto p + \mathbf{v}$.) Affine geometry is the geometry arising from the simply transitive action of the vector space of translations (isomorphic to $\mathbb{R}^n$).

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Recall that an action of a group G on a space X is *simply transitive* if and only if for some (and then necessarily every) $x \in X$, the evaluation map

$$\begin{aligned} G &\longrightarrow X \\ g &\longmapsto g \cdot x \end{aligned}$$

is bijective: that is, for all $x, y \in X$, a unique $g \in G$ takes x to y. Equivalently, the action is both:

- *Transitive:* There is only one orbit, and
- *Free:* No nontrivial element fixes a point.

For further general discussion about group actions, see §A.3.

Definition 1.2.2. Let G be a group. A *G-torsor* is a space X with a simply transitive G-action.

Thus a G-torsor is like the group G, except that the special role of its identity element is "forgotten." Thus all the points are regarded as equivalent. What is remembered is the algebraic structure of the transformations in G which transport uniquely between the points.

Now we give the formal definition of a an affine space:

Definition 1.2.3. An *affine space* is a V-torsor A, where V is a vector space. We call V the *vector space underlying* A, and denote it by $\mathsf{Trans}(\mathsf{A})$, the elements of which are the *translations* of A.

This abstract approach provides the usual coordinates for an affine space, which identifies A with V. (In turn, choosing a basis of V identifies V with k^n.) Namely, choose a basepoint $p_0 \in \mathsf{A}$ which will correspond to the *origin* which identifies with the *zero vector* $\mathbf{0} \in \mathsf{V} \cong \mathsf{k}^n$. Any other point $p \in \mathsf{A}$ relates to p_0 by a unique translation $\tau \in \mathsf{Trans}(\mathsf{A})$ satisfying $p = \tau(p_0)$. (This translation exists because the action is transitive, and is unique because the action is free.) Identifying the transformations $\mathsf{Trans}(\mathsf{A})$ with vectors $\mathbf{v} \in \mathsf{V}$ in the usual coordinates, the vector $\mathbf{v}$ corresponding to τ is just $\mathbf{v} = p - p_0$, and τ is the mapping

$$\begin{aligned} \mathsf{A} &\xrightarrow{\tau} \mathsf{A} \\ p &\longmapsto p + \mathbf{v}. \end{aligned}$$

1.2.2. Affine transformations. Here is the second (more abstract) definition of the notion of a transformation being *affine.* The key point is that the arbitrary choices of basepoints enables "choosing origins" and identifies the affine spaces with their underlying vector spaces.

Affine maps are maps between affine spaces A, A' which are *compatible* with these simply transitive actions of vector spaces. In other words, they

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preserve the structures on A, A' as *torsors.* Denote the underlying vector spaces

$$\mathsf{V} \longleftrightarrow \mathsf{Trans}(\mathsf{A})$$
$$\mathsf{V}' \longleftrightarrow \mathsf{Trans}(\mathsf{A}')$$

respectively Then a continuous map

$$\mathsf{A} \xrightarrow{f} \mathsf{A}'$$

is *affine* if for each $\tau \in \mathsf{Trans}(\mathsf{A})$, there exists a translation $\tau' \in \mathsf{Trans}(\mathsf{A}')$ such that the diagram

$$\begin{array}{ccc} \mathsf{A} & \xrightarrow{f} & \mathsf{A}' \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau'} \\ \mathsf{A} & \xrightarrow{f} & \mathsf{A}' \end{array}$$

commutes, that is, $f \circ \tau = \tau' \circ f$. That is, for each vector $\mathbf{v} \in \mathsf{V}$, there exists a vector $\mathbf{v}' \in \mathsf{V}'$ such that f conjugates translation by $\mathbf{v}$ in A to translation by $\mathbf{v}'$ in A'.

Exercise 1.2.4. Suppose that f is affine as above, conjugating τ to τ'.

- Show that τ' is unique, and therefore f defines a map $\mathsf{V} \longrightarrow \mathsf{V}'$.
- Show that this map is a linear map of vector spaces.

This linear map, which we denote A, is the *linear part* of f, denoted $\mathsf{L}(f)$. Denoting the space of all affine maps $\mathsf{A} \longrightarrow \mathsf{A}'$ by $\mathsf{aff}(\mathsf{A}, \mathsf{A}')$ and the space of all linear maps $\mathsf{V} \longrightarrow \mathsf{V}'$ by $\mathsf{Hom}(\mathsf{V}, \mathsf{V}')$, linear part defines a map

$$\mathsf{aff}(\mathsf{A}, \mathsf{A}') \xrightarrow{\mathsf{L}} \mathsf{Hom}(\mathsf{V}, \mathsf{V}')$$

When $\mathsf{A} = \mathsf{A}'$, the set of affine *endomorphisms* of an affine space A will be denoted by $\mathsf{aff}(\mathsf{A})$ and the group of affine *automorphisms* of A will be denoted $\mathsf{Aff}(\mathsf{A})$. In particular if $f \in \mathsf{aff}(\mathsf{A})$, then $\mathsf{L}(f) \in \mathsf{End}(\mathsf{V})$. Moreover

$$\mathsf{L}(f) \in \mathsf{Aut}(\mathsf{V}) \cong \mathsf{GL}(n, \mathsf{k})$$

if and only if $f \in \mathsf{Aff}(\mathsf{A})$.

The notion of *translational part* involves choosing basepoints $p_0 \in \mathsf{A}$ and $p_0' \in \mathsf{A}'$, respectively. In particular, choice of a basepoint $p_0 \in \mathsf{A}$ identifies $\mathsf{GL}(V)$ as the subgroup of $\mathsf{Aff}(\mathsf{A})$ fixing p_0. More generally, $\mathsf{Hom}(\mathsf{V}, \mathsf{V}')$ identifies with the subspace of $\mathsf{aff}(\mathsf{A}, \mathsf{A}')$ comprising affine maps $\mathsf{A} \to \mathsf{A}'$ taking p_0 to p_0'.

Using the identifications

$$\mathsf{V} \longleftrightarrow \mathsf{A}$$
$$\mathsf{V}' \longleftrightarrow \mathsf{A}'$$

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defined by the respective basepoints, the *translational part* of $f \in \mathsf{aff}(\mathsf{A}, \mathsf{A}')$ is simply the vector $\mathbf{b}' \in \mathsf{V}'$ corresponding to the translation $\tau_f \in \mathsf{Trans}(\mathsf{A}')$ taking $p_0' \in \mathsf{A}'$ to $f(p_0) \in \mathsf{A}'$. Then $(\tau_f)^{-1} \circ f$ maps p_0 to p_0', and is a *linear* map $\mathsf{V} \to \mathsf{V}'$ in the above sense.

Exercise 1.2.5. An arbitrary point $p \in \mathsf{A}$ corresponds to a vector

$$\mathbf{x} \;=\; p - p_0 \in \mathsf{V},$$

that is, translation

$$\tau = \;\tau_{\mathbf{x}} \in \mathsf{Trans}(\mathsf{A})$$

by $\mathbf{x}$ takes p_0 to p. Show that under the above identifications, f corresponds to the map

$$p \;\leftrightarrow \mathbf{x} \quad \longmapsto \quad \mathbf{A}\mathbf{x} + \mathbf{b}' \leftrightarrow f(p).$$

The following suggestive facts are due to Vey [**337**], and foreshadow the discussion of affine Lie groups in Chapter 10.

Exercise 1.2.6. The space $\mathsf{aff}(\mathsf{A}, \mathsf{A}')$ of affine maps $\mathsf{A} \to \mathsf{A}'$ itself is an affine space, with underlying vector space $\mathsf{Hom}(\mathsf{V}, \mathsf{V}')$.

- The Cartesian product $\mathsf{Aff}(\mathsf{A}) \times \mathsf{Aff}(\mathsf{A}')$ acts by composition on $\mathsf{aff}(\mathsf{A}, \mathsf{A}')$, preserving the affine structure.
- More generally, if $\Omega \subset \mathsf{A}$ is a subdomain, show that *affine automorphism group* $\mathsf{Aff}(\Omega)$ is a Lie group with a bi-invariant affine structure (as defined in Chapter 10). Identify the Lie algebra of $\mathsf{Aff}(\mathsf{A})$ with $\mathsf{aff}(\mathsf{A})$.

§1.6 identifies $\mathsf{aff}(\mathsf{A})$ with the Lie algebra of *affine vector fields* on A.

Affine geometry is the study of affine spaces and affine maps between them. If $U \subset \mathsf{A}$ is an open subset, then a map $U \xrightarrow{f} \mathsf{A}'$ is *locally affine* if for each connected component U_i of U, there exists an affine map $f_i \in \mathsf{aff}(\mathsf{A}, \mathsf{A}')$ such that the restrictions of f and f_i to U_i are identical. Note that two affine maps which agree on a nonempty open set are identical.

1.3. The connection on affine space

Now we discuss the structure of an affine space A as a smooth manifold. To analyze the differentiable structure on A, we consider smooth paths in A and their velocity vector fields, which live in the tangent bundle TA. From this we "connect" the tangent spaces to define covariant differentiation enabling us to define acceleration as the covariant derivative of the velocity. Geodesics are curves of zero acceleration.

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1.3.1. The tangent bundle of an affine space. Let $\gamma(t)$ denote a *smooth curve* in A; that is, in coordinates

$$\gamma(t) = \big(x^1(t), \dots, x^n(t)\big)$$

where $x^j(t)$ are smooth functions of the time parameter, which ranges in an interval $[t_0, t_1] \subset \mathbb{R}$. The vector $\gamma(t) - \gamma(t_0)$ corresponds to the unique translation taking $\gamma(t_0)$ to $\gamma(t)$, and lies in the vector space V underlying A. It represents the displacement of the curve γ as it goes from t_0 to t. Define its *velocity vector* $\gamma'(t) \in \mathsf{V}$ as the derivative of this path in the vector space V of translations. It represents the *infinitesimal* displacement of $\gamma(t)$ as t varies.

The set of tangent vectors is a vector space, denoted $\mathsf{T}_p\mathsf{A}$, and naturally identifies with V as follows. If $\mathbf{v} \in \mathsf{V}$ is a vector, then the path $\gamma_{(p,\mathbf{v})}(t)$ defined by:

$$t \mapsto p + t\mathbf{v} = \tau_{t\mathbf{v}}(p) \tag{1.1}$$

is a smooth path with $\gamma(0) = p$ and velocity vector $\gamma'(0) = \mathbf{v}$. Conversely, the above discussion of infinitesimal displacement implies that every smooth path through $p = \gamma(0)$ with velocity $\gamma'(0) = \mathbf{v}$ is tangent to the curve (1.1) as above.

The tangent spaces *linearize* smooth manifolds as follows. Let M, M' be smooth manifolds and

$$M \xrightarrow{f} M'$$

a continuous map. Then f is *differentiable at* $p \in M$ if every infinitesimal displacement $\mathbf{v} \in \mathsf{T}_pM$ maps to an infinitesimal displacement $\mathsf{D}_pf(\mathbf{v}) \in \mathsf{T}_qM'$, where $q = f(p)$. That is, if γ is a smooth curve with $\gamma(0) = q$ and $\gamma'(0) = \mathbf{v}$, then we require that $f \circ \gamma$ is a smooth curve through q at $t = 0$; then we call the new velocity $(f \circ \gamma)'(0)$ the value of the *derivative*

$$\begin{aligned} \mathsf{T}_pM &\xrightarrow{\mathsf{D}_pf} \mathsf{T}_qM' \\ \mathbf{v} &\longmapsto (f \circ \gamma)'(0) \end{aligned}$$

1.3.2. Parallel transport. On an affine space A, all the tangent spaces identify with each other. Namely, if $x, y \in \mathsf{A}$, let $\tau \in \mathsf{Trans}(\mathsf{A})$ be the unique translation taking x to y. (τ corresponds to the vector $y - x$.) The differential $(\mathsf{D}\tau)_x$ maps $\mathsf{T}_x\mathsf{A}$ isomorphically to $\mathsf{T}_y\mathsf{A}$) and we denote this by:

$$\mathsf{T}_x\mathsf{A} \xrightarrow{\mathbb{P}_{x,y}} \mathsf{T}_y\mathsf{A}$$

We call this map *parallel transport* from x to y.

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Exercise 1.3.1. Another construction involves the linear structure of $\mathsf{V} \longleftrightarrow \mathsf{Trans}(\mathsf{A})$. Namely, the action of V by translations identifies the vector space V with $\mathsf{T}_x\mathsf{A}$. Denoting this isomorphism by $\mathsf{V} \xrightarrow{\alpha_x} \mathsf{T}_x\mathsf{A}$, show that

$$\mathbb{P}_{x,y} = \alpha_y \circ (\alpha_x)^{-1}.$$

A vector field $\xi \in \mathsf{Vec}(\mathsf{A})$ is *parallel* if it is invariant under parallel transport. That is, $\mathbb{P}_{x,y}(\xi_x) = \xi_y$ for any $x, y \in \mathsf{A}$. This just means that ξ is a "constant vector field," defined by a constant map $\mathsf{A} \xrightarrow{\mathbf{v}} \mathsf{V}$: as a differential operator

$$\begin{aligned} \mathsf{C}^\infty(\mathsf{A}) &\xrightarrow{\xi} \mathsf{C}^\infty(\mathsf{A}) \\ f &\longmapsto v^i(x)\frac{\partial f}{\partial x^i} \end{aligned}$$

where $\mathbf{v}(x)$ is constant. Thus V identifies with the space of parallel vector fields on A, and is based by the coordinate vector fields

$$\frac{\partial}{\partial x^i} \in \mathsf{Vec}(A),$$

which we abbreviate simply by ∂_i.

Exercise 1.3.2. Show that $\xi \in \mathsf{Vec}(\mathsf{A})$ is parallel if and only if it generates a one-parameter group of translations.

Similarly, the dual vector space V^* identifies with parallel 1-forms as follows. A 1-form (covector field) on A corresponds to a constant map $\mathsf{A} \longrightarrow \mathsf{V}^*$. The basis of parallel covector fields dual to the coordinate basis $\{\partial_1, \dots, \partial_n\}$ of parallel vector fields is denoted $\{dx^1, \dots, dx^n\}$ (as usual).

Exercise 1.3.3. Show that a parallel 1-form is exact, and hence closed.

1.3.3. Acceleration and geodesics. The velocity vector field $\gamma'(t)$ of a smooth curve $\gamma(t)$ is an example of a *vector field along the curve* $\gamma(t)$*:* For each t, the tangent vector $\gamma'(t) \in \mathsf{T}_{\gamma(t)}\mathsf{A}$. Differentiating the velocity vector field raises a significant difficulty: since the values of the vector field live in different vector spaces, we need a way to compare, or to *connect* them. The natural way is use the simply transitive action of the group V of translations of A. That is, suppose that $\gamma(t)$ is a smooth path, and $\xi(t)$ is a vector field along $\gamma(t)$. Let τ_s^t denote the translation taking $\gamma(t+s)$ to $\gamma(t)$, that is, in coordinates:

$$\begin{aligned} \mathsf{A} &\xrightarrow{\tau_s^t} \mathsf{A} \\ p &\longmapsto p + \big(\gamma(t) - \gamma(t+s)\big) \end{aligned}$$

Its differential

$$\mathsf{T}_{\gamma(t+s)}\mathsf{A} \xrightarrow{\mathsf{D}\tau_s^t} \mathsf{T}_{\gamma(t)}\mathsf{A}$$

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then maps $\xi(t+s)$ into $\mathsf{T}_{\gamma(t)}\mathsf{A}$ and the *covariant derivative* $\frac{\mathsf{D}}{dt}\xi(t)$ is the derivative of this smooth path in the *fixed* vector space $\mathsf{T}_{\gamma(t)}\mathsf{A}$:

$$\frac{\mathsf{D}}{dt}\xi(t) := \frac{d}{ds}\bigg|_{s=0} (\mathsf{D}\tau_s^t)\big(\xi(t+s)\big)$$

$$= \lim_{s\to 0} \frac{(\mathsf{D}\tau_s^t)\big(\xi(t+s)\big) - \xi(t)}{s}$$

In this way, define the *acceleration* as the covariant derivative of the velocity:

$$\gamma''(t) := \frac{\mathsf{D}}{dt}\gamma'(t)$$

A curve with zero acceleration is called a *geodesic.*

Exercise 1.3.4. Given a point p and a tangent vector $\mathbf{v} \in \mathsf{T}_p\mathsf{A}$, show that the unique *geodesic* $\gamma(t)$ with

$$\big(\gamma(0), \gamma'(0)\big) = (p, \mathbf{v})$$

is given by (1.1).

In other words, geodesics in A are parametrized curves which are Euclidean straight lines traveling at *constant* speed. However, in affine geometry the *speed* itself in not defined, but "motion along a straight line at constant speed" is affinely invariant (zero acceleration).

This leads to the following important definition:

Definition 1.3.5. Let $p \in \mathsf{A}$ and $\mathbf{v} \in \mathsf{T}_p(\mathsf{A}) \cong \mathsf{V}$. Then the *exponential mapping* is defined by:

$$\mathsf{T}_p\mathsf{A} \xrightarrow{\mathsf{Exp}_p} \mathsf{A}$$
$$\mathbf{v} \longmapsto p + \mathbf{v}.$$

Thus the unique geodesic with initial position and velocity $(p, \mathbf{v})$ equals

$$t \longmapsto \mathsf{Exp}_p(t\mathbf{v}) = p + t\mathbf{v}.$$

1.4. Parallel structures

Many important refinements of affine geometry involve structures which are *parallel.* Parallelism generalizes the notion of "constant" when the targets vary from point to point.

For example, the most familiar geometry is *Euclidean geometry,* extremely rich with metric notions such as distance, angle, area, and volume.

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We have seen that *affine geometry* underlies it with the more primitive notion of *parallelism.* Euclidean geometry arises from affine geometry by introducing a Riemannian structure on A, which is *parallel.*

Parallel vector fields and 1-forms were introduced back in §1.3.2, where parallel vector fields correspond to vectors in V and parallel 1-forms (parallel covector fields) correspond to covectors in V^*. Now we consider parallel tensor fields of higher order.

1.4.1. Parallel Riemannian structures. Let B be an inner product on V and $\mathsf{O}(\mathsf{V};\mathsf{B}) \subset \mathsf{GL}(\mathsf{A})$ the corresponding orthogonal group. Then B defines a flat Riemannian metric on A and the inverse image

$$\mathsf{L}^{-1}(\mathsf{O}(\mathsf{V};\mathsf{B})) \cong \mathsf{O}(\mathsf{V};\mathsf{B}) \cdot \mathsf{Trans}(\mathsf{A})$$

is the full group of isometries, that is, the *Euclidean group.* If B is a nondegenerate indefinite form, then there is a corresponding flat pseudo-Riemannian metric on A and the inverse image $\mathsf{L}^{-1}\big(\mathsf{O}(\mathsf{V};\mathsf{B})\big)$ is the full group of isometries of this pseudo-Riemannian metric.

1.4.2. Similarity geometry. Euclidean space with a parallel *conformal structure* (that is, an infinitesimal notion of *angle*) is a model for *similarity geometry.*[1]

Exercise 1.4.1. Show that an affine automorphism g of Euclidean n-space E^n is conformal (that is, preserves angles) if and only if its linear part is the composition of an orthogonal transformation and scalar multiplication.

Such a transformation will be called a *similarity transformation* and the group of similarity transformations will be denoted $\mathsf{Sim}(\mathsf{E}^n)$. The scalar multiple is called the *scale factor* $\lambda(g) \in \mathbb{R}^\times$ and defines a homomorphism $\mathsf{Sim}(\mathsf{E}^n) \xrightarrow{\lambda} \mathbb{R}^\times$. In general, if $g \in \mathsf{Sim}(\mathsf{E}^n)$, then $\exists! \mathbf{A} \in \mathsf{O}(n)$ and $\mathbf{b} \in \mathbb{R}^n$ such that

$$x \xrightarrow{g} \lambda(g)\mathbf{A}x + \mathbf{b}.$$

Furthermore $\lambda(g)\mathbf{A} = \mathsf{L}(g)$ identifies with $\mathsf{D}g$.

1.4.3. Parallel tensor fields. Any tangent vector $\mathbf{v}_p \in \mathsf{T}_p\mathsf{A}$ extends uniquely to a vector field on A invariant under the group of translations. As we saw in §1.4.1, Euclidean structures are defined by extending an inner product from a single tangent space to all of E.

Dual to parallel vector fields are *parallel* 1*-forms.* Every tangent *covector* $\omega_p \in \mathsf{T}_p^*\mathsf{A}$ extends uniquely to a translation-invariant 1-form.

[1]According to Fried [**135**] Euclidean geometry is affine geometry with a parallel *ruler* and similarity geometry is affine geometry with a parallel *protractor.*

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Exercise 1.4.2. Prove that a parallel 1-form is closed. Express a parallel 1-form in local coordinates.

If $n = \mathsf{dim}(\mathsf{A})$, then an exterior n-form ω must be $f(x)\, dx^1 \wedge \cdots \wedge dx^n$ in local coordinates, where $f \in C^\infty(\mathsf{A})$ is a smooth function. Then ω is parallel if and only if $f(x)$ is constant.

Exercise 1.4.3. Prove that an affine transformation $g \in \mathsf{Aff}(\mathsf{A})$ preserves a parallel volume form if and only if $\det \mathsf{L}(g) = 1$.

Parallel volume forms are discussed more extensively in §1.7

1.4.4. Complex affine geometry. We have been working entirely over $\mathbb{R}$, but it is clear one may study affine geometry over any field. If $\mathsf{k} \supset \mathbb{R}$ is a field extension, then every k-vector space is a vector space over $\mathbb{R}$ and thus every k-affine space is an $\mathbb{R}$-affine space. In this way we obtain more refined geometric structures on affine spaces by considering affine maps whose linear parts are linear over k.

Exercise 1.4.4. Relate 1-dimensional complex affine geometry to 2-dimensional similarity geometry with a fixed orientation.

This structure is another case of a parallel structure on an affine space, as follows. Recall a complex vector space has an underlying structure as a real vector space V. The difference is a notion of scalar multiplication by $\sqrt{-1}$, which is given by a linear map

$$\mathsf{V} \xrightarrow{J} \mathsf{V}$$

such that $J \circ J = -\mathbb{I}$. Such an automorphism is called a *complex structure* on V, and “turns V into” a *complex vector space.*

If M is a manifold, an endomorphism field (that is, a $(1,1)$-tensor field) J where, for each $p \in M$, the value J_p is a complex structure on the tangent space $\mathsf{T}_p M$ is called an *almost complex structure.* Necessarily $\mathsf{dim}(M)$ is even.

Recall that a *complex manifold* is a manifold with an atlas of coordinate charts where coordinate changes are biholomorphic. (Such an atlas is called a *holomorphic atlas.*) Every complex manifold admits an almost complex structure, but not every almost complex structure arises from a holomorphic atlas, except in dimension two.

Exercise 1.4.5. Prove that a complex affine space is the same as an affine space with a parallel almost complex structure.

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1.5. Affine subspaces

Suppose that $\mathsf{A}_1 \overset{\iota}{\hookrightarrow} \mathsf{A}$ is an injective affine map; then we say that $\iota(\mathsf{A}_1)$ (or with slight abuse, ι itself) is an *affine subspace.* If A_1 is an affine subspace then for each $x \in \mathsf{A}_1$ there exists a linear subspace $\mathsf{V}_1 \subset \mathsf{Trans}(\mathsf{A})$ such that A_1 is the orbit of x under V_1. That is, "an affine subspace in a vector space is just a coset (or translate) of a linear subspace $\mathsf{A}_1 = x + \mathsf{V}_1$." An affine subspace of dimension 0 is thus a point and an affine subspace of dimension 1 is a line. The next exercise describes how the quotient of an affine space by an affine subspace has the natural structure of a *vector space.*

Exercise 1.5.1. Let $\mathsf{A}_1 \subset \mathsf{A}$ is an affine subspace; then the cosets $p + \mathsf{A}_1$ are parallel affine subspaces defining equivalence classes for an equivalence relation on A. Let A/A_1 be the corresponding quotient space. Define a vector space structure on A/A_1 so that every affine transformation of A which preserves A_1 induces a linear transformation of A/A_1.

The next exercise describes a natural *affine parameter* along an affine line.

Exercise 1.5.2. Show that if ℓ, ℓ' are (affine) lines and

$$(x, y) \in \ell \times \ell,\ x \neq y$$
$$(x', y') \in \ell' \times \ell',\ x' \neq y'$$

are pairs of distinct points. Then there is a unique affine map $\ell \xrightarrow{f} \ell'$ such that

$$f(x) = x',$$
$$f(y) = y'.$$

If $x, y, z \in \ell$ (with $x \neq y$), then define $[x, y, z]$ to be the image of z under the unique affine map $\ell \xrightarrow{f} \mathbb{R}$ with $f(x) = 0$ and $f(y) = 1$. Show that if $\ell = \mathbb{R}$, then $[x, y, z]$ is given by the formula

$$[x, y, z] = \frac{z - x}{y - x}.$$

1.6. Affine vector fields

A vector field X on A is said to be *affine* if it generates a one-parameter group of affine transformations. Affine vector fields include parallel vector fields and radiant vector fields. Parallel vector fields generate one-parameter groups of translations, and *radiant* vector fields generate one-parameter groups of homotheties. Covariant differentiation provides general criteria characterizing affine vector fields.

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1.6.1. Translations and parallel vector fields. Now we discuss the important case of "constant" vector fields, which play the role of *infinitesimal translations.*

Definition 1.6.1. A vector field X on A is *parallel* if, for every $p, q \in \mathsf{A}$, the values $X_p \in \mathsf{T}_p\mathsf{A}$ and $X_q \in \mathsf{T}_q\mathsf{A}$ are parallel.

Since translation $\tau_{\mathbf{v}}$ by $\mathbf{v} = q - p$ is the unique translation taking p to q, this simply means that the differential $(\mathsf{D}\tau_{\mathbf{v}})_p$ maps X_p to X_q.

Exercise 1.6.2. Let $X \in \mathsf{Vec}(\mathsf{A})$ be a vector field on an affine space A. The following conditions are equivalent:

- X is parallel
- The coefficients of X (in affine coordinates) are constant.
- $\nabla_Y X = 0$ for all $Y \in \mathsf{Vec}(\mathsf{A})$.
- The covariant differential $\nabla X = 0$.
- Show how to identify $\mathsf{Vec}(\mathsf{A})$ with $\mathsf{Map}(\mathsf{A}, \mathsf{V})$ so that X corresponds to an affine map f with linear part $\mathsf{L}(f) = 0$.

The vector space V identifies with the space of parallel vector fields on A, which is an abelian subalgebra of the Lie algebra $\mathsf{Vec}(\mathsf{A})$.

1.6.2. Homotheties and radiant vector fields. Scalar multiplication determine another important class of affine vector fields: the *radiant vector fields,* which are *infinitesimal homotheties.*

Definition 1.6.3. An affine transformation $\phi \in \mathsf{Aff}(\mathsf{A})$ is a *homothety* if it is conjugate by a translation to scalar multiplication $\mathbf{v} \longmapsto \lambda\mathbf{v}$, for some scalar $\lambda \in \mathbb{R}^{\times}$ with $\lambda \neq \pm 1$. An affine vector field is *radiant* if it generates a one-parameter group of homotheties.

Observe that a homothety fixes a unique point $p \in \mathsf{A}$, which we often take to be the origin. The only zero of the corresponding radiant vector field is p. We denote by Rad_p the unique radiate vector field vanishing at p, which, in coordinates, equals:

$$\mathsf{Rad}_p := \sum_{i=1}^{n} (x^i - p^i) \frac{\partial}{\partial x^i}. \tag{1.2}$$

Radiant vector fields are also called *Euler vector fields,* due to their role in Euler's theorem on homogeneous functions: Recall that a function $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ is *homogeneous of degree* m if and only if

$$f(\lambda x) = \lambda^m f(x)$$

for all $\lambda \in \mathbb{R}^+$.

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Exercise 1.6.4. Prove *Euler's theorem* on homogeneous functions: Suppose that Rad is the radiant vector field vanishing at the origin $\mathbf{0}$. Then f is *homogeneous of degree* m if and only if $\mathsf{Rad} f = mf$.

Radiant vector fields play an important role, since important examples of affine manifolds admit radiant vector fields. Furthermore radiant vector fields provide a link between n-dimensional affine manifolds and $(n-1)$-dimensional projective manifolds, through the *radiant suspension* construction discussed in §6.5.3.

Exercise 1.6.5. Let $X \in \mathsf{Vec}(\mathsf{A})$ be a vector field. Then the following conditions are equivalent:

- X is radiant;
- $\nabla_Y X = X$, for all $Y \in \mathsf{Vec}(\mathsf{A})$
- $\nabla X = \mathbb{I}_{\mathsf{A}}$ where $\mathbb{I}_{\mathsf{A}} \in \mathcal{A}^1(\mathsf{A}, \mathsf{T}\mathsf{A})$ is the $\mathsf{T}\mathsf{A}$-valued 1-form on A (solder form) defined in § B.1 corresponding to the identity endomorphism field (defined above);

In particular, Rad generates the one-parameter group of homotheties

$$p \longmapsto b + e^t(p-b)$$

fixing b. Thus a radiant vector field is a special kind of affine vector field. Furthermore Rad generates the center of the isotropy group of $\mathsf{Aff}(\mathsf{A})$ at b, which is conjugate (by translation by b) to $\mathsf{GL}(\mathsf{V})$.

Exercise 1.6.6. Show that the radiant vector fields on A form an affine subspace of $\mathsf{Vec}(\mathsf{A})$ isomorphic to A.

- Show that the sum of a parallel vector field and a radiant vector field and a parallel vector is radiant.
- Define a vector field X to be *ϵ-radiant* if it equals ϵR for a radiant vector filed. Show that X is ϵ-radiant if and only if $\nabla_Y X = \epsilon Y$, $\forall Y \in \mathsf{Vec}(\mathsf{A})$, or, equivalently, $\nabla X = \epsilon X$.
- Let Y be a parallel vector field. Find a family X_ϵ of vector fields such that X_ϵ is ϵ-radiant and
$$\lim_{\epsilon \to 0} X_\epsilon = Y.$$

1.6.3. Affineness criteria. Affine vector fields can be characterized in terms of the covariant differential

$$\mathcal{T}^p(M; \mathsf{T}M) \xrightarrow{\nabla} \mathcal{T}^{p+1}(M; \mathsf{T}M)$$

where $\mathcal{T}^p(M; \mathsf{T}M)$ denotes the space of $\mathsf{T}M$-valued covariant p-tensor fields on M (the tensor fields of type $(1,p)$). Thus $\mathcal{T}^0(M; \mathsf{T}M) = \mathsf{Vec}(M)$ and

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$\mathcal{T}^1(M;\mathsf{T}M)$ comprises endomorphism fields, (alternatively $\mathsf{T}M$-valued 1-forms).

Exercise 1.6.7. X is affine if and only if it satisfies any of the following equivalent conditions:

- For all $Y, Z \in \mathsf{Vec}(\mathsf{A})$,
$$\nabla_Y \nabla_Z X = \nabla_{(\nabla_Y Z)} X.$$
- $\nabla\nabla X = 0$.
- The coefficients of X are affine functions, that is,
$$X = \sum_{i,j=1}^{n} (a^i{}_j x^j + b^i) \frac{\partial}{\partial x^i}$$
for constants $a^i{}_j, b^i \in \mathbb{R}$.

Write
$$\mathsf{L}(X) = \sum_{i,j=1}^{n} a^i{}_j x^j \frac{\partial}{\partial x^i}$$
for the linear part (which corresponds to the matrix $(a^i{}_j) \in \mathsf{gl}(\mathbb{R}^n)$) and
$$X(0) = \sum_{i=1}^{n} b^i \frac{\partial}{\partial x^i}$$
for the translational part (the translational part of an affine vector field is a parallel vector field). We denote the affine transformation corresponding to $X \in \mathsf{Vec}(\mathsf{A})$ by $\hat{X}$. Thus $\hat{X} = \left[\mathsf{L}(X) \;\middle|\; X(0)\right]$.

Exercise 1.6.8. Under this correspondence, covariant derivative corresponds to composition of affine maps (matrix multiplication):
$$\nabla_B A \quad \longleftrightarrow \hat{A}\hat{B}$$

The Lie bracket of two affine vector fields is given by:

- $\mathsf{L}([X,Y]) = [\mathsf{L}(X), \mathsf{L}(Y)] = \mathsf{L}(X)\mathsf{L}(Y) - \mathsf{L}(X)\mathsf{L}(Y)$ (matrix multiplication)

- $[X,Y](0) = \mathsf{L}(X)Y(0) - \mathsf{L}(Y)X(0)$.

In this way the space $\mathsf{aff}(\mathsf{A}) = \mathsf{aff}(\mathsf{A},\mathsf{A})$ of affine endomorphisms of A is a Lie algebra.

Let M be an affine manifold. A vector field $\xi \in \mathsf{Vec}(M)$ is *affine* if in local coordinates ξ appears as a vector field in $\mathsf{aff}(\mathsf{A})$. We denote the space of affine vector fields on an affine manifold M by $\mathsf{aff}(M)$.

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Exercise 1.6.9. Let M be an affine manifold.

(1) Show that $\mathsf{aff}(M)$ is a subalgebra of the Lie algebra $\mathsf{Vec}(M)$.

(2) Show that the identity component of the affine automorphism group $\mathsf{Aut}(M)$ has Lie algebra $\mathsf{aff}(M)$.

(3) If ∇ is the flat affine connection corresponding to the affine structure on M, show that a vector field $\xi \in \mathsf{Vec}(M)$ is affine if and only if, for all $v \in \mathsf{Vec}(M)$,

$$\nabla_\xi v = [\xi, v].$$

1.7. Volume in affine geometry

Although an affine automorphism of an affine space A need not preserve a natural measure on A, Euclidean volume nonetheless does behave rather well with respect to affine maps. The Euclidean volume form ω can almost be characterized affinely by its parallelism: it is invariant under all translations. Moreover two $\mathsf{Trans}(\mathsf{A})$-invariant volume forms differ by a scalar multiple but there is no natural way to normalize. Such a volume form will be called a *parallel volume form.* If $g \in \mathsf{Aff}(\mathsf{A})$, then the distortion of volume is:

$$g^*\omega = \det \mathsf{L}(g) \cdot \omega.$$

Compare Exercise 1.4.3.) Thus although there is no canonically normalized volume or measure there is a natural affinely invariant line of measures on an affine space. The subgroup $\mathsf{SAff}(\mathsf{A})$ of $\mathsf{Aff}(\mathsf{A})$ consisting of volume-preserving affine transformations is the inverse image $\mathsf{L}^{-1}(\mathsf{SL}(\mathsf{V}))$, sometimes called the *special affine group* of A. Here $\mathsf{SL}(\mathsf{V})$ denotes the *special linear group*

$$\mathsf{Ker}\big(\mathsf{GL}(\mathsf{V}) \xrightarrow{\det} \mathbb{R}^\times\big) \;=\; \{g \in \mathsf{GL}(\mathsf{V}) \mid \det(g) = 1\}.$$

1.7.1. Centers of gravity.

Given a finite subset $F \subset \mathsf{A}$ of an affine space, its *center of gravity* or *centroid* $\bar{F} \in \mathsf{A}$ is point associated with F in an affinely invariant way: that is, given an affine map $\mathsf{A} \xrightarrow{\phi} \mathsf{A}'$ we have

$$\overline{\phi(F)} = \phi(\bar{F}).$$

This operation can be generalized as follows.

Theorem 1.7.1. Let μ be a compactly supported probability measure on an affine space A. Then there exists a unique point $\bar{x} \in \mathsf{A}$ (the *centroid* of μ) such that for all affine maps $\mathsf{A} \xrightarrow{f} \mathbb{R}$,

$$f(\bar{x}) = \int_{\mathsf{A}} f\, d\mu \tag{1.3}$$

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Proof. Let $(x^1, \dots, x^n)$ be an affine coordinate system on A. Let $\bar{x} \in \mathsf{A}$ be the points with coordinates $(\bar{x}^1, \dots, \bar{x}^n)$ given by

$$\bar{x}^i = \int_{\mathsf{A}} x^i \, d\mu.$$

This uniquely determines $\bar{x} \in \mathsf{A}$; we must show that (1.3) is satisfied for all affine functions. Suppose $\mathsf{A} \xrightarrow{f} \mathbb{R}$ is an affine function. Then there exist $a_1, \dots, a_n, b$ such that

$$f = a_1 x^1 + \dots + a_n x^n + b$$

and thus

$$\begin{aligned} f(\bar{x}) = a_1 \int_{\mathsf{A}} x^1 \, d\mu + \dots + a_n \int_{\mathsf{A}} x^n \, d\mu \\ + b \int_{\mathsf{A}} d\mu \;\; = \;\; \int_{\mathsf{A}} f \, d\mu \end{aligned}$$

as claimed. □

We denote the centroid of μ by $\mathsf{centroid}(\mu)$.

Now let $C \subset \mathsf{A}$ be a *convex body,* that is, a convex open subset having compact closure. Then C determines a probability measure μ_C on A by

$$\mu_C(X) = \frac{\int_{X \cap C} \omega}{\int_C \omega}$$

where ω is any parallel volume form on A.

Proposition 1.7.2. Let $C \subset \mathsf{A}$ be a convex body. Then the centroid $\bar{C}$ of C lies in C.

Proof. C is the intersection of halfspaces, that is, C consists of all $x \in \mathsf{A}$ such that $f(x) > 0$ for all affine maps

$$\mathsf{A} \xrightarrow{f} \mathbb{R}$$

such that $f|_C > 0$.

If f is such an affine map, then clearly $f(\bar{C}) > 0$. Therefore $\bar{C} \in C$. □

1.7.2. Divergence. If $\xi \in \mathsf{Vec}(M)$, then the infinitesimal distortion of volume is the *divergence* of ξ, defined as the function $\mathsf{div}(\xi)$ such that

$$\mathcal{L}_\xi(\omega) = \mathsf{div}(\xi)\omega$$

where ω is (any) parallel volume form and $\mathcal{L}_\xi$ denotes Lie differentiation with respect to ξ. If, in coordinates $\xi = \xi^i \partial_i$, then

$$\mathsf{div}(\xi) = \partial_i \xi^i$$

(the usual formula).

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Exercise 1.7.3. The Lie algebra of the special affine group $\mathsf{SAff}(\mathsf{A})$ consists of affine vector fields of divergence zero.

1.8. Linearizing affine geometry

Associated to every affine space A is an embedding $\mathcal{A}'$ of A as an *affine hyperplane* in a vector space W as follows.

Exercise 1.8.1. Let A be an affine space over a field k with underlying vector space $\mathsf{V} := \mathsf{Trans}(\mathsf{A})$. Let $\mathsf{W} := \mathsf{V} \oplus \mathsf{k}$ and let $\mathsf{W} \xrightarrow{\psi} \mathsf{k}$ denote linear projection onto the second summand.

- For each $s \in \mathsf{k}$, the group V acts simply transitively on the affine hyperplane $\psi^{-1}(s)$.
  - A identifies with $\psi^{-1}(1)$.
  - V identifies with $\mathsf{Ker}(\psi) = \psi^{-1}(0)$.
- Define a bijective correspondence between n-dimensional affine spaces A and pairs (W, ψ) where W is an $n+1$-dimensional vector space and $\psi \in \mathsf{W}^*$ is a nonzero covector, where A corresponds to $\psi^{-1}(1)$.
- Identify the affine group $\mathsf{Aff}(\mathsf{A})$ with the subgroup of $\mathsf{GL}(\mathsf{W})$ preserving this hyperplane, as well as the stabilizer of ψ.
- If $\left[\mathbf{A} \,\middle|\, \mathbf{b}\right]$ represents an affine transformation with linear part $\mathbf{A}$ and translational part $\mathbf{b}$, show that the corresponding linear transformation of W is represented by the block matrix

$$\begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & 1 \end{bmatrix}$$

where $\mathbf{0}$ is the row vector representing the zero map $\mathsf{V} \to \mathbb{R}$.

In coordinates, $\mathcal{A}'(\mathbf{v}) = \begin{bmatrix} v^1 \\ \vdots \\ v^n \\ 1 \end{bmatrix} \in \mathsf{W}$ and $\psi = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} \in \mathsf{W}^*$. The affine transformation has linear part $\mathbf{A} = \begin{bmatrix} A^1{}_1 & \dots & A^1{}_n \\ \vdots & & \vdots \\ A^n{}_1 & \dots & A^n{}_n \end{bmatrix} \in \mathsf{GL}(\mathsf{V})$ and translational

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part $\mathbf{b} = \begin{bmatrix} b^1 \\ \vdots \\ b^n \end{bmatrix} \in \mathsf{V}$:

$$\mathbf{v} \xmapsto{\left[\mathbf{A} \,\middle|\, \mathbf{b}\right]} \begin{bmatrix} A^1{}_1 & \dots & A^1{}_n \\ \vdots & A^i{}_j & \vdots \\ A^n{}_1 & \dots & A^n{}_n \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^j \\ \vdots \\ v^n \end{bmatrix} + \begin{bmatrix} b^1 \\ \vdots \\ b^i \\ \vdots \\ b^n \end{bmatrix} = \begin{bmatrix} A^1{}_j v^i + b^1 \\ \vdots \\ A^i{}_j v^j + b^i \\ \vdots \\ A^n{}_j v^j + b^n \end{bmatrix}$$

$$\mathcal{A}'(\mathbf{v}) \longmapsto \begin{bmatrix} A^1{}_1 & \dots & A^1{}_n & b^1 \\ \vdots & & \vdots & \\ A^n{}_1 & \dots & A^n{}_n & b^n \\ 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \\ 1 \end{bmatrix} = \begin{bmatrix} A^1{}_i v^i + b^1 \\ \vdots \\ A^n{}_i v^i + b^n \\ 1 \end{bmatrix}.$$

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Chapter 2

Projective geometry

Projective geometry arose historically out of the efforts of Renaissance artists to understand perspective. Imagine a painter looking at a 2-dimensional canvas with one eye closed. The painter's open eye plays the role of the origin in the 3-dimensional vector space W and the canvas plays the role of an affine hyperplane $\mathsf{A} \subset \mathsf{W}$ as in §1.8. As the canvas tilts, the geometry seen by the painter changes. Parallel lines no longer appear parallel (like the railroad tracks described below) and distance and angle are distorted. But lines stay lines and the basic relations of collinearity and concurrence are unchanged. The change in perspective given by "tilting" the canvas is determined by a linear transformation of W, since a point on A is determined completely by the 1-dimensional linear subspace of V containing it. (One must solve systems of linear equations to write down the effect of such transformation.) Projective geometry is the study of points, lines and the incidence relations between them.

Projective space *closes off* affine space — that is, it *compactifies* affine space by adding *ideal points* at infinity. To develop an intuitive feel for projective geometry, consider how points in A^n may go to infinity:

The easiest way to go to infinity in A^n is by following geodesics, since they have zero acceleration. Furthermore two geodesics approach the same ideal point if they are parallel. A good model is railroad tracks running parallel to each other — they meet ideally at the horizon. We thus force parallel lines to intersect by attaching ideal points where the extended parallel lines are to intersect.

The passage between the geometry of P and the algebra of V is a "dictionary" between linear algebra and projective geometry. Linear maps and linear subspaces correspond geometrically to projective maps and projective

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subspaces: inclusions, intersections, and linear spans correspond to incidence relations in projective geometry. Thus projective geometry lets us visually understand linear algebra and linear algebra enables to prove theorems in geometry by calculation.

A good general reference for projective geometry (especially in dimension two) is Coxeter [**103**], as well as Berger [**48**, **49**] and Coxeter [**102**]. More classical treatments are Busemann–Kelly [**73**], Semple–Kneebone [**304**] and Veblen–Young [**333**, **334**].

2.1. Ideal points

Parallelism of lines in A is an equivalence relation. Define an *ideal point* of A as an equivalence class. The *ideal set* of an affine space A is the space $\mathsf{P}_\infty(\mathsf{A})$ of ideal points, with the quotient topology. If $l, l' \subset \mathsf{A}$ are parallel lines, then the point in P_∞ corresponding to their equivalence class is defined as their intersection. So two lines are parallel if and only if they intersect at infinity.

Projective space is defined as the union $\mathsf{P} := \mathsf{A} \cup \mathsf{P}_\infty(\mathsf{A})$, with a suitable topology. The natural structure on P is perhaps most easily seen in terms of an alternate, maybe more familiar, description. Embed A as an affine hyperplane in a vector space $\mathsf{W} \cong \mathsf{V} \oplus \mathsf{k}$ as in §1.8, where $\mathsf{V} = \mathsf{Trans}(\mathsf{A})$ is the vector space underlying A. For example, if

$$p = \begin{bmatrix} p^1 \\ \vdots \\ p^n \end{bmatrix} \in \mathsf{A}^n,$$

then its embedding in $\mathsf{W} = \mathsf{k}^{n+1}$ is the nonzero vector

$$\mathcal{A}'(p) = \begin{bmatrix} p^1 \\ \vdots \\ p^n \\ 1 \end{bmatrix} \in \mathsf{W}.$$

Furthermore the line $\mathsf{k}\mathcal{A}'(p)$ spans meets the affine hyperplane $\mathsf{A} \leftrightarrow \mathsf{k}^n \times \{1\}$ in a single point. Think of A as the "canvas" or *viewing hyperplane,* and the line $\mathsf{k}\mathcal{A}'(p)$ as the line of sight as the point p is viewed from the origin $\mathbf{0} \in \mathsf{W}$ (the eye of the painter). The nonzero elements of this line $\mathsf{k}^\times\mathcal{A}'(p)$ form the *projective equivalence class* $[\mathcal{A}'(p)]$.

Now suppose that p travels to infinity along an affine geodesic $\ell \subset \mathsf{A}$:

$$p(t) := p + t\mathbf{v},$$

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where $t \in \mathsf{k}$ and $\mathbf{v} \in \mathsf{V}$ is a nonzero vector. Denote the corresponding path of vectors in W by

$$\mathbf{w}(t) := \mathcal{A}'\big(p(t)\big) \in \mathsf{W}.$$

Although $\lim_{t\to\infty} \mathbf{w}(t)$ does not exist, the corresponding lines $\mathsf{k} \cdot \mathbf{w}(t)$ converge to the line $\mathsf{k} \cdot \mathbf{v}$ corresponding to $\mathbf{v} \in \mathsf{W}$. This limiting line defines the *ideal point* of the affine line ℓ:

$$\lim_{t\to\infty} [\mathbf{w}(t)] = [\mathbf{v}]$$

This motivates the following fundamental definition:

Definition 2.1.1. Let W denote a vector space over k. The *projective space associated to* W is the space $\mathsf{P}(\mathsf{W})$ of projective equivalence classes $[\mathbf{w}]$ of nonzero vectors $\mathbf{w} \in \mathsf{W}$, with the quotient topology.

Thus a point in $\mathsf{P}(\mathsf{W})$ (a "projective point") corresponds to a *line* (that is, a 1-dimensional linear subspace) in W. If $\mathbf{w} = \begin{bmatrix} w^1 \\ \vdots \\ w^{n+1} \end{bmatrix} \in \mathsf{W}$, the corresponding projective point is

$$p := [\mathbf{w}] = \left[\!\!\left[\begin{matrix} w^1 \\ \vdots \\ w^{n+1} \end{matrix}\right]\!\!\right] \in \mathsf{P}(\mathsf{W})$$

and $w^1, \dots, w^{n+1}$ are the *homogeneous coordinates* of p. Since linear transformations of W preserve lines, $\mathsf{GL}(\mathsf{W}) = \mathsf{Aut}(\mathsf{W})$ acts on $\mathsf{P}(\mathsf{W})$; the induced transformations are the *projective transformations* or *collineations* of $\mathsf{P}(\mathsf{W})$.

Exercise 2.1.2. The action of $\mathsf{GL}(\mathsf{W})$ on $\mathsf{P}(\mathsf{W})$ is *not* effective. Its kernel consists of the group $\mathsf{k}^\times$ of nonzero scalings, which forms the center of $\mathsf{GL}(\mathsf{W})$. The *projective group* or *collineation group* is the quotient

$$\mathsf{PGL}(\mathsf{W}) := \mathsf{GL}(\mathsf{W})/\mathsf{k}^\times$$

which does act effectively.

2.2. Projective subspaces

Returning to the projective geometry of the line ℓ, note that the affine line $\mathbf{w}(t)$ in W lies in the linear 2-plane $\mathsf{span}(p, \mathbf{v}) \subset \mathsf{W}$. The 1-dimensional linear subspaces contained in this linear 2-plane is a *projective line.*

Definition 2.2.1. Let $\mathsf{P} = \mathsf{P}(\mathsf{W})$ be a projective space, and let d be a nonnegative integer. A d-dimensional *projective subspace* S of P is the collection of all projective equivalence classes $[\mathbf{v}]$ of nonzero vectors $\mathbf{v}$ lying in a fixed

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$d+1$-dimensional linear subspace $\mathsf{S} \subset \mathsf{W}$. We write $S = \mathsf{P}(\mathsf{S})$ and call S the *projectivization* of S.

Thus a *projective line* is the projectivization of a linear 2-plane in W and a *projective hyperplane* is the projectivization a linear hyperplane in W.

A linear embedding $S_1 \hookrightarrow S_2 \subset \mathsf{W}$ of linear subspaces induces an embedding of projective subspaces $\mathsf{P}(\mathsf{S}_1) \hookrightarrow \mathsf{P}(\mathsf{S}_2)$ and we say that the subspaces $\mathsf{P}(\mathsf{S}_1)$ and $\mathsf{P}(\mathsf{S}_2)$ are *incident.* Clearly projective transformations preserve the relation of incidence. Conversely, *an incidence-preserving transformation of projective space is a projective transformation.* This is a deep theorem, sometimes called the *fundamental theorem of projective geometry.*

Exercise 2.2.2. Show that the set of ideal points is a projective hyperplane.

Suppose that $S_1, S_2 \subset \mathsf{P}$ are projective subspaces. Write $S_i = \mathsf{P}(\mathsf{S}_i)$ for respective linear subspaces $\mathsf{S}_i \subset \mathsf{W}$. Their sum $\mathsf{S}_1 + \mathsf{S}_2$ is a vector space and denote its projectivization as the projective subspace

$$\mathsf{span}(S_1, S_2) \ := \ \mathsf{P}(\mathsf{S}_1 + \mathsf{S}_2).$$

Exercise 2.2.3. If $S_1 \cap S_2 = \emptyset$, then

$$\mathsf{dim}\big(\mathsf{span}(S_1, S_2)\big) = \mathsf{dim}(S_1) + \mathsf{dim}(S_2) + 1.$$

If S_1 and S_2 are points, then $\mathsf{span}(S_1, S_2)$ is a line, and we use the more familiar notation $\overleftrightarrow{S_1 S_2}$. If S_i are projective subspaces and $S_1 \cap S_2 \neq \emptyset$, althen $S_1 \cap S_2$ is a projective subspace and

$$\mathsf{dim}\big(\mathsf{span}(S_1, S_2)\big) + \mathsf{dim}(S_1 \cap S_2) = \mathsf{dim}(S_1) + \mathsf{dim}(S_2).$$

Evidently $\mathsf{span}(S_1, S_2)\big)$ is the smallest projective subspace containing S_1 and S_2.

2.2.1. Affine patches. Ideal points are only special when projective space is the completion of affine space; by changing the viewing hyperplane, one gets different notions of "ideal." Indeed, every projective point has neighborhoods which are affine subspaces.

Let P be d-dimensional projective space and $H \subset \mathsf{P}$ be a projective hyperplane Then the complement $\mathsf{P} \setminus H$ is an *affine patch* and has the structure as a d-dimensional affine space with underlying vector space V via an *affine chart*

$$\mathsf{V} \xrightarrow[\approx]{\mathcal{A}} \mathsf{P} \setminus H,$$

defined as follows. Write $\mathsf{P} = \mathsf{P}(\mathsf{W})$. Choose a covector $\psi \in \mathsf{W}^*$ such that $H = \mathsf{P}(\mathsf{V})$ where $\mathsf{V} := \mathsf{Ker}(\psi)$. Choose a vector $\mathbf{w}_0 \in \mathsf{W}$ with $\psi(\mathbf{w}_0) = 1$ to

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define an origin in the affine patch. Then

$$\begin{aligned} \mathsf{V} &\xrightarrow{\mathcal{A}(\psi,\mathbf{w}_0)} \mathsf{P}\setminus H \\ \mathbf{v} &\longmapsto [\mathbf{w}_0 + \mathbf{v}] \end{aligned}$$

defines an affine chart on $\mathsf{P}\setminus H$. Compare §1.8.

Writing

$$p = [\mathbf{X}] = \left[\!\!\left[\begin{matrix} X^1 \\ \vdots \\ X^{d+1} \end{matrix}\right]\!\!\right] \in \mathsf{P},$$

some homogeneous coordinate X^i of the nonzero vector $\mathbf{X}$ is nonzero. Let $\psi \in \mathsf{V}^*$ denote the covector corresponding to the homogeneous coordinate X^i. Then corresponding affine patch is defined by $X^i \neq 0$ and has a chart

$$\mathsf{k}^d \xrightarrow{\mathcal{A}^i} \mathsf{P}$$

$$\begin{bmatrix} v^1 \\ \vdots \\ v^d \end{bmatrix} \longmapsto \left[\!\!\left[\begin{matrix} v^1 \\ \vdots \\ v^{i-1} \\ 1 \\ v^i \\ \vdots \\ v^d \end{matrix}\right]\!\!\right]$$

and $p \in \mathsf{A}^{(i)} := \mathcal{A}^{(i)}(\mathsf{V})$. These $d+1$ *coordinate affine patches* define a covering by contractible open sets.

Exercise 2.2.4. Suppose that $1 \leq i \neq j \leq d+1$.

(1) Express the intersection $\mathsf{A}^{(i)} \cap \mathsf{A}^{(j)}$ in terms of the charts $\mathcal{A}^{(i)}, \mathcal{A}^{(j)}$.

(2) Compute the change of coordinates

$$\left(\mathcal{A}^{(i)}\right)^{-1}\left(\mathsf{A}^{(i)} \cap \mathsf{A}^{(j)}\right) \xrightarrow{\left(\mathcal{A}^{(j)}\right)^{-1} \circ \mathcal{A}^{(i)}} \left(\mathcal{A}^{(j)}\right)^{-1}\left(\mathsf{A}^{(i)} \cap \mathsf{A}^{(j)}\right).$$

(3) Let $\mathsf{k} = \mathbb{R}$ and let $a, b \in \mathbb{R}$ with $a < b$. Suppose that

$$(a, b) \xrightarrow{\gamma} \mathsf{V}$$

is a curve such that $\mathcal{A}^{(i)}\big(\gamma(t)\big) \in \mathsf{A}^{(i)}$ and

$$\mathcal{A}^{(j)}\big(\gamma(t)\big) \in \mathsf{A}^{(j)}$$

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for $a < t < b$. Suppose that $\mathcal{A}^{(i)} \circ \gamma$ is a geodesic in $\mathsf{A}^{(i)}$. Show there exists a *reparametrization,* that is, a diffeomorphism

$$(a,b) \xrightarrow[\approx]{\tau} \tau\big((a,b)\big) \subset \mathbb{R}$$

such that the composition $\mathcal{A}^{(j)} \circ \gamma \circ \tau$ is a geodesic in $\mathsf{A}^{(j)}$.

In general, the topology of projective space is complicated. Since it arises from a *quotient* and not a *subset* construction, it is more sophisticated than a subset. Indeed, projective space generally does not arise as a *hypersurface* in Euclidean space. Although P can be covered by $d+1$ contractible open sets, it cannot be covered by fewer contractible open sets. For either $\mathsf{k} = \mathbb{R}$ or $\mathbb{C}$, projective space $\mathsf{P}^d(\mathsf{k})$ is a compact smooth manifold. We summarize some basic facts about the topology.

Exercise 2.2.5. Suppose $\mathsf{k} = \mathbb{R}$. Exhibit $\mathsf{P}^d(\mathbb{R})$ as a quotient of the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ by the antipodal map.[1] Show that $\mathsf{P}^1(\mathbb{R}) \approx \mathbb{S}^1$ and for $d > 1$ the fundamental group $\mathsf{P}^d(\mathbb{R})$ has order two. Show that $\mathsf{P}^d(\mathbb{R})$ is orientable if and only if d is odd.

Exercise 2.2.6. Suppose that $\mathsf{k} = \mathbb{C}$. Exhibit $\mathsf{P}^d(\mathbb{C})$ as a quotient of the unit sphere $\mathbb{S}^{2d-1} \subset \mathbb{C}^{d+1}$ by the group $\mathbb{T}$ of unit complex numbers. Show that $\mathsf{P}^1(\mathbb{C}) \approx \mathbb{S}^2$, and $\mathsf{P}^d(\mathbb{C})$ is simply connected and orientable for all $d \geq 1$.

Exercise 2.2.7. Find a *natural* $\mathbb{S}^1$-fibration $\mathsf{P}^{2d+1}(\mathbb{R}) \longrightarrow \mathsf{P}^d(\mathbb{C})$.

2.3. Projective mappings

Linear mappings $\mathsf{V} \xrightarrow{\phi} \mathsf{W}$ between vector spaces define mappings between the corresponding projective spaces. However, if ϕ is not injective, the corresponding projective map is not defined on all of $\mathsf{P}(\mathsf{V})$. We begin discussing with projective maps defined by injective linear maps, particularly emphasizing *automorphisms,* classically known as *collineations.* Collineations arise from linear automorphisms of the vector space W.

2.3.1. Embeddings and Collineations.

A projective subspace $S \subset \mathsf{P}(\mathsf{W})$ determines a projective map, determined by the linear inclusion $\mathsf{S} \hookrightarrow \mathsf{W}$, where S is the linear subspace of W projectivizing to S.

Exercise 2.3.1. Show that an injective projective map $\mathsf{P}(\mathsf{S}) \longrightarrow \mathsf{P}(\mathsf{W})$ is determined by an injective linear map $\mathsf{S} \longrightarrow \mathsf{W}$ unique up to left-composition with scalar multiplication on S and right-composition with scalar multiplication on W.

[1] This sphere can be described intrinsically in a projectively-invariant way as the *sphere of directions.* Compare §6.2.1.

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A linear automorphism of a vector space W induces an invertible transformation $\mathsf{P}(\mathsf{W}) \xrightarrow{\phi} \mathsf{P}(\mathsf{W})$. We call such a transformation a *collineation* or a *homography.* Evidently a collineation preserves projective subspaces, and the relations between them. An *involution* is a collineation of order two, that is, $\phi = \phi^{-1}$.

Exercise 2.3.2. Show that the projective automorphisms of P form a group $\mathsf{Aut}(\mathsf{P})$ which has the following description. If

$$\mathsf{P} \xrightarrow{f} \mathsf{P}$$

is a projective automorphism, some linear isomorphism

$$\mathsf{V} \xrightarrow{\tilde{f}} \mathsf{V}$$

induces f. Indeed,

$$1 \longrightarrow \mathbb{R}^\times \longrightarrow \mathsf{GL}(\mathsf{V}) \longrightarrow \mathsf{Aut}(\mathsf{P}) \longrightarrow 1$$

is a short exact sequence, where $\mathbb{R}^\times \longrightarrow \mathsf{GL}(\mathsf{V})$ is the inclusion of the group of multiplications by nonzero scalars. This quotient, *the projective general linear group*

$$\mathsf{PGL}(\mathsf{V}) \ := \ \mathsf{PGL}(\mathsf{V}) \ := \mathsf{GL}(\mathsf{V})/\mathbb{R}^\times \cong \mathsf{Aut}(\mathsf{P}^n),$$

is also denoted $\mathsf{PGL}(n+1,\mathbb{R})$ If n is even, then

$$\mathsf{PGL}(n+1,\mathbb{R}) \cong \mathsf{SL}(n+1;\mathbb{R}).$$

If n is odd, then $\mathsf{PGL}(n+1,\mathbb{R})$ has two connected components, and its identity component is doubly covered by $\mathsf{SL}(n+1;\mathbb{R})$ and is isomorphic to $\mathsf{SL}(n+1;\mathbb{R})/\{\pm\mathbb{I}\}$.

Exercise 2.3.3. Let $\mathbb{R}\mathsf{P}^n$ be a real projective space of dimension n, and let ϕ be an involution of $\mathbb{R}\mathsf{P}^n$.

- Suppose n is even. Then $\mathsf{Fix}(\phi)$ is the union of two disjoint projective subspaces of dimensions d_1, d_2 where $d_1 + d_2 = n - 1$.
- Suppose $n = 2m+1$ is odd. Then either:
  - $\mathsf{Fix}(\phi) \neq \emptyset$ and equals the union of two disjoint projective subspaces of dimensions d_1, d_2 where $d_1 + d_2 = n - 1$, or
  - $\mathsf{Fix}(\phi) = \emptyset$, and ϕ leaves invariant an $\mathbb{S}^1$-fibration

$$\mathbb{R}\mathsf{P}^n \longrightarrow \mathbb{C}\mathsf{P}^m.$$

For example, an involution ϕ of $\mathbb{R}\mathsf{P}^2$ has an isolated fixed point p and a disjoint fixed line ℓ. In an affine patch containing p, the involution ϕ looks like a symmetry in p, preserving the local orientation. In contrast, ϕ looks like a reflection in ℓ in an affine patch containing ℓ, reversing the local

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orientation. Since $\mathbb{R}\mathsf{P}^2$ is nonorientable no *global* orientation exists to either preserve or reverse.

Here is an explicit example. The involution defined in homogeneous coordinates by:

$$\mathbb{R}\mathsf{P}^2 \xrightarrow{\iota} \mathbb{R}\mathsf{P}^2$$
$$\begin{bmatrix}\!\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}\!\end{bmatrix} \longmapsto \begin{bmatrix}\!\begin{bmatrix} -X \\ -Y \\ Z \end{bmatrix}\!\end{bmatrix}$$

fixes the point

$$p = \begin{bmatrix}\!\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\!\end{bmatrix}$$

and the projective line defined by $Z = 0$, that is, $\ell = \begin{bmatrix}\!\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\!\end{bmatrix}$. In the affine chart $\mathcal{A}^3$, the isolated fixed point has coordinates $(0, 0)$ and ι appears as the *symmetry*

$$\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \longmapsto \begin{bmatrix} -v^1 \\ -v^2 \end{bmatrix}.$$

and in the affine chart $\mathcal{A}^1$, the fixed line has coordinates $(*, 0)$ and ι appears as the reflection

$$\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \longmapsto \begin{bmatrix} -v^1 \\ v^2 \end{bmatrix}$$

fixing the vertical axis. That a single reflection can appear simultaneously as a symmetry in a point and reflection in a line indicates the topological complexity of P^2: A reflection in a line *reverses* a local orientation about a point on the line, and a symmetry in a point *preserves* a local orientation about the point.

2.3.2. Singular projective mappings. When the linear map ϕ is not injective, then $\mathsf{P}\big(\mathsf{Ker}(\phi)\big)$ is a projective subspace, upon which the projectivization $[\phi]$ of ϕ is not defined. For that reason, we call it the *undefined set* of $[\phi]$, denoting it $\mathbb{U}([\phi])$. If V, V' are vector spaces with associated projective spaces P, P' then a linear map $\mathsf{V} \xrightarrow{\tilde{f}} \mathsf{V}'$ maps lines through 0 to lines through 0. But $\widetilde{f}$ only induces a map $\mathsf{P} \xrightarrow{f} \mathsf{P}'$ if it is injective, since $f(x)$

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can only be defined if $\widetilde{f}(\ \tilde{x}\ ) \neq 0$ (where $\tilde{x}$ is a point of $\Pi^{-1}(x) \subset \mathsf{V} \setminus \{0\}$). Suppose that $\tilde{f}$ is a (not necessarily injective) linear map and let

$$\mathbb{U}(f) = \Pi\big(\mathsf{Ker}(\tilde{f})\big).$$

The resulting *projective endomorphism* of P is defined on the complement $\mathsf{P} \setminus \mathbb{U}(f)$. If $\mathbb{U}(f) \neq \emptyset$, the corresponding projective endomorphism is by definition a *singular projective transformation* of P. If f is singular, its image is a proper projective subspace, called the *range* of f and denoted $\mathcal{R}(f)$.

A projective map $\mathsf{P}_1 \xrightarrow{\iota} \mathsf{P}$ corresponds to a linear map $\mathsf{V}_1 \xrightarrow{\tilde{\iota}} \mathsf{V}$ between the corresponding vector spaces (well-defined up to scalar multiplication). Since ι is defined on all of P_1, $\tilde{\iota}$ is an injective linear map and hence corresponds to an embedding. Such an embedding (or its image) will be called a *projective subspace.* Projective subspaces of dimension k correspond to linear subspaces of dimension $k+1$. (By convention the empty set is a projective space of dimension -1.) Note that the "bad set" $\mathbb{U}(f)$ of a singular projective transformation is a projective subspace. Two projective subspaces of dimensions k, l where $k + l \geq n$ intersect in a projective subspace of dimension at least $k+l-n$. The *rank* of a projective endomorphism is defined to be the dimension of its image.

Exercise 2.3.4. Let P be a projective space of dimension n. Show that the (possibly singular) projective transformations of P form themselves a projective space of dimension $(n+1)^2 - 1$. We denote this projective space by $\mathsf{End}(\mathsf{P})$. Show that if $f \in \mathsf{End}(\mathsf{P})$, then

$$\mathsf{dim}\ \mathbb{U}(f) + \mathsf{rank}(f) = n - 1.$$

Show that $f \in \mathsf{End}(\mathsf{P})$ is nonsingular (in other words, a collineation) if and only if $\mathsf{rank}(f) = n$, that is, $\mathbb{U}(f) = \emptyset$. Equivalently, $\mathcal{R}(f) = \mathsf{P}$.

An important kind of projective endomorphism is a *projection,* also called a *perspectivity.* Let $A^k, B^l \subset \mathsf{P}^n$ be disjoint projective subspaces whose dimensions satisfy $k + l = n - 1$. Define the projection $\Pi = \Pi_{A^k,B^l}$ onto A^k from B^l

$$\mathsf{P}^n - B^l \xrightarrow{\Pi} A^k$$

as follows. For every $x \in \mathsf{P}^n - B^l$ the minimal projective subspace

$$\overleftrightarrow{xB} := \mathsf{span}(\{x\} \cup B^l)$$

containing $\{x\} \cup B^l$ is unique and has dimension $l + 1$. It intersects A^k transversely in a 0-dimensional projective subspace, that is, a unique point $\Pi_{A^k,B^l}(x)$, that is:

$$\{\Pi_{A^k,B^l}(x)\} \ = \ \overleftrightarrow{xB} \cap A^k.$$

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Perspectivities are projective mappings obtained as restrictions of projections:

Exercise 2.3.5. Let $A' \subset \mathsf{P}$ be a projective subspace of dimension k disjoint from B.

- Restriction $\Pi_{A,B}|_{A'}$ is a projective isomorphism $A' \to A$.
- Express an arbitrary projective isomorphism between projective subspaces as a composition of perspectivities.

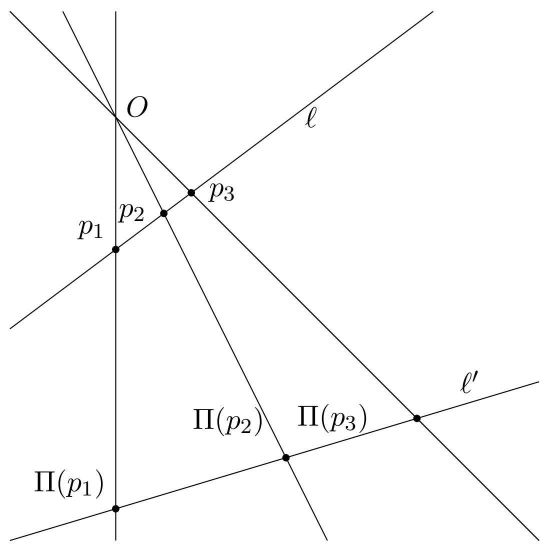

Figure 2.1. A perspectivity $\ell \xrightarrow{\Pi} \ell'$ is a projective mapping between two lines ℓ, ℓ' in the plane. Π is the restriction to ℓ of the projection $\mathsf{P}^2 \setminus \{O\} \twoheadrightarrow \ell'$. The undefined set $\mathbb{U}(\Pi)$ is traditionally called the *center* of the perspectivity.

2.3.3. Locally projective maps. Suppose that P, P' are projective spaces and $U \subset \mathsf{P}$ is an open subset. A map $U \xrightarrow{f} \mathsf{P}'$ is *locally projective* if for each component $U_i \subset U$ there exists a linear map

$$\mathsf{V}(\mathsf{P}) \xrightarrow{\tilde{f}_i} \mathsf{V}(\mathsf{P}')$$

such that the restrictions of $f \circ \Pi$ and $\Pi \circ \tilde{f}_i$ to $\Pi^{-1}U_i$ agree. Locally projective maps (and hence also locally affine maps) satisfy the *Unique Extension Property:* if $U \subset U' \subset \mathsf{P}$ are open subsets of a projective space with U nonempty and U' connected, then any two locally projective maps $f_1, f_2 : U' \longrightarrow \mathsf{P}'$ which agree on U must be identical. (Compare §5.1.1.)

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Exercise 2.3.6. Let $U \subset \mathsf{P}$ be a connected open subset of a projective space of dimension greater than 1. Let $U \xrightarrow{f} \mathsf{P}$ be a local diffeomorphism. Then f is locally projective if and only if for each line $\ell \subset \mathsf{P}$, the image $f(\ell \cap U)$ is a line.

2.4. Affine patches

Let $H \subset \mathsf{P}$ be a projective hyperplane (projective subspace of codimension one). Then the complement $\mathsf{P} \setminus H$ has a natural affine geometry, that is, it is an affine space in a natural way. Indeed the group of projective automorphisms $\mathsf{P} \to \mathsf{P}$ leaving fixed each $x \in H$ and whose differential $\mathsf{T}_x\mathsf{P} \to \mathsf{T}_x\mathsf{P}$ is a volume-preserving linear automorphism is a vector group acting simply transitively on $\mathsf{A} = \mathsf{P} \setminus H$. Moreover the subgroup of $\mathsf{Aut}(\mathsf{P})$ leaving H invariant is $\mathsf{Aff}(\mathsf{A})$. In this way affine geometry *embeds* in projective geometry.

Here is how it looks in terms of matrices. Let $\mathsf{A} = \mathbb{R}^n$; then the affine subspace of

$$\mathsf{V} = \mathsf{Trans}(\mathsf{A}) \oplus \mathbb{R} = \mathbb{R}^{n+1}$$

corresponding to A is $\mathbb{R}^n \times \{1\} \subset \mathbb{R}^{n+1}$, the point of A with *affine* or *inhomogeneous* coordinates $(x^1, \dots, x^n)$ has homogeneous coordinates $[x^1, \dots, x^n, 1]$. Let $f \in \mathsf{Aff}(E)$ be the affine transformation with linear part $A \in \mathsf{GL}(n; \mathbb{R})$ and translational part $\mathbf{b} \in \mathbb{R}^n$, that is, $f(x) = Ax + \mathbf{b}$, is then represented by the $(n+1)$-square matrix

$$\begin{bmatrix} A & \mathbf{b} \\ 0 & 1 \end{bmatrix}$$

where $\mathbf{b}$ is a column vector and 0 denotes the $1 \times n$ zero row vector.

2.4.1. Projective vector fields. In the affine space A^n, let

$$\mathfrak{B}(\mathsf{A}^n) \longrightarrow \mathsf{A}^n$$

denote the *bundle of bases,* more commonly known as the *affine frame bundle* over A^n: its fiber $\mathfrak{B}_p$ over a point $p \in \mathsf{A}^n$ consists of the set of bases for the tangent space $\mathsf{T}_p\mathsf{A}^n$. Using the simply transitive action of $\mathsf{k}^n = \mathsf{Trans}(\mathsf{A}^n)$ on A^n, the total space $\mathfrak{B}(\mathsf{A}^n)$ is a torsor for the affine automorphism group $\mathsf{Aff}(\mathsf{A}^n)$: an affine automorphism is determined uniquely by its action on a basepoint $p_0 \in \mathsf{A}^n$ and a basis $\beta_0 \in \mathfrak{B}_{p_0}$ of $\mathsf{T}_p\mathsf{A}^n$. Every $(p, \beta) \in \mathfrak{B}(\mathsf{A}^n)$ is the image of (p_0, β_0) under a unique element of $\mathsf{Aff}(\mathsf{A}^n)$.

Let $g \in \mathsf{Aut}(\mathsf{P}^n)$ be a projective automorphism. Fix a basepoint p_0 and a basis β_0 of $T_{p_0}\mathsf{A}^n$ such that $g(p_0)$ is not ideal, that is, $g(p_0) \in \mathsf{A}^n$. Let

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$h \in \mathsf{Aff}(\mathsf{A}^n)$ be the unique *affine* automorphism taking (p_0, β_0) to

$$(p, \beta) \;=\; g(p_0, \beta_0).$$

Then $h^{-1} \circ g$ is a projective automorphism fixing p_0 acting identically on

$$\mathsf{T}_{p_0}\mathsf{P}^n = \mathsf{T}_{p_0}\mathsf{A}^n.$$

In the affine chart $\mathcal{A}^{n+1}$ where p_0 is the origin, such a projective transformation is defined in homogeneous coordinates by:

$$\begin{bmatrix}\begin{bmatrix} X^1 \\ \vdots \\ X^n \\ X^{n+1} \end{bmatrix}\end{bmatrix} \longmapsto \begin{bmatrix}\begin{bmatrix} X^1 \\ \vdots \\ X^n \\ \sum_{i=1}^n \xi_i X^i + X^{n+1} \end{bmatrix}\end{bmatrix}$$

for a row vector $\xi^\dagger$ for $\xi \in \mathsf{k}^n$, that is, by the block matrix

$$\begin{bmatrix} \mathbb{I}_n & \mathbf{0} \\ \xi & 1 \end{bmatrix}.$$

In affine coordinates such a transformation is given by

$$(x^1, \dots, x^n) \xrightarrow{g_\xi} \left(\frac{x^1}{1 + \sum_{i=1}^n \xi_i x^i}, \dots, \frac{x^n}{1 + \sum_{i=1}^n \xi_i x^i} \right).$$

Exercise 2.4.1. Show that this group is isomorphic to an n-dimensional vector group (that is, the additive group of an n-dimensional vector space). Show that g_ξ lies in the one-parameter group

$$t \longmapsto g_{t\xi}$$

and the corresponding vector field equals the product

$$-\Big(\sum_{i=1}^n \xi_i x^i\Big)\,\mathsf{Rad}$$

where Rad is the radiant vector field defined in §1.6.2.

Denote the Lie algebra of such vector fields by $\mathfrak{g}_1$, the Lie algebra of parallel vector fields on A^n by $\mathfrak{g}_{-1}$, and the Lie algebra of linear vector fields on A^n by $\mathfrak{g}_0$, show that, for $\lambda = 0, \pm 1$, the subalgebra $\mathfrak{g}_\lambda$ of the Lie algebra $\mathfrak{g}$ of projective vector fields equals the λ-eigenspace of $\mathsf{ad}(\mathsf{Rad})$. Furthermore $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$ where $\mathfrak{g}_\nu := 0$ if $\nu \neq 0, \pm 1$. In particular as a vector space $\mathfrak{g}$ decomposes as a direct sum

$$\mathfrak{g} \;=\; \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \;\cong\; \mathsf{k}^n \oplus \mathsf{gl}(n) \oplus \mathsf{k}^n.$$

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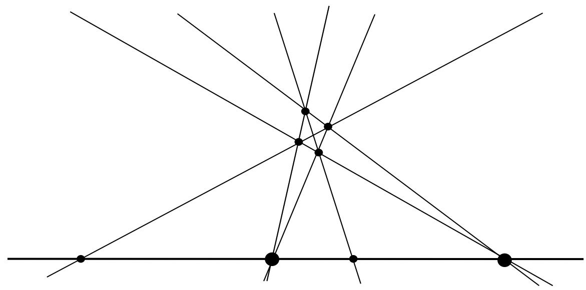

Figure 2.2. The four points on the horizontal line form a harmonic quadruple. Such a quadruple is characterized by having cross-ratio -1.

Exercise 2.4.2. Describe the corresponding group-theoretic decomposition of the projective automorphism group

$$\mathsf{Aut}(\mathsf{P}^n) := \mathsf{PGL}(n+1).$$

2.5. Classical projective geometry

This section surveys standard results in projective geometry. The *fundamental theorem of projective geometry* characterizes projective mappings. The *cross ratio* of a set of four points is the fundamental invariant in 1-dimensional projective geometry. The classical notion of a *harmonic set* is introduced, and is applied to the study of projective reflections and their products.

2.5.1. 1-dimensional reflections. Let ℓ be a projective line containing distinct points x, z. Then there exists a unique reflection[2]

$$\ell \xrightarrow{\rho_{x,z}} \ell$$

whose fixed-point set equals $\{x, z\}$. We say that a pair of points y, w are *harmonic* with respect to x, z if $\rho_{x,z}$ interchanges them. In that case one can show that x, z are harmonic with respect to y, w. Furthermore this relation is equivalent to the existence of lines p, q through x and lines r, s through z such that

$$y = \overleftrightarrow{(p \cap r)(q \cap s)} \cap \ell \tag{2.1}$$

$$z = \overleftrightarrow{(p \cap s)(q \cap r)} \cap \ell \tag{2.2}$$

Compare Figure 2.2.

This leads to a projective-geometry construction of reflection, as follows. Let $x, y, z \in \ell$ be fixed; we seek the harmonic conjugate of y with respect

[2]a *harmonic homology* in classical terminology

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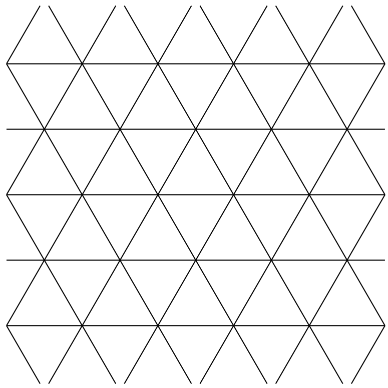

Figure 2.3. Euclidean $(3,3,3)$-triangle tessellation

to x, z, that is, the image $\rho_{x,z}(y)$. Erect arbitrary lines (in general position) p, q through x and a line r through z. Through y draw the line

$$\ell' := \overleftrightarrow{y \quad (r \cap q)}$$

through $r \cap q$; join its intersection with p with z to form a line:

$$s = \overleftrightarrow{z \quad (p \cap \ell')}.$$

Then $\rho_{x,z}(y)$ will be the intersection of ℓ with

$$\overleftrightarrow{(p \cap r) \ (s \cap q)}$$

as in Figure 2.2.

2.5.2. $(3,3,3)$**-triangle tessellations.** Figure 2.5 depicts a projective tessellation of the inside of a triangle by smaller triangles. This tessellation is combinatorially equivalent to the usual Euclidean $(3,3,3)$-tessellation of the plane by equilateral triangles. Around each vertex are six triangles, and the tessellation is invariant under the group generated by Euclidean reflections in the three sides of (any) triangle. These tessellations depend on one real parameter, and can be constructed by an easy projective-geometry construction using only a straightedge.

Exercise 2.5.1. Here is the construction of the tessellation. Begin with a triangle $\triangle$ with vertices v_0, v_1, v_2. Choose a line segment s with one endpoint a vertex (say v_2) and the other endpoint on the opposite edge e_2. Choose two distinct points a, b on s. Connect these points to the other vertices v_0 and v_1.

Draw lines from p, q to the vertices v_0, v_1. These lines (together with s) cut off two triangles inside $\triangle$ with "new" vertices

$$\overleftrightarrow{v_0\ p} \cap \overleftrightarrow{v_1\ q}, \qquad \overleftrightarrow{v_0\ q} \cap \overleftrightarrow{v_1\ p}$$

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which should now be joined to v_2, creating new triangles inside $\triangle$. Iterating this procedure creates the tilings depicted in Figure 2.5.

2.5.3. Rationality.

Exercise 2.5.2. Consider the projective line $\mathsf{P}^1 = \mathbb{R} \cup \{\infty\}$. Show that for every rational number $x \in \mathbb{Q}$ there exists a sequence

$$x_0, x_1, x_2, x_3, \ldots, x_n \in \mathsf{P}^1$$

such that:

- $x = x_n$;
- $\{x_0, x_1, x_2\} = \{0, 1, \infty\}$;
- For each $i \geq 3$, there is a harmonic quadruple (x_i, y_i, z_i, w_i) with

$$y_i, z_i, w_i \in \{x_0, x_1, \ldots, x_{i-1}\}.$$

If x is written in reduced form p/q then what is the smallest n for which x can be reached in this way?

Exercise 2.5.3 (Synthetic arithmetic)**.** Using the above synthetic geometry construction of harmonic quadruples, show how to add, subtract, multiply, and divide real numbers by a straightedge-and-pencil construction. In other words, draw a line l on a piece of paper and choose three points to have coordinates $0, 1, \infty$ on it. Choose arbitrary points corresponding to real numbers x, y. Using only a straightedge (not a ruler!) construct the points corresponding to

$$x + y, x - y, xy, \text{ and } x/y \text{ if } y \neq 0.$$

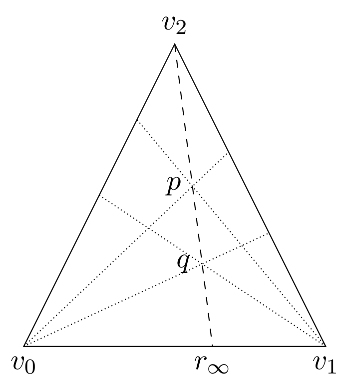

Figure 2.4. Initial configuration for non-Euclidean $(3,3,3)$-tessellation. By iterating the process of joining vertices of the original triangle to interior intersections of lines, one creates tilings like the ones depicted in Figure 2.5.

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2.5.4. Fundamental theorem of projective geometry. One version of what is sometimes called the *fundamental theorem of projective geometry* is that the projective transformations (defined by linear transformations of the associated vector space) are precisely the transformations of projective space which preserve the ternary relation of collinearity (hence *collineations*). Collinearity is a special instance of the set of *incidence relations* between projective subspaces. For example, two distinct projective points p, q are incident to a unique projective line $\overleftrightarrow{p\,q}$ and (p, q, r) is a collinear triple if and only if $r \in \overleftrightarrow{p\,q}$. We do not develop this theory in detail, but refer to the texts of Berger [**49**, **50**] and Coxeter [**102**] for discussion, and in particular the relation with automorphisms of the ground field k.

If $\ell \subset \mathsf{P}$ and $\ell' \subset \mathsf{P}'$ are projective lines, part of the fundamental theorem that for given triples $x, y, z \in \ell$ and $x', y', z' \in \ell'$ of distinct points there exists a unique projective map

$$\begin{aligned} \ell &\xrightarrow{f} \ell' \\ x &\longmapsto x' \\ y &\longmapsto y' \\ z &\longmapsto z' \end{aligned}$$

If $w \in \ell$ then the *cross-ratio* $[w, x, y, z]$ is defined to be the image of w under the unique collineation $l \xrightarrow{f} \mathsf{P}^1$ with

$$\begin{aligned} x &\xmapsto{f} 1 \\ y &\xmapsto{f} 0 \\ z &\xmapsto{f} \infty \end{aligned}$$

If $\ell = \mathsf{P}^1$, then this linear fractional transformation is:

$$f(w) := \frac{w-y}{w-z} \Big/ \frac{x-y}{x-z}$$

so

$$[w, x, y, z] := \frac{w-y}{w-z} \Big/ \frac{x-y}{x-z}, \tag{2.3}$$

thus defining[3] the *cross-ratio*. The cross-ratio extends to quadruples of points on a projective line, of which at least three are distinct.

[3]The literature has several variations of the cross-ratio; the version here is used by Veblen–Young [**333**], Kneebone–Semple[**304**], Coxeter [**104**], and Ahlfors [**6**]. Other versions can be found in Goldman [**155**], Hubbard [**197**], and Ovsienko–Tabachnikov [**284**].

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Exercise 2.5.4. Let σ be a permutation on four symbols. Show that there exists a linear fractional transformation Φ_σ such that

$$[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}] = \Phi_\sigma([x_1, x_2, x_3, x_4]).$$

Determine which permutations leave the cross-ratio invariant.

The cross-ratio is a function of four variables, and therefore transforms under the action of the symmetric group $\mathfrak{S}_4$ on four symbols. The group $\mathfrak{S}_4$ is a split extension

$$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \lhd \mathfrak{S}_4 \twoheadrightarrow \mathfrak{S}_3$$

where the normal subgroup $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ consists of products of disjoint transpositions and a section $\mathfrak{S}_3 \hookrightarrow \mathfrak{S}_4$ corresponds to the inclusion $\{2, 3, 4\} \hookrightarrow \{1, 2, 3, 4\}$.

Exercise 2.5.5. Show that the cross-ratio is invariant under the normal subgroup $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, and transforms under $\mathfrak{S}_3 = \mathsf{Aut}\{2, 3, 4\}$ by the rules, where $z = [z_1, z_2, z_3, z_4]$:

$$\begin{aligned}
[z_1, z_2, z_4, z_3] &= 1/z \\
[z_1, z_3, z_2, z_4] &= 1 - z \\
[z_1, z_3, z_4, z_2] &= 1 - 1/z \\
[z_1, z_4, z_2, z_3] &= 1/(1 - z) \\
[z_1, z_4, z_3, z_2] &= z/(z - 1)
\end{aligned}$$

Using the isomorphism $\mathsf{Aut}(\mathbb{R}\mathsf{P}^1) \cong \mathsf{PGL}(2, \mathbb{R})$, this corresponds to an embedding $\mathfrak{S}_3 \hookrightarrow \mathsf{PGL}(2, \mathbb{Z})$ taking

$$(34) \longmapsto \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(23) \longmapsto \pm \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$(234) \longmapsto \pm \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$(243) \longmapsto \pm \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$(24) \longmapsto \pm \begin{bmatrix} -1 & 0 \\ -1 & +1 \end{bmatrix}$$

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which induces an isomorphism $\mathfrak{S}_3 \cong \mathsf{PGL}(2, \mathbb{Z}/2)$.

A pair $\{w, x\}$ is harmonic with respect to the pair $\{y, z\}$ (in which case we say that (x, y, w, z) is a *harmonic quadruple*) if and only if the cross-ratio $[x, y, w, z] = -1$.

Exercise 2.5.6. When $z = \infty$, the expression for the cross-ratio simplifies:

$$[x, y, w, \infty] := \frac{x - w}{y - w}$$

and defines a fundamental *affine* invariant.

- (x, y, w, ∞) is a harmonic quadruple if and only if y is the midpoint of $\overline{xw}$.
- Suppose $(x, y, w), (x', y', w')$ are ordered triples of distinct points of A^1. Show that

$$[x, y, w, \infty] = [x', y', w', \infty]$$

if and only if $\exists g \in \mathsf{Aff}(\mathsf{A}^1)$ such that

$$x' = g(x)$$
$$y' = g(y)$$
$$w' = g(w).$$

The process of extending a triple of points on P^1 to a harmonic quadruple is equivalent to applying a projective reflection in one pair to the remaining element. This process is called *harmonic subdivision.* Iterated harmonic subdivisions produce a countable dense subsets of P^1 corresponding to the rational numbers $\mathbb{Q} \subset \mathbb{R}$. Such a subset is called a *harmonic net* (Coxeter [**103**], §3.5) or a *net of rationality* in Veblen–Young [**333**],p.84). Compare also Busemann–Kelly [**73**], §I.6. These ideas provide an approach to the fundamental theorem:

Exercise 2.5.7. Let $\mathsf{P}^1 \xrightarrow{f} \mathsf{P}^1$ be a homeomorphism. Show that the following conditions are equivalent:

- f is projective;
- f preserves harmonic quadruples
- f preserves cross-ratios, that is, for all quadruples (x, y, w, z), the cross-ratios satisfy

$$[f(x), f(y), f(w), f(z)] = [x, y, w, z].$$

Determine the weakest hypothesis on f to obtain these conditions.

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2.5.5. Distance via cross-ratios. For later use in §12.1, here are some explicit formulas for cross-ratios on 1-dimensional spaces $\mathbb{R}^+$ and the interval $\mathbf{I} = [-1, 1]$. Their infinitesimal forms yield the Poincaré metrics on $\mathbb{R}^+$ and $\mathsf{int}(\mathbf{I})$.

Exercise 2.5.8. (Parameters on the positive ray and unit intervals)

$$[0, e^a, e^b, \infty] \;=\; [-1, \tanh(a/2), \tanh(b/2), 1] \;=\; e^{b-a}$$

2.5.6. Products of reflections. If ϕ, ψ are collineations, each of which fix a point $O \in \mathsf{P}^2$, their composition $\phi\psi = \phi \circ \psi$ fixes O. In particular its derivative

$$\mathsf{D}(\phi\psi)_O = (\mathsf{D}\phi)_O \circ (\mathsf{D}\psi)_O$$

acts linearly on the tangent space $\mathsf{T}_O\mathsf{P}$. We consider the case where ϕ, ψ are reflections in P^2. As in Exercise 2.3.3, a reflection ϕ is completely determined by its set $\mathsf{Fix}(\phi)$ which consists of a point p_ϕ and a line ℓ_ϕ such that $p_\phi \notin \ell_\phi$. Define

$$O := \ell_\phi \cap \ell_\psi$$

and P^*_O the projective line whose points are the lines incident to O.

Exercise 2.5.9. Let ρ denote the cross-ratio of the four lines

$$\ell_\phi,\ \overleftrightarrow{Op_\phi},\ \overleftrightarrow{Op_\psi},\ \ell_\psi$$

as elements of P^*_O.

- The linear automorphism

$$\mathsf{T}_O\mathsf{P}^2 \xrightarrow{\mathsf{D}(\phi\psi)_O} \mathsf{T}_O\mathsf{P}^2$$

  of the tangent space $\mathsf{T}_O\mathsf{P}^2$ leaves invariant a positive definite inner product g_O on $\mathsf{T}_O\mathsf{P}^2$.
- Furthermore $\mathsf{D}(\phi\psi)_O$ represents a rotation of angle θ in the tangent space $\mathsf{T}_O(\mathsf{P})$ with respect to g_O if and only if

$$\rho = \frac{1}{2}(1 + \cos\theta)$$

  for $0 < \theta < \pi$.
- $\mathsf{D}(\phi\psi)_O$ represents a rotation of angle π (that is, an involution) if and only if $p_\phi \in \ell_\psi$ and $p_\psi \in \ell_\phi$.[4]

[4]In other words, the four lines ℓ_ϕ, $\overleftrightarrow{Op_\phi}$, $\overleftrightarrow{Op_\psi}$, ℓ_ψ form a harmonic quadruple.

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2.6. Asymptotics of projective transformations

We shall be interested in the singular projective transformations since they occur as limits of nonsingular projective transformations. The collineation group $\mathsf{Aut}(\mathsf{P})$ of $\mathsf{P} = \mathsf{P}^n$ is a large noncompact group which is naturally embedded in the projective space $\mathsf{End}(\mathsf{P})$ as an open dense subset as in Exercise 2.3.4. Thus understanding precisely what it means for a sequence of collineations to converge to a (possibly singular) projective transformation is crucial.

This leads to an algebraic description of the classical notion of limit sets of discrete groups isometries of hyperbolic space (see §2.6.2). These ideas were used by Kuiper [**232**, **234**, **235**] and later Benzéctri [**46**] who credits them to Myrberg [**278**].

In general collineations do not preserve a metric on P, although P admits many metrics; for example the Fubini–Study metric discussed in §3.2.2. Being compact, two such metrics determine the same *uniform structure* on P, enabling the definition of uniform convergence of mappings defined on subsets of P.

A *singular projective transformation* of P is a projective map f defined on the complement of a projective subspace $\mathbb{U}(f) \subset \mathsf{P}$, called the *undefined subspace* of f and taking values in a projective subspace $\mathcal{R}(f) \subset \mathsf{P}$, called the *range* of f. Furthermore

$$\dim \mathsf{P} \;=\; \dim \mathbb{U}(f) \;+\; \dim \mathcal{R}(f) \;+\; 1.$$

Proposition 2.6.1. Let $g_m \in \mathsf{Aut}(\mathsf{P})$ be a sequence of collineations of P and let $g_\infty \in \mathsf{End}(\mathsf{P})$. Then the sequence g_m converges to g_∞ in $\mathsf{End}(\mathsf{P})$ if and only if the restrictions $g_m|_K$ converge uniformly to $g_\infty|_K$ for all compact sets $K \subset \mathsf{P} - \mathbb{U}(g_\infty)$.

Convergence in $\mathsf{End}(\mathsf{P})$ may be described as follows: Let $\mathsf{P} = \mathsf{P}(\mathsf{V})$ where $\mathsf{V} \cong \mathbb{R}^{n+1}$ is a vector space. Then $\mathsf{End}(\mathsf{P})$ is the projective space associated to the vector space $\mathsf{End}(\mathsf{V})$ of $(n+1)$-square matrices. If $a = (a^i_j) \in \mathsf{End}(\mathsf{V})$ is such a matrix, let

$$\|a\| \;=\; \sqrt{\mathsf{Tr}(aa^\dagger)} \;=\; \sqrt{\sum_{i,j=1}^{n+1} |a^i_j|^2}$$

denote its Euclidean norm; projective endomorphisms then correspond to matrices a with $\|a\| = 1$, uniquely determined up to the antipodal map $a \mapsto -a$. The following lemma will be useful in the proof of Proposition 2.6.1.

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Lemma 2.6.2. Let V, V' be vector spaces and let $\mathsf{V} \xrightarrow{\widetilde{f}_n} \mathsf{V}'$ be a sequence of linear maps converging to $\mathsf{V} \xrightarrow{\widetilde{f}_\infty} \mathsf{V}'$. Let $\widetilde{K} \subset \mathsf{V}$ be a compact subset of $\mathsf{V} \setminus \mathsf{Ker}(\widetilde{f}_\infty)$. Define:

$$\mathsf{V} \xrightarrow{f_i} \mathsf{V}'$$
$$x \longmapsto \frac{\widetilde{f}_i(x)}{\|\widetilde{f}_i(x)\|}.$$

Then f_n converges uniformly to f_∞ on $\widetilde{K}$ as $n \longrightarrow \infty$.

Here uniform convergence is defined by a Euclidean metric on V. Although linear transformations generally do not preserve a Euclidean metric, the uniform structure is preserved, enabling the definition of *uniform convergence.*

Proof. Choose $C > 0$ such that $C \leq \|\widetilde{f}_\infty(x)\|$for $x \in \widetilde{K}$. For $\epsilon > 0$, $\exists N$ such that if $n > N$, then $\forall x \in \widetilde{K}$,

$$\|\widetilde{f}_\infty(x) - \widetilde{f}_n(x)\| < \frac{C\epsilon}{2},$$
$$\left|1 - \frac{\|\widetilde{f}_n(x)\|}{\|\widetilde{f}_\infty(x)\|}\right| < \frac{\epsilon}{2}. \tag{2.4}$$

Let $x \in \widetilde{K}$. Then

$$\begin{aligned}
\Big\| f_n(x) - f_\infty(x) \Big\| &= \left\| \frac{\widetilde{f}_n(x)}{\|\widetilde{f}_n(x)\|} - \frac{\widetilde{f}_\infty(x)}{\|\widetilde{f}_\infty(x)\|} \right\| \\
&= \frac{1}{\|\widetilde{f}_\infty(x)\|} \quad \left\| \frac{\|\widetilde{f}_\infty(x)\|}{\|\widetilde{f}_n(x)\|} \widetilde{f}_n(x) - \widetilde{f}_\infty(x) \right\| \\
&\leq \frac{1}{\|\widetilde{f}_\infty(x)\|} \quad \Bigg(\quad \left\| \frac{\|\widetilde{f}_\infty(x)\|}{\|\widetilde{f}_n(x)\|} \widetilde{f}_n(x) - \widetilde{f}_n(x) \right\| \\
&\qquad\qquad + \quad \|\widetilde{f}_n(x) - \widetilde{f}_\infty(x)\| \quad \Bigg) \\
&= \left|1 - \frac{\|\widetilde{f}_n(x)\|}{\|\widetilde{f}_\infty(x)\|}\right| \quad + \quad \frac{1}{\|\widetilde{f}_\infty(x)\|} \|\widetilde{f}_n(x) - \widetilde{f}_\infty(x)\| \\
&< \quad \frac{\epsilon}{2} + C^{-1}\Big(\frac{C\epsilon}{2}\Big) \; = \; \epsilon \qquad \Big(\text{by } (2.4)\Big),
\end{aligned}$$

completing the proof of Lemma 2.6.2. ☐

Proof of Proposition 2.6.1. Suppose g_m is a sequence of locally projective maps defined on a connected domain $\Omega \subset \mathsf{P}$ converging uniformly on

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all compact subsets of Ω to a map

$$\Omega \xrightarrow{g_\infty} \mathsf{P}'.$$

Lift g_∞ to a linear transformation $\widetilde{g}_\infty$ of norm 1, and lift g_m to linear transformations $\widetilde{g}_m$, also linear transformations of norm 1, converging to $\widetilde{g}_\infty$. Then

$$g_m \longrightarrow g_\infty$$

in $\mathsf{End}(\mathsf{P})$. Conversely if $g_m \longrightarrow g_\infty$ in $\mathsf{End}(\mathsf{P})$ and

$$K \subset \mathsf{P} - \mathbb{U}(g_\infty),$$

choose lifts as above and $\widetilde{K} \subset\subset V$ such that $\Pi(\widetilde{K}) = K$. By Lemma 2.6.2,

$$\widetilde{g_m}/\|\widetilde{g_m}\| \rightrightarrows \widetilde{g_\infty}/\|\widetilde{g_\infty}\|$$

on $\widetilde{K}$. Hence $g_m|_K \rightrightarrows g_\infty|_K$, completing the proof of Proposition 2.6.1.

□

2.6.1. Some examples. Let us consider a few examples of this convergence. Consider the case first when $n = 1$. Let $\lambda_m \in \mathbb{R}$ be a sequence converging to $+\infty$ and consider the projective transformations given by the diagonal matrices

$$g_m = \begin{bmatrix} \lambda_m & 0 \\ 0 & (\lambda_m)^{-1} \end{bmatrix}$$

Then $g_m \longrightarrow g_\infty$ where g_∞ is the singular projective transformation corresponding to the matrix

$$g_\infty = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

— this singular projective transformation is undefined at $\mathbb{U}(g_\infty) = \{[0,1]\}$; every point other than $[0,1]$ is sent to $[1,0]$. It is easy to see that a singular projective transformation of P^1 is determined by the ordered pair of points $\big(\mathbb{U}(f), \mathcal{R}(f)\big)$. Note that in the next example, the two points $\mathbb{U}(\phi_\infty), \mathcal{R}(\phi_\infty)$ coincide.

Exercise 2.6.3. Consider the sequence of projective transformations of P^1

$$\phi_n(x) := \frac{x}{1 - nx}, \text{ as } n \longrightarrow +\infty$$

- Show that the pointwise limit equals the constant function 0:

$$\lim_{n\to\infty} \phi_n(x) = 0, \qquad \forall x \in \mathsf{P}^1.$$

- Show that ϕ_n does *not* converge uniformly on any subset $S \subset \mathsf{P}^1$ which contains an infinite subsequence $s_j > 0$ with $s_j \searrow 0$ as $j \longrightarrow \infty$.

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- Show that ϕ_n *does* converge to the singular projective transformation ϕ_∞ defined on the complement of $\mathbb{U}(\phi_\infty) = \{0\}$ having constant value 0.
- Use this idea, together with the techniques involved in the proof of Lemma 2.6.2, to prove a statement converse to Proposition 2.6.1:

  If $g_n \in \mathsf{Aut}(\mathsf{P})$ is a sequence of projective transformations converging to a singular projective transformation $g_\infty \in \mathsf{End}(\mathsf{P})$, then, for any open subset $S \subset \mathsf{P}$ which meets $\mathbb{U}(g_\infty)$, the restrictions $g_n|_S$ do not converge uniformly.

2.6.2. Relation to limit sets. A discrete subgroup $\Gamma < \mathsf{PGL}(2, \mathbb{C})$ acts properly on H^3. If $x \in \mathsf{H}^3$, then its orbit $\Gamma \cdot x$ is discrete in H^3. However the action of Γ extends to the compactification $\mathsf{H}^3 \cup \mathbb{C}\mathsf{P}^1$, where $\mathbb{C}\mathsf{P}^1 = \partial\mathsf{H}^3$. Since $\mathsf{H}^3 \cup \mathbb{C}\mathsf{P}^1$ is compact, $\Gamma \cdot x$ has accumulation points in $\mathsf{H}^3 \cup \mathbb{C}\mathsf{P}^1$. Denote by $\Lambda(\Gamma, x)$ the set of accumulation points.

Exercise 2.6.4. Show that if $x, y \in \mathsf{H}^3$, then $\Lambda(\Gamma, x) = \Lambda(\Gamma, y)$.

This set is called the *limit set* of Γ and denoted $\Lambda(\Gamma)$. If its cardinality exceeds two, then it is the unique Γ-invariant closed subset of $\mathbb{C}\mathsf{P}^1$.

Exercise 2.6.5. If H^3/Γ is compact, then $\Lambda(\Gamma) = \mathbb{C}\mathsf{P}^1$.

Compare Kapovich [**210**], Mumord–Series–Wright [**277**], Marden [**251**], or Thurston [**324**].

Exercise 2.6.6. (1) The projective group $\mathsf{PGL}(2, \mathbb{R}) = \mathsf{Aut}(\mathbb{R}\mathsf{P}^1)$ is an open dense subset of $\mathsf{End}(\mathbb{R}\mathsf{P}^1) \approx \mathbb{R}\mathsf{P}^3$. Its complement naturally identifies with the Cartesian product $\mathbb{R}\mathsf{P}^1 \times \mathbb{R}\mathsf{P}^1$ under the correspondence

$$\begin{aligned} \mathsf{End}(\mathbb{R}\mathsf{P}^1) \setminus \mathsf{Aut}(\mathbb{R}\mathsf{P}^1) &\longleftrightarrow \mathbb{R}\mathsf{P}^1 \times \mathbb{R}\mathsf{P}^1 \\ [f] &\longleftrightarrow \big(\mathbb{U}(f), \mathcal{R}(f)\big). \end{aligned}$$

(2) Prove the analogous statements for $\mathsf{PGL}(2, \mathbb{C})$ and $\mathbb{C}\mathsf{P}^1$, that is, when $\mathbb{R}$ is replaced by $\mathbb{C}$.

(3) Show that if $\Gamma < \mathsf{PGL}(2, \mathbb{C})$ is a nonelementary discrete subgroup with limit set $\Lambda \subset \mathbb{C}\mathsf{P}^1$, then $\overline{\Gamma}\backslash\Gamma$ identifies with $\Lambda \times \Lambda \subset \mathbb{C}\mathsf{P}^1 \times \mathbb{C}\mathsf{P}^1$.

2.6.3. Higher-dimensional projective maps. More interesting phenomena arise when $n = 2$. Let $g_m \in \mathsf{Aut}(\mathsf{P}^2)$ be a sequence of diagonal matrices

$$\begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \mu_m & 0 \\ 0 & 0 & \nu_m \end{bmatrix}$$

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where $0 < \lambda_m < \mu_m < \nu_m$ and $\lambda_m \mu_m \nu_m = 1$. Corresponding to the three eigenvectors (the coordinate axes in $\mathbb{R}^3$) are three fixed points

$$p_1 = [1, 0, 0], \quad p_2 = [0, 1, 0], \quad p_3 = [0, 0, 1].$$

They span three invariant lines

$$l_1 = \overleftrightarrow{p_2\, p_3}, \quad l_2 = \overleftrightarrow{p_3\, p_1}, \quad l_3 = \overleftrightarrow{p_1\, p_2}.$$

Since $0 < \lambda_m < \mu_m < \nu_m$, the collineation has a repelling fixed point at p_1, a saddle point at p_2 and an attracting fixed point at p_3. Points on l_2 near p_1 are repelled from p_1 faster than points on l_3 and points on l_2 near p_3 are attracted to p_3 more strongly than points on l_1.

Suppose that g_m does not converge to a nonsingular matrix; it follows that $\nu_m \longrightarrow +\infty$ and $\lambda_m \longrightarrow 0$ as $m \longrightarrow \infty$ (after extracting a subsequence). Suppose that $\mu_m/\nu_m \longrightarrow \rho$; then g_m converges to the singular projective transformation g_∞ determined by the matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which, if $\rho > 0$, has undefined set $\mathbb{U}(g_\infty) = p_1$ and range l_1; otherwise $\mathbb{U}(g_\infty) = l_2$ and $\mathcal{R}(g_\infty) = p_3$.

2.6.4. Limits of similarity transformations. Convergence to singular projective transformations is perhaps easiest for translations of affine space, or, more generally, Euclidean isometries.

Exercise 2.6.7. Suppose $g_m \in \mathsf{Isom}(\mathsf{E}^n)$ is a divergent sequence of Euclidean isometries. Show that $\exists p \in \mathsf{P}^{n-1}_\infty$ and a subsequence g_{m_k}, that $g_{m_k}|_K \rightrightarrows p$ for every compact $K \subset \mathsf{E}^n$.

Indeed the boundary of the translation group V of A^n is the projective space P^{n-1}_∞. More generally the boundary of $\mathsf{Isom}(\mathsf{E}^n)$ identifies with P^{n-1}_∞.

Exercise 2.6.8. Suppose $g_m \in \mathsf{Sim}(\mathsf{E}^n)$ is a divergent sequence of similarities of Euclidean space. Then $\exists$ a subsequence g_{m_k}, and a point

$$p \in \mathsf{E}^n \coprod \mathsf{P}^{n-1}_\infty$$

such that one of three possibilities occurs:

- $p \in \mathsf{P}^{n-1}_\infty$ and $g_{m_k}|_K \rightrightarrows p, \ \forall K \subset\subset \mathsf{E}^n$;
- $p \in \mathsf{P}^{n-1}_\infty$ and $\exists q \in \mathsf{E}^n$ such that

$$g_{m_k}|_K \rightrightarrows p,$$
$$g_{m_k}^{-1}|_K \rightrightarrows q, \qquad \forall K \subset\subset \mathsf{E}^n \setminus \{q\};$$

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- $p \in \mathsf{E}^n_\infty$ and

$$g_{m_k}|_K \rightrightarrows p, \ \forall K \subset\subset \mathsf{E}^n.$$

The scale factor homomorphism $\mathsf{Sim}(\mathsf{E}^n) \xrightarrow{\lambda} \mathbb{R}^+$ defined in §1.4.1 of Chapter 1 controls the asymptotics of linear similarities. The two latter cases occur when $\lim_{k\to\infty} \lambda(g_{m_k}) = \infty$ and $\lim_{k\to\infty} \lambda(g_{m_k}) = 0$, respectively.

These results will be used in Fried's classification of closed similarity manifolds in §11.4 .

2.6.5. Normality domains. Convergence to singular projective transformations closely relates to the notion of *normality,* introduced by Kulkarni–Pinkall [**237**], and extending the classical notion of *normal families* in complex analysis. Let G be a group acting on a space X strongly effectively. A point $x \in X$ is a *point of normality* with respect to G if and only if x admits an open neighborhood W such the set of restrictions

$$G|_W := \{g|_W \mid g \in G\}$$

is a precompact subset of $\mathsf{Map}(W, X)$ with respect to the compact-open topology on $\mathsf{Map}(W, X)$. (This means that $G|_W$ is a *normal family* in the sense of Montel.) Denote the set of points of normality by $\mathsf{Nor}(G, X)$. Clearly $\mathsf{Nor}(G, X)$ is a G-invariant open subset of X, called the *normality domain.*

Proposition 2.6.9. Suppose that $\Gamma < \mathsf{Aut}(\mathsf{P})$ is a discrete group of collineations of a projective space P. Let $\overline{\Gamma} \subset \mathsf{End}(\mathsf{P})$ denote its closure in the set of singular projective transformations. Then the normality domain $\mathsf{Nor}(\Gamma, \mathsf{P})$ consists of the complement

$$\mathcal{U}_\Gamma := \mathsf{P} \setminus \bigcup_{\overline{\gamma} \in \overline{\Gamma}} \mathbb{U}(\overline{\gamma})$$

in P of the union $\bigcup_{\overline{\gamma} \in \overline{\Gamma}} \mathbb{U}(\overline{\gamma})$.

Proof. Observe first that $\bigcup_{\overline{\gamma} \in \overline{\Gamma}} \mathbb{U}(\overline{\gamma})$ is compact since each projective subspace $\mathbb{U}(\overline{\gamma})$ is compact and the parameter space $\overline{\Gamma}$ is compact. In particular $\overline{\Gamma}$ is closed so its complement

$$\mathcal{U}_\Gamma := \mathsf{P} \setminus \bigcup_{\overline{\gamma} \in \overline{\Gamma}} \mathbb{U}(\overline{\gamma})$$

is open. We claim that $\mathcal{U}_\Gamma = \mathsf{Nor}(\Gamma, \mathsf{P})$.

We first show any point $x \in \mathcal{U}_\Gamma$ is a point of normality. To this end, we show that the set of restrictions $\Gamma|_{\mathcal{U}_\Gamma}$ is precompact in $\mathsf{Map}(\mathcal{U}_\Gamma, \mathsf{P})$. This follows immediately from Proposition 2.6.1 as follows. Consider an infinite

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sequence $\gamma_n \in \Gamma$. Proposition 2.6.1 ensures a subsequence γ_n and a singular projective transformation $\overline{\gamma}_\infty \in \mathsf{End}(\mathsf{P})$, such that

$$\gamma_n|_K \rightrightarrows \gamma_\infty|_K, \qquad \forall K \subset\subset \mathfrak{U}_\Gamma,$$

as desired. (Since $K \cap \mathbb{U}(\gamma_\infty) = \emptyset$, the restriction $\gamma_\infty|_K$ is defined.)

Conversely suppose that $x \in \mathbb{U}(\overline{\gamma})$ for some $\overline{\gamma} \in \overline{\Gamma}$. Choose a sequence $g_n \in \Gamma$ converging to $\overline{\gamma}$, and a precompact open neighborhood $S \ni x$. By Exercise 2.6.3, the restrictions $g_n|_S$ do not converge uniformly, and the restrictions to the closure $\overline{S} \subset \mathsf{P}$ do not converge uniformly. Thus $x \notin \mathsf{Nor}(\Gamma, \mathsf{P})$ as claimed. $\square$

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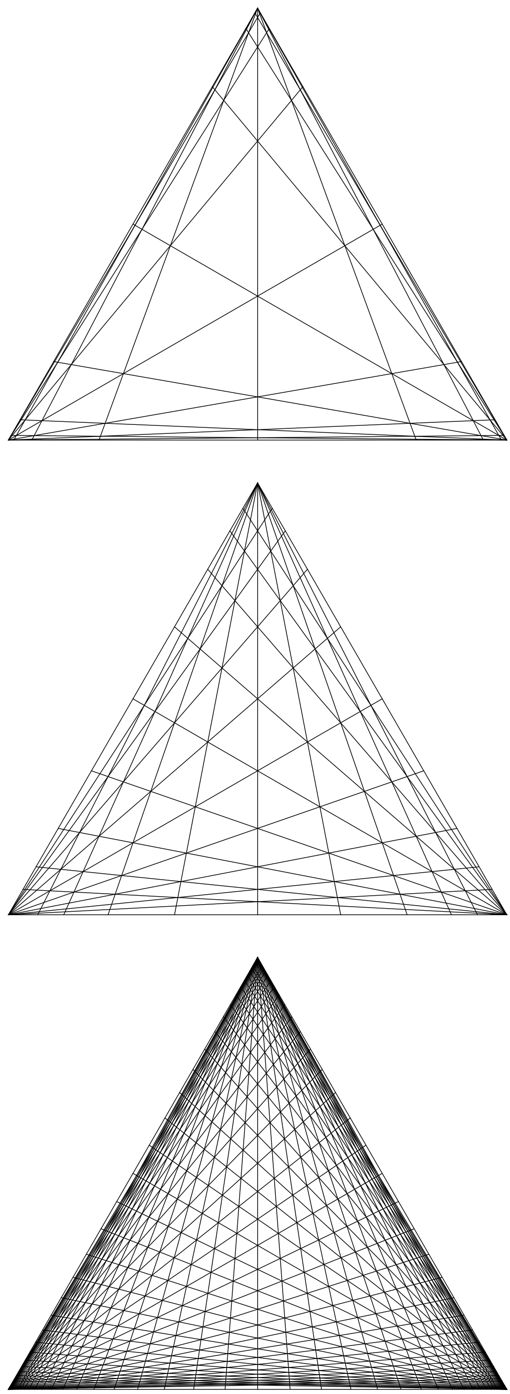

Figure 2.5. Non-Euclidean tessellations by equilateral triangles

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Chapter 3

Duality and Non-Euclidean geometry

The axiomatic development of projective geometry enjoys a basic symmetry: In P^2, a pair of distinct points lie on a unique line and a pair of distinct lines meet in a unique point. Consequently any statement about the geometry of P^2 can be *dualized* by replacing "point" by "line", "line" by "point", "collinear" with "concurrent", etc. — as long as it is done in a *completely consistent fashion.*

One of the oldest nontrivial theorems of projective geometry is Pappus's theorem (300 A.D.), asserting that if $l, l' \subset \mathsf{P}^2$ are distinct lines and $A, B, C \in l$ and $A', B', C' \in l'$ are triples of distinct points, then the three points

$$\overleftrightarrow{AB'} \cap \overleftrightarrow{A'B}, \quad \overleftrightarrow{BC'} \cap \overleftrightarrow{B'C}, \quad \overleftrightarrow{CA'} \cap \overleftrightarrow{C'A}$$

are collinear. Dual to Pappus's theorem is the following: if $p, p' \in \mathsf{P}^2$ are distinct points and a, b, c are distinct lines all passing through p and a', b', c' are distinct lines all passing through p', then the three lines

$$\overleftrightarrow{(a \cap b') \quad (a' \cap b)}, \quad \overleftrightarrow{(b \cap c') \quad (b' \cap c)}, \quad \overleftrightarrow{(c \cap a') \quad (c' \cap a)}$$

are concurrent. (According to [**102**], Hilbert observed that Pappus's theorem is equivalent to the commutative law of multiplication.)

This chapter introduces the projective models for elliptic and hyperbolic geometry through the classical notion of *polarities.* We begin by working

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over both $\mathbb{R}$ and $\mathbb{C}$, but specialize to the case $\mathsf{k} = \mathbb{R}$ when we discuss important cases of polarities:

- *Elliptic* polarities, corresponding to definite symmetric bilinear forms, and leading to elliptic (spherical) geometry;
- *Hyperbolic* polarities, corresponding to Lorentzian bilinear forms, and leading to hyperbolic geometry;
- *Null* polarities, corresponding to nondegeneraate skew-symmetric bilinear forms, and leading to contact projective geometry (in odd dimensions). (Compare Semple and Kneebone [**304**].)

See Ratcliffe [**293**] for a good elementary treatment of the Beltrami–Klein model for hyperbolic geometry, as well as Thurston [**324**], §2.3. The various derivations of hyperbolic geometry are aided by symmetry. The Beltrami–Klein metric generalizes to the Hilbert metric on convex domains. For an extensive modern discussion of Hilbert geometry, see Papadopoulos–Troyanov [**287**]. These, in turn, are special cases of intrinsic metrics, which we introduce here, and discuss later in the context of affine and projective structures on manifolds.

3.1. Dual projective spaces

In terms of our projective geometry/linear algebra dictionary, projective duality translates into duality between vector spaces as follows. Let $\mathsf{P} = \mathsf{P}(\mathsf{W})$ be a projective space associated to the vector space W. A nonzero linear functional

$$\mathsf{W} \xrightarrow{\psi} \mathsf{k}$$

defines a projective hyperplane H_ψ in P; two such functionals define the same hyperplane if and only if they are *projectively equivalent,* that is, they differ by scalar multiplication by a nonzero scale factor. Equivalently, they determine the same line in the vector space W^* dual to W.

The *projective space* P^* *dual to* P consists of lines in the dual vector space W^*, which correspond to hyperplanes in P. The line joining two points in P^* corresponds to the intersection of the corresponding hyperplanes in P, and a hyperplane in P^* corresponds to a point in P.

Exercise 3.1.1. Show that an n-dimensional projective space enjoys a natural correspondence

$$\{k - \text{dimensional subspaces of } \mathsf{P}\} \longleftrightarrow \{l - \text{dimensional subspaces of } \mathsf{P}^*\}$$

where $k + l = n - 1$.

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Since vector spaces of the same dimension are isomorphic, a projective space P is projectively isomorphic to the dual of P^*, but *in many different ways.*

Let $\mathsf{P} \xrightarrow{f} \mathsf{P}'$ be a projective map. Then for each hyperplane $H' \subset \mathsf{P}'$ the preimage $f^{-1}(H')$ is a hyperplane in P. This defines a projective map $(\mathsf{P}')^* \xrightarrow{f^*} \mathsf{P}^*$.

Here is the detailed construction. Suppose that f is defined by a linear mapping of vector spaces $\mathsf{W} \xrightarrow{F} \mathsf{W}'$, where P, P' projectivize W, W' respectively. Since f is defined on all of P, the linear map F is injective. The projective hyperplane H corresponds to the linear hyperplane

$$S := \mathsf{Ker}(\psi) \subset \mathsf{W},$$

for some nonzero covector $\psi \in \mathsf{W}^*$. Since $F^{-1}(S) = \mathsf{Ker}(\psi \circ F)$, the preimage $f^{-1}(H')$ is the projective hyperplane defined by the covector $\psi \circ F \in (\mathsf{W}')^*$.

Exercise 3.1.2. If f is the projectivization of a linear map $\mathsf{W} \xrightarrow{F} \mathsf{W}'$, show that f^* is the projectivization of the dual map $(\mathsf{W}')^* \xrightarrow{F^*} \mathsf{W}^*$ given by matrix transpose. That is, represent F by a matrix M by choosing bases of W and W' respectively. Show that the matrix representing F^* in the respective dual bases of W^* and $(\mathsf{W}')^*$ is the transpose $M^\dagger$ of the matrix M.

3.2. Correlations and polarities

Definition 3.2.1. Let P be a projective space. A *correlation* of P is a projective map $\mathsf{P} \longrightarrow \mathsf{P}^*$. Since projective maps which are defined on *all* of projective space are invertible, correlations are necessarily *isomorphisms.*

That is, a correlation associates to every projective point p a projective hyperplane $C(p)$ in such a way to *preserve incidences:* if $p, q, r \in \mathsf{P}$ are distinct projective points, then p, q, r are collinear if and only if the projective subspace $C(p) \cap C(q) \cap C(r)$ has codimension two (not three).

Exercise 3.2.2. Let W be a vector space such that $\mathsf{P} = \mathsf{P}(\mathsf{W})$. Correlations of P identify with projective equivalence classes of nondegenerate bilinear forms

$$\mathsf{W} \times \mathsf{W} \longrightarrow \mathsf{k},$$

or, equivalently, linear isomorphisms $\mathsf{W} \longrightarrow \mathsf{W}^*$.

In linear algebra, *reflexivity* means that the natural linear map

$$\mathsf{W} \longrightarrow (\mathsf{W}^*)^*$$

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is an isomorphism of vector spaces. The *dual* of a linear map $\mathsf{W} \to \mathsf{V}$ is a linear map $\mathsf{V}^* \to \mathsf{W}$, and hence reflexivity implies that every correlation $\mathsf{P} \to \mathsf{P}^*$ determines an *inverse* correlation

$$\mathsf{P} \xrightarrow{\cong} \left(\mathsf{P}^*\right)^* \longrightarrow \mathsf{P}^*.$$

Exercise 3.2.3. Using the matrix representation of bilinear forms determined by a basis, identify the matrix operation corresponding to inversion of correlations.

3.2.1. Example: Elliptic geometry. Here is an example of a correlation, corresponding to the standard Euclidean inner product on $\mathsf{W} = \mathbb{R}^3$. This correlation associates a point $p = [\mathbf{v}] \in \mathbb{R}\mathsf{P}^2$ the projective line p^* corresponding to the orthogonal complement $\mathbf{v}^\perp$. The corresponding linear isomorphism $\mathsf{W} \longrightarrow \mathsf{W}^*$ is the usual *transpose* operation, interchanging column vectors and row vectors.

Exercise 3.2.4. p and p^* are never incident.

An important property of this correlation is that it is *self-inverse* in the following sense:

Exercise 3.2.5. Let $p \longmapsto p^*$ be the correlation defined as above. If $\ell, m \in \mathsf{P}^*$ are distinct projective lines, with

$$\ell = p^*,\ m = q^*,$$

for respective projective points p, q, then

$$(l \cap m)^* \ = \ \overleftrightarrow{p\,q}.$$

Definition 3.2.6. A self-inverse correlation is called a *polarity.*

3.2.2. Elliptic polarties and elliptic geometry. Our interest in projective correlations stems from their use to define models of non-Euclidean geometry.

Our first example is the Fubini–Study metric on $\mathbb{R}\mathsf{P}^n$, making $\mathbb{R}\mathsf{P}^2$ into a model of the elliptic plane. Elliptic geometry arises from the polarity defined by the Euclidean inner product on $\mathbb{R}^n$. We describe how the correlation c defines a distance d on $\mathbb{R}\mathsf{P}^2$, making $(\mathbb{R}\mathsf{P}^2, \mathsf{d})$ into a metric space, called the *elliptic plane.* It is a basic example of *non-Euclidean geometry.*

Here is how to define the Fubini–Study distance $\mathsf{d}(p,q)$ between two points $p, q \in \mathbb{R}\mathsf{P}^2$. For $p = q$, their distance will be zero. Otherwise, p, q span a unique projective line $\ell := \overleftrightarrow{p\,q}$, and we extend (p, q) to a quadruple on l and compute its cross-ratio. Since p and p^* (respectively q and q^*) are not incident, $p^* \cap \ell$ and $q^* \cap \ell$ are points on ℓ, denoted p', q' respectively. Define:

$$\mathsf{d}(p,q) := \tan^{-1}\sqrt{[p,q,p',q']}.$$

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Exercise 3.2.7. $(\mathbb{R}\mathsf{P}^2, \mathsf{d})$ is a metric space satisfying:

- $\mathsf{SO}(3)$ acts isometrically and transitively on this metric space.
- (Busemann–Kelly [**73**], Exercise 38.1, p. 237):
  This metric space arises from a Riemannian structure, defined in an affine patch with coordinates $(x, y) \in \mathbb{R}^2$, by the metric tensor
  $$\frac{dx^2 \;+\; dy^2 \;+\; \big(x\,dy \;-\; y\,dx\big)^2}{x^2 + y^2 + 1}.$$
- Relate this metric space to the Euclidean unit sphere in $\mathbb{R}^3$.
- The Gaussian curvature equals $+1$.
- The geodesics are exactly the projective lines.

This construction extends to higher dimensions, and the metric geometry is *Elliptic Geometry.* The corresponding Riemannian metric is called the *Fubini–Study metric.*

In 1866 Beltrami showed that the only Riemannian metrics on domains in P^n where the geodesics are straight line segments are (up to a collineation and change of scale factor) Euclidean metrics, the Fubini–Study (elliptic) metric and the Beltrami–Klein (hyperbolic) metric discussed in §3.3. Hilbert's fourth problem was to determine all metric space structures on domains in P^n whose geodesics are straight line segments. There are many unusual such metrics, see Busemann–Kelly [**73**], and Pogorelov [**290**]. For more general discussion compare Coxeter [**104**] and Goldman [**155**],§1.3.

3.2.3. Polarities. The correlation defining elliptic geometry is an example of a *polarity,* which is self-inverse in the sense of Exercise 3.2.5. Polarities give rise to more general and fascinating geometries, the most important being *hyperbolic non-Euclidean geometry.*

We discuss polarities in general. For expository simplicity, we henceforth restrict to the case $\mathsf{k} = \mathbb{R}$, although the complex case is quite interesting and basic.

Using the dictionary between projective geometry and linear algebra, one sees that if W is the vector space corresponding to $\mathsf{P} = \mathsf{P}(\mathsf{W})$, then $\mathsf{P}^* = \mathsf{P}(\mathsf{W}^*)$ and a correlation θ is realized as a linear isomorphism $\mathsf{W} \xrightarrow{\widetilde{\theta}} \mathsf{W}^*$, which is uniquely determined up to homotheties. Linear maps $\mathsf{W} \xrightarrow{\widetilde{\theta}} \mathsf{W}^*$, correspond to bilinear forms

$$\mathsf{W} \times \mathsf{W} \xrightarrow{\mathsf{B}_{\tilde{\theta}}} \mathbb{R}$$

under the correspondence

$$\widetilde{\theta}(v)(w) = \mathsf{B}_{\tilde{\theta}}(v, w)$$

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and $\widetilde{\theta}$ is an isomorphism if and only if $\mathsf{B}_{\tilde{\theta}}$ is nondegenerate. Thus correlations can be interpreted analytically as projective equivalence classes of nondegenerate bilinear forms.

Exercise 3.2.8. A correlation θ is a polarity (that is, θ is self-inverse) if and only if a corresponding bilinear form B_θ is either symmetric or skew-symmetric.

Let θ be a polarity on P. A point $p \in \mathsf{P}$ is *conjugate* if it is incident to its polar hyperplane, that is, if $p \in \theta(p)$. By our dictionary we see that the conjugate points of a polarity correspond to *null vectors* of the associated quadratic form, that is, to nonzero vectors $v \in \mathsf{W}$ such that $\mathsf{B}_\theta(v, v) = 0$. A polarity is said to be *elliptic* if it admits no conjugate points.

The polarity of § 3.2.1 which associates to a point $p = [\mathbf{v}] \in \mathsf{P}^2$ the line $\mathsf{P}(\mathbf{v}^\perp) \subset \mathsf{P}^2$ is an elliptic polarity (compare Exercise 3.2.4). A polarity is *hyperbolic* if conjugate points exist, but not every point is a conjugate point. At the other extreme, a polarity is *null* if and only if every point is conjugate.

Exercise 3.2.9. Null polarities of a projective space P correspond to nondegenerate skew-symmetric bilinear forms on the vector space W, where $\mathsf{P} = \mathsf{P}(\mathsf{W})$. A projective space P admits a null polarity if and only if $\dim \mathsf{P}$ is odd.

Exercise 6.6.1 develops the theory of geometric structures (called *contact projective structures*) related to null polarities.

3.2.4. Quadrics. The set of conjugate points of a hyperbolic polarity forms a *quadric.*

As in §3.2.2 an elliptic polarity describes the structure of elliptic geometry on $\mathbb{R}\mathsf{P}^n$. In particular it induces a Riemannian structure on $\mathbb{R}\mathsf{P}^n$, namely the *Fubini–Study metric.* This construction readily generalizes to polarities which are not null, and correspond to nondegenerate symmetric bilinear forms, which are *indefinite.* The isomorphism class of such a bilinear form is determined by its *signature* (p, q) where p, q are positive integers with $p + q = n + 1$. A standard example is $\mathsf{B}_{p,q}$, defined by:

$$(x, y) \xmapsto{\mathsf{B}_{p,q}} x^1y^1 + \cdots + x^py^p - x^{p+1}y^{p+1} - \cdots - x^{p+q}y^{p+q}$$

corresponding to the quadratic form

$$x \longmapsto (x^1)^2 + \cdots + (x^p)^2 - (x^{p+1})^2 - \cdots - (x^{p+q})^2.$$

Exercise 3.2.10. Let $\mathsf{B} = \mathsf{B}_\theta$ be a nondegenerate symmetric bilinear form on the vector space space W of signature (p, q) where $p+q = n+1 = \dim(\mathsf{W})$.

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Consider the quadric hypersurfaces:

$$\mathbf{Q}^- := \{\mathbf{w} \in \mathsf{W} \mid \mathsf{B}(\mathbf{w},\mathbf{w}) = -1\}$$
$$\mathbf{Q}^0 := \{\mathbf{w} \in \mathsf{W} \mid \mathsf{B}(\mathbf{w},\mathbf{w}) = 0\}$$
$$\mathbf{Q}^+ := \{\mathbf{w} \in \mathsf{W} \mid \mathsf{B}(\mathbf{w},\mathbf{w}) = 1\}.$$

Their respective projectivizations in $\mathsf{P} = \mathsf{P}(\mathsf{W})$ decompose P as a disjoint union:

$$\mathsf{P} \;=\; \mathsf{P}\big(\mathbf{Q}^-\big) \amalg \mathsf{P}\big(\mathbf{Q}^0\big) \amalg \mathsf{P}\big(\mathbf{Q}^+\big).$$

- If $p = 0$, then B is positive definite, and $\mathsf{P}\big(\mathbf{Q}^-\big) = \mathsf{P}\big(\mathbf{Q}^0\big) = \emptyset$ and $\mathsf{P}\big(\mathbf{Q}^+\big) = \mathsf{P}$.
- If $0 < p, q < n+1$, then B is indefinite and $\mathsf{P}\big(\mathbf{Q}^-\big), \mathsf{P}\big(\mathbf{Q}^0\big), \mathsf{P}\big(\mathbf{Q}^+\big)$ are each connected and nonempty. Indeed,
$$\mathsf{P}\big(\mathbf{Q}^0\big) \approx S^{p-1} \times S^{q-1}$$
if $p, q > 1$ and
$$\mathsf{P}\big(\mathbf{Q}^0\big) \approx S^{q-1}$$
if $p = 1$.
- If $p = n+1$, then B is negative definite, $\mathsf{P}\big(\mathbf{Q}^+\big) = \mathsf{P}\big(\mathbf{Q}^0\big) = \emptyset$ and $\mathsf{P}\big(\mathbf{Q}^-\big) = \mathsf{P}$.

Proposition 3.2.11. $\mathsf{P}\big(\mathbf{Q}^+\big)$ admits a $\mathsf{PO}(p,q)$-invariant pseudo-Riemannian structure of signature $(p-1,q)$. Similarly, $\mathsf{P}\big(\mathbf{Q}^-\big))$ admits a $\mathsf{PO}(p,q)$-invariant pseudo-Riemannian structure of signature $(p, q-1)$.

Proof. The symmetric bilinear form $\mathbf{Q}$ on W induces an invariant pseudo-Riemannian structure on $\mathbf{Q}^+$. Let $\mathbf{v} \in \mathbf{Q}^+$. Then the tangent space $\mathsf{T}_{\mathbf{v}}\mathbf{Q}^+$ identifies with the orthogonal complement $\mathbf{v}^\perp \subset \mathsf{W}$. The restriction of the pseudo-Riemannian structure to $\mathbf{v}^\perp$ has signature $(p-1,q)$ and is evidently $\mathsf{PO}(p,q)$-invariant. The case of $\mathbf{Q}^-$ is completely analogous. $\square$

The set of conjugate points of a polarity θ is the quadric $\mathsf{P}\big(\mathbf{Q}^0_\theta\big)$, comprising points $[\mathbf{w}] \in \mathsf{P}$ with $\mathsf{B}_\theta(\mathbf{w},\mathbf{w}) = 0$.

The quadric $\mathbf{Q}$ determines the polarity θ as follows.

For brevity we consider only the case $p = 1$, in which case the complement $\mathsf{P} \setminus \mathbf{Q}$ has two components, a convex component

$$\Omega = \{[x^0 : x^1 : \cdots : x^n] \mid (x^0)^2 - (x^1)^2 - \cdots - (x^n)^2 < 0\}$$

and a nonconvex component

$$\Omega^\dagger = \{[x^0 : x^1 : \cdots : x^n] \mid (x^0)^2 - (x^1)^2 - \cdots - (x^n)^2 > 0\}$$

diffeomorphic to the total space of the tautological line bundle over P^{n-1} (for $n = 2$ this is a Möbius band). If $x \in \mathbf{Q}$, let $\theta(x)$ denote the hyperplane

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tangent to $\mathbf{Q}$ at x. If $x \in \Omega^\dagger$ the points of $\mathbf{Q}$ lying on tangent lines to $\mathbf{Q}$ containing x all lie on a hyperplane which is $\theta(x)$. If $H \in \mathsf{P}^*$ is a hyperplane which intersects $\mathbf{Q}$, then either H is tangent to $\mathbf{Q}$ (in which case $\theta(H)$ is the point of tangency) or there exists a cone tangent to $\mathbf{Q}$ meeting $\mathbf{Q}$ in $\mathbf{Q} \cap H$ — the vertex of this cone will be $\theta(H)$. If $x \in \Omega$, then there will be no tangents to $\mathbf{Q}$ containing x, but by representing x as an intersection $H_1 \cap \cdots \cap H_n$, we obtain $\theta(x)$ as the hyperplane containing $\theta(H_1), \dots, \theta(H_n)$.

Exercise 3.2.12. Show that $\mathsf{P} \xrightarrow{\theta} \mathsf{P}^*$ is projective.

Observe that a polarity on P of signature (p, q) determines, for each non-conjugate point $x \in \mathsf{P}$ a unique reflection R_x which preserves the polarity. The group of collineations preserving such a polarity is the *projective orthogonal group* $\mathsf{PO}(p, q)$, that is, the image of the orthogonal group $\mathsf{O}(p, q) \subset \mathsf{GL}(n+1, \mathbb{R})$ under the projectivization homomorphism

$$\mathsf{GL}(n+1, \mathbb{R}) \longrightarrow \mathsf{PGL}(n+1, \mathbb{R})$$

having kernel the scalar matrices $\mathbb{R}^\times \subset \mathsf{GL}(n+1, \mathbb{R})$. Let

$$\Omega = \{\Pi(v) \in \mathsf{P} \mid \mathsf{B}(v, v) < 0\};$$

then projection from the origin identifies Ω with the quotient of the hyperquadric

$$\{v \in \mathbb{R}^{p,q} \mid \mathsf{B}(v, v) = -1\}$$

by $\{\pm\mathbf{1}\}$. The induced pseudo-Riemannian structure has signature $(q, p-1)$ and constant nonzero curvature. In particular if $(p, q) = (1, n)$ then Ω models hyperbolic n-space H^n in the sense that the group of isometries of H^n are represented precisely as the group of collineations of P^n preserving Ω^n. In this model, geodesics are the intersections of projective lines in P with Ω. More generally, intersections of projective subspaces with Ω define totally geodesic subspaces.

Consider the case that $\mathsf{P} = \mathsf{P}^2$. Points "outside" Ω correspond to geodesics in H^2. If $p_1, p_2 \in \Omega^\dagger$, then

$$\overleftrightarrow{p_1 p_2} \cap \Omega \neq \emptyset$$

if and only if the geodesics $\theta(p_1), \theta(p_2)$ are ultra-parallel in H^2; in this case $\theta(\overleftrightarrow{p_1 p_2})$ is the geodesic orthogonal to both $\theta(p_1), \theta(p_2)$. (Geodesics $\theta(p)$ and l are orthogonal if and only if $p \in l$.) Furthermore $\overleftrightarrow{p_1 p_2}$ is tangent to Q if and only if $\theta(p_1)$ and $\theta(p_2)$ are parallel. For more information on this model for hyperbolic geometry, see [**102**] or [**323**], §2. This model for non-Euclidean geometry seems to have first been discovered by Cayley in 1858.

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3.3. Projective model of hyperbolic geometry

The case when $q = 1$ is fundamentally important. Then $\mathsf{P}\big(\mathbf{Q}^-\big)$ is equivalent to the *unit ball* $\mathbb{B} \subset \mathbb{R}^n$ defined by

$$\|x\|^2 \;=\; x\cdot x \;=\; \sum_{i=1}^{n} (x^i)^2 < 1.$$

with the induced Riemannian metric[1]

$$\begin{aligned} ds^2_{\mathbb{B}} &= \frac{-4}{\sqrt{1-\|x\|^2}} d^2\sqrt{1-\|x\|^2} \\ &= \frac{-4}{1-\|x\|^2}\Big\{ \big(1-\|x\|^2\big) dx\cdot dx \;+ \big(\; x\cdot dx\big)^2 \Big\} \\ &= \frac{4}{\big(1-\|x\|^2\big)^2} \;\; \sum_{i=1}^{n} \big(x^i dx^i\big)^2 + \big(1-\|x\|^2\big)^2 \big(dx^i\big)^2 \qquad (3.1) \end{aligned}$$

defines a complete $\mathsf{PO}(n,1)$-invariant Riemannian structure of constant curvature -1 on $\mathbb{B}$. The resulting Riemannian manifold is *(real) hyperbolic space* H^n.

In 1894 Hilbert discovered a beautiful general construction of the distance function on the underlying metric space involving projective geometry. Suppose $x, y \in \mathbb{B} \subset \mathsf{P}$ are distinct points. They span a unique projective line $\overleftrightarrow{xy} \subset \mathsf{P}$. Then $\overleftrightarrow{xy}$ meets $\partial\mathbb{B}$ in two points x_0, y_0 as in Figure 3.1. Then the cross-ratio $[x, y, x_0, y_0]$ is defined and the *Hilbert distance*

$$\mathsf{d}(x,y) := \log[x, y, x_0, y_0]$$

makes $(\mathbb{B}, \mathsf{d})$ into a metric space.

Exercise 3.3.1. Show that this metric space underlies the Riemannian structure defined above. Show that its group of isometries is $\mathsf{PO}(n,1)$ and acts transitively not just on H^n but on its unit tangent bundle.

This metric is analogous to the construction of the Fubini–Study metric given in Exercise 3.2.7. Its generalizes to arbitrary properly convex domains, as will be discussed in the next chapter.

3.3.1. The hyperbolic plane. Due to its fundamental role, we discuss the projective model of the hyperbolic plane H^2 in detail:

Exercise 3.3.2. Let H^2 denote the upper halfplane $\mathbb{R}\times\mathbb{R}^+$ with the Poincaré metric

$$\mathsf{g} = y^{-2}\big(dx^2 + dy^2\big)$$

[1] d^2 denotes the *Hessian,* defined in §B.2.

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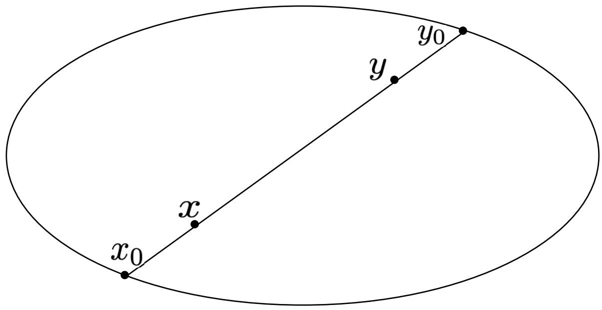

Figure 3.1. The *Beltrami–Klein* projective model of hyperbolic space. The Hilbert metric on the convex domain bounded by a quadric in projective space is defined in terms of cross-ratio. This metric is a Riemannian metric of constant negative curvature and is projectively flat (its geodesics are Euclidean line segments).

in coordinates $x \in \mathbb{R}, y \in \mathbb{R}^+$. Our model for the Lorentzian vector space $\mathbb{R}^{2,1}$ is the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of traceless 2×2 real matrices with the quadratic form

$$\frac{1}{2}\mathsf{Tr}\begin{bmatrix} a & b \\ c & -a \end{bmatrix}^2 = a^2 + bc = a^2 + \Big(\frac{b+c}{2}\Big)^2 - \Big(\frac{b-c}{2}\Big)^2.$$

The mapping

$$\mathsf{H}^2 \xrightarrow{\mathcal{I}} \mathfrak{sl}(2,\mathbb{R}) \cap \mathsf{SL}(2,\mathbb{R}) \subset \mathbb{R}^{2,1} \tag{3.2}$$

$$(x,y) \longmapsto y^{-1}\begin{bmatrix} x & -(x^2+y^2) \\ 1 & -x \end{bmatrix}$$

isometrically embeds H^2 as the component of the hypersphere in $\mathbb{R}^{2,1}$

$$\left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \;\middle|\; a^2 + bc = -1, \quad c > 0 \right\}.$$

Composition $\mathsf{P}\circ\mathcal{I}$ with projectivization isometrically embeds H^2 in projective space, as above. $\mathsf{P}\circ\mathcal{I}$ is equivariant, mapping the isometry group $\mathsf{PGL}(2,\mathbb{R})$ of H^2 isomorphically onto the projective automorphism group $\mathsf{PO}(2,1) \cong \mathsf{SO}(2,1)$. For any point $z \in \mathsf{H}^2$, the matrix $\mathcal{I}(z)$ defines the *symmetry* about z, that is, the orientation-preserving involutive isometry of H^2 fixing z.

Appendix B.6 derives the Levi–Civita connection of this metric.

3.3.2. The upper halfspace model of hyperbolic 3-space. Hyperbolic 3-space is fundamentally important as well. Its group of orientation-preserving isometries is isomorphic to $\mathsf{PGL}(2,\mathbb{C}) \cong \mathsf{PSL}(2,\mathbb{C})$ under a local

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Table 3.1. Multiplication table for the quaternionic multiplication for the basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ of $\mathbb{H}$.

	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	$-\mathbf{1}$	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$-\mathbf{1}$

isomorphism

$$\mathsf{PSL}(2, \mathbb{C}) \longrightarrow \mathsf{O}(3, 1).$$

(See Appendix F for a construction of such a local isomorphism.) It admits a useful model as the *upper halfspace* in the division algebra of *quaternions.*

3.3.2.1. *Quaternions.* Recal that the (Hamilton) quaternions are defined as a 4-dimensional real vector space $\mathbb{H}$ with basis denoted $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Table 3.1 describes quaternion multiplication in terms of this basis.[2] Distributivity of multiplication (that is, bilinearity) extends the multiplication of this basis to the full algebra $\mathbb{H}$.

$\mathbf{1}$ is a two-sided identity element, and $\mathbb{H}$ is an associative (but *not* commutative!) division algebra over $\mathbb{R}$. We write

$$q := r\mathbf{1} + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = r\mathbf{1} + \mathbf{v}$$

where $r, x, y, z \in \mathbb{R}$ are scalars and

$$\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \in \mathbb{R}^3$$

is a vector. *Quaternionic conjugation*

$$\begin{aligned} \mathbb{H} &\longrightarrow \mathbb{H} \\ q = r\mathbf{1} + \mathbf{v} &\longmapsto \overline{q} := r\mathbf{1} - \mathbf{v} \end{aligned}$$

is an *anti-automorphism,* that is, $\overline{q_1 q_2} = \overline{q_2 q_1}$. Its fixed set is thus a subalgebra, the image of the embedding

$$\begin{aligned} \mathbb{R} &\hookrightarrow \mathbb{H} \\ r\mathbf{1} &\longmapsto r\mathbf{1} + 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}. \end{aligned}$$

[2]The product of basis elements $b_m b_n$ is the entry in the m-th row and n-th column.

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(In fact, $\mathbb{R} = \mathbb{R}\mathbf{1}$ is the *center* of $\mathbb{H}$.) In particular the *real part* is the projection

$$\mathbb{H} \xrightarrow{\Re} \mathbb{R}$$
$$q \longmapsto r = \frac{1}{2}(q + \overline{q})$$

A quaternion is *pure* if its real part is zero, and pure quaternions identify with the vector space $\mathbb{R}^3$. Multiplication of pure quaternions corresponds to the operations of dot and cross product of vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1\mathbf{v}_2 \;=\; -(\mathbf{v}_1 \cdot \mathbf{v}_2)\,\mathbf{1} + \mathbf{v}_1 \times \mathbf{v}_2.$$

Furthermore

$$q\overline{q} = \|q\|^2 = r^2 + \|\mathbf{v}\|^2 \geq 0$$

and $q\overline{q} > 0$ if $q \neq 0$. Thus $\mathbb{H}$ is a *division algebra,* with inversion of a nonzero element define by:

$$q^{-1} \;:=\; \|q\|^{-2}\,\overline{q}.$$

3.3.2.2. *Hyperbolic* 3*-space.* The embedding

$$\mathbb{C} \hookrightarrow \mathbb{H}$$
$$r\mathbf{1} + x\mathbf{i} \longmapsto r\mathbf{1} + x\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

makes $\mathbb{H}$ into a (left) vector space over $\mathbb{C}$. Define the *upper halfspace* H^3 as the subset of $\mathbb{H}$ consisting of $z + h\mathbf{j}$, where $z \in \mathbb{C}$ and $h > 0$.

Exercise 3.3.3. Let

$$g := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathsf{SL}(2, \mathbb{C}),$$

that is, $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$.

- Show that
$$g(w) := (aw + b)(cw + d)^{-1}$$
defines a (left) action of $\mathsf{SL}(2, \mathbb{C})$ on $\mathbb{H}$. Interpret this action as a projective action over $\mathbb{H}$.
- Show that H^3 is invariant under this action.
- Find a Riemannian metric on H^3 upon which this group acts as its group of orientation-preserving isometries.
- Prove that the subspace $\mathsf{H}^2 \subset \mathsf{H}^3$ defined by $\Im(z) = 0$ (that is, $\mathbb{R} \times \mathbb{R}^+$) is an isometrically embedded hyperbolic plane. Determine its group of isometries.

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- Show that
$$z + h\mathbf{j} \longmapsto \overline{z} + h\mathbf{j}$$
is an isometry of H^3 fixing H^2 (reflection in H^2) and given by quaternion conjugation by
$$q \longmapsto -\mathbf{i}\,\overline{q}\,\mathbf{i}.$$
- Show that the *symmetry* in the point $\mathbf{j} \in \mathsf{H}^3$ is given by the quaternionic formula
$$q \longmapsto \mathbf{i}\,\overline{q}^{-1}\,\mathbf{i}.$$

For the relationship between this model and a the Beltrami–Klein model $\mathsf{H}^3 \hookrightarrow \mathbb{R}\mathsf{P}^3$, see Appendix F.

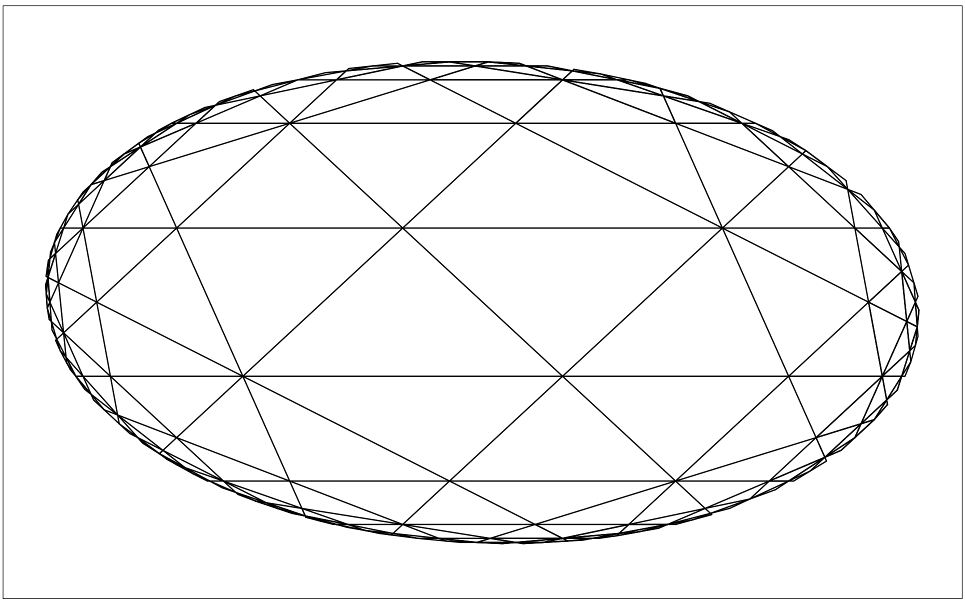

Figure 3.2. Projective model of a $(3, 3, 4)$-triangle tesselation of H^2

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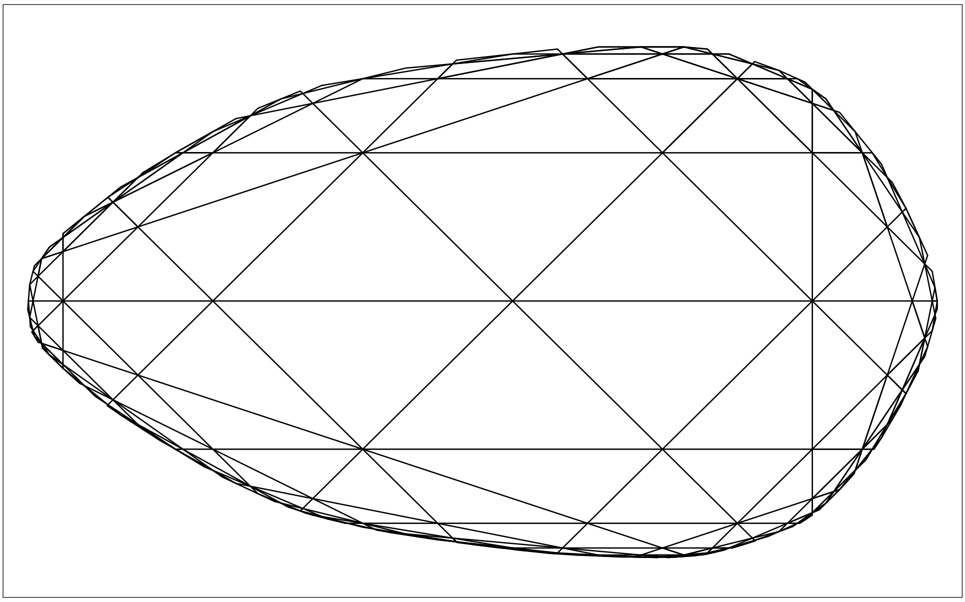

Figure 3.3. Projective deformation of hyperbolic $(3, 3, 4)$-triangle tessellation

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Chapter 4

Convexity

The Beltrami–Klein projective model of hyperbolic geometry extends to geometries defined on properly convex domains in $\mathbb{R}\mathsf{P}^n$. This chapter concerns this notion of convexity. After surveying some basic examples, we describe the *Hilbert metric,* a projective-geometry construction which includes the Beltrami–Klein hyperbolic metric. This metric is Riemannian only in this special case, and is generally *Finsler,* that is, arises from families of norms on the tangent spaces.[1] From there we describe another construction, due to Vinberg [**340**], which leads to a natural Riemannian structure. We use this structure to give new proofs of several results of Benzécri [**46**], on spaces of convex bodies in projective space. These results lead to regularity properties for domains arising from convex $\mathbb{R}\mathsf{P}^n$-manifolds.

A convex domain $\Omega \subset \mathsf{P}$ is said to be *quasi-homogeneous* if $\mathsf{Aut}(\Omega)$ acts syndetically, that is, the quotient $\Omega/\mathsf{Aut}(\Omega)$ is compact (but not necessarily Hausdorff).[2] If the action is *proper,* (that is, the quotient is Hausdorff), then the domain is said to be *divisible.* We are particularly interested in these compactness criteria. For reasons of space, we do not discuss the currently very active field of *finite volume* properly convex manifolds, but refer to relatively recent work by Marquis [**255–257**], Cooper–Long–Tillmann [**100**, **101**] Ballas–Danciger–Lee–Marquis [**22**], Choi [**95**] and Ballas–Cooper–Leitner [**21**].

Later in §12.2 we revisit these ideas in the more general setting of the *intrinsic metrics* introduced by Carathéodory and Kobayashi, in the analogous context of complex manifolds. Our treatment of the projective theory closely follows Kobayashi [**220–222**] and Vey [**337**].

[1] In this case it arises from a single norm; see §12.

[2] Benzécri calls such domains *balayable,* translated as "sweepable."

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4.1. Convex domains and cones

Let W be a real vector space. Recall that a subset $\Omega \subset \mathsf{W}$ is *convex* if, whenever $x, y \in \Omega$, the line segment $\overline{xy} \subset \Omega$. Equivalently, W is closed under *convex combinations:* if $\mathbf{w}_1, \dots, \mathbf{w}_m \in \mathsf{W}$ and $t_1, \dots, t_m \in \mathbb{R}$ satisfy $t_i \geq 0$ and

$$t_1 + \dots + t_m = 1,$$

then

$$t_1\mathbf{w}_1 + \dots + t_m\mathbf{w}_m \ \in \ \Omega.$$

A convex domain $\Omega \subset \mathsf{W}$ is *properly convex* if and only if it contains no complete affine line. A domain $\Omega \subset \mathsf{W}$ is a *cone* if and only if it is invariant under the group $\mathbb{R}^+$ of positive homotheties, that is, scalar multiplication by positive real numbers. A *sharp*[3] *convex cone* is a convex cone which is properly convex.

For example, W itself and the upper half-space

$$\mathbb{R}^n \times \mathbb{R}^+ = \{(x^0, \dots, x^n) \in \mathsf{W} \mid x^0 > 0\}$$

are both convex cones but neither is sharp. The *positive orthant*

$$(\mathbb{R}^+)^{n+1} := \{(x^0, \dots, x^n) \in \mathsf{W} \mid x^i > 0 \text{ for } i = 0, 1, \dots, n\}$$

and the positive light-cone

$$C_{n+1} = \{(x^0, \dots, x^n) \in \mathsf{W} \mid x^0 > 0 \text{ and } -(x^0)^2 + (x^1)^2 + \dots (x^n)^2 < 0\}$$

are both sharp convex cones.

Exercise 4.1.1. Define the *parabolic convex domain* as

$$\mathcal{P} \ := \ \left\{(x, y) \in \mathsf{A}^2 \mid y > x^2\right\} \tag{4.1}$$

- Show that $\mathcal{P}$ is a properly convex affine domain but not affinely equivalent to a cone.
- Describe the group of affine automorphisms of $\mathcal{P}$.
- Describe the group of projective automorphisms of $\mathcal{P}$.

Exercise 4.1.2. Show that the set $\mathfrak{P}_n(\mathbb{R})$ of all positive definite symmetric $n \times n$ real matrices is a sharp convex cone in the $n(n+1)/2$-dimensional vector space W of $n \times n$ symmetric matrices. Describe the group of affine transformations of W preserving $\mathfrak{P}_n(\mathbb{R})$.

Exercise 4.1.3. Suppose $\mathsf{W}_1, \mathsf{W}_2$ are vector spaces and $\Omega_i \subset \mathsf{W}_i$ are sharp convex cones for $i = 1, 2$. Then $\Omega_1 \times \Omega_2 \subset \mathsf{W}_1 \times \mathsf{W}_2$ is a sharp convex cone.

[3] In French, *saillant*

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Homogeneous convex cones were classified by Vinberg [**340**] in terms of algebras he calls *clans* (see §10.5.6) and the above examples are some basic cases of his algebraic construction.

Convex affine domains have the structure of principal $\mathbb{R}^k$-bundles over sharp convex cones:

Proposition 4.1.4. Let $\Omega \subset \mathsf{V}$ be an open convex cone in a vector space. Then there exists a unique linear subspace $\mathsf{W} \subset \mathsf{V}$ such that:

- Ω is invariant under translation by vectors in W (that is, Ω is W-*invariant;*)
- There exists a sharp convex cone $\Omega_0 \subset \mathsf{V}/\mathsf{W}$ such that $\Omega = \pi_{\mathsf{W}}^{-1}(\Omega_0)$ where $\pi_{\mathsf{W}} : \mathsf{V} \longrightarrow \mathsf{V}/\mathsf{W}$ denotes the linear projection.

Proof. Let
$$\mathsf{W} = \{w \in \mathsf{V} \mid x + tw \in \Omega,\ \forall x \in \Omega, t \in \mathbb{R}\}.$$
Then W is a linear subspace of V and Ω is W-invariant. Let
$$\Omega_0 = \pi_{\mathsf{W}}(\Omega) \subset \mathsf{V}/\mathsf{W};$$
then $\Omega = \pi_{\mathsf{W}}^{-1}(\Omega_0)$. We must show that Ω_0 is properly convex. To this end we can immediately reduce to the case $\mathsf{W} = 0$. Suppose that Ω contains a complete affine line $\{y + tw \mid t \in \mathbb{R}\}$ where $y \in \Omega$ and $w \in \mathsf{V}$. Then for each $s, t \in \mathbb{R}$
$$x_{s,t} = \frac{s}{s+1}x + \frac{1}{s+1}\Big(y + stw\Big) \in \Omega$$
whence
$$\lim_{s\to\infty} x_{s,t} = x + tw \in \bar{\Omega}.$$
Thus $x + tw \in \bar{\Omega}$ for all $t \in \mathbb{R}$. Since $x \in \Omega$ and Ω is open and convex, $x + tw \in \Omega$ for all $t \in \mathbb{R}$ and $w \in \mathsf{W}$ as claimed. □

4.1.1. Half-spaces and supporting hyperplanes. Let $\Omega \subset \mathsf{A}$ be a proper convex domain in an affine space A.

Proposition 4.1.5. Ω is an intersection of open half-spaces.

Here is a sketch of the proof, using the Hahn–Banach theorem. As stated in Theorem 11.4.1 of Berger [**48**], the Hahn–Banach theorem asserts that every affine subspace (for example a point) disjoint from Ω extends to an affine hyperplane disjoint from Ω. Since Ω is proper, $\mathsf{A} \setminus \Omega$ is nonempty. Choose a point $p \in \mathsf{A} \setminus \Omega$. The Hahn–Banach theorem guarantees an affine hyperplane $H \subset \mathsf{V}$ containing p disjoint from Ω. The two components of its complement $\mathsf{V} \setminus H$ are halfspaces. Since Ω is connected, one of them contains Ω.

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Writing $\mathcal{W}$ for the set of halfspaces $W \supset \Omega$, we prove that

$$\Omega = \bigcap_{W \in \mathcal{W}} W.$$

To this end, suppose that $p \in \left(\bigcap_{W \in \mathcal{W}} W\right) \setminus \Omega$. Applying the Hahn–Banach theorem again guarantees a half-space containing Ω but not containing p, a contradiction. Thus Ω is the intersection of open half-spaces.

The set $\mathcal{W}$ is partially ordered by inclusion. The boundary of a minimal open halfspace containing Ω is a hyperplane, called a *supporting hyperplane* for Ω. Proposition 11.5.2 of Berger [**48**] implies that at every point of $\partial\Omega$ is a supporting hyperplane for Ω.

4.1.2. Convexity in projective space. Convexity is somewhat more subtle in projective space P. First observe that convexity is invariant under translations, and thus invariant under affine transformations. Say that a domain $\Omega \subset \mathsf{P}$ is *convex* if and only if Ω lies in some affine patch $\mathsf{A} \subset \mathsf{P}$, and is a convex subset of A.

Exercise 4.1.6. Show that this notion is independent of the choice of A. Equivalently, if $\mathsf{P} = \mathsf{P}(\mathsf{W})$ and

$$\mathsf{W} \setminus \{\mathbf{0}\} \xrightarrow{\Pi} \mathsf{P}$$

denotes projectivization, then Ω is convex if $\Omega = \Pi(\Omega')$ for some convex cone $\Omega' \subset \mathsf{W}$.

A domain $\Omega \subset \mathsf{P}$ is *properly convex* if and only if a *sharp* properly convex cone $\Omega' \subset \mathsf{W}$ exists such that $\Omega = \Pi(\Omega')$. Equivalently, Ω is properly convex if and only if a hyperplane $H \subset \mathsf{P}$ exists such that $\overline{\Omega}$ is a convex subset of the affine space $\mathsf{P} \setminus H$. If $\Omega \subset \mathsf{P}$ is properly convex, then its intersection $\Omega \cap \mathsf{P}'$ with any projective subspace $\mathsf{P}' \subset \mathsf{P}$ is either empty or properly convex in P'. In particular every projective line intersecting Ω meets $\partial\Omega$ in exactly two points.

Since $\Omega' \subset \mathsf{W} \setminus \{\mathbf{0}\}$ is convex, Ω must be disjoint from at least one hyperplane H in P. (In particular P is itself *not* convex, by our definition.) Equivalently $\Omega \subset \mathsf{P}$ is convex if a hyperplane $H \subset \mathsf{P}$ exists such that Ω is a convex set in the complementary affine space $\mathsf{P} \setminus H$.

Exercise 4.1.7. Find two convex domains in A^2 which are not affinely isomorphic, but which are projectivizations of the same convex cone in $\mathbb{R}^3$.

For studying convex subsets of projective space, passing to the double covering space, the *sphere of directions* (defined in §6.2.1) is useful, especially for calculations.

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4.2. The Hilbert metric

In 1894 Hilbert introduced a projectively invariant metric $d = d_\Omega$ on any properly convex domain $\Omega \subset \mathsf{P}$ as follows. This was introduced in §3.3 as an explicit form of the metric on H^n in the Beltrami–Klein model. After reviewing its definition and basic properties, we discuss the other basic example of an open simplex, in which case the metric does not arise from a Riemannian structure. A simple example is Vey's semisimplicity Theorem 4.3.1, which we prove in a special case (used later in §12 to classify completely incomplete affine structures, following Kobayashi [**222**] and Vey [**338, 339**]).

4.2.1. Definition and basic properties. Let $x, y \in \Omega$ be a pair of distinct points; then the line $\overleftrightarrow{xy}$ meets $\partial\Omega$ in two points which we denote by x_∞, y_∞ (the point closest to x will be x_∞, etc). The *Hilbert distance*

$$\mathsf{d} = \mathsf{d}_\Omega^{\mathsf{Hilb}}$$

between x and y in Ω will be defined as the logarithm of the cross-ratio of this quadruple:

$$\mathsf{d}(x,y) = \log[x_\infty, x, y, y_\infty]$$

(where the cross-ratio is defined in (2.3)). Clearly $\mathsf{d}(x,y) \geq 0$ and $\mathsf{d}(x,y) = \mathsf{d}(y,x)$. Since Ω contains no complete affine line, $x_\infty \neq y_\infty$, and $\mathsf{d}(x,y) > 0$ if $x \neq y$.

Similarly

$$\Omega \times \Omega \xrightarrow{\mathsf{d}} \mathbb{R}$$

is *proper, or finitely compact:* that is, for each $x \in \Omega$ and $r > 0$, the closed "r-ball"

$$B_r(x) = \{y \in \Omega \mid \mathsf{d}(x,y) \leq r\}$$

is compact. Once the triangle inequality is established, the completeness of the metric space (Ω, d) follows. The triangle inequality results from the convexity of Ω, although we deduce it by showing that the Hilbert metric agrees with the general intrinsic metric introduced by Kobayashi [**222**]. Thus we enforce the triangle inequality as part of the construction of the metric. In this metric the geodesics are represented by straight lines.

By Exercise 3.3.1, the Hilbert metric on a quadric domain agrees with the Beltrami–Klein Riemannian structure. The other fundamental example is the open simplex, which is the projectivization of an *orthant* in $\mathbb{R}^n$ (for example the *positive orthant* $(\mathbb{R}^+)^n \subset \mathbb{R}^n$). When $n = 3$, the projective domain is just a triangle.

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4.2.2. The Hilbert metric on a triangle. Let $\triangle \subset \mathsf{P}^2$ denote a domain bounded by a triangle. Then the balls in the Hilbert metric are hexagonal regions. (In general if Ω is a convex k-gon in P^2 then the unit balls in the Hilbert metric will be interiors of l-gons where $l \leq 2k$.)

Exercise 4.2.1. (Unit balls in the Hilbert metric)

- Prove that $\mathsf{Aut}(\triangle)$ is conjugate to a finite extension of the group of diagonal matrices with positive eigenvalues.
- Deduce that $\mathsf{Aut}(\triangle)$ acts transitively on $\triangle$.
- Conclude that all the unit balls are isometric.

Recall the construction from §2.5.2 to illustrate the Hilbert geometry of $\triangle$. (Compare Figure 2.5.) Start with a triangle $\triangle$ and choose line segments ℓ_1, ℓ_2, ℓ_3 from an arbitrary point $p_1 \in \triangle$ to the vertices v_1, v_2, v_3 of $\triangle$. Choose another point p_2 on ℓ_1, say, and form lines ℓ_4, ℓ_5 joining it to the remaining vertices. Let

$$\rho = \log \left| \left[v_1,\ p_1,\ p_2,\ \ell_1 \cap \overleftrightarrow{v_2\ v_3} \right] \right|$$

where $[,,,]$ denotes the cross-ratio of four points on ℓ_1. The lines ℓ_4, ℓ_5 intersect ℓ_2, ℓ_3 in two new points which we call p_3, p_4. Join these two points to the vertices by new lines ℓ_i which intersect the old ℓ_i in new points p_i. In this way one generates infinitely many lines and points inside $\triangle$, forming a configuration of smaller triangles T_j inside $\triangle$. For each p_i, the union of the T_j with vertex p_i is a convex hexagon which is a Hilbert ball in $\triangle$ of radius ρ. Note that this configuration is combinatorially equivalent to the tessellation of the plane by congruent equilateral triangles. Indeed, this tessellation of $\triangle$ arises from an action of a (3,3,3)-triangle group by collineations and converges (in an appropriate sense) to the Euclidean equilateral-triangle tessellation as $\rho \longrightarrow 0$.

Exercise 4.2.2. Let $\triangle := \{(x, y) \in \mathbb{R}^2 \mid x, y > 0\}$ be the positive quadrant. Then the Hilbert distance is given by

$$\mathsf{d}((x,y),(x',y')) = \log \max \left\{ \frac{x}{x'}, \frac{x'}{x}, \frac{y}{y'}, \frac{y'}{y}, \frac{xy'}{x'y}, \frac{x'y}{xy'} \right\}.$$

- Show that the unit balls are hexagons.
- For any two points $p, p' \in \triangle$, show that there are infinitely many geodesics joining p to p'.
- Show that there are even non-smooth polygonal curves from p to p' having minimal length.

Daryl Cooper has called such a Finsler metric a *hex-metric,* since the unit balls are hexagons.

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4.3. Vey's semisimplicity theorem

The following theorem is due to Vey [**338, 339**]. Recall that a discrete subgroup $\Gamma < \mathsf{Aut}(\Omega)$ *divides* a domain Ω if and only if Γ acts properly on Ω and the quotient Ω/Γ is compact. Say that Ω is *divisible* if and only if $\exists\Gamma < \mathsf{Aut}(\Omega)$ dividing Ω.

Theorem 4.3.1. Let V be a real vector space and $\Omega \subset \mathsf{V}$ a divisible sharp convex cone. Then the action of $\mathsf{Aut}(\Omega)$ is semisimple, that is, any $\mathsf{Aut}(\Omega)$-invariant linear subspace $\mathsf{W} < \mathsf{V}$, there exists an $\mathsf{Aut}(\Omega)$-invariant complementary linear subspace. In particular a unique decomposition

$$\mathsf{V} \;=\; \bigoplus_{i=1}^{r} \mathsf{V}_i$$

exists, with sharp convex cones $\Omega_i \subset \mathsf{V}_i$ such that

$$\Omega \;=\; \prod_{i=1}^{r} \Omega_i$$

and the action of $\mathsf{Aut}(\Omega_i)$ on V_i is irreducible.

Corollary 4.3.2. Suppose $\Omega \subset \mathsf{A}^n$ is a properly convex divisible domain. Then Ω is a sharp convex cone.

The parabolic region $\mathcal{P}$ defined in (4.1) is an example of a properly convex domain which is quasi-homogeneous but not divisible.[4] In particular Ω is *not* a cone.

We don't prove all of Theorem 4.3.1 here, but just the special case when W is assumed to be a supporting hyperplane for Ω. This is all which is needed to deduce Corollary 4.3.2, although the general case is not much harder. Our treatment is based on Hoban [**195**].

Proof of Corollary 4.3.2. Suppose that $\Omega \subset \mathsf{A}$ is a properly convex domain with $\Gamma < \mathsf{Aut}(\Omega)$ dividing Ω. Embed A as an affine hyperplane in a vector space V. Let $\psi \in \mathsf{V}^*$ be a linear functional such that $\mathsf{A} = \psi^{-1}(1)$. Then

$$\Omega' \;:=\; \big\{\omega \in \mathsf{V} \;\big|\; \psi(\omega) > 0 \text{ and } \psi(\omega)^{-1}\omega \in \Omega\big\}$$

is sharp convex cone. The Γ-action on A extends to a linear action on V preserving Ω'. This linear action extends to $\Gamma \times \mathbb{R}^+$, where $\mathbb{R}^+$ acts by homotheties. The linear hyperplane

$$\mathsf{W} := \mathsf{Ker}(\psi) = \psi^{-1}(0)$$

[4] Subgroups $\Gamma < \mathsf{Aut}(\Omega)$ exist whose actions are syndetic but not proper. The quotient $\Gamma\backslash\Omega$ is compact but not Hausdorff .

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supports Ω' in the sense of §4.1.1. Furthermore, taking $\lambda > 1$, the product $\Gamma' := \Gamma \times \langle\lambda\rangle$ divides Ω' and preserves W.

By the special case of Theorem 4.3.1 when W is a supporting hyperplane, $\exists L < \mathsf{V}$ which is Γ-invariant and

$$\mathsf{V} \;=\; \mathsf{W} \oplus L.$$

Now L is a line which intersects the affine hyperplane A in a point p_0, and p_0 is fixed by Γ. Then Ω is an open convex cone with vertex p_0. ☐

The proof of the special case of Theorem 4.3.1] uses the following general standard fact about metric spaces, given in Burago–Burago–Ivanov [**68**], Theorem 1.6.15.

Lemma 4.3.3. Let X be any compact metric space. Then any distance nonincreasing surjective continuous map $X \to X$ is an isometry.

Let $\Omega^* \subset \mathsf{V}^*$ be the cone dual to Ω, as in §4.4. If W is a linear hyperplane which supports Ω, then the annihilator $\mathsf{Ann}(\mathsf{W}) < \mathsf{V}^*$ is a line lying in $\partial\Omega^*$. It suffices to find a Γ-invariant complementary linear hyperplane $H < \mathsf{V}^*$.

Let L be a line which is invariant under Γ and intersects nontrivially with $\overline{\Omega}$. We define a Γ-invariant map $\Omega \xrightarrow{s} \partial\Omega$ as follows. For $x \in \Omega$, let

$$\Omega_x := \Omega \cap (x + L).$$

Since Ω is sharp, Ω_x is a ray, with endpoint $s(x)$ on the boundary of Ω. Evidently s commutes with Γ. Then, for $t \in \mathbb{R}$,

$$c_t(x) \;:=\; s(x) + e^t\big(x - s(x)\big)$$

defines a one parameter group of homeomorphisms of Ω and $s \circ c_t = s$.

Lemma 4.3.4. c_t does not increase Hilbert distance:

$$\mathsf{d}^{\mathsf{Hilb}}\big(c_t(p), c_t(q)\big) \le \mathsf{d}^{\mathsf{Hilb}}(p, q) \tag{4.2}$$

Proof. To prove (4.2), choose $p, q \in \Omega$ as in Figure 4.1. Clearly $c_t(p) - p$ and $c_t(q) - q$ are both in L. Let a and b denote the intersections of $\overleftrightarrow{pq}$ with $\partial\Omega$. Let c and d denote the two points of the intersection of $\partial\Omega$ with

$$\overleftrightarrow{c_t(p)\ c_t(q)}\,.$$

Now the intersections of $a + L$ and $b + L$ respectively with $\overleftrightarrow{c_t(p)\ c_t(q)}$ are two points, denoted $\widehat{a}$ and $\widehat{b}$. Since Ω is convex,

$$\overline{\widehat{a}\widehat{b}} \subset \overline{cd},$$

hence

$$[c, c_t(p), c_t(q), d] \le [\widehat{a}, c_t(p), c_t(q), \widehat{b}] = [a, p, q, b].$$

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The last equality is due to the invariance of the cross ratio under projective transformations (in particular this is a perspectivity). Since cross ratio is nonincreasing, $\mathsf{d}^{\mathsf{Hilb}}$ is nonincreasing as well, concluding the proof of Lemma 4.3.4. □

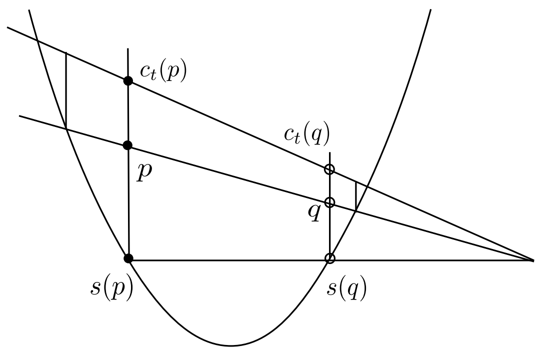

Figure 4.1. Projection does not increase Hilbert distance. This picture depicts the affine plane containing the points p, q and the line $p + L$. This figure is a bit misleading here since the domain pictured is not a cone, and c_t is actually strictly decreasing for this domain. For a less misleading picture, see Figure 4.2 below.

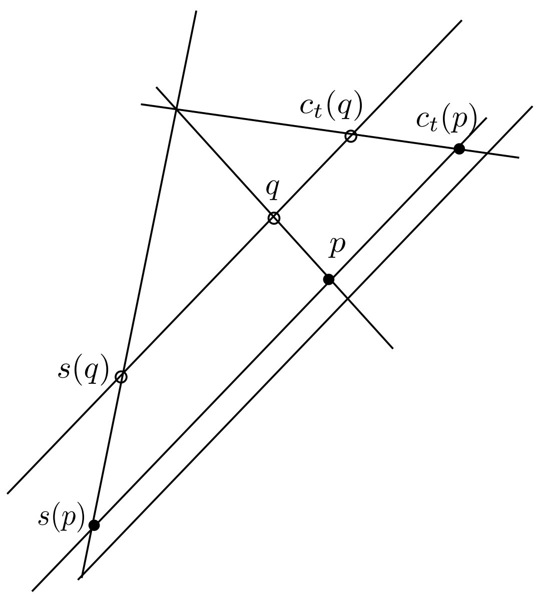

Figure 4.2. c_t is an isometry of the cone for all t.

Since s commutes with Γ, the map c_t defines a map $M \xrightarrow{C_t} M$ on the quotient $M = \Gamma\backslash\Omega$, which inherits the structure of a metric space since

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Γ divides Ω. Furthermore C_t doesn't increase $\mathsf{d}_M^{\mathsf{Hilb}}$-distances. Thus C_t is a distance nonincreasing homeomorphism of $M = \Gamma\backslash\Omega$, and Lemma 4.3.3 implies C_t is an isometry. It follows *c_t is a local isometry on Ω.*

Lemma 4.3.5. $s(\Omega)$ is convex.

Proof. Suppose $\exists x, y \in \Omega$ such that the line segment

$$\overline{s(x)\ s(y)} \not\subset s(\Omega) \subset \partial\Omega.$$

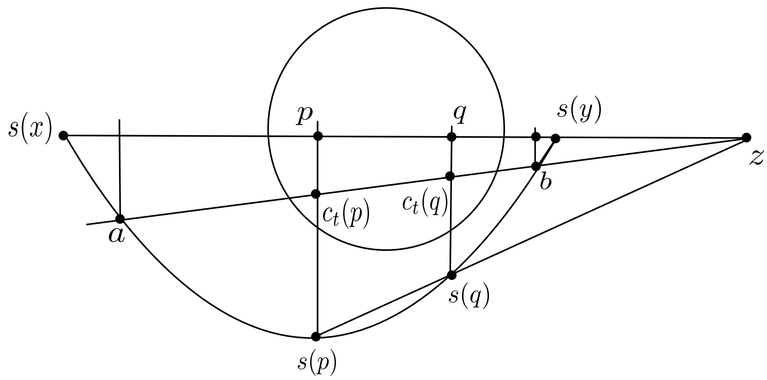

Figure 4.3. $s(\Omega)$ is convex.

Suppose

$$p \in \overline{s(x)\ s(y)} \setminus s(\Omega).$$

Let N be a neighborhood of p on which c_t is an isometry, and choose

$$q \in (\overline{s(x)\ s(y)}) \cap N.$$

Choose $t < 0$ such that both $c_t(p)$ and $c_t(q)$ lie in N. Let a, b be the intersections of $\overleftrightarrow{c_t(p)\ c_t(q)}$ with $\partial\Omega$. Write

$$\begin{aligned} \widehat{a} &= (a + L) \ \cap \ \overleftrightarrow{p\ q} \\ \widehat{b} &= (b + L) \ \cap \ \overleftrightarrow{p\ q}. \end{aligned}$$

Clearly

$$\widehat{a} \in \overline{s(x)\ s(y)},$$

but $\widehat{a} \neq s(x)$: otherwise $a = s(x)$ and $t = 0$. Similarly

$$\widehat{b} \in \overline{s(x)\ s(y)},$$

but $\widehat{b} \neq s(y)$.

Thus $\overline{\widehat{a}\ \widehat{b}} \not\subset \overline{s(x)\ s(y)}$, whence

$$[\widehat{a}, p, q, \widehat{b}] \ < \ [s(x), p, q, s(y)].$$

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Furthermore, a perspectivity maps $(a, c_t(p), c_t(q), b)$ to $(\widehat{a}, p, q, \widehat{b})$, so

$$[a, c_t(p), c_t(q), b] \;=\; [\widehat{a}, p, q, \widehat{b}] \;<\; [s(x), p, q, s(y)].$$

Therefore $\mathsf{d}^{\mathsf{Hilb}}(c_t(p), c_t(q)) \;<\; \mathsf{d}^{\mathsf{Hilb}}(p, q)$. This is a contradiction since c_t is an isometry. (Compare Figure 4.3.) □

Now $s(\Omega)$ is a convex set contained in the boundary of Ω, and generates a hyperplane H. Now $\mathsf{V} = L + H$ and this sum is actually a direct sum: $x - s(x) \in L$ for any $x \in \mathsf{V}$, so

$$\begin{aligned} x = \;& \big(x - s(x)\big) + s(x) \\ \in \;& \quad L \quad\;\; \oplus H. \end{aligned}$$

Hence $\mathsf{V} = L \oplus H$.

Since s commutes with Γ, the hyperplane H is Γ-invariant. This completes the proof of the special case of Theorem 4.3.1.

4.4. The Vinberg metric

This section develops the remarkable geometry of a sharp convex cone Ω in terms of a natural Riemannian structure defined in terms of the cone Ω^* dual to Ω. The Riemannian metric is the logarithmic Hessian of an natural analytic function introduced by Vinberg [**340**], the *Vinberg characteristic function.* The logarithmic differential of the characteristic function defines a natural closed 1-form α whose covariant differential is the metric. Shima [**306**] calls the α the *Koszul 1-form.*

Theorem 4.4.1. Let $\Omega \subset \mathsf{V}$ be a sharp convex cone. Then there exists a real analytic $\mathsf{Aff}(\Omega)$-invariant closed 1-form α on Ω whose covariant differential $\nabla\alpha$ is an $\mathsf{Aff}(\Omega)$-invariant Riemannian metric on Ω. Furthermore

$$\alpha(\mathsf{Rad}_{\mathsf{V}}) = -n < 0$$

where $\mathsf{Rad}_{\mathsf{V}}$ is the radiant vector field on V.

The proof of Theorem 4.4.1 occupies the rest of this section.

4.4.1. The dual of a sharp convex cone. Suppose that $\Omega \subset \mathsf{V}$ is a sharp convex cone. Define its *dual cone:*

$$\Omega^* \;:=\; \{\psi \in \mathsf{V}^* \mid \psi(x) > 0,\, \forall x \in \overline{\Omega}\}$$

where V^* is the vector space dual to V.

Lemma 4.4.2. Let $\Omega \subset \mathsf{V}$ be a properly convex open cone. Then its dual cone Ω^* is a properly convex open cone.

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Proof. Clearly Ω^* is a convex cone. We show that Ω^* is nonempty, properly convex and open. If Ω^* contains a line, then $\psi_0, \lambda \in \mathsf{V}^*$ exist such that $\lambda \neq 0$ and $\psi_0 + t\lambda \in \Omega^*$ for all $t \in \mathbb{R}$, that is, $\forall x \in \Omega$,

$$\psi_0(x) + t\lambda(x) > 0$$

for each $t \in \mathbb{R}$. Let $x \in \Omega$; then necessarily $\lambda(x) = 0$. Otherwise, if $\lambda(x) \neq 0$, then $t \in \mathbb{R}$ exists with

$$\psi_0(x) + t\lambda(x) \leq 0,$$

a contradiction. Thus Ω^* is properly convex. The openness of Ω^* follows from the proper convexity of Ω. Since Ω is properly convex, its projectivization $\mathsf{P}(\Omega)$ is a properly convex domain; in particular its closure lies in an open ball in an affine subspace A of P and thus the set of hyperplanes in P disjoint from $\mathsf{P}(\Omega)$ is open. It follows that $\mathsf{P}(\Omega^*)$, and hence Ω^*, is open. □

Lemma 4.4.3. The canonical isomorphism $\mathsf{V} \longrightarrow \mathsf{V}^{**}$ maps Ω onto Ω^{**}.

Proof. Identify V^{**} with V. Clearly, $\Omega \subset \Omega^{**}$. Since both Ω and and Ω^{**} are open convex cones, either $\Omega = \Omega^{**}$ or $\partial\Omega \cap \Omega^{**} \neq \emptyset$. Let $y \in \partial\Omega \cap \Omega^{**}$. Let $H \subset \mathsf{V}$ be a supporting hyperplane for Ω at y. Then the covector $\psi \in \mathsf{V}^*$ defining H vanishes at y and is positive on Ω. Thus $\psi \in \Omega^*$. However, $y \in \Omega^{**}$ implies $\psi(y) > 0$, a contradiction. □

4.4.2. Vinberg's characteristic function. Choose a parallel volume form $d\psi$ on V^*. Define the *characteristic function* f of the sharp convex cone Ω by the following integral

$$\Omega \xrightarrow{f} \mathbb{R} \tag{4.3}$$
$$f(x) := \int_{\Omega^*} e^{-\psi(x)} d\psi$$

over the dual cone Ω^*. This function and its derivatives yields a canonical Riemannian geometry on Ω invariant under the automorphism group $\mathsf{Aff}(\Omega)$. Furthermore it produces a canonical diffeomorphism $\Omega \longrightarrow \Omega^*$. (Note that replacing the parallel volume form $d\psi$ by another one $c\, d\psi$ replaces the characteristic function f by its constant multiple $c\, f$. Thus $\Omega \xrightarrow{f} \mathbb{R}$ is well-defined only up to scaling.)

For example in the 1-dimensional case, where

$$\Omega = \mathbb{R}_+ \subset \mathsf{V} = \mathbb{R},$$

the dual cone equals $\Omega^* = \mathbb{R}_+$ and the characteristic function equals

$$\Omega \xrightarrow{f} \mathbb{R}$$
$$ax \longmapsto \int_0^\infty e^{-\psi(x)} d\psi = \frac{1}{x}.$$

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We begin by showing the integral (4.3) converges for $x \in \Omega$. For $x \in \mathsf{V}$ and $t \in \mathbb{R}$ consider the hyperplane cross-section

$$\mathsf{V}_x^*(t) = \{\psi \in \mathsf{V}^* \mid \psi(x) = t\}$$

and let

$$\Omega_x^*(t) = \Omega^* \cap \mathsf{V}_x^*(t).$$

For each $x \in \Omega$ we obtain a decomposition

$$\Omega^* = \bigcup_{t>0} \Omega_x^*(t)$$

and for each $s > 0$ there is a diffeomorphism

$$\begin{aligned} \Omega_x^*(t) &\xrightarrow{h_s} \Omega_x^*(st) \\ \psi &\longmapsto s\psi \end{aligned}$$

and obviously $h_s \circ h_t = h_{st}$. We decompose the volume form $d\psi$ on Ω^* as

$$d\psi = d\psi_t \wedge dt$$

where $d\psi_t$ is an $(n-1)$-form on $\mathsf{V}_x^*(t)$. Now the volume form $(h_s)^* d\psi_{st}$ on $\Omega_x^*(t)$ equals $s^{n-1} d\psi_t$. Thus:

$$\begin{aligned} f(x) &= \int_0^\infty \left(e^{-t} \int_{\Omega_x^*(t)} d\psi_t \right) dt \\ &= \int_0^\infty e^{-t} t^{n-1} \left(\int_{\Omega_x^*(1)} d\psi_1 \right) dt \\ &= (n-1)! \, \mathsf{vol}_{n-1}(\Omega_x^*(1)) < \infty \end{aligned} \tag{4.4}$$

since $\Omega_x^*(1)$ is a bounded subset of $\mathsf{V}_x^*(1)$. Since

$$\mathsf{vol}_{n-1}(\Omega_x^*(n)) = n^{n-1} \mathsf{vol}_{n-1}(\Omega_x^*(1)),$$

the formula in (4.4) implies:

$$f(x) = \frac{n!}{n^n} \mathsf{vol}_{n-1}(\Omega_x^*(n)) \tag{4.5}$$

Now we show that f is analytic.

Lemma 4.4.4. Let

$$\Omega_{\mathbb{C}} \; := \; \Omega + \sqrt{-1}\, \mathsf{V} \subset \; \mathsf{V} \otimes_{\mathbb{R}} \mathbb{C}$$

denote the *tube domain* over Ω. Then the integral defining $f(z)$ converges absolutely for every $z \in \Omega_{\mathbb{C}}$.

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Proof. Let $z = x + iy \in \Omega_{\mathbb{C}}$ so that $x \in \Omega$ and $y \in \mathsf{V}$. Choose some $x_0 \in \Omega$ so that $x \in x_0 + \Omega$. Then

$$|f(x+iy)| \leq \int e^{\psi(x)} d\psi \leq \int e^{\psi(x_0)} d\psi = f(x_0)$$

and the integral for $f(z)$ converges absolutely for any $z \in \Omega_{\mathbb{C}}$. □

Therefore $\Omega \xrightarrow{f} \mathbb{R}$ extends to a holomorphic function $\Omega_{\mathbb{C}} \to \mathbb{C}$, implying f is real analytic.

Lemma 4.4.5. The function $f(x) \longrightarrow +\infty$ as $x \longrightarrow \partial\Omega$.

Proof. Consider a sequence $\{x_n\}_{n>0}$ in Ω converging to $x_\infty \in \partial\Omega$. Then the functions

$$\Omega^* \xrightarrow{F_k} \mathbb{R}$$
$$\psi \mapsto e^{-\psi(x_k)}$$

$(k = 1, 2, \dots)$ are nonnegative functions converging uniformly to F_∞ on every compact subset of Ω^* so that

$$\liminf f(x_k) = \liminf \int_{\Omega^*} F_k(\psi) d\psi \geq \int_{\Omega^*} F_\infty(\psi) d\psi.$$

Suppose that $\psi_0 \in \mathsf{V}^*$ defines a supporting hyperplane to Ω at x_∞; then $\psi_0(x_\infty) = 0$. Let $K \subset \Omega^*$ be a closed ball; then $K + \mathbb{R}_+\psi_0$ is a cylinder in Ω^* with cross-section

$$K_c \ := \ \psi_0 + cK+$$

lying in a hyperplane with nonempty interior.

$$\begin{aligned} \int_{\Omega^*} F_\infty(\psi) d\psi &\geq \int_{K+\mathbb{R}_+\psi_0} e^{-\psi(x_\infty)} d\psi \\ &\geq \int_{K_1} \left(\int_0^\infty dt \right) e^{-\psi(x_\infty)}\, d\psi_1 = \infty \end{aligned}$$

where $d\psi_1$ is a volume form on K_1. □

Lemma 4.4.6. If $\gamma \in \mathsf{Aff}(\Omega) \subset \mathsf{GL}(\mathsf{V})$ is an automorphism of Ω, then

$$f \circ \gamma = \det(\gamma)^{-1} \cdot f \tag{4.6}$$

In other words, if dx is a parallel volume form on V, then $f(x)\, dx$ defines an $\mathsf{Aff}(\Omega)$-invariant volume form on Ω.

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Proof.

$$\begin{aligned} f(\gamma x) &= \int_{\Omega^*} e^{-\psi(\gamma x)}\ d\psi \\ &= \int_{\gamma^{-1}\Omega^*} e^{-\psi(x)}\ \gamma^* d\psi \\ &= \int_{\Omega^*} e^{-\psi(x)} (\det\gamma)^{-1}\ d\psi \\ &= (\det\gamma)^{-1} f(x) \end{aligned}$$

□

4.4.3. The metric tensor. For any function $\Omega \xrightarrow{f} \mathbb{R}$, the logarithmic Hessian

$$d^2 \log f = \nabla d \log f = \nabla\alpha$$

is a symmetric 2-form on Ω and equals:

$$d^2(\log f) = \nabla(f^{-1}df) = f^{-1}d^2 f - (f^{-1}df)^2.$$

Furthermore the value of $d^2 f(x) \in \mathsf{Sym}^2\mathsf{T}^*_x\Omega$ on a pair

$$(X,Y) \in \mathsf{T}_x\Omega \times \mathsf{T}_x\Omega = \mathsf{V}\times\mathsf{V}$$

equals

$$\int_{\Omega^*} \psi(X)\psi(Y)e^{-\psi(x)} d\psi$$

Proposition 4.4.7. $d^2 \log f$ is positive definite and defines an $\mathsf{Aff}(\Omega)$-invariant Riemannian metric on Ω.

Proof. The proof uses the usual L^2 inner product and norm on $(\Omega^*, d\psi)$, which we denote by $\langle,\rangle_2$ and $\|\ \|_2$ respectively. When $X \in \mathsf{T}_x\Omega$ is a nonzero tangent vector, the functions

$$\begin{aligned} \psi &\longmapsto e^{-\psi(x)/2}, \\ \psi &\longmapsto \psi(X)e^{-\psi(x)/2} \end{aligned}$$

on Ω^* are not proportional. Applying the Schwarz inequality,

$$\begin{aligned} f(x)^2\ \left(d^2\log f(x)\right)(X,X)\ &=\ \int_{\Omega^*} e^{-\psi(x)}\ d\psi \int_{\Omega^*} \psi(X)^2 e^{-\psi(x)}\ d\psi \\ &\qquad - \left(\int_{\Omega^*} \psi(X)e^{-\psi(x)}\ d\psi\right)^2 \\ &=\ \|e^{-\psi(x)/2}\ \|_2^2\ \ \|\psi(X)e^{-\psi(x)/2}\ \|_2^2 \\ &\qquad - \langle e^{-\psi(x)/2},\ \psi(X)e^{-\psi(x)/2}\ \rangle_2^2 \\ &> 0 \end{aligned}$$

deduce that$d^2 \log f$ is positive definite as claimed. □

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4.4.4. The covector field. Using the trivialization of the cotangent bundle $\mathsf{T}^*\Omega \cong \Omega \times \mathsf{V}^*$, the Koszul 1-form α of Theorem 4.4.1 arises from an $\mathsf{Aff}(\Omega)$-equivariant mapping $\Omega \xrightarrow{\Phi} \mathsf{V}^*$, which admits the following geometric description.

Theorem 4.4.8. Let $x \in \Omega$. Then

$$\Phi(x) = \mathsf{centroid}\big(\Omega^*_x(n)\big). \tag{4.7}$$

where $\Phi = -d\log f$.

Proof. Since $\det(\gamma)$ is a constant, it follows from (4.6) that $\log f$ transforms under γ by the additive constant $\log\det(\gamma)^{-1}$ and thus

$$\alpha = d\log f = f^{-1}df$$

is an $\mathsf{Aff}(\Omega)$-invariant closed 1-form on Ω. Furthermore, taking γ to be the homothety

$$x \xmapsto{h_s} sx,$$

implies:

$$f \circ h_s = s^{-n} \cdot f,$$

which by differentiation with respect to s yields:

$$\alpha(\mathsf{Rad}_{\mathsf{V}}) = -n.$$

Let $X \in \mathsf{T}_x\Omega \cong \mathsf{V}$ be a tangent vector; then $df(x) \in \mathsf{T}^*_x\Omega$ maps

$$X \mapsto -\int_{\Omega^*} \psi(X) e^{-\psi(x)} d\psi.$$

Using the identification $\mathsf{T}^*_x\Omega \cong \mathsf{V}^*$, the 1-from $-d\log f$ defines a mapping to covectors

$$\Omega \xrightarrow{\Phi} \mathsf{V}^*$$
$$x \mapsto -d\log f(x).$$

As a linear functional, $\Phi(x)$ maps $X \in \mathsf{V}$ to

$$\frac{\int_{\Omega^*} \psi(X) e^{-\psi(x)}\ d\psi}{\int_{\Omega^*} e^{-\psi(x)}\ d\psi}$$

so if $X \in \Omega$, the restriction of $\Phi(x)$ to Ω is positive. Thus $\Omega \xrightarrow{\Phi} \Omega^*$.

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Decomposing the volume form on Ω^* yields:

$$\begin{aligned}\Phi(x) &= \frac{\int_0^\infty e^{-t}t^n \left(\int_{\Omega_x^*(1)} \psi_1\, d\psi_1\right) dt}{\int_0^\infty e^{-t}t^{n-1} \left(\int_{\Omega_x^*(1)} d\psi_1\right) dt} \\ &= n\, \frac{\int_{\Omega_x^*(1)} \psi_1\, d\psi_1\, dt}{\int_{\Omega_x^*(1)} d\psi_1\, dt} \\ &= n\ \mathsf{centroid}(\Omega_x^*(1)).\end{aligned}$$

where the centroid is defined in §1.7.1. Now

$$\begin{aligned}\Phi(x) &= n\ \mathsf{centroid}(\Omega_x^*(1)) \\ &= \mathsf{centroid}(h_n\Omega_x^*(1)) \\ &= \mathsf{centroid}(\Omega_x^*(n))\end{aligned}$$

proving (4.7). □

We characterize the linear functional $\Phi(x) \in \Omega^*$ quite simply as follows. Since $\Phi(x)$ is parallel to $df(x)$, each of its level hyperplanes is parallel to the tangent plane of the level set S_x of $\Omega \xrightarrow{f} \mathbb{R}$ containing x. Note that $\Phi(x)(x) = n$.

Proposition 4.4.9. The tangent space to the level set S_x of $\Omega \xrightarrow{f} \mathbb{R}$ at x equals $\Phi(x)^{-1}(n)$.

This characterization yields the following result:

Theorem 4.4.10. $\Omega \xrightarrow{\Phi} \Omega^*$ is bijective.

Proof. Let $\psi_0 \in \Omega^*$ and let

$$Q_0 := \{z \in \mathsf{V} \mid \psi_0(z) = n\}.$$

Then the restriction of $\log f$ to the affine hyperplane Q_0 is a convex function which approaches $+\infty$ on $\partial(Q_0 \cap \Omega)$. Therefore the restriction $f|_{Q_0\cap\Omega}$ has a unique critical point x_0, which is necessarily a minimum. Then $T_{x_0}S_{x_0} = Q_0$ from which Proposition 4.4.9 implies that $\Phi(x_0) = \psi_0$. Furthermore, if $\Phi(x) = \psi_0$, then $f|_{Q_0\cap\Omega}$ has a critical point at x so $x = x_0$. Therefore $\Omega \xrightarrow{\Phi} \Omega^*$ is bijective as claimed. □

If $\Omega \subset \mathsf{V}$ is a properly convex cone and Ω^* is its dual, then let $\Phi_{\Omega^*} : \Omega^* \longrightarrow \Omega$ be the diffeomorphism $\Omega^* \longrightarrow \Omega^{**} = \Omega$ defined above. If $x \in \Omega$, then $\psi = (\Phi^*)^{-1}(x)$ is the unique $\psi \in \mathsf{V}^*$ such that:

- $\psi(x) = n$;
- The centroid of $\Omega \cap \psi^{-1}(n)$ equals x.

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The duality isomorphism $\mathsf{GL}(\mathsf{V}) \longrightarrow \mathsf{GL}(\mathsf{V}^*)$ (given by inverse transpose of matrices) defines an isomorphism $\mathsf{Aff}(\Omega) \longrightarrow \mathsf{Aff}(\Omega^*)$. Let

$$\Omega \xrightarrow{\Phi_\Omega} \Omega^*,$$

$$\Omega^* \xrightarrow{\Phi_{\Omega^*}} \Omega^{**} = \Omega$$

be the duality maps for Ω and Ω^* respectively. Vinberg points out in [**340**] that, in general, the composition

$$\Omega \xrightarrow{\Phi_{\Omega^*} \circ \Phi_\Omega} \Omega$$

is not the identity. However, if Ω is *homogeneous* (that is, $\mathsf{Aff}(\Omega) \subset \mathsf{GL}(\mathsf{V})$ acts transitively on Ω), then $\Phi_{\Omega^*} \circ \Phi_\Omega = \mathbb{I}$:

4.4.5. Homogeneous convex cones.

Proposition 4.4.11 (Vinberg [**340**])**.** Let $\Omega \subset \mathsf{V}$ be a homogeneous properly convex cone. Then Φ_{Ω^*} and Φ_Ω are inverse maps $\Omega^* \longleftrightarrow \Omega$.

Proof. Let $x \in \Omega$ and $Y \in \mathsf{V} \cong \mathsf{T}_x\Omega$ be a tangent vector. Denote the value of the canonical Riemannian metric $\nabla\alpha = d^2 \log f$ at x by:

$$\mathsf{T}_x\Omega \times \mathsf{T}_x\Omega \xrightarrow{g_x} \mathbb{R}$$

Then the differential of $\Omega \xrightarrow{\Phi_\Omega} \Omega^*$ at x equals the composition

$$\mathsf{T}_x\Omega \xrightarrow{\widetilde{g}} \mathsf{T}_x^*\Omega \cong \mathsf{V}^* \cong \mathsf{T}_{\Phi(x)}\Omega^*$$

where $\mathsf{T}_x\Omega \xrightarrow{\widetilde{g}_x} \mathsf{T}_x^*\Omega$ is the linear isomorphism corresponding to g_x and the second isomorphism

$$\mathsf{T}_x^*\Omega \cong \mathsf{V}^* \cong \mathsf{T}_{\Phi(x)}\Omega^*$$

arises from parallel translation. Taking directional derivative of the equation

$$\alpha_x(\mathsf{Rad}_x) = -n$$

with respect to $Y \in \mathsf{V} \cong \mathsf{T}_x\Omega$ yields:

$$\begin{aligned} 0 =& (\nabla_Y\alpha)(\mathsf{Rad}) + \alpha(\nabla_Y\mathsf{Rad}) \\ =& \, g_x(\mathsf{Rad}_x, Y) + \alpha_x(Y) \\ =& \, g_x(x, Y) - \Phi(x)(Y). \end{aligned} \tag{4.8}$$

Let $\Omega \xrightarrow{f_\Omega} \mathbb{R}$ and $\Omega^* \xrightarrow{f_{\Omega^*}} \mathbb{R}$ be the characteristic functions for Ω and Ω^* respectively. Then $(f_\Omega)(x)\, dx$ is a volume form on Ω invariant under $\mathsf{Aff}(\Omega)$ and $(f_{\Omega^*})(\psi)\, d\psi$ is a volume form on Ω^* invariant under the induced action of $\mathsf{Aff}(\Omega)$ on Ω^*.

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Moreover $\Omega \xrightarrow{\Phi} \Omega^*$ is equivariant with respect to the isomorphism $\mathsf{Aff}(\Omega) \longrightarrow \mathsf{Aff}(\Omega^*)$. Therefore the tensor field on Ω defined by

$$\begin{aligned} f_\Omega(x)\,dx \ \otimes\ (f_{\Omega^*} \circ \Phi)\,(x)\,d\psi \\ \in \wedge^n \mathsf{T}_x\Omega\omega \otimes \wedge^n T_{\Phi(x)}\Omega^* \\ \cong \wedge^n \mathsf{V} \otimes \wedge^n \mathsf{V}^* \end{aligned}$$

is $\mathsf{Aff}(\Omega)$-invariant. Since the parallel tensor field $dx \otimes d\psi \in \wedge^n\mathsf{V} \otimes \wedge^n \mathsf{V}^*$ is invariant under all of $\mathsf{Aff}(\mathsf{V})$, the coefficient

$$h(x) = f_\Omega(x)\big(f_{\Omega^*} \circ \Phi\big)(x) \tag{4.9}$$

is an $\mathsf{Aff}(\Omega)$-invariant function on Ω. Since Ω is homogeneous, h is constant.

Differentiating $\log h$ using (4.9),

$$0 = d\log h = d\log f_\Omega(x) + d\log(f_{\Omega^*} \circ \Phi)(x).$$

Since $d\log f_{\Omega^*}(\psi) = \Phi_{\Omega^*}(\psi)$,

$$\begin{aligned} 0 &= -\Phi(x)(Y) + \Phi_{\Omega^*}(d\Phi(Y)) \\ &= -\Phi(x)(Y) + g_x(Y, \Phi_{\Omega^*} \circ \Phi_\Omega(x)) \end{aligned}$$

Combining this equation with (4.8) yields:

$$\Phi_{\Omega^*} \circ \Phi_\Omega(x) = x$$

as desired. □

Thus, if Ω is a homogeneous cone, then $\Phi(x) \in \Omega^*$ is the centroid of the cross-section $\Omega^*_x(n) \subset \Omega^*$ in V^*.

4.5. Benzécri's compactness theorem

Let $\mathsf{P} = \mathsf{P}(\mathsf{V})$ and $\mathsf{P}^* = \mathsf{P}(\mathsf{V}^*)$ be the associated projective spaces. Then the projectivization $\mathsf{P}(\Omega) \subset \mathsf{P}$ of Ω is by definition a *properly convex domain* and its closure $K = \overline{\mathsf{P}(\Omega)}$ a *convex body*. Then the *dual convex body* K^* equals the closure of the projectivization $\mathsf{P}(\Omega^*)$ consisting of all hyperplanes $H \subset \mathsf{P}$ such that $\bar\Omega \cap H = \emptyset$. A *pointed convex body* consists of a pair (K, x) where K is a convex body and x is an interior point of K. Let $H \subset \mathsf{P}$ be a hyperplane and $\mathsf{A} = \mathsf{P} \setminus H$ its complementary affine space. We say that the pointed convex body (K, u) is *centered relative to* A (or H) if u is the centroid of K in the affine geometry of A.

Proposition 4.5.1. Let (K, u) be a pointed convex body in a projective space P. Then there exists a hyperplane $H \subset \mathsf{P}$ disjoint from K such that in the affine space $\mathsf{A} = \mathsf{P} \setminus H$, the centroid of $K \subset E$ equals u.

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Proof. Let $\mathsf{V} = \mathsf{V}(\mathsf{P})$ be the vector space corresponding to the projective space P and let $\Omega \subset \mathsf{V}$ be a sharp convex cone whose projectivization is the interior of K. Let $x \in \Omega$ be a point corresponding to $u \in \mathsf{int}(K)$. Let

$$\Omega^* \xrightarrow{\Phi_{\Omega^*}} \Omega$$

be the duality map for Ω^* and let $\psi = (\Phi_{\Omega^*})^{-1}(y)$. Then the centroid of the cross-section

$$\Omega_\psi(n) = \{x \in \Omega \mid \psi(x) = n\}$$

in the affine hyperplane $\psi^{-1}(n) \subset \mathsf{V}$ equals y. Let $H = \mathsf{P}(\mathsf{Ker}(\psi))$ be the projective hyperplane in P corresponding to ψ; then projectivization defines an affine isomorphism

$$\psi^{-1}(n) \longrightarrow \mathsf{P} \setminus H$$

mapping $\Omega_\psi(n) \longrightarrow K$. Since affine maps preserve centroids, it follows that (K, u) is centered relative to H. □

Thus every pointed convex body (K, u) is centered relative to a unique affine space containing K.

Here is the 1-dimensional case:

Exercise 4.5.2. Let $K \subset \mathbb{R}\mathsf{P}^1$ be a closed interval $[a, b] \subset \mathbb{R}$ and $a < x < b$ an interior point. Then x is the midpoint of $[a, b]$ relative to the "hyperplane" H obtained by projectively reflecting x with respect to the pair $\{a, b\}$:

$$H = R_{[a,b]}(x) = \frac{(a+b)x - 2ab}{2x - (a+b)}$$

An equivalent version of Proposition 4.5.1 involves using collineations to "move a pointed convex body" into affine space to center it:

Proposition 4.5.3. Let $K \subset \mathsf{A}$ be a convex body in an affine space and let $x \in \mathsf{int}(\mathsf{A})$ be an interior point. Let $\mathsf{P} \supset \mathsf{A}$ be the projective space containing A. Then there exists a collineation $\mathsf{P} \xrightarrow{g} \mathsf{P}$ such that:

- $g(K) \subset \mathsf{A}$;
- $(g(K), g(x))$ is centered relative to A.

The 1-dimensional version is really just the fundamental theorem of projective geometry: if $[a, b]$ is a closed interval with interior point x, then the unique collineation mapping

$$a \mapsto -1$$
$$x \mapsto 0$$
$$b \mapsto 1$$

centers $[a, b]$ at x.

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Proposition 4.5.4. Let $K_i \subset \mathsf{A}$ be convex bodies ($i = 1, 2$) in an affine space A with respective centroids u_i. Suppose that $\mathsf{P} \xrightarrow{g} \mathsf{P}$ is a collineation such that $g(K_1) = K_2$ and $g(u_1) = u_2$. Then g is an affine automorphism of A, that is, $g(\mathsf{A}) = \mathsf{A}$.

Proof. Let V be a vector space containing A as an affine hyperplane and let Ω_i be the sharp convex cones in V whose projectivizations are $\mathsf{int}(K_i)$. By assumption there exists a linear map $\mathsf{V} \xrightarrow{\widetilde{\mathsf{g}}} \mathsf{V}$ and points $x_i \in \Omega_i$ mapping to $u_i \in K_i$ such that $\widetilde{\mathsf{g}}(\Omega_1) = \Omega_2$ and $\widetilde{\mathsf{g}}(x_1) = x_2$. Let $S_i \subset \Omega_i$ be the level set of the Vinberg characteristic function

$$\Omega_i \xrightarrow{f_i} \mathbb{R}$$

containing x_i. Since (K_i, u_i) is centered relative to A, it follows that the tangent plane

$$\mathsf{T}_{x_i} S_i = \mathsf{A} \subset \mathsf{V}.$$

Since the construction of the characteristic function is linearly invariant, it follows that $\widetilde{\mathsf{g}}(S_1) = S_2$. Moreover

$$\widetilde{\mathsf{g}}(\mathsf{T}_{x_1} S_1) = \mathsf{T}_{x_2} S_2,$$

that is, $\widetilde{\mathsf{g}}(\mathsf{A}) = \mathsf{A}$ and $g \in \mathsf{Aff}(\mathsf{A})$ as desired. □

4.5.1. Convex bodies in projective space. Let $\mathfrak{C}(\mathsf{P})$ denote the set of all convex bodies in P, with the topology induced from the Hausdorff metric on the set of all closed subsets of P (which itself is induced from the Fubini–Study metric on P; see §3.2.2). Let

$$\mathfrak{C}_*(\mathsf{P}) = \{(K, x) \in \mathfrak{C}(\mathsf{P}) \times \mathsf{P} \mid x \in \mathsf{int}(K)\}$$

be the corresponding set of pointed convex bodies, with a topology induced from the product topology on $\mathfrak{C}(\mathsf{P}) \times \mathsf{P}$. The collineation group G acts continuously on $\mathfrak{C}(\mathsf{P})$ and on $\mathfrak{C}_*(\mathsf{P})$.

Recall that an action of a group Γ on a space X is *syndetic* if $\exists K \subset\subset X$ such that $\Gamma K = X$ (Gottschalk–Hedlund [**171**]) Furthermore the action is *proper* if the corresponding map

$$\begin{aligned} \Gamma \times X &\longrightarrow X \times X \\ (\gamma, x) &\longmapsto (\gamma x, x) \end{aligned}$$

is a proper map (inverse images of compact subsets are compact). See §A.2 for discussion of elementary properties of group actions.

Theorem 4.5.5 (Benzécri)**.** The collineation group G acts properly and syndetically on $\mathfrak{C}_*(\mathsf{P})$. In particular the quotient $\mathfrak{C}_*(\mathsf{P})/G$ is a compact Hausdorff space.

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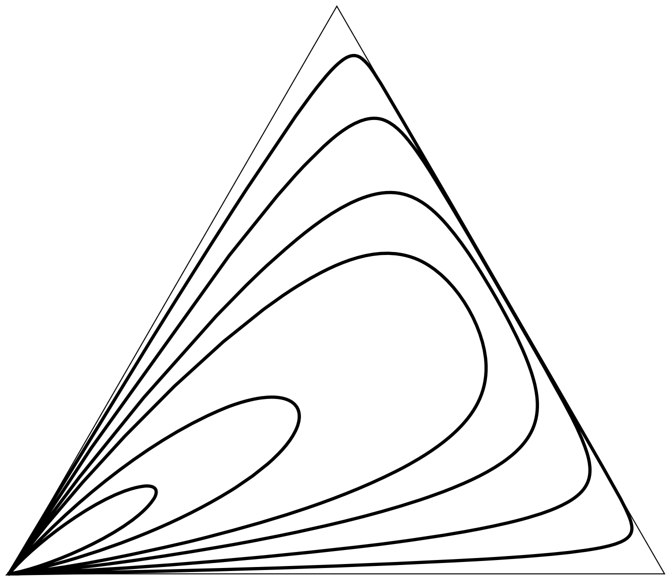

Figure 4.4. A sequence of projectively equivalent convex domains with a corner converging to a triangle with the corner as a vertex.

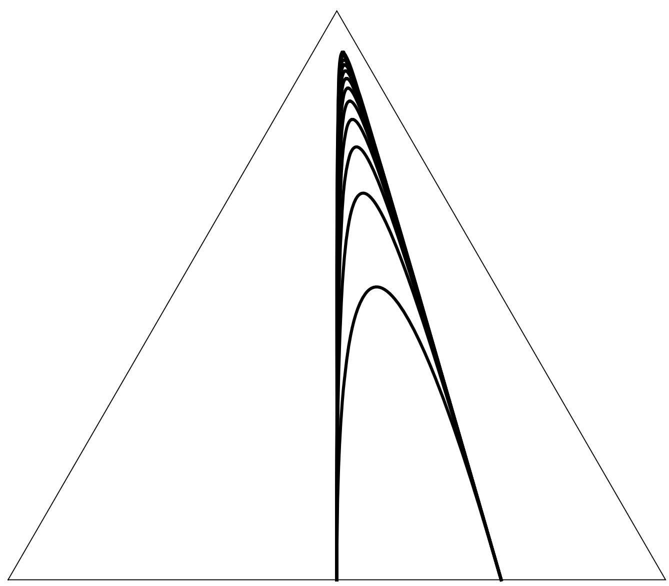

Figure 4.5. A sequence of projectively equivalent convex domains with a flat part of the boundary converging to a triangle having the flat part as a side.

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4.5.2. How convex bodies can degenerate. While the quotient $\mathfrak{C}_*(\mathsf{P})/G$ is Hausdorff, the space of equivalence classes of convex bodies $\mathfrak{C}(\mathsf{P})/G$ is generally *not* Hausdorff. Here are three basic examples.

4.5.2.1. *Corners.* Suppose that Ω is a properly convex planar domain whose boundary is not C^1 at a point x_1. Then $\partial\Omega$ has a "corner" at x_1 and we may choose homogeneous coordinates so that $x_1 = [1 : 0 : 0]$ and $\bar{\Omega}$ lies in the domain

$$\triangle = \{[x : y : z] \in \mathbb{R}\mathsf{P}^2 \mid x, y, z > 0\}$$

in such a way that $\partial\Omega$ is tangent to $\partial\triangle$ at x_1. Under the one-parameter group of collineations defined by

$$g_t = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{bmatrix}$$

as $t \longrightarrow +\infty$, the domains $g_t\Omega$ converge to $\triangle$. (Compare Figure 4.4.) Then the G-orbit of $\bar{\Omega}$ in $\mathfrak{C}(\mathsf{P})$ is not closed. The corresponding equivalence class of $\bar{\Omega}$ is not a closed point in $\mathfrak{C}(\mathsf{P})/G$ unless Ω was already a triangle.

4.5.2.2. *Flats.* Similarly suppose that Ω is a properly convex planar domain which is not *strictly convex,* that is, its boundary contains a nontrivial line segment σ. (We suppose that σ is a maximal line segment contained in $\partial\Omega$.) As above, we may choose homogeneous coordinates so that $\Omega \subset \triangle$ and such that $\bar{\Omega} \cap \bar{\triangle} = \bar{\sigma}$ and σ lies on the line $\{[x : y : 0] \mid x, y \in \mathbb{R}\}$. As $t \longrightarrow +\infty$ the image of Ω under the collineation

$$g_t = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

converges to a triangle region with vertices $\{[0 : 0 : 1]\} \cup \partial\sigma$. As above, the equivalence class of $\bar{\Omega}$ in $\mathfrak{C}(\mathsf{P})/G$ is not a closed point in $\mathfrak{C}(\mathsf{P})/G$ unless Ω is a triangle.

4.5.2.3. *Osculating conics.* As a final example, consider a properly convex planar domain Ω with C^1 boundary such that there exists a point $u \in \partial\Omega$ such that $\partial\Omega$ is C^2 at u, but not "flat", that is, $\partial\Omega$ contains no open line segment containing u. In that case a conic C *osculates* $\partial\Omega$ at u, that is, C agrees with $\partial\Omega$ to second order at u. Choose homogeneous coordinates such that $u = [1 : 0 : 0]$ and

$$C = \{[x : y : z] \mid xy + z^2 = 0\}.$$

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Then as $t \longrightarrow +\infty$ the image of Ω under the collineation

$$g_t = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

converges to the convex region

$$\{[x : y : z] \mid xy + z^2 < 0\}$$

bounded by C. As above, the equivalence class of $\bar{\Omega}$ in $\mathfrak{C}(\mathsf{P})/G$ is not a closed point in $\mathfrak{C}(\mathsf{P})/G$ unless $\partial\Omega$ is a conic.

In summary:

Proposition 4.5.6. Suppose $\bar{\Omega} \subset \mathbb{R}\mathsf{P}^2$ is a convex body whose equivalence class $[\bar{\Omega}]$ is a closed point in $\mathfrak{C}(\mathsf{P})/G$. Suppose that $\partial\Omega$ is neither a triangle nor a conic. Then $\partial\Omega$ is a C^1 strictly convex curve which is nowhere C^2.

The forgetful map $\mathfrak{C}_*(\mathsf{P}) \xrightarrow{\Pi} \mathfrak{C}(\mathsf{P})$ which forgets the point of a pointed convex body is the composition of the inclusion $\mathfrak{C}_*(\mathsf{P}) \hookrightarrow \mathfrak{C}(\mathsf{P}) \times \mathsf{P}$ with Cartesian projection $\mathfrak{C}(\mathsf{P}) \times \mathsf{P} \longrightarrow \mathfrak{C}(\mathsf{P})$.

Theorem 4.5.7 (Benzécri)**.** Let $\Omega \subset \mathsf{P}$ is a properly convex domain such that there exists a subgroup $\Gamma \subset \mathsf{Aut}(\Omega)$ which acts syndetically on Ω. Then the corresponding point $[\bar{\Omega}] \in \mathfrak{C}(\mathsf{P})/G$ is closed.

All but the continuous differentiability of the boundary in the following result was originally proved in Kuiper [**235**] using a somewhat different technique; the C^1 statement is due to Benzécri [**46**] as well as the proof given here.

Corollary 4.5.8. Suppose that $M = \Omega/\Gamma$ is a convex $\mathbb{R}\mathsf{P}^2$-manifold such that $\chi(M) < 0$. Then either the $\mathbb{R}\mathsf{P}^2$-structure on M is a hyperbolic structure or the boundary $\partial\Omega$ of its universal covering is a C^1 strictly convex curve which is nowhere C^2.

Proof. Apply Proposition 4.5.6 to Theorem 4.5.7. ☐

Proof of Theorem 4.5.7 assuming Theorem 4.5.5. Let Ω be a properly convex domain with an automorphism group $\Gamma \subset \mathsf{Aff}(\Omega)$ acting syndetically on Ω. It suffices to show that the G-orbit of $\{\bar{\Omega}\}$ in $\mathfrak{C}(\mathsf{P})$ is closed, which is equivalent to showing that the G-orbit of

$$\Pi^{-1}(\{\bar{\Omega}\}) = \{\bar{\Omega}\} \times \Omega$$

in $\mathfrak{C}_*(\mathsf{P})$ is closed. In turn, this is equivalent to showing that the image of

$$\{\bar{\Omega}\} \times \Omega \subset \mathfrak{C}_*(\mathsf{P})$$

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under the quotient map $\mathfrak{C}_*(\mathsf{P}) \longrightarrow \mathfrak{C}_*(\mathsf{P})/G$ is closed. Let $K \subset \Omega$ be a compact subset such that $\Gamma K = \Omega$; then $\{\bar\Omega\} \times K$ and $\{\bar\Omega\} \times \Omega$ have the same image in $\mathfrak{C}_*(\mathsf{P})/\Gamma$ and hence in $\mathfrak{C}_*(\mathsf{P})/G$. Hence it suffices to show that the image of $\{\bar\Omega\} \times K$ in $\mathfrak{C}_*(\mathsf{P})/G$ is closed. Since K is compact and the composition

$$K \longrightarrow \{\bar\Omega\} \times K \hookrightarrow \{\bar\Omega\} \times \Omega \subset \mathfrak{C}_*(\mathsf{P}) \longrightarrow \mathfrak{C}_*(\mathsf{P})/G$$

is continuous, it follows that the image of K in $\mathfrak{C}_*(\mathsf{P})/G$ is compact. By Theorem 4.5.5 , $\mathfrak{C}_*(\mathsf{P})/G$ is Hausdorff and hence the image of K in $\mathfrak{C}_*(\mathsf{P})/G$ is closed, as desired. The proof of Theorem 4.5.7 (assuming Theorem 4.5.5) is now complete. □

Now we prove Theorem 4.5.5. Choose a fixed hyperplane $H_\infty \subset \mathsf{P}$ and let $\mathsf{A} = \mathsf{P} \setminus H_\infty$ be the corresponding affine patch and $\mathsf{Aff}(\mathsf{A})$ the group of affine automorphisms of A. Let $\mathfrak{C}(\mathsf{A}) \subset \mathfrak{C}(\mathsf{P})$ denote the set of convex bodies $K \subset \mathsf{E}$, with the induced topology. (Note that the $\mathfrak{C}(\mathsf{A})$ is a complete metric space with respect to the Hausdorff metric induced from the Euclidean metric on E and we may use this metric to define the topology on $\mathfrak{C}(\mathsf{A})$. The inclusion map $\mathfrak{C}(\mathsf{A}) \hookrightarrow \mathfrak{C}(\mathsf{P})$ is continuous, although not uniformly continuous.

4.5.3. Reduction to the affine case.

Theorem 4.5.9. Let $\mathsf{A} \subset \mathsf{P}$ be an affine patch in projective space. Then the map

$$\begin{aligned} \mathfrak{C}(\mathsf{A}) &\xrightarrow{\iota} \mathfrak{C}_*(\mathsf{P}) \\ K &\longmapsto \big(K, \mathsf{centroid}(K)\big) \end{aligned}$$

is equivariant with respect to the inclusion $\mathsf{Aff}(\mathsf{A}) \longrightarrow G$ and the corresponding homomorphism of topological transformation groupoids[5]

$$\big(\mathfrak{C}(\mathsf{A}), \mathsf{Aff}(\mathsf{A})\big) \xrightarrow{\iota} (\mathfrak{C}_*(\mathsf{P}), G)$$

is an equivalence of groupoids.

Proof. The surjectivity of $\mathfrak{C}(\mathsf{A})/\mathsf{Aff}(\mathsf{A}) \xrightarrow{\iota_*} \mathfrak{C}_*(\mathsf{P})/G$ follows immediately from Proposition 4.4.11 and the bijectivity of

$$\mathsf{Hom}(a, b) \xrightarrow{\iota_*} \mathsf{Hom}(\iota(a), \iota(b))$$

follows immediately from Proposition 4.5.3. □

Thus the proof of Proposition 4.5.4 reduces (via Lemma A.3.1 and Theorem 4.5.9) to the following:

Theorem 4.5.10. $\mathsf{Aff}(\mathsf{A})$ acts properly and syndetically on $\mathfrak{C}(\mathsf{A})$.

[5] §A.3 discusses the terminology of transformation groupoids.

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Let $\mathcal{E} \subset \mathfrak{C}(\mathsf{A})$ denote the subspace of ellipsoids in A; the affine group $\mathsf{Aff}(\mathsf{A})$ acts transitively on $\mathcal{E}$ with isotropy group the orthogonal group — in particular this action is proper. Suppose $K \in \mathfrak{C}(\mathsf{A})$ is a convex body. Then an ellipsoid $\mathsf{ell}(K) \in \mathcal{E}$ exists, *the ellipsoid of inertia*[6] *of* K, such that for each affine map $\psi : \mathsf{A} \longrightarrow \mathbb{R}$ such that $\psi(\mathsf{centroid}(K)) = 0$ the moments of inertia satisfy:

$$\int_K \psi^2 dx = \int_{\mathbf{ell}(K)} \psi^2 dx$$

Proposition 4.5.11. Taking the ellipsoid-of-inertia of a convex body

$$\mathfrak{C}(\mathsf{A}) \xrightarrow{\mathsf{ell}} \mathcal{E}$$

defines an $\mathsf{Aff}(\mathsf{A})$-equivariant proper retraction of $\mathfrak{C}(\mathsf{A})$ onto $\mathcal{E}$.

Proof of Theorem 4.5.10 assuming Proposition 4.5.11.
Since $\mathsf{Aff}(\mathsf{A})$ acts properly and syndetically on $\mathcal{E}$ and ell is a proper map, Therefore $\mathsf{Aff}(\mathsf{A})$ acts properly and syndetically on $\mathfrak{C}(\mathsf{A})$. □

Proof of Proposition 4.5.11. The map ell is $\mathsf{Aff}(\mathsf{A})$-invariant and continuous. Since $\mathsf{Aff}(\mathsf{A})$ acts transitively on $\mathcal{E}$, it suffices to show that a single fiber $\mathsf{ell}^{-1}(E)$ is compact for $E \in \mathcal{E}$. Assume that E is the unit sphere centered at the origin 0. Since the collection of compact subsets of A which lie between two compact balls is a compact subset of $\mathfrak{C}(\mathsf{A})$, Proposition 4.5.11 follows from: □

Proposition 4.5.12. For each n there exist constants $0 < r(n) < R(n)$ such that every convex body $K \subset \mathbb{R}^n$ whose centroid is the origin and whose ellipsoid-of-inertia is the unit sphere satisfies

$$B_{r(n)}(O) \subset K \subset B_{R(n)}(O).$$

Cooper–Long–Tillmann [**100**] call such an affine chart a *Benzécri chart,* and in [**101**] provide an algorithm for computing a Benzécri chart. The proof of Proposition 4.5.12 is based on:

Lemma 4.5.13. Let $K \subset \mathsf{A}$ be a convex body with centroid O. Suppose that l is a line through O which intersects ∂K in the points X, X'. Then

$$\frac{1}{n} \leq \frac{\mathsf{d}(O, X)}{\mathsf{d}(O, X')} \leq n. \tag{4.10}$$

Proof. Choose an affine map $\mathsf{A} \xrightarrow{\psi} \mathbb{R}$ such that $\psi(X) = 0$ and $\psi^{-1}(1)$ is a supporting hyperplane for K at X'; then $\psi(x) \leq 1$ for all $x \in K$.

[6] Arnold [**9**], §6 Goldstein [**170**], p.155 discuss of moments of inertia and the ellipsoid of inertia.

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We claim that:

$$\psi(O) \leq \frac{n}{n+1}. \tag{4.11}$$

For $t \in \mathbb{R}$ let

$$\begin{aligned} \mathsf{A} &\xrightarrow{h_t} \mathsf{A} \\ x &\longmapsto t(x-X)+X \end{aligned}$$

be the homothety fixing X having strength t. We compare the linear functional ψ with the "polar coordinates centered at X" on K defined by the map

$$\begin{aligned} [0,1]\times\partial K &\xrightarrow{F} K \\ (t,\mathbf{s}) &\mapsto h_t\mathbf{s} \end{aligned}$$

which is bijective on $(0,1]\times\partial K$ and collapses $\{0\}\times\partial K$ onto X. Thus a well-defined function $K \xrightarrow{\mathbf{t}} [0,1]$ exists, such that $\forall x \in K$,

$$x \;=\; F(\mathbf{t}(x),\mathbf{s})$$

for some $\mathbf{s}\in\partial K$. Since $\psi(\mathbf{s}) \leq 1$, applying ψ to

$$x = h_{\mathbf{t}(x)}(\mathbf{s}) = \mathbf{t}(x)(\mathbf{s}-X)+X,$$

implies

$$\psi(x) \;=\; \mathbf{t}(x)\psi(\mathbf{s}) \;\leq\; \mathbf{t}(x). \tag{4.12}$$

Let $\mu = \mu_K$ denote the probability measure supported on K defined by

$$\mu(S) = \frac{\int_{S\cap K} dx}{\int_K dx}.$$

There exists a measure ν on ∂K such that for each measurable function $f : \mathsf{A} \longrightarrow \mathbb{R}$

$$\int f(x)d\mu(x) = \int_{t=0}^{1}\int_{\mathbf{s}\in\partial K} f(t\mathbf{s})t^{n-1}d\nu(\mathbf{s})dt,$$

that is, $F^*d\mu = t^{n-1}d\nu\wedge dt$.

The first moment of $K \xrightarrow{\mathbf{t}} [0,1]$ is:

$$\bar{\mathbf{t}}(K) = \int_K \mathbf{t}\,d\mu = \frac{\int_K \mathbf{t}\,d\mu}{\int_K d\mu} = \frac{\int_0^1 t^n \int_{\partial K} d\nu\,dt}{\int_0^1 t^{n-1}\int_{\partial K} d\nu\,dt} = \frac{n}{n+1}.$$

Since the value of the affine function ψ on the centroid equals the first moment of ψ on K,

$$0 < \psi(O) = \int_K \psi\,d\mu \leq \int_K \mathbf{t}\,d\mu = \frac{n}{n+1},$$

by (4.12). This proves (4.11).

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Now the distance function on the line $\overleftrightarrow{XX'}$ is affinely related to the linear functional ψ, that is, there exists a constant $c > 0$ such that for $x \in \overleftrightarrow{XX'}$ the distance $\mathsf{d}(X, x) = c|\psi(x)|$; since $\psi(X') = 1$ it follows that

$$\psi(x) = \frac{\mathsf{d}(X, x)}{\mathsf{d}(X, X')}$$

and since $\mathsf{d}(O, X) + \mathsf{d}(O, X') = \mathsf{d}(X, X')$ it follows that

$$\frac{\mathsf{d}(O, X')}{\mathsf{d}(O, X)} = \frac{\mathsf{d}(X, X')}{\mathsf{d}(O, X)} - 1 \geq \frac{n+1}{n} - 1 = \frac{1}{n}.$$

This gives the second inequality of (4.10). The first inequality follows by reversing the roles of X, X'. □

Proof of Proposition 4.5.12. Let $X \in \partial K$ be a point at minimum distance from the centroid O; then there exists a supporting hyperplane H at x which is orthogonal to $\overleftrightarrow{OX}$ and let $\mathsf{A} \xrightarrow{\psi} \mathbb{R}$ be the corresponding linear functional of unit norm. Let $a = \psi(X) > 0$ and $b = \psi(X') < 0$; Proposition 4.5.11 implies $-b \leq na$.

We claim that $|\psi(x)| \leq na$ for all $x \in K$. To this end let $x \in K$; we may assume that $\psi(x) > 0$ since $-na \leq \psi(X')$.

Furthermore we may assume that $x \in \partial K$. Let $z \in \partial K$ be the other point of intersection of $\overleftrightarrow{Ox}$ with ∂K; then $\psi(z) < 0$. Applying Lemma 4.5.13,

$$\frac{1}{n} \leq \frac{\mathsf{d}(O, z)}{\mathsf{d}(O, x)},$$

which implies that

$$\frac{1}{n} \leq \frac{|\psi(z)|}{|\psi(x)|}$$

(since the linear functional ψ is affinely related to signed distance along $\overleftrightarrow{Ox}$). Since $0 > \psi(z) \geq -a$, it follows that $|\psi(x)| \leq na$ as claimed.

As $\|\psi\| = 1$, the moment of inertia w_n of ψ on the unit sphere is independent of ψ. Since

$$w_n \;=\; \int_K \psi^2 d\mu \;\leq\; \int_K n^2 a^2 d\mu \;=\; n^2 a^2,$$

$a \geq \sqrt{w_n}/n$. Taking $r(n) = \sqrt{w_n}/n$ implies that K contains the $r(n)$-ball centered at O.

To obtain the upper bound, observe that if C is a right circular cone with vertex X, altitude h and base a sphere of radius ρ and $C \xrightarrow{t} [0, h]$ is the altitudinal distance from the base, then the integral

$$\int_C t^2 d\mu \;=\; 2h^3 \rho^{n-1} v_{n-1}$$

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where v_{n-1} denotes the $(n-1)$-dimensional volume of the unit $(n-1)$-ball. Let $X \in \partial K$ and C be a right circular cone with vertex X and base an $(n-1)$-dimensional ball of radius $r(n)$. We have just seen that K contains $B_{r(n)}(O)$; it follows that $K \supset C$. Let $K \xrightarrow{t} \mathbb{R}$ be the unit-length linear functional vanishing on the base of C; then

$$t(X) = h = \mathsf{d}(O, X).$$

Its second moment is

$$w_n = \int_K t^2 d\mu \geq \int_C t^2 d\mu = \frac{2h^3 r(n)^{n-1} v_{n-1}}{(n+2)(n+1)n}$$

and thus it follows that

$$OX = h \leq R(n)$$

where

$$R(n) = \left(\frac{(n+2)(n+1)nw_n}{2r(n)^{n-1}v_{n-1}} \right)^{\frac{1}{3}}$$

as desired. The proof is now complete. ☐

Exercise 4.5.14. The volume of the unit ball in $\mathbb{R}^n$ equals:

$$v_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)} = \begin{cases} \pi^{n/2}/(n/2)! & \text{for } n \text{ even} \\ 2^{(n+1)/2}\pi^{(n-1)/2}/(1 \cdot 3 \cdot 5 \cdots n) & \text{for } n \text{ odd} \end{cases}$$

Its moments of inertia are:

$$w_n = \begin{cases} v_n/(n+2) & \text{for } n \text{ even} \\ 2v_n/(n+2) & \text{for } n \text{ odd} \end{cases}$$

Based on Cooper–Long–Tillmann [**100**], explicit formulas are given in Casella–Tate–Tillmann[**82**] for the size of a *Benzécri chart.*

4.6. Quasi-homogeneous and divisible domains

Corollary 4.5.8 of Benzécri's Theorem 4.5.7 provides sharp information on the geometry of a convex domain Ω on which $\mathsf{Aut}(\Omega)$ acts syndetically. A convex domain $\Omega \subset \mathsf{P}$ is said to be *quasi-homogeneous* if $\mathsf{Aut}(\Omega$ acts syndetically, that is, the quotient $\Omega/\mathsf{Aut}(\Omega)$ is compact (although not necessarily Hausdorff). If the action is *proper,* then the domain is said to be *divisible.* Although for simplicity we have only discussed the 2-dimensional situation, much is known in this case, especially through a series of papers of Benoist, starting with his paper *Automorphismes des cônes convexes* [**38**], and continuing into his series of four *Convex divisibles* papers [**40–43**].[7]

[7]For an excellent treatment (in English) of these results, background and further advances (at the time), see Benoist [**44**].

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Benoist begins by characterizing automorphism groups of convex cones by the dynamical notion of *proximality.* A projective action of Γ on P is *proximal* if $\exists\gamma \in \Gamma$ with a unique attracting fixed point. From this he finds restrictions on which Zariski closures $\mathbb{A}(\Gamma)$ can occur.

A crucial idea is that when Ω is *divisible,* the natural geodesic flow defined by the Hilbert metric is an *Anosov flow,* that is, it preserves a decomposition of the tangent bundle as a direct sum of (1) the line bundle generating the flow; (2) a subbundle of tangent vectors exponentially expanded by the flow (with respect to a fixed Riemannian structure); (3) a subbundle of tangent bundle exponentially contracted by the flow. From this Benoist deduces that Γ is a hyperbolic group in the sense of Gromov. Furthermore $\partial\Omega$ is $C^{1+\alpha}$ for some $\alpha > 0$.

When Ω is *not* strictly convex (such as the triangle) much of this breaks down. However, as Benzécri noted, Theorem 4.5.7 implies that if Ω is not strictly convex, then Ω contains a properly embedded triangle. A simple example[8] is a deformation of a Coxeter group Γ_0 built on a regular ideal tetrahedron in $\mathsf{H}^3 \subset \mathbb{R}\mathsf{P}^3$. Deformations Γ_t exist of this (noncompact) convex $\mathbb{R}\mathsf{P}^3$-orbifold where the cusps of Γ_t, where $t > 0$, are the regular $(3,3,3)$-triangle tessellation of a triangle discussed in §4.2.2 and depicted in Figure 2.5. In $\mathbb{R}\mathsf{P}^3$, these cusp groups preserve a projective hyperplane $H_t \subset \mathbb{R}\mathsf{P}^3$ By adding reflections

$$R_1(t), R_2(t), R_3(t), R_4(t)$$

in these hyperplanes one creates projective Coxeter groups Γ_t^* acting properly and syndetically on a properly convex domain Ω_t^* which is *not* strictly convex. This is analogous to deforming the cusps in Thurston's hyperbolic Dehn surgery (compare Ratcliffe [**293**],§10.5). The quotient of Ω_t^* by a torsionfree finite-index subgroup of Γ_t^* is a closed 3-manifold with incompressible tori corresponding to the cusps of Γ_0.

Benoist gives a comprehensive description of such divisble 3-domains in [**43**], and relates the non-strict convexity of the boundary to incompressible tori and the JSJ-decomposition of the quotient convex $\mathbb{R}\mathsf{P}^3$-manifolds. See Choi–Hodgson–Lee [**93**] and Marquis [**258**] for further discussion of divisible convex domains arising from Coxeter groups in $\mathbb{R}\mathsf{P}^3$.

In a series of papers [**202–205**], Kyeonghee Jo investigates the differentiability of the boundary of a convex quasi-homogeneous domain. She shows that under various assumptions on the differentiability of $\partial\Omega$, such a domain must be homogeneous. For example, if Ω is strictly convex quasi-homogeneous, then its boundary is at least C^1, but if it is C^2 except at a

[8]The author gave a lecture on this example at a regional meeting of the American Mathematical Society on October 30, 1982.

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finite set of points, then it must be an ellipsoid. (Other characterizations of ellipsoids among quasi-homogeneous convex domains are due to Socié–Méthou [**313**] and Colbois–Verovic [**97**].)

Kapovich [**211**] gives examples of strictly convex divisible domains in all dimensions ≥ 4 whose quotients are the non-locally symmetric negatively curved manifolds first discovered by Gromov and Thurston [**176**].

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Part 2

Geometric manifolds

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Chapter 5

Locally homogeneous geometric structures

Let M be a manifold and let X be a space with a transitive action of a Lie group G. Then in the spirit of Klein's Erlangen program [**216**], (G, X) defines a *geometry:* namely, the objects in X which are invariant under G. (A recent discussion of the Erlangen program is [**201**].)

We want to impart this geometry to M by a system of coordinate charts taking coordinate patches in M to open subsets of X, in such a way that the coordinate changes on overlapping patches are locally restrictions of transformations coming from G. Such a coordinate atlas defines a (G, X)-*structure* on M, and we call M a (G, X)-*manifold.*

The general notion of defining a structure on a manifold by an atlas of local charts is that of a *pseudogroup.* These are defined by a collection $\mathcal{G}$ of homeomorphisms between open subsets of a topological space S satisfying several natural conditions: $\mathcal{G}$ contains the identity $\mathbb{I}_S$ and is closed under restrictions to open subsets, inversion and composition (where defined). Furthermore, if $U = \bigcup_\alpha U_\alpha$ and $g_\alpha, g_\beta \in \mathcal{G}$ are defined on U_α and U_β respectively, such that the restrictions

$$g_\alpha|_{U_\alpha} = g_\beta|_{U_\beta},$$

then $\exists g \in \mathcal{G}$ defined on U restricting to g_α on U_α. See, for example, Kobayashi–Nomizu [**224**], pp.1–2 for further discussion.

For example, if (G, X) is affine or projective geometry, the corresponding global object is an *affine structure* or *projective structure* on M. (Such structures are also called "affinely flat structures," "flat affine structures," "flat projective structures," etc. An affine structure on a manifold is the

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same thing as a flat torsion-free affine connection, and a projective structure is the same thing as a flat normal projective connection (see Sharpe [**305**], Chern–Griffiths [**86**] Kobayashi [**219**] or Hermann [**190**] for the theory of projective connections). We shall refer to a projective structure modeled on $\mathbb{R}\mathsf{P}^n$ *an* $\mathbb{R}\mathsf{P}^n$*-structure;* a manifold with an $\mathbb{R}\mathsf{P}^n$-structure will be called an $\mathbb{R}\mathsf{P}^n$*-manifold.*

In many cases of interest, there may be a readily identifiable geometric object on X whose stabilizer is G, and modeling a manifold on (G, X) may be equivalent to a geometric object locally equivalent to the G-invariant geometric object on X. Perhaps the most important such object is a locally homogeneous *Riemannian metric.* For example if X is a simply connected Riemannian manifold of constant curvature K and G is its group of isometries, then locally modeling M on (G, X) is equivalent to giving M a Riemannian metric of curvature K. (This idea can be vastly extended, for example to cover indefinite metrics, locally homogeneous metrics whose curvature is not necessarily constant, etc.) In particular Riemannian metrics of constant curvature are special cases of (G, X)-structures on manifolds.

Thurston [**324**] gives a detailed discussion of some of the pseudogroups defining structures on 3-manifolds.

5.1. Geometric atlases

Let G be a Lie group acting transitively on a manifold X. Let $U \subset X$ be an open set and let $U \xrightarrow{f} X$ be a smooth map. We say that f is *locally-G* if U admits a covering by open sets U_α and elements $g_\alpha \in G$ (where α lies in an index set A) such that

$$f|_{U_\alpha} = g_\alpha|_{U_\alpha}$$

for each $\alpha \in A$. (Of course f will have to be a local diffeomorphism.) The collection of open subsets of X, together with locally-G maps defines a pseudogroup upon which can model structures on manifolds as follows.

A (G, X)-*atlas* on M is a pair $(\mathcal{U}, \Phi)$ where

$$\mathcal{U} := \{U_\alpha \mid \alpha \in A\},$$

is an open covering of M and

$$\Phi = \left\{U_\alpha \xrightarrow{\phi_\alpha} X\right\}_{U_\alpha \in \mathcal{U}}$$

is a collection of coordinate charts such that for each pair

$$(U_\alpha, U_\beta) \in \mathcal{U} \times \mathcal{U}$$

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the restriction of $\phi_\alpha \circ (\phi_\beta)^{-1}$ to $\phi_\beta(U_\alpha \cap U_\beta))$ is locally-G. A (G, X)-*structure* on M is a maximal (G, X)-atlas and a (G, X)-*manifold* is a manifold together with a (G, X)-structure on it.

A (G, X)-manifold has an underlying real analytic structure, since the action of G on X is real analytic.

This notion of a map being *locally*-G has already been introduced for locally affine and locally projective maps.

Suppose that M and N are two (G, X)-manifolds and $M \xrightarrow{f} N$ is a map. Then f is a (G, X)-*map* if for each pair of charts

$$U_\alpha \xrightarrow{\phi_\alpha} X, \quad V_\beta \xrightarrow{\psi_\beta} X,$$

for M and N respectively, the restriction

$$\psi_\beta \circ f \circ \phi_\alpha^{-1}\Big|_{\phi_\alpha\left(U_\alpha \cap f^{-1}(V_\beta)\right)}$$

is locally-G. In particular (G, X)-maps are necessarily local real analytic diffeomorphisms. Clearly the set of (G, X)-automorphisms $M \longrightarrow M$ forms a group, which we denote by $\mathsf{Aut}_{(G,X)}(M)$ or just $\mathsf{Aut}(M)$ when the context is clear.

Exercise 5.1.1. Let N be a (G, X)-manifold and $M \xrightarrow{f} N$ a local diffeomorphism.

- There is a unique (G, X)-structure on M for which f is a (G, X)-map.
- Every covering space of an (G, X)-manifold has a canonical (G, X)-structure.
- Conversely, suppose M is a (G, X)-manifold upon which a discrete subgroup $\Gamma \subset \mathsf{Aut}_{(G,X)}(M)$ acts properly and freely. Then M/Γ is an (G, X)-manifold and the quotient mapping

  $$M \longrightarrow M/\Gamma$$

  is a (G, X)-covering space.

5.1.1. The pseudogroup of local mappings. The fundamental example of a (G, X)-manifold is X itself. Evidently any open subset $\Omega \subset X$ has a (G, X)-structure (with only one chart—the inclusion $\Omega \hookrightarrow X$). Locally-G maps satisfy the following *Unique Extension Property:* If $U \subset X$ is a nonempty connected open subset, and $U \xrightarrow{f} X$ is locally-G, then there exists a unique element $g \in G$ restricting to f. (Real analyticity implies uniqueness when U is nonempty.)

A crucial point for this pseudogroup of local mappings is that every local mapping is the restriction of a map $X \to X$ defined *globally* on X. This

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is exactly why the pseudogroup of local biholomorphisms, used to define a holomorphic atlas on a complex manifold is *not* a geometric atlas in this sense.[1]

Here is another perspective on a (G, X)-atlas. First regard M as a quotient space of the disjoint union

$$\mathfrak{U} = \coprod_{\alpha \in A} U_\alpha$$

by the equivalence relation $\sim$ defined by intersection of patches. A point $u \in U_\alpha \cap U_\beta$ determines corresponding elements

$$u_\alpha \in U_\alpha \subset \mathfrak{U}$$
$$u_\beta \in U_\beta \subset \mathfrak{U}$$

and we define the equivalence relation on $\mathfrak{U}$ by: $u_\alpha \sim u_\beta$.

Now the *coordinate change*

$$\phi_\beta(U_\alpha \cap U_\beta) \xrightarrow{\phi_\alpha \circ (\phi_\beta)^{-1}} \phi_\alpha(U_\alpha \cap U_\beta)$$

is locally-G. By the unique extension property, its restriction to each connected component of $\phi_\alpha(U_\alpha)$ agrees with the restriction of a *unique* element of G to each connected component of $\phi_\beta(U_\alpha \cap U_\beta)$. Thus it corresponds to a *locally constant map:*

$$U_\alpha \cap U_\beta \xrightarrow{g_{\alpha\beta}} G \tag{5.1}$$

We can alternatively define the (G, X)-manifold M as the quotient of the disjoint union

$$\mathfrak{U}_\Phi := \coprod_{\alpha \in A} \phi_\alpha(U_\alpha)$$

by the equivalence relation $\sim_\Phi$ defined as:

$$\phi_\alpha(u_\alpha) \sim_\Phi g_{\alpha\beta}\big(\phi_\beta(u_\beta)\big)$$

for $u \in U_\alpha \cap U_\beta$ notated as above. That $\sim_\Phi$ is an equivalence relation follows from the *cocycle identities*

$$\begin{aligned} g_{\alpha\alpha}(u_\alpha) &= 1 \\ g_{\alpha\beta}(u_\beta) g_{\beta\alpha}(u_\alpha) &= 1 \\ g_{\alpha\beta}(u_\beta) g_{\beta\gamma}(u_\gamma) g_{\gamma\alpha}(u_\alpha) &= 1 \end{aligned} \tag{5.2}$$

whenever $u \in U_\alpha,\ u \in U_\alpha \cap U_\beta,\ u \in U_\alpha \cap U_\beta \cap U_\gamma$, respectively.

This rigidity property is a distinguishing feature of the kind of geometric structures considered here. However, many familiar pseudogroup structures lack this kind of rigidity:

[1]Otherwise, one could prove that $\mathbb{S}^2$ is *not* a complex manifold.

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Exercise 5.1.2. Show that the following pseudogroups do *not* satisfy the unique extension property:

- C^r local diffeomorphisms between open subsets of $\mathbb{R}^n$, when $r = 0, 1, \ldots, \infty, \omega$.
- Local biholomorphisms between open subsets of $\mathbb{C}^n$.
- Smooth diffeomorphisms between open subsets of a domain $\Omega \subset \mathbb{R}^n$ preserving an exterior differential form on Ω.

5.1.2. (G, X)-automorphisms. Now we discuss the *automorphisms* of a structure locally modeled on (G, X).

If $\Omega \subset X$ is connected, nonempty, and open, then a (G, X)-automorphism $\Omega \xrightarrow{f} \Omega$ is the restriction of a unique element $g \in G$ preserving Ω, that is:

$$\mathsf{Aut}_{(G,X)}(\Omega) \;\cong\; \mathsf{Stab}_G(\Omega) \;=\; \{g \in G \mid g(\Omega) = \Omega\}.$$

Now suppose that $M \xrightarrow{\phi} \Omega$ is a local diffeomorphism onto a domain $\Omega \subset X$. There is a homomorphism

$$\mathsf{Aut}_{(G,X)}(M) \xrightarrow{\phi_*} \mathsf{Aut}_{(G,X)}(\Omega)$$

whose kernel consists of all maps $M \xrightarrow{f} M$ making the diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi} & \Omega \\ {\scriptstyle f}\big\downarrow & & \big\| \\ M & \xrightarrow[\phi]{} & \Omega \end{array}$$

commute.

Exercise 5.1.3. Find examples where:

- ϕ_* is surjective but not injective;
- ϕ_* is injective but not surjective.

5.2. Development, holonomy

There is a useful globalization of the coordinate charts of a geometric structure in terms of the universal covering space and the fundamental group. The coordinate atlas $\{U_\alpha\}_{\alpha \in A}$ is replaced by a universal covering space $\widetilde{M} \longrightarrow M$ with its group of deck transformations π. In the first approach, M is the quotient space of the disjoint union $\coprod_{\alpha \in A} U_\alpha$, and in the second it is the quotient space of $\widetilde{M}$ by the group action π. The coordinate charts $U_\alpha \xrightarrow{\psi_\alpha} X$ are replaced by a globally defined map $\widetilde{M} \xrightarrow{\mathsf{dev}} X$.

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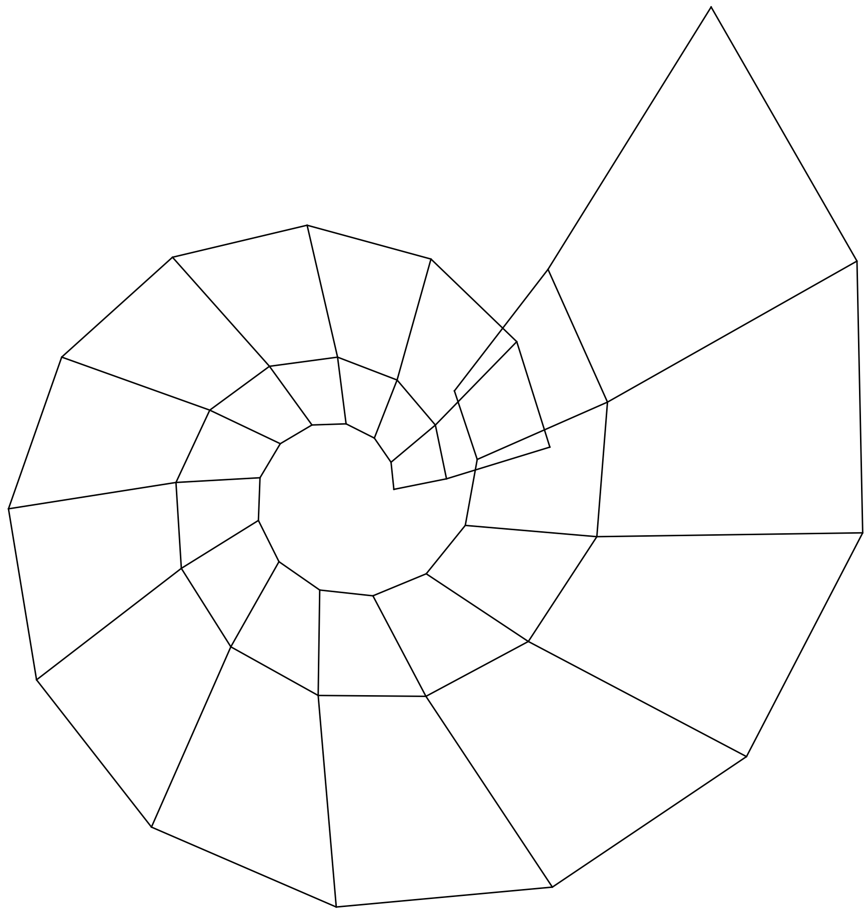

Figure 5.1. Development of incomplete affine structure on a torus

This process of development originated with Élie Cartan and generalizes the notion of a developable surface in E^3. If $S \hookrightarrow \mathsf{E}^3$ is an embedded surface of zero Gaussian curvature, then for each $p \in S$, the exponential map at p defines an isometry of a neighborhood of 0 in the tangent plane T_pS, and corresponds to rolling the tangent plane $\mathsf{T}_p(S)$ on S without slipping. In particular every curve in S starting at p lifts to a curve in T_pS starting at $0 \in \mathsf{T}_pS$. For a Euclidean manifold, this globalizes to a local isometry of the universal covering $\widetilde{S} \longrightarrow \mathsf{E}^2$, called by Élie Cartan the *development* of the surface (along the curve). The metric structure is actually subordinate to the affine connection, as this notion of development really only involves the construction of *parallel transport.*

Later this was incorporated into the notion of a *fiber space,* as discussed by Ehresmann in the 1950 conference [**322**]. The collection of coordinate changes of a (G, X)-manifold M defines a fiber bundle $\mathcal{E}_M \longrightarrow M$ with fiber

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X and structure group G. The fiber over $p \in M$ of the associated principal bundle

$$\mathfrak{P}_M \xrightarrow{\Pi_{\mathfrak{P}}} M$$

consists of all possible germs of (G, X)-coordinate charts at p. The fiber over $p \in M$ of $\mathcal{E}_M$ consists of all possible *values* of (G, X)-coordinate charts at p. Assigning to the germ at p of a coordinate chart $U \xrightarrow{\psi} X$ its value

$$x = \psi(p) \in X$$

defines a mapping

$$(\mathfrak{P}_M)_p \longrightarrow (\mathcal{E}_M)_p.$$

Working in a local chart, the fiber over a point in $(\mathcal{E}_M)_p$ corresponding to $x \in X$ consists of all the different germs of coordinate charts ψ taking $p \in M$ to $x \in X$. This mapping identifies with the quotient mapping of the natural action of the stabilizer $\mathsf{Stab}(G, x) \subset G$ of $x \in X$ on the set of germs.

For Euclidean manifolds, $(\mathfrak{P}_M)_p$ consists of all *affine orthonormal frames,* that is, pairs (x, F) where $x \in \mathsf{E}^n$ is a point and F is an orthonormal basis of the tangent space $\mathsf{T}_x\mathsf{E}^n \cong \mathbb{R}^n$. For an affine manifold, $(\mathfrak{P}_M)_p$ consists of all *affine frames:* pairs (x, F) where now F is *any* basis of $\mathbb{R}^n$.

5.2.1. Construction of the developing map. Let M be a (G, X)-manifold. Choose a universal covering space

$$\tilde{M} \xrightarrow{\mathbf{\Pi}} M$$

and let $\pi = \pi_1(M)$ be the corresponding fundamental group. The covering projection $\mathbf{\Pi}$ induces an (G, X)-structure on $\tilde{M}$ upon which π acts by (G, X)-automorphisms. The Unique Extension Property has the following important consequence.

Proposition 5.2.1. Let M be a simply connected (G, X)-manifold. Then there exists a (G, X)-map $M \xrightarrow{f} X$.

It follows that the (G, X)-map f completely determines the (G, X)-structure on M, that is, the geometric structure on a simply-connected manifold is "pulled back" from the model space X. The (G, X)-map f is called a *developing map* for M and enjoys the following uniqueness property. If $M \xrightarrow{f'} X$ is another (G, X)-map, then there exists an(G, X)-automorphism ϕ of M and an element $g \in G$ such that

$$\begin{array}{ccc} M & \xrightarrow{f'} & X \\ \phi\big\downarrow & & \big\downarrow g \\ M & \xrightarrow[f]{} & X \end{array}$$

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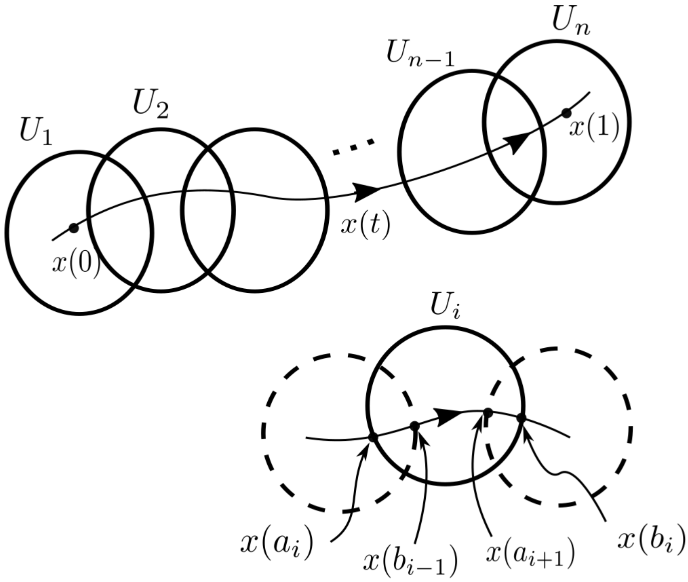

Figure 5.2. Extending a coordinate chart to a developing map

Proof of Proposition 5.2.1. Choose a basepoint $x_0 \in M$ and a coordinate patch U_0 containing x_0. For $x \in M$, we define $f(x)$ as follows. Choose a path $\{x_t\}_{0 \le t \le 1}$ in M from x_0 to $x = x_1$. Cover the path by coordinate patches U_i (where $i = 0, \ldots, n$) such that $x_t \in U_i$ for $t \in (a_i, b_i)$ where

$$a_0 < 0 < a_1 < b_0 < a_2 < b_1 < a_3 < b_2 < \\ \cdots < a_{n-1} < b_{n-2} < a_n < b_{n-1} < 1 < b_n$$

Let $U_i \xrightarrow{\psi_i} X$ be an (G, X)-chart and let $g_i \in G$ be the unique transformation such that $g_i \circ \psi_i$ and ψ_{i-1} agree on the component of $U_i \cap U_{i-1}$ containing the curve $\{x_t\}_{a_i < t < b_{i-1}}$. Let

$$f(x) = g_1 g_2 \ldots g_{n-1} g_n \psi_n(x);$$

we show that f is indeed well-defined. The map f does not change if the cover is refined. Suppose that a new coordinate patch U' is "inserted between" U_{i-1} and U_i. Let $\{x_t\}_{a' < t < b'}$ be the portion of the curve lying inside U' so

$$a_{i-1} < a' < a_i < b_{i-1} < b' < b_i.$$

Let $U' \xrightarrow{\psi'} X$ be the corresponding coordinate chart and let $h_{i-1}, h_i \in G$ be the unique transformations such that ψ_{i-1} agrees with $h_{i-1} \circ \psi'$ on the component of $U' \cap U_{i-1}$ containing $\{x_t\}_{a' < t < b_{i-1}}$ and ψ' agrees with $h_i \circ \psi_i$ on the component of $U' \cap U_i$ containing $\{x_t\}_{a_i < t < b'}$. By the Unique Extension

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Property $h_{i-1}h_i = g_i$ and it follows that the corresponding developing map

$$\begin{aligned} f(x) &= g_1 g_2 \dots g_{i-1} h_{i-1} h_i g_{i+1} \dots g_{n-1} g_n \psi_n(x) \\ &= g_1 g_2 \dots g_{i-1} g_i g_{i+1} \dots g_{n-1} g_n \psi_n(x) \end{aligned}$$

is unchanged. Thus the developing map as so defined is independent of the coordinate covering, since any two coordinate coverings have a common refinement.

Next we claim the developing map is independent of the choice of path. Since M is simply connected, any two paths from x_0 to x are homotopic. Every homotopy can be broken up into a succession of "small" homotopies, that is, homotopies such that there exists a partition

$$0 = c_0 < c_1 < \dots < c_{m-1} < c_m = 1$$

such that during the course of the homotopy the segment $\{x_t\}_{c_i < t < c_{i+1}}$ lies in a coordinate patch. It follows that the expression defining $f(x)$ is unchanged during each of the small homotopies, and hence during the entire homotopy. Thus f is independent of the choice of path.

Since f is a composition of a coordinate chart with a transformation $X \longrightarrow X$ coming from G, it follows that f is a (G, X)-map. The proof of Proposition 5.2.1 is complete. $\square$

If M is an arbitrary (G, X)-manifold, then we may apply Proposition 5.2.1 to a universal covering space $\tilde{M}$. We obtain the following basic result:

Theorem 5.2.2 (Development Theorem)**.** Let M be a (G, X)-manifold with universal covering space $\tilde{M} \xrightarrow{\mathbf{\Pi}} M$ and group of deck transformations

$$\pi = \pi_1(M) \subset \mathsf{Aut}\Big(\widetilde{M} \xrightarrow{\mathbf{\Pi}} M\Big)$$

Then $\exists$ a pair (dev, h) consisting of a (G, X)-map $\widetilde{M} \xrightarrow{\mathsf{dev}} X$ and a homomorphism $\pi \xrightarrow{h} G$ such that for each $\gamma \in \pi$,

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\mathsf{dev}} & X \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle h(\gamma)} \\ \widetilde{M} & \xrightarrow[\mathsf{dev}]{} & X \end{array}$$

commutes. Furthermore if (dev', h') is another such pair, then $\exists g \in G$ such that $\mathsf{dev}' = g \circ \mathsf{dev}$ and $h'(\gamma) = \mathsf{Inn}(g) \circ h(\gamma)$ for all $\gamma \in \pi$. That is, the

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diagram

$$\begin{array}{ccccc} \widetilde{M} & \xrightarrow{\mathsf{dev}} & X & \xrightarrow{g} & X \\ \gamma\big\downarrow & & \big\downarrow \mathsf{hol}(\gamma) & & \big\downarrow \mathsf{hol}'(\gamma) \\ \widetilde{M} & \xrightarrow[\mathsf{dev}]{} & X & \xrightarrow[g]{} & X \end{array}$$

commutes.

We call such a pair (dev, h) a *development pair*, and the homomorphism h the *holonomy representation.* (It is the holonomy of a flat connection on a principal G-bundle over M associated to the (G, X)-structure.) The developing map globalizes the coordinate charts of the manifold and the holonomy representation globalizes the coordinate changes. In this generality the Development Theorem is due to C. Ehresmann [**122**] in 1936.

5.2.2. Role of the holonomy group.

The image of the holonomy representation is the "smallest" subgroup $\Gamma \subset G$ such that M admits a (Γ, X)-*structure*, that is, an atlas when the coordinate changes are *locally* Γ-:

Exercise 5.2.3. Let M be a (G, X)-manifold with development pair (dev, h).

- Find a (G, X)-atlas for M such that the coordinate changes $g_{\alpha\beta}$ lie in Γ.
- Suppose that $N \longrightarrow M$ is a covering space. Show that there exists a (G, X)-map $N \longrightarrow X$ if and only if the holonomy representation restricted to $\pi_1(N) \hookrightarrow \pi_1(M)$ is trivial.

Thus the *holonomy covering space* $\hat{M} \longrightarrow M$ — the covering space of M corresponding to the kernel of h — is the "smallest" covering space of M for which a developing map is "defined." The holonomy group

$$\mathsf{hol}(\pi) = \Gamma \subset G$$

is the "smallest" subgroup of G for which there is a compatible (G, X)-atlas, where the coordinate changes lie in Γ.

5.2.3. Extending geometries.

A geometry may *contain* or *refine* another geometry. In this way one can pass from structures modeled on one geometry to structures modeled on a geometry containing it. Let (G, X) and (G', X') be homogeneous spaces and let $X \xrightarrow{\Phi} X'$ be a local diffeomorphism which is equivariant with respect to a homomorphism $\phi : G \to G'$ in the following

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sense: for each $g \in G$ the diagram

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & X' \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle \phi(g)} \\ X & \xrightarrow[\Phi]{} & X' \end{array}$$

commutes. Hence locally-G maps determine locally-G'-maps on $\mathfrak{X}'$ and a (G, X)-structure on M induces a (G, X'')-structure on M in the following way. Let $U_\alpha \xrightarrow{\psi_\alpha} X$ be an (G, X)-chart; the composition

$$U_\alpha \xrightarrow{\Phi \circ \psi_\alpha} X'$$

defines a (G, X'')-chart.

Exercise 5.2.4. Suppose that (G, X) and (G', X') represent a pair of geometries for which there exists a pair (Φ, ϕ) as in §5.2.3. Show that if M is a (G, X)-manifold with development pair (dev, h), then

$$(\Phi \circ \mathsf{dev}, \phi \circ h)$$

is a development pair for the induced (G, X'')-structure on M.

5.2.4. Simple applications of the developing map.

Exercise 5.2.5. Suppose that M is a closed manifold with finite fundamental group.

- If X is noncompact then M admits no (G, X)-structure.
- If X is compact and simply connected show that every (G, X)-manifold is (G, X)-isomorphic to a quotient of X by a finite subgroup of G.

(Hint: if M and N are manifolds of the same dimension, $M \xrightarrow{f} N$ is a local diffeomorphism and M is closed, show that f must be a covering space.)

As a consequence, a closed affine manifold must have infinite fundamental group and every $\mathbb{R}\mathsf{P}^n$-manifold with finite fundamental group is a quotient of S^n by a finite group (and hence a spherical space form).

Exercise 5.2.6. Let M be an (G, X)-manifold with developing pair (dev, h) and holonomy group Γ. Suppose $\Omega \subset X$ is a Γ-invariant open subset.

- $\mathsf{dev}^{-1}(\Omega)$ is a a π-invariant open subset of $\widetilde{M}$;
- Its image
$$M_\Omega := \Pi\big(\mathsf{dev}^{-1}(\Omega)\big)$$
is an open subset of M;

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- Each connected component of $\Pi^{-1}(M_\Omega) \subset \widetilde{M}$ is a connected component of $\mathsf{dev}^{-1}(\Omega)$.
- $M_\Omega \subset M$ depends only on the pair (Γ, Ω) and is independent of the choice of universal covering space $\widetilde{M} \longrightarrow M$ and developing map $\widetilde{M} \xrightarrow{\mathsf{dev}} X$.

This will be used later in §14.2.

5.3. The graph of a geometric structure

A (G, X)-structure can be described by a coordinate atlas or a developing map. However, neither description involves a mapping directly defined on the manifold. Therefore we introduce the *fiber bundle associated to a* (G, X)-*structure.* We replace the model space X by a fiber bundle $\mathcal{E}_M \longrightarrow M$ with fiber X and structure group G in the sense of Steenrod [**317**]. It plays a role analogous to the tangent bundle of a smooth manifold. It admits a *flat structure,* that is a foliation $\mathcal{F}$ transverse to the fibration. (This is equivalent to a reduction of structure group from the Lie group G to G given the discrete topology.) The (G, X)-structure corresponds to a section $\mathcal{D}_M$ which is transverse to both $\mathcal{F}$ and to the fibration.[2] This *developing section* plays the role of the zero-section of the tangent bundle of a differentiable manifold. Indeed, its normal bundle inside $\mathcal{E}_M$ is isomorphic to the tangent bundle $\mathsf{T}M$ of M. It is obtained as the *graph* of the collection Φ of coordinate charts. The flat bundle $\mathcal{E}_M$ is the natural "home" in which $\mathcal{D}_M$ lives. Compare Figure 5.3.

5.3.1. The tangent (G, X)-bundle.

The total space $\mathcal{E}_M$ of this bundle is obtained from the disjoint union

$$\mathfrak{U}_X := \coprod_{\alpha \in A} U_\alpha \times X$$

of trivial X-bundles.

Now suppose $U_\alpha, U_\beta \in \mathcal{U}$ are coordinate patches. Introduce an equivalence relation $\sim_X$ on $\mathfrak{U}_X$ by:

$$(u_\alpha, x) \sim_X \big(u_\beta, g_{\alpha\beta}(u_\beta)x\big)$$

where $g_{\alpha\beta}$ is the cocycle introduced in (5.1). The cocycle identities (5.2) imply that $\sim_X$ is an equivalence relation. The projections

$$U_\alpha \times X \longrightarrow U_\alpha$$

are trivial X-bundles and define a trivial X-bundle

$$\mathfrak{U}_X \longrightarrow \mathfrak{U}$$

[2] A smooth section of a smooth fibration is necessarily transverse to the fibration.

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compatible with the equivalence relations $\sim_X, \sim$. The corresponding mapping of quotient spaces

$$\begin{array}{ccc} \mathcal{E}_M & = & \mathfrak{U}_X/\sim_X \\ & & \Big\downarrow \Pi \\ M & = & \mathfrak{U}/\sim \end{array}$$

is a locally trivial X-bundle with structure group G.

Furthermore the structure group is really G *with the discrete topology,* since the transition functions

$$U_\alpha \cap U_\beta \xrightarrow{g_{\alpha\beta}} G$$

are locally constant. This implies that the foliation of the total space $\mathcal{E}_M$ with local leaves (sometimes called *plaques*) $U_\alpha \times \{x\}$ piece together to define the leaves of a foliation $\mathcal{F}$ of $\mathcal{E}_M$. (Compare Steenrod [**317**].)

Exercise 5.3.1.

- Show that for every leaf $L \subset \mathcal{E}_M$ of $\mathcal{F}$, the restriction $\Pi|_L$ is a covering space $L \longrightarrow M$.
- If M is simply connected, then $(\mathcal{E}_M, \mathcal{F})$ is *trivial,* that is, isomorphic to $M \times X$ with the *trivial foliation,* namely the one with leaves $M \times \{x\}$, where $x \in X$.

It follows that the flat (G, X)-bundle $(\mathcal{E}_M, \mathcal{F})$ arises from a representation $\pi_1(M) \xrightarrow{\mathsf{h}} G$ as follows.

The group $\pi_1(M)$ admits a (left-)action on the trivial bundle $\widetilde{M} \times X$ by:

$$(\widetilde{p}, x) \xmapsto{\gamma} \big(\widetilde{p}\gamma^{-1}, \mathsf{h}(\gamma)x\big)$$

where

$$\begin{aligned} \widetilde{M} \times \pi_1(M) &\longrightarrow \widetilde{M} \\ (\widetilde{p}, \gamma) &\longmapsto \widetilde{p}\gamma \end{aligned}$$

denotes the (right-) action of $\pi_1(M)$ by deck transformations. Then $\mathcal{E}_M$ identifies as the quotient $(\widetilde{M} \times X)/\pi_1(M)$, that is as the *fiber product* $\widetilde{M} \times_{\mathsf{h}} X$. Furthermore h is unique up to the action of $\mathsf{Inn}(G)$ by left-composition. We call $\mathsf{h} \in \mathsf{Hom}\big(\pi_1(M), G\big)$ the *holonomy representation* of the flat (G, X)-bundle $(\mathcal{E}_M, \mathcal{F})$.

5.3.2. Developing sections. Just as $(\mathcal{E}_M, \mathcal{F})$ globalizes the coordinate changes, its *transverse section* $\mathcal{D}_M$ globalizes the coordinate atlas Φ.

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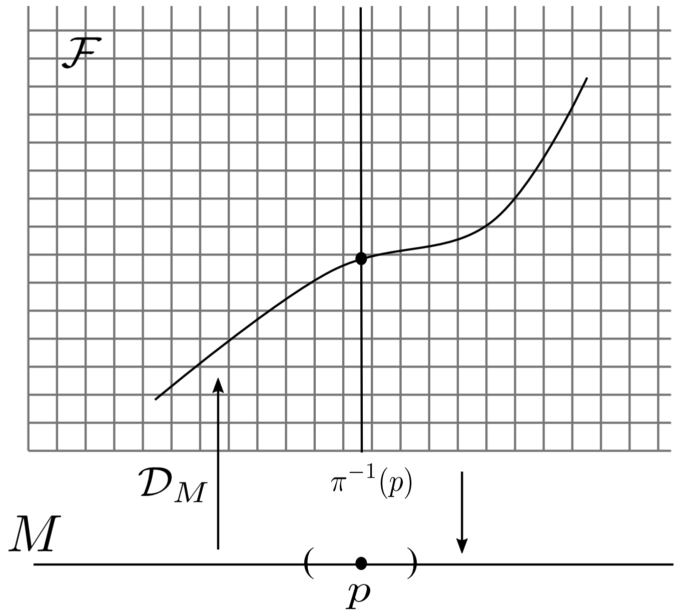

Figure 5.3. The graph of a developing map is an $\mathcal{F}$-transverse section. Here the fibers are drawn as vertical lines and the leaves of $\mathcal{F}$ as horizontal lines. The developing section is transverse to $\mathcal{F}$ as well as the fibration. Locally it is the *graph* of the developing map dev.

When M is a single coordinate patch, then $\mathcal{E}_M$ is just the product $M \times X$ and $\mathcal{E}_M \longrightarrow M$ is just the Cartesian projection $M \times X \longrightarrow M$. A section of $\mathcal{E}_M \longrightarrow M$ is just the graph of a map $M \xrightarrow{f} X$:

$$\begin{aligned} M &\xrightarrow{\mathsf{graph}(f)} M \times X \;\cong\; \mathcal{E}_M \\ p &\longmapsto \big(p, f(p)\big). \end{aligned}$$

The coordinate atlas/developing map defines a section $\mathcal{D}_M$ of $\mathcal{E}_M \to M$ which is transverse to the two complementary foliations of $\mathcal{E}_M$:

- As a section, it is necessarily transverse to the foliation of $\mathcal{E}_M$ by fibers;
- The nonsingularity of the coordinate charts/developing map implies this section is transverse to the horizontal foliation $\mathcal{F}_M$ of $\mathcal{E}_M$ defining the flat structure.

This picture of an Ehresmann structure will be used in defining the *deformation space* $\mathsf{Def}_{(G,X)}(\Sigma)$ in Chapter 7, §7.2.

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5.4. Developing sections for $\mathbb{R}\mathsf{P}^1$-manifolds

Figure 5.4, Figure 5.5, and Figure 5.6 depict developing sections for various $\mathbb{R}\mathsf{P}^1$-manifolds. M and X are both homeomorphic to S^1, and we represent M as a horizontal closed interval with endpoints identified. Similarly X is represented as a vertical closed interval with endpoints identified. Thus the total space $\mathcal{E}_M$ is represented by a square, where the left and right edges are identified by parallel (horizontal) translation. The projection Π is just horizontal projection, with fibers are vertical line segments. The leaves of $\mathcal{F}_M$ are drawn so that they are identified by the parallel translation.

Figure 5.4 and Figure 5.5 depict structures with trivial holonomy. The leaves, represented by horizontal lines (lines of slope 0), are all closed sections corresponding to the singular structure with "constant developing map." Figure 5.4 depicts the canonical structure $\mathbb{R}\mathsf{P}^1$; the developing section is the line of slope 1, the graph of the identity map $\mathbb{R}\mathsf{P}^1 \to \mathbb{R}\mathsf{P}^1$.

For any $m \in \mathbb{Z}$ a line segment of slope m (and some of its horizontal translates) describes a section s. We have already discussed the cases $m = 0, 1$. If $m \neq 0$, the section is transverse to $\mathcal{F}$. Replacing m by $-m$ gives a section inducing the opposite orientation, so that the line of slope -1 (the other diagonal of the square) depicts the developing section for an oppositely oriented manifold; explicitly, the developing section is the graph of a reflection of $\mathbb{R}\mathsf{P}^1$ (an involution which reverses orientation).

Figure 5.5 depicts the developing section for the double covering of $\mathbb{R}\mathsf{P}^1$.

Figure 5.6 depicts a structure with elliptic holonomy. In this case the foliation is a linear foliation of the torus. The leaves are drawn as lines of positive slope $m = 1/3$. Each leaf projects to M by a triple covering. The "diagonal" section s_1 and the "horizontal" section s_2 are both $\mathcal{F}$-transverse and define $\mathbb{R}\mathsf{P}^1$-structures.

Figure 5.7 depicts structures with hyperbolic holonomy η. The two fixed points of η on $\mathbb{R}\mathsf{P}^1$ determine two closed leaves, "horizontal" sections. In the picture, these are represented by the top/bottom edges of the square and the diameter halfway up. The depicted section s_1 is horizontal and misses these two horizontal sections; the corresponding developing map misses $\mathsf{Fix}(\eta)$ and corresponds to the (Hopf) affine structure. The depicted section s_2 crosses both constant horizontal sections, and the corresponding developing map is onto.

Figure 5.9 is similar, except now the holonomy η is parabolic. Corresponding to the single fixed point of η is a horizontal closed leaf, represented in this picture as the top/bottom sides of the square. The complete Euclidean structure is represented by the $\mathcal{F}$-transverse horizontal section

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s_1 and misses this '"constant" section. The section s_2 is $\mathcal{F}$-transverse and corresponds to a structure with surjective developing map.

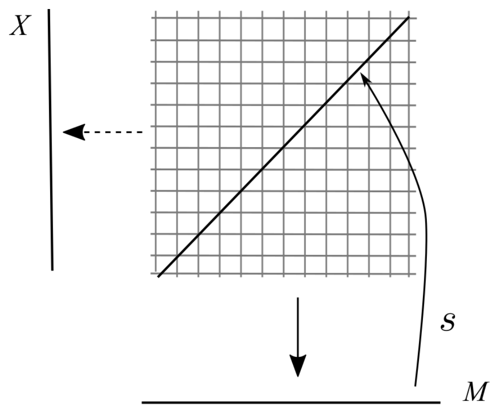

Figure 5.4. Developing section for the canonical projective closed 1-manifold $\mathbb{R}\mathsf{P}^1$. This is just the graph of the identity map $\mathbb{R}\mathsf{P}^1 \to \mathbb{R}\mathsf{P}^1$ which is (essentially) the developing map.

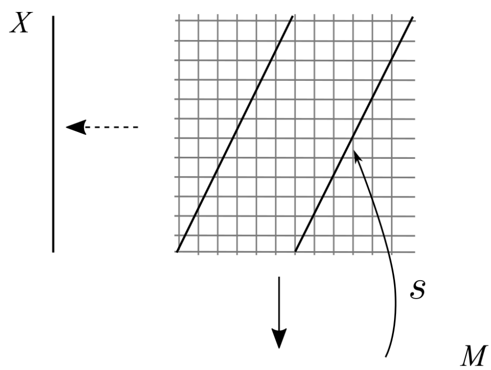

Figure 5.5. Developing section for other projective closed 1-manifolds with trivial holonomy. The depicted structure corresponds to the double covering of $\mathbb{R}\mathsf{P}^1$.

5.5. The classification of geometric 1-manifolds

The basic general question concerning geometric structures on manifolds is the following: *Given a topological manifold* Σ *and a geometry* (G, X)*, determine whether a* (G, X)*-structure on* Σ *exists and if so, to classify all* (G, X)*-structures on* Σ*.* Ideally, one would like a *deformation space*, a topological space whose points correspond to isomorphism classes of (G, X)-manifolds.

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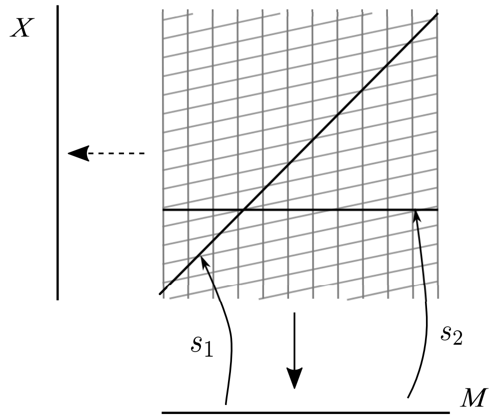

Figure 5.6. Developing sections for projective closed 1-manifolds with elliptic holonomy.

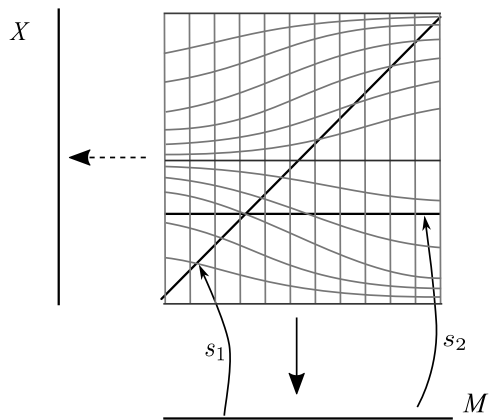

Figure 5.7. Developing sections for closed $\mathbb{R}\mathsf{P}^1$-manifolds with hyperbolic holonomy. The transverse section missing both closed leaves (drawn as the thick horizontal line) corresponds to the Hopf affine structure $\mathbb{R}^+/\langle\lambda\rangle$. The section of the singular "constant" structure is a closed leaf (represented by the thin horizontal line).

As an exercise to illustrate these general ideas, we classify geometric manifolds in dimension one. We consider the three geometries

$$\mathsf{E}^1 \xrightarrow{\cong} \mathsf{A}^1 \hookrightarrow \mathsf{P}^1$$

in increasing order. Euclidean manifolds are affine manifolds, which in turn are projective manifolds. Thus we classify $\mathbb{R}\mathsf{P}^1$-manifolds. (Compare Kuiper [**233**], Goldman [**146**], Baues [**33**].)

Let M be a connected 1-manifold. There are two cases:

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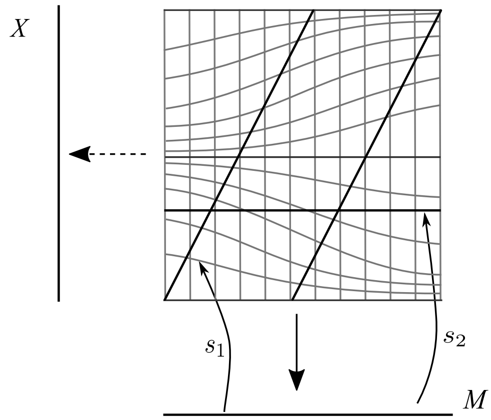

Figure 5.8. Developing sections for projective closed 1-manifolds with hyperbolic holonomy which cross the closed leaf twice. It is obtained by grafting $\mathbb{R}\mathsf{P}^1$ with a Hopf structure $\mathbb{R}^+/\langle\lambda\rangle$.

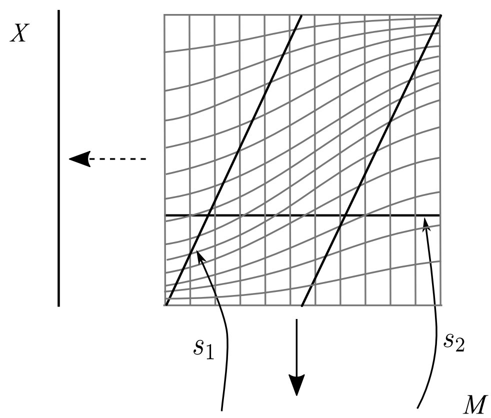

Figure 5.9. Developing sections for projective closed 1-manifolds with parabolic holonomy.

- M is noncompact, in which case M is homeomorphic (diffeomorphic) to a line ($M \approx \mathbb{R}$);
- M is compact, in which case M homeomorphic (diffeomorphic) to a circle ($M \approx S^1$).

In particular M is simply connected $\Longleftrightarrow$ M is noncompact and otherwise $\pi_1(M) \cong \mathbb{Z}$.

5.5.1. Compact Euclidean 1-manifolds and flat tori. The cyclic group $\mathbb{Z}$ acts by translations on $\mathsf{E}^1 \cong \mathbb{R}$. The quotient

$$E_1 := \mathsf{E}^1/\mathbb{Z} \cong \mathbb{R}/\mathbb{Z}$$

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is a compact Euclidean 1-manifold. The Euclidean metric on $\mathbb{R}$ induces a flat Riemannian structure on the quotient $\mathbb{R}/\mathbb{Z}$ which has length 1.

More generally, choose $\ell > 0$. Then the quotient

$$E_\ell := \mathsf{E}^1/\ell\mathbb{Z} \cong \mathbb{R}/\ell\mathbb{Z}$$

is a compact Euclidean 1-manifold which has length ℓ. Different choices of ℓ determine different isometry classes of Euclidean 1-manifolds but E_1 is *affinely isomorphic* to E_ℓ by the affine map $x \longmapsto \ell x$. In other words, if $\ell \neq 1$, then E_1 and E_ℓ are inequivalent Euclidean manifolds but equivalent affine manifolds.

Exercise 5.5.1. Let $M = E_\ell$. Show that the total space of $\mathcal{E}_M$ identifies with the quotient of $\mathbb{R}^2$ by the diagonally embedded $\mathbb{Z}$ acting by translations:

$$(x, y) \longmapsto (x + n, y + n\ell)$$

for $n \in \mathbb{Z}$, the fibration is induced by the projection

$$(x, y) \longmapsto x,$$

the foliation induced by horizontal lines $\mathbb{R} \times \{y\}$, and the developing section $\mathcal{D}_M$ by the graph

$$x \longmapsto (x, \ell x).$$

When these structures are regarded as $\mathbb{R}\mathsf{P}^1$-manifolds, $\mathcal{E}_M$ acquires an extra (horizontal) closed leaf. This leaf (corresponding to the ideal point of $\mathbb{R}\mathsf{P}^1$) is disjoint from $\mathcal{D}_M$.

These manifolds generalize to one of the most basic classes of closed geometric manifolds, namely the *flat tori.* Let $\Lambda \subset \mathbb{R}^n$ be a *lattice,* that is the additive subgroup of $\mathbb{R}^n$ generated by a basis. Then Λ acts by translations, so the quotient $\mathbb{R}^n/\Lambda$ is a compact Euclidean manifold. Bieberbach proved that *every* compact Euclidean manifold is finitely covered by a flat torus.

Exercise 5.5.2. Since Λ is a normal subgroup of $\mathbb{R}^n$, a flat torus is also an abelian Lie group. Show that this algebraic structure is compatible with the geometric structure: the Euclidean structure on $\mathbb{R}^n/\Lambda$ is invariant under multiplications. (Since $\mathbb{R}^n$ is commutative, left-multiplications and right-multiplications coincide.)

5.5.2. Compact affine 1-manifolds and Hopf circles.

A compact affine manifold is either a Euclidean manifold as above, or given by the following construction. Let $\lambda \neq 1$ and consider the cyclic group $\langle\lambda\rangle \cong \mathbb{Z}$ acting by homotheties on A^1:

$$x \longmapsto \lambda^n x$$

Then

$$A_\lambda := \mathbb{R}^+/\langle\lambda\rangle$$

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is a compact affine 1-manifold.

Exercise 5.5.3. Show that different values of λ yield inequivalent affine structures, and no A_λ is affinely equivalent to E_ℓ. However show that, for λ, λ' the developing maps for A_λ and A'_λ are *topologically conjugate* by a homeomorphism $\mathsf{A}^1 \longrightarrow \mathsf{A}^1$ and the developing maps for A_λ and E_ℓ are conjugate by a homeomorphism $\mathbb{R}^+ \longrightarrow \mathbb{R} \cong \mathsf{E}^1$.

We call these latter affine 1-manifolds *Hopf circles,* since these are the 1-dimensional cases of *Hopf manifolds* discussed in §6.2.

5.5.2.1. *Geodesics.* Hopf circles model incomplete closed geodesics on affine manifolds. The affine parameter on a Hopf circle is paradoxical. A particle moving with zero acceleration seems to be accelerating so rapidly that in finite time it "runs off the edge of the manifold." Here is an explicit calculation:

The geodesic on A^1 defined by

$$t \longmapsto 1 + t(\lambda^{-1} - 1),$$

begins at 1, and in time

$$t_\infty := 1 + \lambda^{-1} + \lambda^{-2} + \cdots = (1 - \lambda^{-1})^{-1} < \infty$$

reaches 0. It defines a closed incomplete closed geodesic $p(t)$ on M starting at $p(0) = p_0$. The lift

$$(-\infty, t_\infty) \xrightarrow{\widetilde{p}} \widetilde{M}$$

satisfies

$$\mathsf{dev}\big(\widetilde{p}(t)\big) = 1 + t(\lambda^{-1} - 1),$$

which uniquely specifies the geodesic $p(t)$ on M. It is a geodesic since its velocity

$$p'(t) = (\lambda^{-1} - 1)\partial_x$$

is constant (parallel). However $p(t_n) = p_0$ for

$$t_n := \frac{1 - \lambda^{-n}}{1 - \lambda^{-1}} = 1 + \lambda^{-1} + \cdots + \lambda^{1-n}$$

and as viewed in M, seems to go "faster and faster" through each cycle. By time $t_\infty = \lim_{n\to\infty} t_n$, it seems to "run off the manifold:" the geodesic is only defined for $t < t_\infty$. The apparent paradox is that $p(t)$ has *zero acceleration:* it would have "constant speed" if "speed" were only defined.

Exercise 5.5.4. Show that these affine structures are *invariant* affine structures on the Lie group S^1, namely, that the translation on the group S^1 is affine. (Since S^1 is abelian, both left- and right-translation agree.)

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These are the only examples of compact affine 1-manifolds, although there are projective manifolds which have the "same" holonomy homomorphisms, defined by *grafting*; see §5.5.5.

5.5.3. Classification of projective 1-manifolds. To simplify matters, we pass to the universal covering $X = \widetilde{\mathbb{RP}^1}$, which is homeomorphic to $\mathbb{R}$ and has covering group

$$\mathsf{Aut}\big(\widetilde{\mathbb{RP}^1} \to \mathbb{RP}^1\big) \;\cong\; \mathbb{Z}.$$

The collineation group $\mathsf{Aut}\big(\mathbb{RP}^1)\big) \cong \mathsf{PGL}(2,\mathbb{R})$ lifts to the universal covering group $\widetilde{\mathsf{PGL}(2,\mathbb{R})}$. Its identity component $\widetilde{\mathsf{PSL}(2,\mathbb{R})}$ (which we denote by H^0) enjoys a central extension

$$\mathbb{Z} \rightarrowtail \widetilde{\mathsf{PSL}(2,\mathbb{R})} \twoheadrightarrow \;\mathsf{PSL}(2,\mathbb{R})$$

with kernel $\mathsf{Aut}\big(\widetilde{\mathbb{RP}^1} \to \mathbb{RP}^1\big) = \mathsf{center}\big(\widetilde{\mathsf{PSL}(2,\mathbb{R})}\big)$.

For brevity we denote $\widetilde{\mathsf{PGL}(2,\mathbb{R})}$ and $\widetilde{\mathsf{PSL}(2,\mathbb{R})}$ by H and H^0 respectively. After a brief description of noncompact $\mathbb{RP}^1$-manifolds, we classify closed $\mathbb{RP}^1$-manifolds in terms of nontrivial H-conjugacy classes in H^0.

5.5.3.1. *Noncompact* 1-*manifolds.* Suppose that Σ is a connected noncompact $\mathbb{RP}^1$-manifold (and thus diffeomorphic to an open interval). Then a developing map

$$\Sigma \approx \mathbb{R} \xrightarrow{\mathsf{dev}} \mathbb{R} \approx X$$

is necessarily an embedding of Σ onto an open interval in X. Given two such embeddings

$$\Sigma \xrightarrow{f} X, \quad \Sigma \xrightarrow{f'} X$$

whose images are equal, then $f' = f \circ j$ for a diffeomorphism $\Sigma \xrightarrow{f} \Sigma$. Thus two $\mathbb{RP}^1$-structures on Σ which have equal developing images are isomorphic. Thus the classification of $\mathbb{RP}^1$-structures on Σ is reduced to the classification of H-equivalence classes of intervals $J \subset X$. Choose a diffeomorphism

$$X \approx \mathbb{R} \approx (-\infty, \infty);$$

an interval in X is determined by its pair of endpoints in $[-\infty, \infty]$. Since H acts transitively on X, an interval J is either bounded in X or projectively equivalent to X itself or one component of the complement of a point in X. Suppose that J is bounded. Then either the endpoints of J project to the same point in $\mathbb{RP}^1$ or to different points. In the first case, let $N > 0$ denote the degree of the map

$$J/\partial J \longrightarrow \mathbb{RP}^1$$

induced by dev; in the latter case choose an interval J^+ such that the restriction of the covering projection $X \longrightarrow \mathbb{RP}^1$ to J^+ is injective and the union

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$J \cup J^+$ is an interval in X whose endpoints project to the same point in $\mathbb{R}\mathsf{P}^1$. Let $N > 0$ denote the degree of the restriction of the covering projection to $J \cup J^+$. Since H acts transitively on pairs of distinct points in $\mathbb{R}\mathsf{P}^1$, it follows easily that bounded intervals in X are determined up to equivalence by H by the two discrete invariants: whether the endpoints project to the same point in $\mathbb{R}\mathsf{P}^1$ and the positive integer N. It follows that every (G, X)-structure on Σ is (G, X)-equivalent to one of the following types. We shall identify X with the real line and group of deck transformations of $X \longrightarrow \mathbb{R}\mathsf{P}^1$ with the group of integer translations.

- A complete (G, X)-manifold (that is, $\Sigma \xrightarrow{\mathsf{dev}} X$ is a diffeomorphism);
- $\Sigma \xrightarrow{\mathsf{dev}} X$ is a diffeomorphism onto one of two components of the complement of a point in X, for example, $\mathbb{R}^+ = (0, \infty)$.
- dev is a diffeomorphism onto an interval $(0, N)$ where $N > 0$ is a positive integer;
- dev is a diffeomorphism onto an interval $(0, N + \frac{1}{2})$.

5.5.3.2. *Compact* 1*-manifolds.* Suppose that Σ is a compact 1-manifold and choose a basepoint $x_0 \in \Sigma$. Let

$$\pi = \pi_1(\Sigma, x_0) \cong \mathbb{Z}$$

be the corresponding fundamental group of Σ and let $\gamma \in \pi$ be a generator. We claim that the conjugacy class of $\mathsf{h}(\gamma) \in H$ completely determines the structure. Choose a lift J of $\Sigma \setminus \{x_0\}$ to $\widetilde{\Sigma}$ to serve as a fundamental domain for π. Then J is an open interval in $\widetilde{\Sigma}$ with endpoints y_0 and y_1. After choosing a developing map $\widetilde{\Sigma} \xrightarrow{\mathsf{dev}} X$, a holonomy representation $\pi \xrightarrow{\mathsf{h}} H$,

$$\mathsf{dev}(y_1) = \mathsf{h}(\gamma)\mathsf{dev}(y_0).$$

Now suppose that dev' is a developing map for another structure with holonomy conjugate to this one. By applying an element of H which commutes with $h(\gamma)$ we may assume that $\mathsf{dev}(y_0) = \mathsf{dev}'(y_0)$ and that $\mathsf{dev}(y_1) = \mathsf{dev}'(y_1)$. Furthermore a homeomorphism $J \xrightarrow{\phi} J$ exists such that

$$\mathsf{dev}' = \mathsf{dev} \circ \phi$$

This homeomorphism lifts to a homeomorphism $\widetilde{\Sigma} \xrightarrow{\widetilde{\phi}} \widetilde{\Sigma}$ taking dev to dev'. Conversely, suppose that $\eta \in H$ is orientation-preserving (this means simply that η lies in the identity component of H) and is not the identity. Then $\exists x_0 \in X$ which is not fixed by η; let $x_1 = \eta x_0$. There exists a homeomorphism $J \longrightarrow X$ taking the endpoints y_i of J to x_i for $i = 0, 1$. This homeomorphism extends to a developing map $\widetilde{\Sigma} \xrightarrow{\mathsf{dev}} X$. In summary:

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Theorem 5.5.5. A compact $\mathbb{R}\mathsf{P}^1$-manifold is either projectively equivalent to exactly one of the following:

- A Hopf circle $\mathbb{R}^+/\langle\lambda\rangle$;
- A Euclidean 1-manifold $\mathbb{R}/\mathbb{Z}$;
- A quotient of the universal covering $\widetilde{\mathbb{R}\mathsf{P}^1}$ by a cyclic group.

The first two cases are the affine 1-manifolds, and are homogeneous. The last case contains homogeneous structures if the holonomy is elliptic.

Exercise 5.5.6. Determine all automorphisms of each of the above list of $\mathbb{R}\mathsf{P}^1$-manifolds.

Corollary 5.5.7. Let Σ be a connected closed 1-manifold. Then the set of isomorphism classes of $\mathbb{R}\mathsf{P}^1$-structures on Σ is in bijective correspondence with the set

$$\big(H^0 \setminus \{1\}\big)/\mathsf{Inn}(H)$$

of H-conjugacy classes in the set $H^0 \setminus \{1\}$ of elements of H^0 not equal to the identity.

Exercise 5.5.8. Show that the quotient topology on $\big(H^0 \setminus \{1\}\big)/\mathsf{Inn}(H)$ is *not* Hausdorff. (Hint: Find a nontrivial element in H^0 whose H^0-conjugacy class contains the identity element in its closure.)

5.5.4. Homogeneous affine structures. As observed in Exercise 5.5.4, a closed 1-dimensional affine manifold M has the extra structure as an *affine Lie group:* M is a Lie group isomorphic to the circle $\mathbb{R}/\mathbb{Z}$ and the operations of left-translation and right-translation are affine. (Since M is abelian, these two operations are identical.) In particular the universal covering $\widetilde{M}$ inherits an affine Lie group structure (isomorphic to $\mathbb{R}$). By forming products one obtains affine Lie group structures on the 2-dimensional Lie group $\mathbb{R}^2$.

Exercise 5.5.9. Affine Lie group structures on $\mathbb{R}^2$.

- Show that the products of affine Lie group structures on $\mathbb{R}$ give three nonequivalent affine Lie group structures on $G = \mathbb{R}^2$. If $\Lambda < G$ is a lattice, then G/Λ is an affine Lie group isomorphic to the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$. Find such a structure which is *not* affinely equivalent to a product of closed affine 1-manifolds.
- Find two other affine Lie group structures on G.
- Prove that these five structures are the only affine Lie group structures on G.

In § 10.1 and § 10.4, these structures will be identified with 2-dimensional commutative associative algebras over $\mathbb{R}$. and will be generalized to left-invariant affine structures on (possibly noncommutative) Lie groups.

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Every *homogeneous* affine structure on $\mathbb{T}^2$ is obtained by this construction. The other affine structures are obtained by the radiant suspension construction of Exercise 6.2.6; compare Baues [**33**] for more information on the affine structures on $\mathbb{T}^2$.

5.5.5. Grafting. Another approach to the classification is through the operation of *grafting,* developed in Goldman [**151**] in this generality. Let M_1, M_2 be two (G,X)-manifolds with two-sided hypersurfaces $V_i \subset M_i$ respectively. Suppose that each V_i has a tubular neighborhood U_i and with an (G,X)-isomorphism $U_1 \xrightarrow{f} U_2$. Then the complement $M_i \setminus V_i$ is the interior of a manifold-with-boundary $M_i|V_i$ with two boundary components V_i', V_i'' and an identification map $M_i|V_i \twoheadrightarrow M_i$ which identify $V_i' \longleftrightarrow V_i''$ to V_i

Exercise 5.5.10. The restriction of the isomorphism f to $V_i \subset M_i$ induces identifications $V_1' \longleftrightarrow V_2''$ and $V_2' \longleftrightarrow V_1''$ which defines an equivalence relation $\sim$ on the disjoint union $M_1|V_1 \sqcup M_2|V_2$. Then the quotient space

$$M := \Big(M_1|V_1 \sqcup M_2|V_2\Big) \Big/ \sim$$

inherits a unique (G,X)-structure such that the natural inclusions $M_i \setminus V_i \hookrightarrow M$ are (G,X)-maps.

This construction applies in dimension one, giving all compact $\mathbb{R}\mathsf{P}^1$-manifolds.

Exercise 5.5.11. If M is a closed $\mathbb{R}\mathsf{P}^1$-manifold with hyperbolic or parabolic holonomy, the following conditions are equivalent:

- dev is surjective;
- M is not homogeneous;
- The developing image $\mathsf{dev}(\widetilde{M})$ contains at least one fixed point of the holonomy;
- M is obtained by grafting a homogeneous (affine) 1-manifold with the model $\mathbb{R}\mathsf{P}^1$-manifold M_0 (given by an isomorphism $M_0 \cong \mathbb{R}\mathsf{P}^1$).

5.6. Affine structures on closed surfaces

If M is a closed orientable surface with an affine structure, then it is homeomorphic to a 2-torus (Benzécri [**45**], §9.1). Furthermore it is a quotient of one of the four possibilities:

- A^2 itself, in which case M is *complete;*
- An open halfplane $\mathbb{R} \times \mathbb{R}^+$,
- An open quadrant $\mathbb{R}^+ \times \mathbb{R}^+$;
- The universal covering of the complement of a point in A^2.

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The classification of *convex* affine structures on closed 2-manifolds is due to Kuiper [**234**]. The general classification of affine structures on tori is due to Nagano–Yagi [**279**] and Arrowsmith–Furness [**141**]. Affine structures on Klein bottles are classified by Arrowsmith–Furness [**10**]. Compare Baues [**33**].

When M^2 is a closed affine surface obtained as a quotient $\Gamma\backslash\Omega$ where $\Omega \subset \mathsf{A}^2$ is a subdomain, then the holonomy homomorphism h is injective and it determines the structure. Furthermore its image is a discrete subgroup of $\mathsf{Aff}(\mathsf{A}^2)$ which acts properly on Ω.

Exercise 5.6.1. Find examples of pairs of closed affine surfaces M_1, M_2 and a homeomorphism $M_1 \xrightarrow{f} M_2$ such that:

- $M_1 = \Gamma\backslash\Omega$, where $\Omega \subset \mathsf{A}^2$ is a subdomain as above;
- M_2 is a quotient of the universal covering of $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, such that the respective holonomy homomorphisms h_i satisfy:

$$h_1 \;=\; h_2 \circ f_*$$

  (In other words, M_1 and M_2 "have the same holonomy.")
- Determine for which domains Ω this phenomenon occurs.

Another consequence of the classification of closed affine 2-manifolds is a developing map

$$\widetilde{M} \xrightarrow{\mathsf{dev}} \mathsf{A}^2$$

is always a covering space onto its image. However, when the context is expanded to projective structures on the torus, this is longer true, due to the examples of Smillie [**310**] and Sullivan and Thurston [**320**]; see Chapter 13 and Chapter 14.

The classification of affine structures on surfaces is discussed in detail in Chapter 9, Chapter 10, and extended to projective structures in Chapter 14.

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Pictures of affine structures on $\mathbb{T}^2$

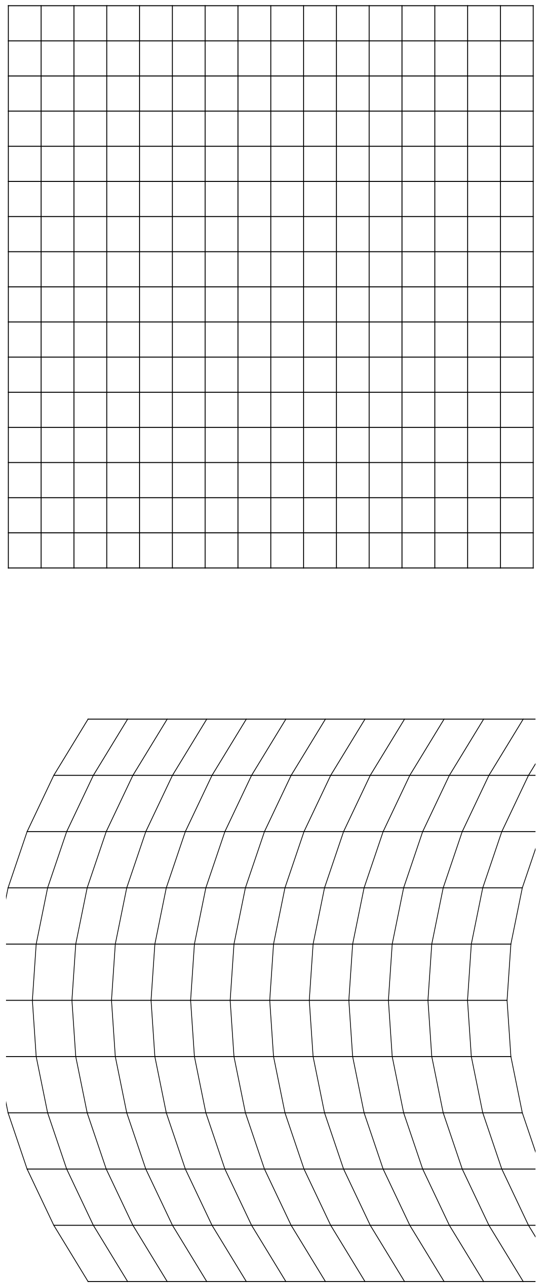

Figure 5.10. Complete affine structures on $\mathbb{T}^2$. The first is a tiling of the Euclidean plane by rectangles, defining a Euclidean structure on the torus. The second is a non-Riemannian structure, depicted as tiling of the affine plane by quadrilaterals.

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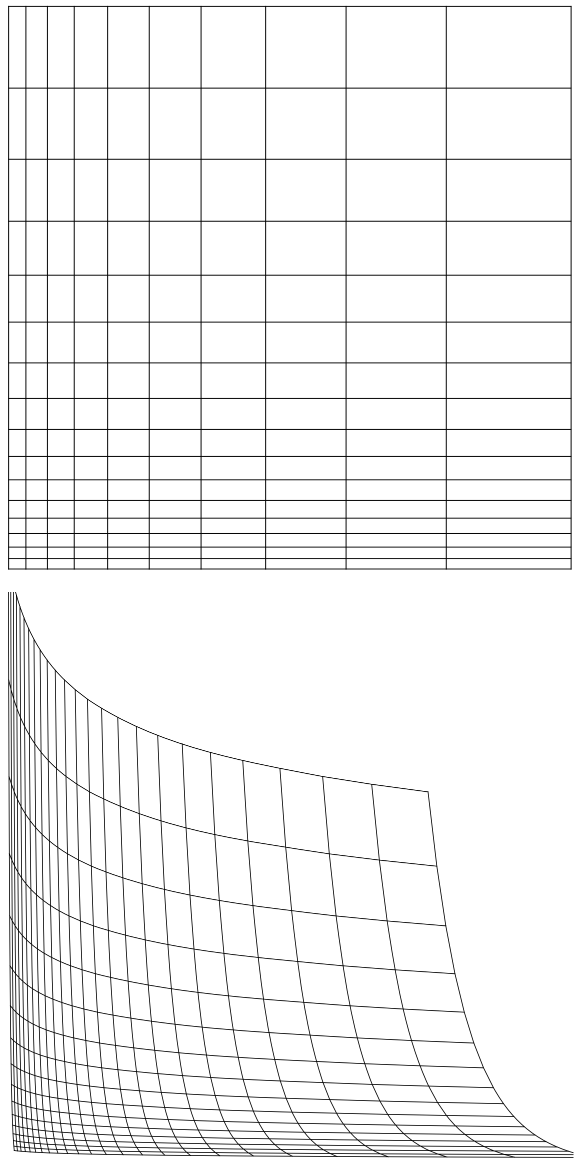

Figure 5.11. Hyperbolic affine structures on $\mathbb{T}^2$. The first depicts a product $\mathbb{R}^+/\langle 2\rangle \times \mathbb{R}^+/\langle 3\rangle$, the second is a more general deformation.

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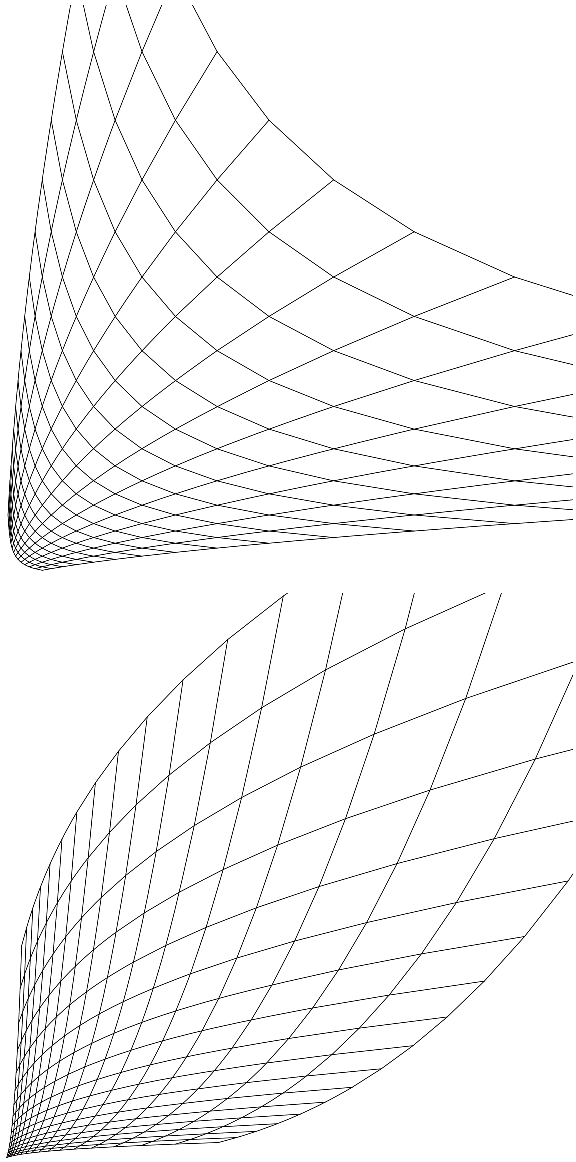

Figure 5.12. Some more hyperbolic affine structures on $\mathbb{T}^2$

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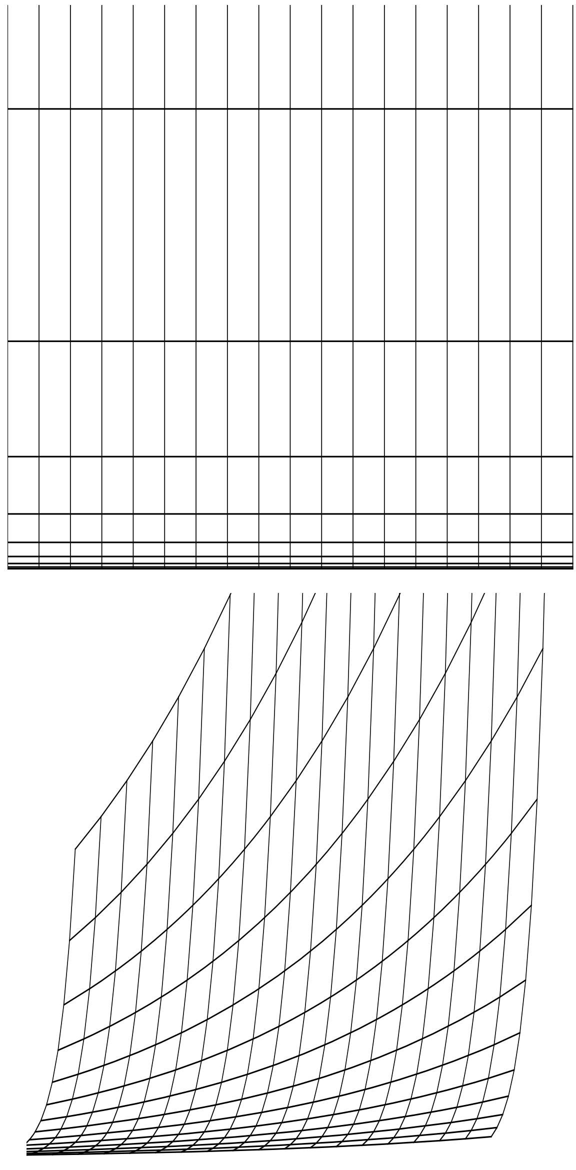

Figure 5.13. Non-radiant incomplete affine structure on T^2; the development is onto a halfplane $\mathbb{R} \times \mathbb{R}^+$. The first is a product of a complete affine 1-manifold $\mathbb{R}/\mathbb{Z}$ with a Hopf circle $\mathbb{R}^+/\langle\lambda\rangle$. The second is a parallel suspension of an automorphism of the Hopf circle.

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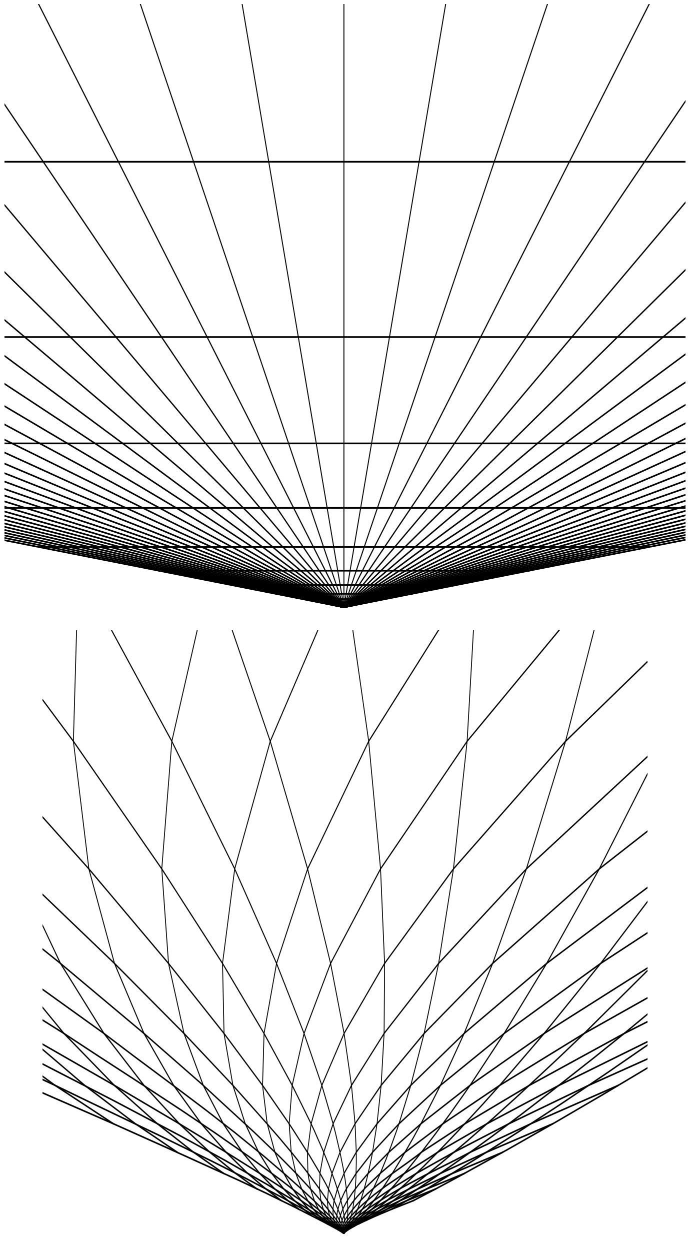

Figure 5.14. Radiant incomplete affine structure on T^2; the development is onto a halfplane $\mathbb{R} \times \mathbb{R}^+$. The first is a radiant suspension of the identity automorphism of a complete affine 1-manifold $\mathbb{R}/\mathbb{Z}$; the flowlines of the radiant flow are all closed. The second is a more general radiant suspension.

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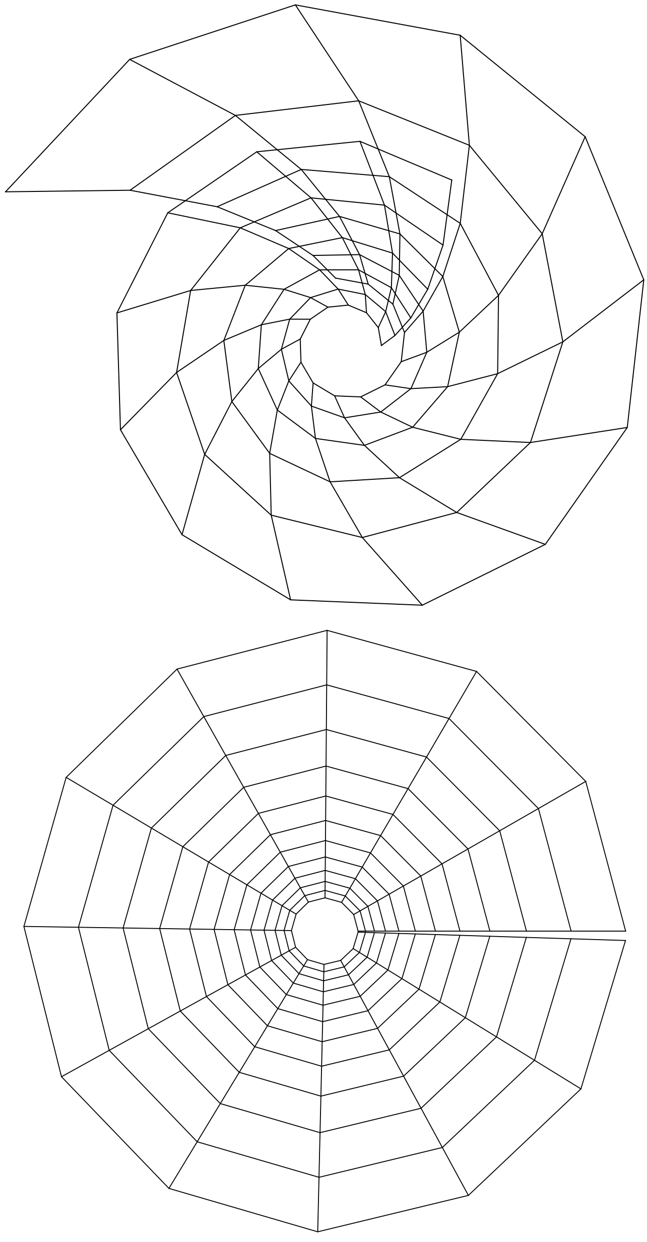

Figure 5.15. Incomplete complex-affine structures (similarity structures) on $\mathbb{T}^2$; In general the holonomy group is a dense subgroup of the multiplicative group $\mathbb{C}^\times$, and the developing map lifts to a diffeomorphism of $\widetilde{M}$ to the universal covering space $\widetilde{\mathbb{C}^\times}$. Using the lifted complex exponential map $\mathbb{C} \xrightarrow{\widetilde{\exp}} \widetilde{\mathbb{C}^\times}$, these structures can be identified with $\widetilde{\exp}\big(\mathbb{C}/\Lambda\big)$, where $\Lambda < \mathbb{C}$ is a lattice.

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Chapter 6

Examples of geometric structures

This section introduces examples of geometric manifolds in dimensions greater than one. The theory of Lie groups and their homogeneous spaces organized the abundance of classical geometries. This *algebraicization of geometry* clarified the relationship between various geometric structures. We give several general constructions to pass from one geometric structure to another. These constructions provide a rich class of geometric structures on manifolds.

We begin with general remarks on these constructions, which include the inclusion of homogeneous subdomains, Cartesian products, mapping tori and homogeneous fibrations. Then we study parallel structures in affine geometry, generalizing the construction of Euclidean geometry as (flat) Riemannian geometry. From our viewpoint, a Euclidean structure is just a *parallel Riemannian structure* on an affine manifold. This is the first example of *extending* a geometry, where the model space X is fixed (in this case an affine space) but the automorphism group G is reduced or enlarged. Many of these examples arise as mapping tori of isometries of Euclidean manifolds, constructed as *parallel suspensions.*

Many important examples arise from the embedding one model space in another model space as a *homogeneous subdomain.* We call this process the *refinement* of geometries. One example is the inclusion of affine space in projective space; in this way every affine structure inherits a projective structure. Another example is the (real-projective model of hyperbolic geometry, whereby every hyperbolic structure is a projective structure. Similarly the complex-projective model (in dimension two) of hyperbolic geometry implies

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that every hyperbolic surface has a natural $\mathbb{C}\mathsf{P}^1$-structure. It follows that every surface admits an $\mathbb{R}\mathsf{P}^2$-structure (respectively a $\mathbb{C}\mathsf{P}^1$-structure).

Another important homogeneous subdomain of affine space A is the complement $\mathsf{A}^n \setminus \{p\}$ of a point p. It naturally identifies with the complement $\mathbb{R}^n \setminus \{\mathbf{0}\}$ which has automorphism group $\mathsf{GL}(n, \mathbb{R})$. Affine structures with holonomy in $\mathsf{GL}(n, \mathbb{R}) < \mathsf{Aff}(n, \mathbb{R})$ are called *radiant* and have many special properties. Hopf manifolds — quotients of $\mathsf{A}^n \setminus \{p\}$ by cyclic groups — are basic examples of radiant affine manifolds, Radiant affine n-manifolds closely relate to $\mathbb{R}\mathsf{P}^{n-1}$-manifolds.

After discussing the general philosophy of "refining" geometries, and the basic examples of enlarging the structure on a fixed model space (§6.1.1) and inclusions of homogeneous subdomains (§6.1.2) we describe in detail the basic example of Hopf manifolds and some of their modifications (§6.2). Hopf manifolds are diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$, and are the simplest closed affine manifolds which are *not* aspherical.

Returning to the general theory, we discuss further methods of obtaining new structures from old, including Cartesian products, and pulling back fibered structures. This leads to several *suspension constructions,* including structures on mapping tori (§6.3.1). Parallel suspensions (discussed in §6.3.2) give a rich class of *complete* affine manifolds when applied to affine automorphisms of complete affine manifolds. Radiant suspensions (discussed in §6.5.3) give radiant affine structures on mapping tori of projective automorphisms of projective manifolds.

§6.4 expounds the theory of Euclidean manifolds from the vantage point of these constructions. §6.5 expounds the theory of radiant affine manifolds. This chapter concludes (§6.6) with an account of contact projective manifolds.

6.1. Refining geometries and structures

Suppose that (G', X') and (G, X) are different geometries, $X \xrightarrow{\phi} X'$ is a local diffeomorphism which is equivariant with respect to a homomorphism $G \xrightarrow{\Phi} G'$ in the following sense: For all $g \in G$, the diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle \Phi(g)} \\ X & \xrightarrow[\phi]{} & X' \end{array}$$

commutes, that is, $\Phi(g) \circ \phi = \phi \circ g$. In this case we say that (G, X) *refines* (G', X').

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By composing the developing map (or the coordinate charts) of an (G, X)-manifold with ϕ, one sees that: *Every* (G, X)*-structure on a manifold* M *induces an* (G', X')*-structure on* M. We say that the (G, X)-structure *refines* the (G', X')-structure on M.

6.1.1. Parallel structures in affine geometry. An important case occurs when $X' = X$ and Φ is the identity map. In that case (G', X') has additional properties that are preserved by G'. A basic example is the refinement of affine geometry by Euclidean geometry.

The refinement of affine geometry by Euclidean geometry occurs by imposing conditions on the linear holonomy $\mathsf{L}(\Gamma) < \mathsf{GL}(\mathbb{R}^n)$, and one obtains refinements of affine geometry involving parallel structures. When $\mathsf{L}(G)$ acts by isometries of an inner product B on V, one obtain *flat pseudo-Riemannian manifolds;* these are just affine structures with parallel pseudo-Riemannian structures. In particular if $\mathsf{L}(\Gamma) < \mathsf{O}(n-1, 1)$, one obtains *flat Lorentzian manifolds.*

Affine structures with a parallel almost complex structure (as in §1.4.4) have linear holonomy in $\mathsf{GL}(m, \mathbb{C}) < \mathsf{GL}(2m, \mathbb{R})$, and are *complex-affine* structures.

Many classification results fit in this context. For example Bieberbach's structure theorem implies that every Euclidean structure on a closed manifold refines to a $(\Upsilon \ltimes \mathbb{R}^n, \mathsf{E}^n)$-structure where Υ is a finite subgroup of $\mathsf{O}(n)$ and $\mathbb{R}^n$ the vector space of translations, as will be discussed in §6.4. The classification of similarity manifolds[1] asserts every similarity structure on a closed manifold refines to either a Euclidean structure or a structure modeled on $(\mathbb{R}^+ \times \mathsf{O}(n), \mathbb{R}^n \setminus \{\mathbf{0}\})$. The latter manifolds are finitely covered by Hopf manifolds.

6.1.2. Homogeneous subdomains. Projective models for affine geometry and various non-Euclidean geometries are examples of this construction, when ϕ embeds X as an open set in X'. For example, every affine structure determines a projective structure, using the embedding

$$\big(\mathsf{A}^n, \mathsf{Aff}(\mathsf{A}^n)\big) \hookrightarrow \big(\mathsf{P}^n, \mathsf{Aut}(\mathsf{P}^n)\big)$$

of affine geometry in projective geometry.

The polarities discussed in § 3.2.3 provide further examples. For example, elliptic-geometry structures arise as projective structures whose holonomy preserve an elliptic polarity — these identify with Riemannian structures of constant positive curvature. In that case ϕ is the identity map and the elliptic structure refines the projective structure. Similarly, *contact*

[1] This was announced by Kuiper [**231**] with an erroneous proof. Correct proofs were found much later, independently by Fried [**135**] and Reisher–Vaisman [**331**]. (Compare §11.4.)

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projective structures arise on manifolds when the holonomy preserves a null polarity, as in Exercise 3.2.9. They only occur on odd-dimensional manifolds, and the structure group is $\mathsf{PSp}(2m, \mathbb{R}) < \mathsf{PGL}(2m)$ and are briefly discussed in §6.6.

Hyperbolic structures arise from projective structures whose holonomy preserves a hyperbolic polarity of index 1, that is, $G = \mathsf{PO}(n, 1) < \mathsf{PGL}(n + 1, \mathbb{R})$. In this case ϕ embeds H^n as the convex region bounded by the invariant quadric $\mathbf{Q}$. *de Sitter geometry* arises when X is the nonconvex component dS^n of $\mathsf{P}^n \setminus \mathbf{Q}$. By Calabi–Markus [**74**], closed manifolds do not admit structures modeled on $\big(\mathsf{PO}(n, 1), \mathsf{dS}^n\big)$.

Using the Klein model of hyperbolic geometry

$$\big(\mathsf{H}^n, \mathsf{PO}(n, 1)\big) \hookrightarrow \big(\mathsf{P}^n, \mathsf{Aut}(\mathsf{P}^n)\big)$$

every hyperbolic structure determines a projective structure. Using the inclusion of the projective orthogonal group $\mathsf{PO}(n+1) \subset \mathsf{PGL}(n+1; \mathbb{R})$ one sees that every elliptic structure determines a projective structure. Since every surface admits an elliptic, Euclidean or hyperbolic structure, we have the following:

Theorem 6.1.1. Every surface admits an $\mathbb{R}\mathsf{P}^2$-structure.

Similarly, the Poincaré model of 2-dimensional hyperbolic geometry

$$\big(\mathsf{H}^2, \mathsf{PGL}(2, \mathbb{R})\big) \hookrightarrow \big(\mathbb{C}\mathsf{P}^1, \mathsf{PGL}(2, \mathbb{C})\big)$$

embeds the hyperbolic plane in complex-projective 1-dimensional geometry, and every hyperbolic structure on a surface determines a $\mathbb{C}\mathsf{P}^1$-structure. Similarly flat tori and $\mathbb{S}^2$ are $\mathbb{C}\mathsf{P}^1$-manifolds,

Theorem 6.1.2. Every surface admits a $\mathbb{C}\mathsf{P}^1$-structure.

6.2. Hopf manifolds

The basic example of an incomplete affine structure on a closed manifold is a *Hopf manifold.*[2] Consider the domain

$$\Omega \;:=\; \mathbb{R}^n \setminus \{\mathbf{0}\}.$$

The group $\mathbb{R}^+$ of *positive homotheties* (that is, scalar multiplications) acts on Ω properly and freely. Indeed, there is an $\mathbb{R}^+$-equivariant homeomorphism

$$\begin{aligned} \Omega &\xrightarrow{\;h\;} \mathbb{R}^+ \times \mathbb{S}^{n-1} \\ \mathbf{v} &\longmapsto \big(\|\mathbf{v}\|, \mathbf{v}/\|\mathbf{v}\|\big) \end{aligned} \tag{6.1}$$

[2]The 1-dimensional cases were introduced in §5.5.2.

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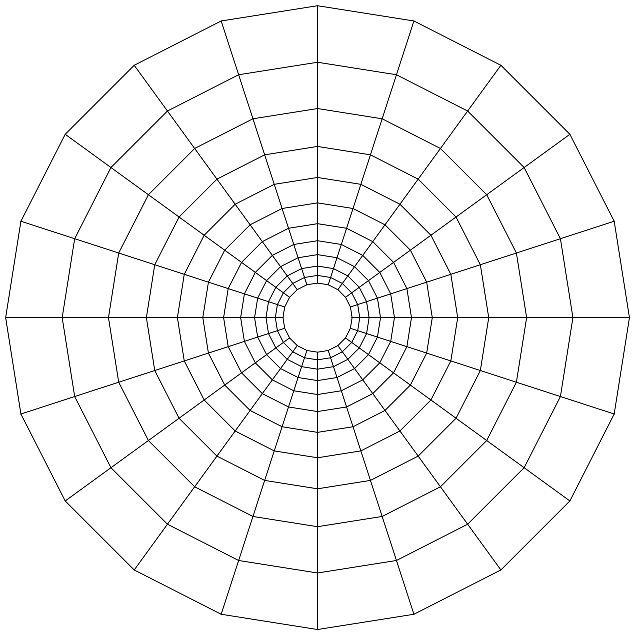

Figure 6.1. The development of a 2-dimensional Hopf manifold, whose holonomy is a pure homothety. This is an incomplete similarity manifold.

where $\mathbb{R}^+$ acts by multiplication on the first factor and identically on the second. Clearly the affine structure on Ω is incomplete. If $\lambda \in \mathbb{R}$ and $\lambda > 1$, then the cyclic group $\langle\lambda\rangle$ is a discrete subgroup of $\mathbb{R}^+$ and the quotient $\Omega/\langle\lambda\rangle$ is a compact *incomplete* affine manifold M. We shall denote this manifold by Hopf^n_λ.

A geodesic whose tangent vector "points" at the origin will be incomplete; on the manifold M the affinely parametrized geodesic will circle around with shorter and shorter period until in a finite amount of time it will "run off" the manifold, as was discussed in §5.5.2.1. The general theory of completeness of affine manifolds will be discussed in Chapter 8.

If $n = 1$, then M consists of two disjoint copies of the *Hopf circle* $\mathbb{R}^+/\langle\lambda\rangle$. This manifold is an incomplete closed geodesic.[3] For $n > 1$, then M is connected and is diffeomorphic to the product $\mathbb{S}^1 \times \mathbb{S}^{n-1}$. For $n > 2$ both the holonomy homomorphism and the developing map are injective.

If $n = 2$, then M is a torus whose holonomy homomorphism maps $\pi_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}$ onto the cyclic group $\langle\lambda\rangle$. Note that $\widetilde{M} \xrightarrow{\mathsf{dev}} \mathbb{R}^2$ is neither injective nor surjective, although it is a covering map onto its image. For $k \geq 1$ let $\pi^{(k)} \subset \pi$ be the unique subgroup of index k which intersects $\mathsf{Ker}(h) \cong \mathbb{Z}$

[3]Indeed, every incomplete closed geodesic on an affine manifold is isomorphic to a Hopf circle.

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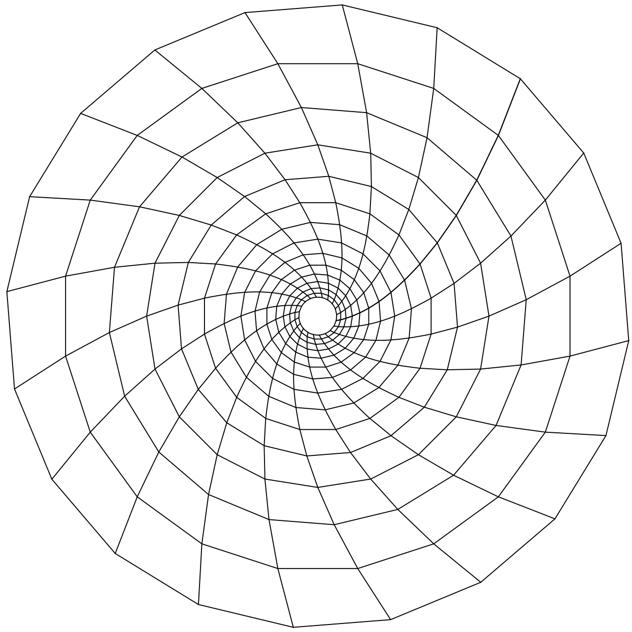

Figure 6.2. Development of a complex 2-dimensional Hopf manifold, whose holonomy is a similarity transformation which is not a pure homothety. It can also be described as a radiant suspension of an elliptic collineation (that is, a rotation) of $\mathbb{R}\mathsf{P}^1$.

in a subgroup of index k. Let $M^{(k)}$ denote the corresponding covering space of M. Then $M^{(k)}$ is another closed affine manifold diffeomorphic to a torus whose holonomy homomorphism is a surjection of $\mathbb{Z} \oplus \mathbb{Z}$ onto $\langle \lambda \rangle$.

Exercise 6.2.1. Show that for $k \neq l$, the two affine manifolds $M^{(k)}$ and $M^{(l)}$ are not isomorphic. (Hint: consider the invariant defined as the least number of breaks of a broken geodesic representing a simple closed curve on M whose holonomy is trivial.) Thus two different affine structures on the same manifold can have the same holonomy homomorphism.

Exercise 6.2.2. Suppose that $\lambda < -1$. Then $M = (\mathbb{R}^n - \{0\})/\langle \lambda \rangle$ is an incomplete compact affine manifold doubly covered by Hopf^n_λ. What is M topologically?

Exercise 6.2.3. Let $A \in \mathsf{GL}(n, \mathbb{R})$ be a *linear expansion,* that is a linear map all of whose eigenvalues have modulus > 1. Suppose that A preserves orientation, that is, $\det(A) > 0$. Then for every $\lambda > 1$, find a homeomorphism

$$\mathbb{R}^n \xrightarrow{\phi} \mathbb{R}^n$$

such that $\phi\big(A(\mathbf{v})\big) = \lambda\phi(\mathbf{v})$. Show that $\big(\mathbb{R}^n \backslash \{\mathbf{0}\}\big)/\langle A \rangle$ is a closed incomplete affine manifold homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$.

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What can you say if $\det(A) < 0$?

Hopf manifolds play an important role in relating projective structures and affine structures on closed manifolds.

6.2.1. The sphere of directions. An important example is the following, which in many contexts is a more useful model space than projective space.

Definition 6.2.4. Let V be an $\mathbb{R}$-vector space with origin $\mathbf{0}$. Define the *sphere of directions* in V as the quotient space of $\mathsf{V} \setminus \{\mathbf{0}\}$ by the group $\mathbb{R}^+$ of positive scalar multiplications, and denote it by:

$$\mathbb{S}(\mathsf{V}) \;:= \big(\mathsf{V} \setminus \{\mathbf{0}\}\big)/\mathbb{R}^+$$

If $\mathsf{V} = \mathbb{R}^{n+1}$, write $\mathbb{S}^n := \mathbb{S}(\mathsf{V})$.

Exercise 6.2.5. Let $\mathsf{V} = \mathbb{R}^{n+1}$ and $G = \mathsf{GL}(n + 1; \mathbb{R})$ its group of linear automorphisms.

- Show that $\mathbb{S}(\mathsf{V}) \approx \mathbb{S}^n$ by explicitly constructing a section of the principal $\mathbb{R}^+$-bundle defined by the quotient $\mathsf{V} \setminus \{\mathbf{0}\} \longrightarrow \mathbb{S}(\mathsf{V})$.
- Construct an explicit double covering $\mathbb{S}(\mathsf{V}) \longrightarrow \mathsf{P}(\mathsf{V})$, realizing the sphere of directions as the universal covering space of projective space.
- Show that the action of the collineation group $\mathsf{PGL}(n + 1, \mathbb{R})$ lifts to the linear action of $\mathsf{GL}(n + 1, \mathbb{R})$ on $\mathbb{S}^n$ and compute the kernel of the action of $\mathsf{GL}(n + 1, \mathbb{R})$ on $\mathbb{S}^n$.

This construction relates to the Hopf manifolds Hopf^n_λ as follows. For each $\lambda > 1$, form the quotient by the cyclic subgroup $\langle\lambda\rangle < \mathbb{R}^+$ rather than all of $\mathbb{R}^+$. The resulting quotient map is a principal $\mathbb{R}^+/\langle\lambda\rangle$-fibration

$$\mathsf{Hopf}^{n+1}_\lambda \longrightarrow \mathbb{S}^n$$

which is $\mathsf{GL}(n + 1, \mathbb{R})$-equivariant.

6.2.2. Radiant affine 2-manifolds. A small modification of the construction of 2-dimensional Hopf manifolds leads to the classification of radiant affine structures on closed 2-manifolds, and, with the classification of affine Lie group structures on the 2-torus, to the full list of closed affine 2-manifolds. Benzécri's theorem 9.1.1 implies every closed affine 2-manifold is homeomorphic to a torus or a Klein bottle. By passing to a covering space we can reduce to affine structures on a 2-torus $\mathbb{T}^2$. The following geometric construction generalizes the construction of a Hopf torus.

These examples are obtained as follows. (Compare Figure 6.3.) Begin with a linear expansion A of $\mathbb{R}^2$. The cyclic group $\langle A\rangle$ acts properly on $\Omega := \mathbb{R}^n \setminus \{0\})$ with quotient M_A homeomorphic to a torus, as in Exercise 6.2.3.

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Here is an explicit description of the development, which will be necessary to describe the modifications needed for all inhomogeneous affine tori. Choose a circle $\mathcal{C}$ centered at $\mathbf{0} \in \mathbb{R}^2$. Then $\mathcal{C}$ and its image $\mathcal{C}' := A(\mathcal{C})$ cobound an annulus $\mathcal{A} \subset \Omega$, which is a fundamental domain for the action of the holonomy group $\langle A \rangle$ on Ω.

Choose a point $\widetilde{x}_0 \in \mathcal{C}$ and an arc

$$\widetilde{x}_0 \overset{\widetilde{a}}{\rightsquigarrow} A(\widetilde{x}_0)$$

in $\mathcal{A}$. Split $\mathcal{A}$ along $\widetilde{a}$ obtaining a quadrilateral $\square$ with four sides:

S1: $\mathcal{C}$ split along $\widetilde{x}_0$;

S2: The original arc $\widetilde{a}$;

S3: $A(\mathcal{C})$ split along $A(\widetilde{x}_0)$;

S4: Another arc corresponding to $\widetilde{a}$.

The annulus $\mathcal{A}$ is the quotient of $\square$ by an identification b which identifies sides S2 and S4. The torus M_A is the quotient of $\mathcal{A}$ by A, which induces an identification of sides S1 and S3.

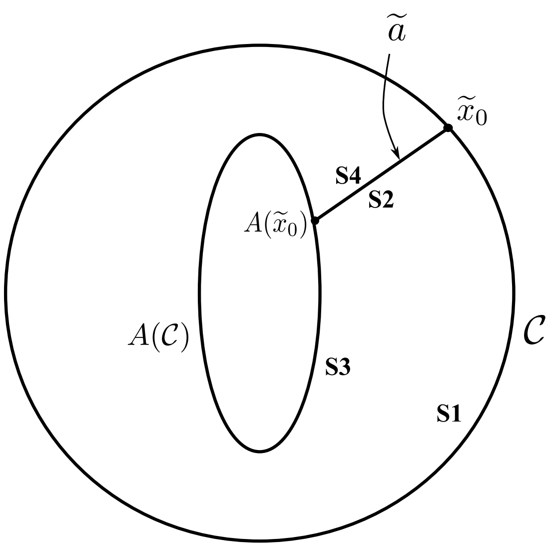

Figure 6.3. An annulus which is combinatorially a quadrilateral, with identification of its sides. Two of the four sides of the quadrilateral are the outer and inner circles. The other sides are already identified in the plane, corresponding to an arc from the outer boundary to the inner boundary.

The image x_0 of the vertex $\widetilde{x}_0$ of $\square$ serves as a basepoint in M_A, and the fundamental group $\pi_1(M_A, x_0)$ is free abelian. Relative homotopy classes of

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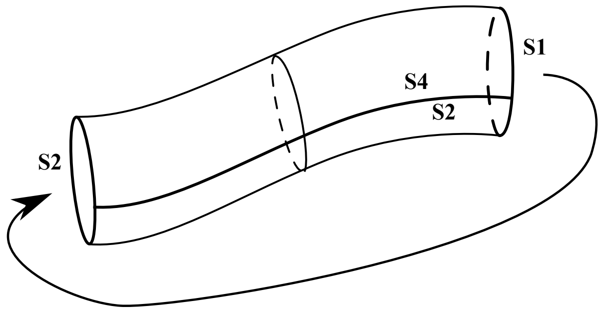

Figure 6.4. Identifying the two boundary components of an annulus.

the based loops corresponding to S1 and S2 define a pair of generators a, b of $\pi_1(M_A, x_0)$.

A model for the universal covering space $\widetilde{M_A}$ of M_A is then the quotient of $\square \times \pi_1(M_A, x_0)$ by identifications described above. The mapping of $\square$ into Ω extending the embedding on the interior of $\square$ generates a developing map $\widetilde{M_A} \twoheadrightarrow \Omega$. The corresponding holonomy homomorphism h maps a to A and b to the identity.

One can modify this construction in various ways. One modification involves passing to an n-fold covering space with the "same holonomy." That is, one passes to the covering space $\widetilde{M}/\langle a, b^n\rangle$ which unwinds in the direction with trivial holonomy. These manifolds are all quotients of the n-fold covering space $\Omega^{(n)}$ of Ω.

All of these holonomy groups are cyclic. The developing map factors through the covering space $\Omega^{(n)} \twoheadrightarrow \Omega$.

However we can modify these structures so that the holonomy of b is nontrivial, and find examples where the holonomy homomorphism is *injective.* Choose an affine transformation β which commutes with A; since $\mathbf{0}$ is the unique point fixed by A, the affine transformation β is necessarily *linear.* We can replace Ω by the quotient Ω_β of $\square \times \pi$ by identifications generated by taking S1 to S2 by β. Equivalently, Ω_β is the quotient of the universal covering space $\widetilde{\Omega}$ by the cyclic group $\langle b \circ \widetilde{\beta}\rangle$ where $\widetilde{\beta}$ is the mapping on $\widetilde{\Omega}$ induced by β.

Since A commutes with β, it defines an affine automorphism A_β of Ω_β, and the quotient

$$M_{A,\beta} := \Omega_\beta/\langle A_\beta\rangle$$

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is a radiant affine torus with holonomy homomorphism

$$\pi \xrightarrow{\mathsf{h}} \mathsf{Aff}(\mathsf{A}^2)$$
$$a \longmapsto A$$
$$b \longmapsto \beta$$

Exercise 6.2.6. Express these manifolds as radiant suspensions[4] of automorphisms of closed $\mathbb{R}\mathsf{P}^1$-manifolds. (Compare Exercise 6.2.3.)

Every closed orientable affine 2-manifold which is *not* covered by an affine Lie group is one of these manifolds.

Exercise 6.2.7. Find an example of an affine Lie group which *is* one of these manifolds.

Exercise 6.2.8. Characterize which complete affine structures on the 2-torus are mapping tori of affine automorphisms of the complete affine manifold $\mathbb{R}/\mathbb{Z}$.

These turn out to be the only affine structures that are *not homogeneous:* indeed every other affine structure on a 2-torus is an affine commutative Lie group.

6.2.3. Hopf tori. There is another point of view concerning Hopf manifolds in dimension two. We may explicitly represent the 2-torus $\mathbb{T}^2$ as a quotient $\mathbb{C}/\Lambda$ where $\Lambda \subset \mathbb{C}$ is a lattice. The complex exponential map $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times$ is a universal covering space having the property that

$$\exp \circ\, \tau(z) \;=\; e^z \cdot \,\exp$$

where $\tau(z)$ denotes translation by $z \in \mathbb{C}$. For various choices of lattices Λ, the exponential map

$$\tilde{M} = \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times$$

is a developing map for a (complex) affine structure on M with holonomy homomorphism

$$\pi \cong \Lambda \xrightarrow{\exp} \exp(\Lambda) \hookrightarrow \mathbb{C}^\times \subset \mathsf{Aff}(\mathbb{C})$$

We denote this affine manifold by $\exp(\mathbb{C}/\Lambda)$; it is an incomplete complex affine 1-manifold or equivalently an incomplete similarity 2-manifold.[5] Every compact incomplete orientable similarity manifold is equivalent to an $\exp(\mathbb{C}/\Lambda)$ for a unique lattice $\Lambda \subset \mathbb{C}$. Taking $\Lambda \subset \mathbb{C}$ to be the lattice generated by $\log \lambda$ and $2\pi i$ we obtain the Hopf manifold Hopf^2_λ. More generally the lattice generated by $\log \lambda$ and $2k\pi i$ corresponds to the k-fold covering space of Hopf^2_λ described above.

[4]Radiant suspensions will be defined in §6.5.3

[5]Figures 6.3 and 6.4 depict the combinatorial identifications for obtaining $\mathbb{T}^2$ in this way.

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Finite cyclic quotients of Hopf^2_λ can be thought of as "fractional" covering spaces of the Hopf manifold obtained from the lattice generated by $\log\lambda$ and $2\pi i/n$ for $n > 1$. These manifolds admit n-fold covering spaces by Hopf^2_λ.

The affine manifold $M = \exp(\mathbb{C}/\Lambda)$ admits no closed geodesics if and only if $\Lambda \cap \mathbb{R} = \{0\}$. Note that the exponential map $\mathbb{C}/\Lambda \longrightarrow M$ defines an biholomorphism of Riemann surfaces which is *n*ot an isomorphism of affine manifolds. We already saw this in the case of Hopf circles in §5.5.2 where the (real) exponential map defines an analytic diffeomorphism between the complete Euclidean manifold $\mathbb{R}/\ell\mathbb{Z}$ and the incomplete affine manifold $\mathbb{R}^+/\langle e^\lambda\rangle$.[6]

6.3. Cartesian products and fibrations

Another way of constructing new structures from old ones is through Cartesian products. The following is due to Benzécri [**46**].

Exercise 6.3.1 (Products of affine manifolds)**.** Let M^m, N^n be affine manifolds.

(1) Show that the Cartesian product $M^m \times N^n$ has a natural affine structure.
(2) Show that $M \times N$ is complete if and only if both M and N are complete.
(3) Show that $M \times N$ is radiant if and only if both M and N are radiant.

For projective structures, the situation is somewhat different:

Exercise 6.3.2. Let $M_1, \dots, M_r$ be manifolds with projective structures.

(1) Find the simplest examples where $M_1 \times \cdots \times M_r$ admits *no* projective structure.
(2) Let $\mathbb{T}^{r-1}$ be an $r-1$-dimensional torus. Find a projective structure on the Cartesian product $M_1 \times \cdots \times M_r \times \mathbb{T}^{r-1}$.

Closely related to Cartesian products is the operation of pulling back geometric structures by equivariant fibrations.

One can also *pull back* geometric structures by *fibrations* of geometries as follows. Let (G, X) be a homogeneous space and suppose that $X' \xrightarrow{\phi} X$ is a fibration with fiber F and that $G' \xrightarrow{\Phi} G$ is a homomorphism such that

[6] These are all isomorphisms of the corresponding affine Lie groups.

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for each $g' \in G'$ the diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X' \\ \phi\big\downarrow & & \big\downarrow\phi \\ X' & \xrightarrow[\Phi(g')]{} & X' \end{array}$$

commutes.

Suppose that M is a (G, X)-manifold, with a universal covering space $\widetilde{M} \xrightarrow{\mathbf{\Pi}} M$ with group of deck transformations π and developing pair (dev, h). Then the pullback $\mathsf{dev}^*\phi$ is an F-fibration $\widehat{M'} \longrightarrow \widetilde{M}$. The induced map $M' \xrightarrow{\mathsf{dev}'} X'$ is a local diffeomorphism and thus a developing map for an (G', X')-structure on $\widehat{M'}$. We summarize these maps in the following commutative diagram:

$$\begin{array}{ccc} \widehat{M'} & \xrightarrow{\mathsf{dev}'} & X' \\ \big\downarrow & & \big\downarrow\phi \\ \widetilde{M} & \xrightarrow[\mathsf{dev}]{} & X \end{array}$$

Suppose that the holonomy representation $\pi \xrightarrow{h} G$ lifts to $\pi \xrightarrow{h'} G'$. (In general the question of whether h lifts will be detected by certain invariants in the cohomology of M.) Then h' defines an extension of the action of π on $\widetilde{M}$ to $\widehat{M'}$ by (G', X')-automorphisms. Since the action of π on $\widehat{M'}$ is proper and free, the quotient $M' = \widehat{M'}/\pi$ is an (G', X')-manifold. Moreover the fibration $\widehat{M'} \longrightarrow \widetilde{M}$ descends to an F-fibration $M' \longrightarrow M$.

6.3.1. Mapping tori and suspensions. Many examples can be understood in terms of two *suspension* or *mapping torus* constructions. Indeed most 2-dimensional affine manifolds arise in this way. These constructions, which produce affine n-manifolds, are the *parallel suspension* of an affine automorphism of an affine $n-1$-manifold and the *radiant suspension* of a projective automorphism of a projective $n-1$-manifold.

Namely, let M be a smooth manifold and $M \xrightarrow{f} M$ a diffeomorphism. The mapping torus of f is defined to be the quotient $N = \mathbf{N}_f(M)$ of the product $M \times \mathbb{R}$ by the $\mathbb{Z}$-action defined by

$$(x, t) \xmapsto{n} (f^{-n}x, t + n)$$

It follows that dt defines a nonsingular closed 1-form ω on N tangent to the fibration

$$N \xrightarrow{t} \mathbb{S}^1 \approx \mathbb{R}/\mathbb{Z}.$$

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Furthermore, the vector field $\frac{\partial}{\partial t}$ on $M \times \mathbb{R}$ defines a vector field S_f on N, the *suspension* of the diffeomorphism $M \xrightarrow{f} M$. The dynamics of f mirrors the dynamics of S_f: there is a natural correspondence between the orbits of f and the trajectories of S_f. The embedding $M \hookrightarrow M \times \{0\}$ is transverse to the vector field S_f and each trajectory of S_f meets M. Such a hypersurface is called a *cross-section* to the vector field. Given a cross-section M to a flow $\{\xi_t\}_{t\in\mathbb{R}}$, then (after possibly reparametrizing $\{\xi_t\}_{t\in\mathbb{R}}$), the flow can be recovered as a suspension. Namely, given $x \in M$, let $f(x)$ equal $\xi_t(x)$ for the smallest $t > 0$ such that $\xi_t(x) \in M$, that is, the first-return map or Poincaré map for $\{\xi_t\}_{t\in\mathbb{R}}$ on M. For the theory of cross-sections to flows see Fried [**136**].

Exercise 6.3.3. Show that if N fibers over $\mathbb{S}^1$, then N is diffeomorphic to a mapping torus.

A fibration $N \xrightarrow{t} \mathbb{S}^1$ induces a everywhere nonzero closed 1-form with integral periods. Conversely, if N admits such a closed 1-form, then it arises from a fibration over $\mathbb{S}^1$. Tischler [**327**] proved that a closed manifold which admits an everywhere nonzero closed 1-form fibers over $\mathbb{S}^1$.

6.3.2. Parallel suspensions. Let M be an affine manifold and $f \in \mathsf{Aff}(M)$ an affine automorphism. We shall define an affine manifold $\mathbf{N}_f$ with a parallel vector field S_f, parallel 1-form ω_f and *affine* cross-section $\Sigma \hookrightarrow M$ to S_f such that the corresponding Poincaré map is f.

Let $M \times \mathsf{A}^1$ be the Cartesian product with the product affine structure and let $M \times \mathsf{A}^1 \xrightarrow{t} \mathsf{A}^1$ be an affine coordinate on the second factor. Then the map

$$\begin{aligned} M \times \mathsf{A}^1 &\xrightarrow{\tilde{f}} M \times \mathsf{A}^1 \\ (x,t) &\longmapsto (f^{-1}(x), t+1) \end{aligned}$$

is affine and generates a free proper $\mathbb{Z}$-action on $M \times \mathsf{A}^1$, which t-covers the action of $\mathbb{Z}$ on $\mathsf{A}^1 \cong \mathbb{R}$ by translation. Let N be the corresponding quotient affine manifold. Then $\frac{\partial}{\partial t}$ is a parallel vector field on $M \times \mathsf{A}^1$ invariant under $\tilde{f}$ and thus defines a parallel vector field S_f on N. Similarly the parallel 1-form dt on $M \times \mathsf{A}^1$ defines a parallel 1-form ω_f on M for which $\omega_f(S_f) = 1$. For each $t \in \mathsf{A}^1/\mathbb{Z}$, the inclusion $M \times \{0\} \hookrightarrow N$ defines a cross-section to S_f, and is an affine submanifold of N. We call (M, S_f) the *parallel suspension* or *affine mapping torus* of the affine automorphism (Σ, f).

Exercise 6.3.4. Express the complete affine structures on the 2-torus as parallel suspensions of affine automorphisms of the complete affine manifold $\mathbb{R}/\mathbb{Z}$.

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The following exercise generalizes this construction by replacing $\mathsf{A}^1/\mathbb{Z}$ by an arbitrary affine manifold B.

Exercise 6.3.5. Suppose that B and F are affine manifolds and that

$$\pi_1(B) \xrightarrow{\phi} \mathsf{Aff}(F)$$

is an action of $\pi_1(B)$ on F by affine automorphisms. The flat F-bundle over B with holonomy ϕ is defined as the quotient of $\widetilde{B} \times F$ by the diagonal action of $\pi_1(B)$ given by deck transformations on $\widetilde{B}$ and by ϕ on F. Show that the total space E is an affine manifold such that the fibration $E \longrightarrow B$ is an affine map and the flat structure (the foliation of E induced by the foliation of $\widetilde{B} \times N$ by leaves $\widetilde{B} \times \{y\}$, for $y \in F$) is an affine foliation.

6.3.3. Radiant suspensions. The second construction involves the fibration of $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ over $\mathbb{R}\mathsf{P}^n$. Namely, the quotient of $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ by scalar multiplication by $\mathbb{R}^\times$ defines a principal $\mathbb{R}^\times$-fibration

$$\mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \longrightarrow \mathbb{R}\mathsf{P}^n$$

equivariant with respect to the projectivization homomorphism

$$\mathsf{GL}(n+1, \mathbb{R}) \longrightarrow \mathsf{PGL}(n+1, \mathbb{R}) = \mathsf{Aut}(\mathbb{R}\mathsf{P}^n).$$

Pulling back this fibration by the developing map of an $\mathbb{R}\mathsf{P}^n$-manifold M yields a radiant affine structure on the total space of a principal $\mathbb{R}^\times$-bundle over $\widetilde{M}$.

The general construction[7] of §6.3 applies to construct an affine structure on the total space M' of a principal $\mathbb{R}^+$-bundle over M with holonomy representation $\tilde{h}$. The radiant vector field $\mathsf{Rad}_{M'}$ generates the (fiberwise) action of $\mathbb{R}^+$; this action of $\mathbb{R}^\times$ on M' is affine, given locally in affine coordinates by homotheties.

Since $\mathbb{R}^+$ is contractible, every principal $\mathbb{R}^+$-bundle is trivial (although there is in general no preferred trivialization). Choose any $\lambda > 1$; then the cyclic group $\langle\lambda\rangle \subset \mathbb{R}^+$ acts properly and freely on M' by affine transformations. We denote the resulting affine manifold by M'_λ and observe that it is homeomorphic to $M \times \mathbb{S}^1$. (Alternatively, one may work directly with the Hopf manifold $\mathsf{Hopf}_\lambda^{n+1}$ and its $\mathbb{R}^\times$-fibration $\mathsf{Hopf}_\lambda^{n+1} \longrightarrow \mathbb{R}\mathsf{P}^n$.) We thus obtain:

Proposition 6.3.6 (Benzécri [**46**], §2.3.1)**.** Suppose that M is an $\mathbb{R}\mathsf{P}^n$-manifold. Let $\lambda > 1$. Then $M \times \mathbb{S}^1$ admits a radiant affine structure for which the trajectories of the radiant vector field are all closed geodesics each affinely isomorphic to the Hopf circle $\mathbb{R}^+/\langle\lambda\rangle$.

[7]This construction is due to Benzécri [**46**] where he calls the manifolds *variétés coniques affines.* He observes there that this construction defines an embedding of the category of $\mathbb{R}\mathsf{P}^n$-manifolds into the category of $(n+1)$-dimensional affine manifolds.

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Since every (closed) surface admits an $\mathbb{R}\mathsf{P}^2$-structure, we obtain:

Corollary 6.3.7 (Benzécri [**46**]). Let Σ be a closed surface. Then $\Sigma \times \mathbb{S}^1$ admits an affine structure.

If Σ is a closed hyperbolic surface, the affine structure on $M = \Sigma \times \mathbb{S}^1$ can be described as follows. A developing map maps the universal covering of M onto the convex cone

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 < 0, z > 0\}$$

which is invariant under the identity component G of $\mathsf{SO}(2,1)$. The group $G \times \mathbb{R}^+$ acts transitively on Ω with isotropy group $\mathsf{SO}(2)$. Choosing a hyperbolic structure on Σ determines an isomorphism of $\pi_1(\Sigma)$ onto a discrete subgroup Γ of G; then for each $\lambda > 1$, the group $\Gamma \times \langle \lambda \rangle$ acts properly and freely on Ω with quotient the compact affine 3-manifold M.

Exercise 6.3.8. A $\mathbb{C}\mathsf{P}^n$-structure is a geometric structure modeled on complex projective space $\mathbb{C}\mathsf{P}^n$ with coordinate changes locally from the projective group $\mathsf{PGL}(n+1;\mathbb{C})$). Let M be a $\mathbb{C}\mathsf{P}^n$-manifold.

- Show that there is a $\mathbb{T}^2$-bundle over M which admits a complex affine structure and an $\mathbb{S}^1$-bundle over M which admits an $\mathbb{R}\mathsf{P}^{2n+1}$-structure.
- Show that this is a *contact* $\mathbb{R}\mathsf{P}^{2n+1}$-structure as defined in Exercise 6.6.1.

Compare Guichard–Wienhard [**180**].

Proposition 6.3.9. Let M be a compact $\mathbb{R}\mathsf{P}^n$-manifold whose holonomy homomorphism lifts to

$$\pi_1(M) \xrightarrow{\widetilde{h}} \mathsf{GL}(n+1,\mathbb{R}).$$

Choose a scalar $\lambda \in \mathbb{R} \setminus \{\pm 1\}$ and and a projective automorphism $f \in \mathsf{Aut}(M)$ preserving $\widetilde{h}$. Then $\exists$ a radiant affine manifold

$$\big(N_{f,\lambda,\widetilde{h}}, \mathsf{Rad}\big)$$

and a cross-section

$$S \overset{\iota}{\hookrightarrow} N_f$$

to Rad_{N_f} such that the Poincaré map for ι equals $\iota^{-1} \circ f \circ \iota$. Furthermore the fibers are pairs of Hopf circles $\mathbb{R}^\times/\langle\lambda\rangle$.

In other words, the mapping torus of a projective automorphism of an compact $\mathbb{R}\mathsf{P}^n$-manifold admits a radiant affine structure.

Exercise 6.3.10. Let Σ be a closed surface. Then $\Sigma \times \mathbb{S}^1$ admits an affine structure.

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Proof. Let $\mathbb{S}^n$ be the double covering of $\mathbb{R}\mathsf{P}^n$ and let

$$\mathbb{R}^{n+1} \setminus \{0\} \xrightarrow{\Phi} \mathbb{S}^n$$

be the corresponding principal $\mathbb{R}^+$-fibration. Let E be the principal $\mathbb{R}^+$-bundle over M constructed in §6.2.1 and choose a section $M \xrightarrow{\sigma} E$. Let $\{\xi_t\}_{t\in\mathbb{R}}$ be the radiant flow on E. Denote by $\{\widetilde{\xi_t}\}_{t\in\mathbb{R}}$ the radiant flow on the induced principal $\mathbb{R}^+$-bundle $\widetilde{E}$ over $\widetilde{M}$.

Let (dev', h') be a development pair for $\widetilde{E}$. Then f lifts to an affine automorphism $\tilde{f}$ of $\widetilde{E}$. Furthermore $\exists$ a projective automorphism $g \in \mathsf{GL}(n+1;\mathbb{R})/\mathbb{R}^+$ of $\mathbb{S}^n$ such that

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{\mathsf{dev}'} & \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \\ {\scriptstyle \tilde{f}}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \widetilde{E} & \xrightarrow[\mathsf{dev}']{} & \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \end{array}$$

commutes. Choose a compact set $K \subset \widetilde{E}$ such that

$$\pi_1(M) \cdot K = \widetilde{E}.$$

Let $\tilde{K} \subset \widetilde{N}$ be the image of K under a lift of σ to a section $\widetilde{M} \longrightarrow \widetilde{N}$. Then

$$\tilde{K} \ \cap \ \tilde{f}\tilde{\xi}_t(\tilde{K}) = \emptyset$$

whenever $t > t_0$, for some t_0. It follows that the affine automorphism $\xi_t\tilde{f}$ generates a free and proper affine $\mathbb{Z}$-action on N for $t > t_0$. We denote the quotient by N. In terms of the trivialization of $N \longrightarrow M$ arising from σ, the quotient of N by this $\mathbb{Z}$-action is diffeomorphic to the mapping torus of f. Furthermore the section σ defines a cross-section $S \hookrightarrow N$ to Rad_N whose Poincaré map corresponds to f. □

We call the radiant affine manifold (M, Rad_M) the *radiant suspension* of the pair (Σ, f).

Exercise 6.3.11. Express the Hopf manifolds of Exercise 6.2.3 as radiant suspensions of the automorphism of $\mathbb{S}^{n-1}$ given by the linear expansion A of $\mathbb{R}^n$.

Hopf manifolds provide another example of a refined geometric structure, which arise in the classification of similarity structures on closed manifolds (§11.4).

Exercise 6.3.12. Let $X = \mathsf{E}^n \setminus \{\mathbf{0}\}$ and $G \subset \mathsf{Sim}(\mathsf{E}^n)$ the stabilizer of $\mathbf{0}$. Let M be a compact (G, X)-manifold with holonomy group $\Gamma \subset G$.

- Prove that $G \cong \mathbb{R}^+ \times \mathsf{O}(n)$.
- Find a G-invariant Riemannian metric g_0 on X.

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   – Is $\mathsf{g_0}$ the Hessian of a function?[8]
   – Is the lift of $\mathsf{g_0}$ to the universal cover the Hessian of a function?
- Suppose that $n > 2$. Prove that $M \cong \Gamma\backslash X$, and that M admits a finite covering space isomorphic to a Hopf manifold.
- Suppose $n = 2$. Find an example where M is *not* isomorphic to the quotient $\Gamma\backslash X$.

6.4. Closed Euclidean manifolds

Recall that a *flat torus* is a Euclidean manifold of the form E^n/Γ, where Γ is a *lattice* of translations. We can regard flat tori as (G, X)-manifolds where both X and G are the same vector space, and G is acting on X by translation. In fact, every closed (G, X)-manifold is a flat torus.

Bieberbach's structure theorem is essentially a qualitative structure theorem *classifying* closed Euclidean manifolds. It states that every closed Euclidean manifold is *finitely covered* by a flat torus. That is, given a closed Euclidean manifold $M = \mathsf{E}^n/\Gamma$, there is a flat torus N and a finite subgroup $\hat{\Upsilon} \subset \mathsf{Isom}(N)$ such that $\hat{\Upsilon}$ acts freely on N and M is isometric to the quotient manifold $N/\hat{\Upsilon}$. Furthermore the degree of the covering (which equals the cardinality $\#\hat{\Upsilon}$) is bounded in terms of the dimension. For an extensive discussion see Charlap [**84**].

From the viewpoint of enlarging and refining geometric structures, this result may be stated as follows. The finite group $\hat{\Upsilon} \cong \mathsf{L}(\Gamma)$ the linear holonomy group, which faithfully represents as a subgroup $\Upsilon < \mathsf{O}(n) < \mathsf{Isom}(\mathsf{E}^n)$. Let $\Upsilon\mathsf{V}$ be the subgroup of $\mathsf{Isom}(\mathsf{E}^n)$ generated by the translation group V and Υ. Then Bieberbach's structure theorem can be restated as follows:

Theorem 6.4.1. Every closed $\big(\mathsf{Isom}(\mathsf{E}^n), \mathsf{E}^n\big)$-manifold has a $(\Upsilon\mathsf{V}, \mathsf{E}^n)$ structure for some finite subgroup $\Upsilon \subset \mathsf{O}(n)$.

Furthermore only finitely many isomorphism classes of Υ exist in any given dimension. This implies that, for given n, closed Euclidean n-manifolds fall into only finitely many topological types. In contrast infinitely many topological types of closed flat affine (indeed flat Lorentzian) exist in dimension 3 (Auslander–Markus [**13**], see §8.6.2.2).

The first example of a closed Euclidean manifold which is not a flat torus is a *Euclidean Klein bottle.* One can easily construct it as the *parallel suspension* of a free isometric orientation-reversing involution of the *Euclidean circle* $\mathsf{E}^1/\mathbb{Z}$.

[8]Hessians of functions are defined in §B.2.

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Exercise 6.4.2. Compute the affine holonomy group of this complete affine surface. Show that the surface has the same rational homology as $\mathbb{S}^1$.

In general affine automorphisms of affine manifolds can display quite complicated dynamics and thus the flows of parallel vector fields and radiant vector fields can be similarly complicated. For example, any element of $\mathsf{GL}(2;\mathbb{Z})$ acts affinely on the flat torus $\mathbb{R}^2/\mathbb{Z}^2$; the most interesting of these are the hyperbolic elements of $\mathsf{GL}(2;\mathbb{Z})$ which determine Anosov diffeomorphisms on the torus. Their suspensions thus determine Anosov flows on affine 3-manifolds which are generated by parallel or radiant vector fields. Indeed, it can be shown (Fried [**137**]) that every Anosov automorphism of a nilmanifold M can be made affine for some complete affine structure on M.

As a simple example of this we consider the linear diffeomorphism of the two-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ defined by a hyperbolic element $A \in \mathsf{GL}(2;\mathbb{Z})$. The parallel suspension of A is the complete affine 3-manifold $\mathbb{R}^3/\Gamma$ where $\Gamma \subset \mathsf{Aff}(\mathbb{R}^3)$ consists of the affine transformations

$$\left[\begin{array}{cc|c} A^n & 0 & p \\ 0 & 1 & n \end{array}\right]$$

where $n \in \mathbb{Z}$ and $p \in \mathbb{Z}^2$. Since A is conjugate in $\mathsf{SL}(2;\mathbb{R})$ to a diagonal matrix with reciprocal eigenvalues, Γ is conjugate to a discrete cocompact subgroup of the subgroup of $\mathsf{Aff}(\mathbb{R}^3)$

$$G = \left\{ \left[\begin{array}{ccc|c} e^u & 0 & 0 & s \\ 0 & e^{-u} & 0 & t \\ 0 & 0 & 1 & u \end{array}\right] \;\middle|\; s,t,u \in \mathbb{R} \right\}$$

which acts simply transitively. Since there are infinitely many conjugacy classes of hyperbolic elements in $\mathsf{SL}(2;\mathbb{Z})$ (for example the matrices

$$\begin{bmatrix} n+1 & n \\ 1 & 1 \end{bmatrix}$$

for $n > 1, n \in \mathbb{Z}$ are all non-conjugate), there are infinitely many isomorphism classes of discrete groups Γ. Louis Auslander observed that there are infinitely many homotopy classes of compact complete affine 3-manifolds — in contrast to the theorem of Bieberbach that in each dimension there are only finitely many homotopy classes of compact flat Riemannian manifolds.

Notice that each of these affine manifolds possesses a parallel Lorentz metric and hence is a flat Lorentz manifold. (Auslander–Markus [**13**]). Here is the algebraic construction:

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Exercise 6.4.3. Suppose A is a 2×2 real matrix of determinant d and trace t. Let $\mathsf{J} = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Then

$$B := \mathsf{J}A - A^{\dagger}\mathsf{J}$$

is a symmetric matrix of determinant $4d - t^2$. In particular if $d \neq 0$ and $t \neq \pm 2$, then A is an isometry of the quadratic form defined by B.

6.4.1. A Euclidean $\mathbb{Q}$-homology 3-sphere. Here is an interesting example in dimension 3, which we denote by $\mathbb{S}^3_{\mathbb{Q}}$. This closed Euclidean 3-manifold is *not* a mapping torus.

Consider the group $\Gamma \subset \mathsf{Isom}(\mathsf{E}^3)$ generated by the three isometries

$$A = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$B = \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$C = \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

and Γ is a discrete group of Euclidean isometries which acts properly and freely on $\mathbb{R}^3$ with quotient a compact 3-manifold M. Furthermore there is a short exact sequence

$$\mathbb{Z}^3 \cong \langle A^2, B^2, C^2, ABC \rangle \hookrightarrow \Gamma \xrightarrow{\mathsf{L}} \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

We denote the quotient $\Gamma \backslash \mathsf{E}^3$ by $\mathbb{S}^3_{\mathbb{Q}}$. It is a Euclidean manifold, which has a regular $\mathbb{Z}/2 \oplus \mathbb{Z}/2$-covering space by a torus, and hence admits a complete affine structure.

Exercise 6.4.4. Let $M^3 = \Gamma \backslash \mathsf{E}^3$ as above.

- Show that M^3 has the same rational homology as $\mathbb{S}^3$.
- Compute $H_*(M^3, \mathbb{Z})$.
- Prove that every closed Euclidean 3-manifold is either a parallel suspension of an isometry of a closed Euclidean 2-manifold or is isometric to M^3.

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Later we will see that every affine structure on $\mathbb{S}^3_{\mathbb{Q}}$ must be complete, and indeed a Euclidean structure as above.

6.5. Radiant affine manifolds

The general construction[9] of §6.3 applies to construct an affine structure on the total space M' of a principal $\mathbb{R}^+$-bundle over M with holonomy representation $\tilde{h}$. The radiant vector field $\mathsf{Rad}_{M'}$ generates the (fiberwise) action of $\mathbb{R}^+$; this action of $\mathbb{R}^\times$ on M' is affine, given locally in affine coordinates by homotheties.

Since $\mathbb{R}^+$ is contractible, every principal $\mathbb{R}^+$-bundle is trivial (although there is in general no preferred trivialization). Choose any $\lambda > 1$; then the cyclic group $\langle\lambda\rangle \subset \mathbb{R}^+$ acts properly and freely on M' by affine transformations. We denote the resulting affine manifold by M'_λ and observe that it is homeomorphic to $M \times \mathbb{S}^1$. (Alternatively, one may work directly with the Hopf manifold $\mathsf{Hopf}^{n+1}_\lambda$ and its $\mathbb{R}^\times$-fibration $\mathsf{Hopf}^{n+1}_\lambda \longrightarrow \mathbb{R}\mathsf{P}^n$.) We thus obtain:

Proposition 6.5.1 (Benzécri [**46**], §2.3.1)**.** Suppose that M is an $\mathbb{R}\mathsf{P}^n$-manifold. Let $\lambda > 1$. Then $M \times \mathbb{S}^1$ admits a radiant affine structure for which the trajectories of the radiant vector field are all closed geodesics each affinely isomorphic to the Hopf circle $\mathbb{R}^+/\langle\lambda\rangle$.

Since every (closed) surface admits an $\mathbb{R}\mathsf{P}^2$-structure, we obtain:

Corollary 6.5.2 (Benzécri [**46**])**.** Let Σ be a closed surface. Then $\Sigma \times \mathbb{S}^1$ admits an affine structure.

If Σ is a closed hyperbolic surface, the affine structure on $M = \Sigma \times \mathbb{S}^1$ can be described as follows. A developing maps the universal covering of M onto the convex cone

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 < 0, z > 0\}$$

which is invariant under the identity component G of $\mathsf{SO}(2, 1)$. The group $G \times \mathbb{R}^+$ acts transitively on Ω with isotropy group $\mathsf{SO}(2)$. Choosing a hyperbolic structure on Σ determines an isomorphism of $\pi_1(\Sigma)$ onto a discrete subgroup Γ of G; then for each $\lambda > 1$, the group $\Gamma \times \langle\lambda\rangle$ acts properly and freely on Ω with quotient the compact affine 3-manifold M.

Exercise 6.5.3. A $\mathbb{C}\mathsf{P}^n$-structure is a geometric structure modeled on complex projective space $\mathbb{C}\mathsf{P}^n$ with coordinate changes locally from the project ive group $\mathsf{PGL}(n + 1; \mathbb{C})$. Let M be a $\mathbb{C}\mathsf{P}^n$-manifold.

[9] This construction is due to Benzécri [**46**] where he calls the manifolds *variétés coniques affines.* He observes there that this construction defines an embedding of the category of $\mathbb{R}\mathsf{P}^n$-manifolds into the category of $(n + 1)$-dimensional affine manifolds.

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- Show that there is a $\mathbb{T}^2$-bundle over M which admits a complex affine structure and an $\mathbb{S}^1$-bundle over M which admits an $\mathbb{R}\mathsf{P}^{2n+1}$-structure.
- Show that this is a *contact* $\mathbb{R}\mathsf{P}^{2n+1}$-structure as defined in Exercise 6.6.1.

Compare Guichard–Wienhard [**180**].

6.5.1. Geometric structures transverse to a foliation. Suppose that $\mathfrak{F}$ is a foliation of a manifold M; then $\mathfrak{F}$ is locally defined by an atlas of smooth submersions $U \longrightarrow \mathbb{R}^q$ for coordinate patches U. An (G,X)-*atlas transverse to* $\mathfrak{F}$ is defined to be a collection of coordinate patches U_α and coordinate charts

$$U_\alpha \xrightarrow{\psi_\alpha} X$$

such that for each pair (U_α, U_β) and each component $C \subset U_\alpha \cap U_\beta$ there exists an element $g_C \in G$ such that

$$g_C \circ \psi_\alpha = \psi_\beta$$

on C. An (G,X)-*structure transverse to* $\mathfrak{F}$ is a maximal (G,X)-*atlas transverse to* $\mathfrak{F}$. Consider an (G,X)-structure transverse to $\mathfrak{F}$; then an immersion $\Sigma \xrightarrow{f} M$ which is transverse to $\mathfrak{F}$ induces an (G,X)-structure on Σ.

A foliation $\mathfrak{F}$ of an affine manifold is said to be *affine* if its leaves are parallel affine subspaces (that is, totally geodesic subspaces). It is easy to see that transverse to an affine foliation of an affine manifold is a natural affine structure. In particular if M is an affine manifold and ζ is a parallel vector field on M, then ζ determines a 1-dimensional affine foliation which thus has a transverse affine structure. Moreover if Σ is a cross-section to ζ, then Σ has a natural affine structure for which the Poincaré map $\Sigma \longrightarrow \Sigma$ is affine.

Exercise 6.5.4. Show that the Hopf manifold Hopf^n_λ has an affine foliation with one closed leaf if $n > 1$ (two if $n = 1$) and its complement consists of two Reeb components. Figure 6.5 depicts a Reeb foliation in a covering space of the torus where the two closed leaves are the vertical lines, and the nearby leaves spiral towards it in the quotient. In this case it's defined by a nonsingular vector field defining a *non-proper* $\mathbb{R}$-action, and the leaf space is a non-Hausdorff 1-manifold.

6.5.2. The radiant vector field. A Hopf manifold is the prototypical example of a *radiant affine manifold.* Many properties of Hopf manifolds are shared by radiant structures, and involve the radiant vector field. For example, a closed radiant affine manifold M is always incomplete, and a radiant vector field is always nonsingular. Therefore $\chi(M) = 0$.

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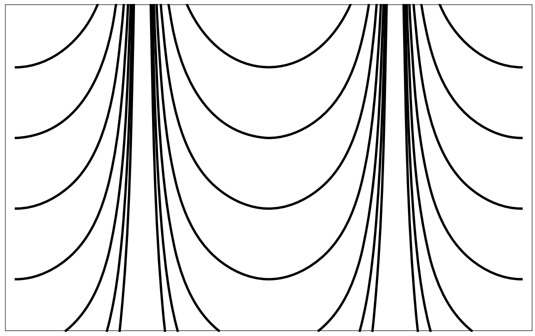

Figure 6.5. The Reeb foliation has a closed leaf for which nearby leaves spiral towards it. This picture is drawn in a covering space of the torus, by parallel identifications of the bounding rectangle.

In this section we discuss general properties of radiant affine structures. Recall, from §1.6.2, that a vector field Rad on an affine manifold M is *radiant* if it is locally equivalent to the *Euler vector field,*

$$\mathsf{Rad}_{\mathbf{0}} := \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i} \tag{6.2}$$

Exercise 1.6.5 gives alternate characterizations of radiance.

Proposition 6.5.5. Let M be an affine manifold with development pair $(\mathsf{dev}, \mathsf{h})$. The following conditions are equivalent:

- The affine holonomy group $\Gamma = \mathsf{h}(\pi)$ fixes a point in A (by conjugation we may assume this fixed point is the origin $\mathbf{0} \in \mathsf{V}$);
- M is isomorphic to a $(\mathsf{V}, \mathsf{GL}(\mathsf{V}))$-manifold;
- M possesses a radiant vector field Rad_M.

If Rad_M is a radiant vector field on M, we shall often refer to the pair (M, Rad_M) as well as a *radiant affine structure.* Then a Γ-invariant radiant vector field $\mathsf{Rad}_{\mathsf{A}}$ on A exists, such that

$$\Pi^* \mathsf{Rad}_M = \mathsf{dev}^* \mathsf{Rad}_{\mathsf{A}}.$$

Theorem 6.5.6. Suppose M is a closed affine manifold. The developing image $\mathsf{dev}(\widetilde{M})$ contains no fixed points of the affine holonomy.[10]

Corollary 6.5.7. Let (M, Rad_M) be a closed radiant affine manifold.

- M is incomplete.
- The radiant vector field Rad_M is nonsingular.
- The Euler characteristic $\chi(M) = 0$.

[10]In other words, M is isomorphic to a $(\mathsf{GL}(\mathsf{V}), \mathsf{V} \setminus \{\mathbf{0}\})$-manifold.

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Proof of Theorem 6.5.6. Choosing affine coordinates $(x^1, \dots, x^n)$ and a developing pair $(\mathsf{dev}, \mathsf{h})$, we may assume that $\mathbf{0}$ is fixed by the affine holonomy $\mathsf{h}\big(\pi_1(M)\big)$, so that $\mathsf{Rad}_{\mathsf{A}} = \mathsf{Rad}_{\mathbf{0}}$ as in (6.2) above. It generates the radiant flow

$$x \xrightarrow{\Psi_t} e^t xk.$$

We prove that $\mathbf{0} \notin \mathsf{dev}(\widetilde{M})$.

We find a vector field $\widetilde{\mathsf{Rad}} \in \mathsf{Vec}(\widetilde{M})$ which is Π-related to $\mathsf{Rad}_{\mathsf{A}}$. Since dev is a local diffeomorphism, the pullback

$$\widetilde{\mathsf{Rad}} := \mathsf{dev}^*(\mathsf{Rad}_{\mathsf{A}})$$

is defined by (0.1). Let $\widetilde{\Phi_t}$ be the corresponding local flow. By the Naturality of Flows (Lee [**244**], Theorem 9.13),

$$\mathsf{dev}\,\widetilde{\Phi_t}(x) \;=\; e^t\,\mathsf{dev}(x).$$

M is radiant, so h preserves $\mathsf{Rad}_{\mathsf{A}}$ and therefore $\widetilde{\mathsf{Rad}}$ is $\pi_1(M)$-invariant. It follows that $\exists \mathsf{Rad}_M \in \mathsf{Vec}(M)$ which is Π-related to $\widetilde{\mathsf{Rad}}$:

$$\Pi^* \mathsf{Rad}_M \;=\; \widetilde{\mathsf{Rad}}.$$

Since M is closed, Rad_M integrates to a global flow $M \xrightarrow{\Pi_t} M$, defined $\forall t \in \mathbb{R}$. Exercise 6.5.8 (below) implies that $\widetilde{\mathsf{Rad}}$ is complete and integrates to a flow

$$\widetilde{M} \xrightarrow{\widetilde{\Phi_t}} \widetilde{M}$$

such that

$$\Pi \circ \widetilde{\Phi_t} \;= \Phi_t$$

Let $M_{\mathbf{0}} := \Pi\big(\mathsf{dev}^{-1}(\mathbf{0})\big)$. Since Π and dev are local diffeomorphisms and $\mathbf{0} \in \mathsf{A}$ is $\mathsf{h}\big(\pi_1(M)\big)$-invariant and discrete, $M_{\mathbf{0}} \subset M$ is discrete. Compactness of M implies that $M_{\mathbf{0}}$ is finite. We show $M_{\mathbf{0}}$ is empty.

Since the only zero of $\mathsf{Rad}_{\mathsf{A}}$ is the origin $\mathbf{0}$, the vector field Rad_M is nonsingular on the complement of $M_{\mathbf{0}}$ Choose a neighborhood U of $M_{\mathbf{0}}$, each component of which develops to a small ball B about $\mathbf{0}$ in A. Let $K \subset\subset \widetilde{M}$ be a compact set such that the saturation $\Pi(K) = M$; then $\exists N \gg 0$ such that

$$e^{-t}(\mathsf{dev}(K)) \subset B$$

for $t \geq N$. Thus $\widetilde{\Phi_t}(K) \subset B$ for $t \leq -N$. Thus U is an attractor for the flow of $-\mathsf{Rad}_M$, that is, $\Phi_{-t}(M) \subset U$ for $t \gg N$. Consequently $M \xrightarrow{\Phi_N} U$ deformation retracts the closed manifold M onto U. Since a closed manifold is not homotopy-equivalent to a finite set, this contradiction implies $M_{\mathbf{0}} = \emptyset$. Therefore $\mathbf{0} \notin \mathsf{dev}(\widetilde{M})$, as desired. □

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Exercise 6.5.8. Let $M \xrightarrow{f} N$ be a local diffeomorphism between smooth manifolds, and $\xi \in \mathsf{Vec}(M), \eta \in \mathsf{Vec}(N)$ be f-related vector fields. Suppose that f is a covering space. Then ξ is complete if and only if η is complete.

Theorem 6.5.9. Let M be a compact radiant manifold. Then M cannot have parallel volume. In other words a closed manifold cannot support a $\big(\mathbb{R}^n, \mathsf{SL}(n;\mathbb{R})\big)$-structure.

Proof. Let $\omega_{\mathsf{A}} = dx^1 \wedge \cdots \wedge dx^n$ be a parallel volume form on A and let ω_M be the corresponding parallel volume form on M, that is, $\Pi^*\omega_M = \mathsf{dev}^*\omega_{\mathsf{A}}$. The interior product

$$\eta_M := \frac{1}{n}\iota_{\mathsf{Rad}_M}(\omega_M)$$

is an $(n-1)$-form on M. Since

$$d\iota_{\mathsf{Rad}_{\mathsf{A}}}(\omega_{\mathsf{A}}) = n\omega_{\mathsf{A}},$$

$d\eta_M = \omega_M$. However, ω_M is a volume form on M and

$$0 < \mathsf{vol}(M) = \int_M \omega_M = \int_M d\eta_M = 0,$$

a contradiction. ☐

Intuitively, the main idea in the proof above is that the radiant flow on M expands the parallel volume uniformly. “Conservation of volume” implies that a compact manifold cannot support both a radiant vector field and a parallel volume form.

Exercise 6.5.10. The first Betti number of a closed radiant affine manifold is always positive. (Hint: Compare Exercise 11.1.2.)

6.5.3. Radiant suspensions. Let (N, Rad_N) be a radiant affine manifold of dimension +1. Transverse to Rad_N is an $\mathbb{R}\mathsf{P}^n$-structure, as follows. In local affine coordinates the trajectories of Rad_N are rays through the origin in $\mathbb{R}^{n+1}$ and projectivization maps coordinate patches submersively into $\mathbb{R}\mathsf{P}^n$. In particular, if S is an n-manifold and $S \xrightarrow{f} N$ is transverse to Rad_N, then f determines an $\mathbb{R}\mathsf{P}^n$-structure on S. The next proposition describes how to reverse this construction.

Proposition 6.5.11. Let M be a compact $\mathbb{R}\mathsf{P}^n$-manifold and $f \in \mathsf{Aut}(M)$ a projective automorphism. Then there exists a radiant affine manifold $(N_f, \mathsf{Rad}_{N_f})$ and a cross-section $S \overset{\iota}{\hookrightarrow} N_f$ to Rad_{N_f} such that the Poincaré map for ι equals $\iota^{-1} \circ f \circ \iota$.

In other words, the mapping torus of a projective automorphism of an compact $\mathbb{R}\mathsf{P}^n$-manifold admits a radiant affine structure.

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Proof. Let $\mathbb{S}^n$ be the double covering of $\mathbb{R}\mathsf{P}^n$ and let

$$\mathbb{R}^{n+1} \setminus \{0\} \xrightarrow{\Phi} \mathbb{S}^n$$

be the corresponding principal $\mathbb{R}^+$-fibration. Let E be the principal $\mathbb{R}^+$-bundle over M constructed in §6.2.1 and choose a section $M \xrightarrow{\sigma} E$. Let $\{\xi_t\}_{t\in\mathbb{R}}$ be the radiant flow on E. Denote by $\{\widetilde{\xi_t}\}_{t\in\mathbb{R}}$ the radiant flow on the induced principal $\mathbb{R}^+$-bundle $\widetilde{E}$ over $\widetilde{M}$.

Let (dev', h') be a development pair for $\widetilde{E}$. Then f lifts to an affine automorphism $\tilde{f}$ of $\widetilde{E}$. Furthermore $\exists$ a projective automorphism $g \in \mathsf{GL}(n+1;\mathbb{R})/\mathbb{R}^+$ of $\mathbb{S}^n$ such that

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{\mathsf{dev}'} & \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \\ \tilde{f}\big\downarrow & & \big\downarrow g \\ \widetilde{E} & \xrightarrow[\mathsf{dev}']{} & \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \end{array}$$

commutes. Choose a compact set $K \subset \widetilde{E}$ such that

$$\pi_1(M) \cdot K = \widetilde{E}.$$

Let $\tilde{K} \subset \widetilde{N}$ be the image of K under a lift of σ to a section $\widetilde{M} \longrightarrow \widetilde{N}$. Then

$$\tilde{K} \ \cap \ \tilde{f}\widetilde{\xi_t}(\tilde{K}) = \emptyset$$

whenever $t > t_0$, for some t_0. It follows that the affine automorphism $\xi_t\tilde{f}$ generates a free and proper affine $\mathbb{Z}$-action on N for $t > t_0$. We denote the quotient by N. In terms of the trivialization of $N \longrightarrow M$ arising from σ, the quotient of N by this $\mathbb{Z}$-action is diffeomorphic to the mapping torus of f. Furthermore the section σ defines a cross-section $S \hookrightarrow N$ to Rad_N whose Poincaré map corresponds to f. ☐

We call the radiant affine manifold (M, Rad_M) the *radiant suspension* of the pair (Σ, f).

Exercise 6.5.12. Express the Hopf manifolds of Exercise 6.2.3 as radiant suspensions of the automorphism of $\mathbb{S}^{n-1}$ given by the linear expansion A of $\mathbb{R}^n$.

6.5.3.1. *Radiant similarity manifolds.* Hopf manifolds provide another example of a refined geometric structure, which arises in the classification of similarity structures on closed manifolds (§11.4).

Exercise 6.5.13. Let $X = \mathsf{E}^n \setminus \{\mathbf{0}\}$ and $G \subset \mathsf{Sim}(\mathsf{E}^n)$ the stabilizer of $\mathbf{0}$. Let M be a compact (G, X)-manifold with holonomy group $\Gamma \subset G$.

- Prove that $G \cong \mathbb{R}^+ \times \mathsf{O}(n)$.
- Find a G-invariant Riemannian metric $\mathsf{g_0}$ on X.

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 – Is $\mathsf{g_0}$ the Hessian of a function?[11]
 – Is the lift of $\mathsf{g_0}$ to the universal cover the Hessian of a function?

- Suppose that $n > 2$. Prove that $M \cong \Gamma\backslash X$, and that M admits a finite covering space isomorphic to a Hopf manifold.
- Suppose $n = 2$. Find an example where M is *not* isomorphic to the quotient $\Gamma\backslash X$.

6.5.3.2. *Cross-sections to the radiant flow.* A natural question is whether every closed radiant affine manifold is a radiant suspension. A radiant affine manifold (M, Rad) is a radiant suspension if and only if the flow of Rad admits a cross-section. Fried [**134**, **138**] constructed a closed affine 6-manifold with diagonal holonomy whose radiant flow admits no cross-section. Choi [**90**] (using work of Barbot [**23**]) proves that every radiant affine 3-manifold is a radiant suspension, and therefore is either a Seifert 3-manifold covered by a product $F \times \mathbb{S}^1$, where F is a closed surface, a nilmanifold or a hyperbolic torus bundle.

In dimensions 1 and 2 all closed radiant manifolds are radiant suspensions. When M is *hyperbolic,* that is, a quotient of a sharp convex cone (see Chapter 12), the existence of the *Koszul* 1*-form* implies that M is a radiant suspension.

6.6. Contact projective structures

Contact projective structures arise on manifolds M^{2m-1} when the holonomy preserves a null polarity, as in Exercise 3.2.9; these structure correspond to the subgroup $\mathsf{PSp}(2m, \mathbb{R}) < \mathsf{PGL}(2m)$.

Exercise 6.6.1. A null polarity θ on P defines a *contact structure* ξ on P. The contact hyperplane $\xi_p \subset \mathsf{T}_p\mathsf{P}$ at $p \in \mathsf{P}$ is just the tangent plane $\mathsf{T}_p\big(\theta(p)\big)$. Conversely, the polar hyperplane $\theta(p)$ is the unique projective hyperplane in P whose tangent space at p equals ξ_p.

A manifold modeled on the projective space with a contact structure arising from a null polarity may be called a *contact projective manifold.* This is a refinement of projective geometry, where the model space is P itself.

Exercise 6.6.2. Let M be a $\mathbb{C}\mathsf{P}^1$-manifold. Show that there is a natural $\mathbb{S}^1$-bundle over M with a natural contact projective structure.

The classification problem for general geometric structures is discussed in detail in §7. As above, a contact projective manifold has an underlying contact structure. Gray's stability theorem (see for example Eliashberg–Mishachev [**124**], §9.5.2, p.95) asserts that contact structures fall into a

[11] Hessians of functions are defined in §B.2.

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discrete set of isotopy classes. Thus it may be more appropriate to fix a closed contact manifold (N^{2d+1}, ξ) and look for contact $\mathbb{R}\mathsf{P}^{2d+1}$-structures compatible with ξ.

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Chapter 7

Classification

Given a topology Σ and a geometry (G, X), how does one determine the various ways of putting (G, X)-structures on Σ? This chapter discusses how to organize the geometric structures on a fixed topology. This is the general *classification problem* for (G, X)-structures.

7.1. Marking geometric structures

We begin with two more familiar and classical cases:

- The moduli space of flat tori;
- The classification of marked Riemann surfaces by *Teichmüller space*.

The latter is only analogous to our classification problem, but plays an important role, both historically and technically, in the study of locally homogeneous structures.

7.1.1. Marked Riemann surfaces. The prototype of this classification problem is the classification of Riemann surfaces of genus g. The *Riemann moduli space* is a space $\mathfrak{M}_g$ whose points correspond to the biholomorphism classes of genus g Riemann surfaces. It admits the structure of a quasiprojective complex algebraic variety. In particular it is a Hausdorff space, with a singular differentiable structure.

In general the set of (G, X)-structures on Σ will not have such a nice structure. The natural space will in general not be Hausdorff, so we must expand our point of view. To this end, we introduce additional structures, called *markings,* such that the marked (G, X)-structures admit a more tractable classification. As before, the prototype for this classification is the Riemann moduli space $\mathfrak{M}_g$, which can be understood as the quotient of the

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Teichmüller space $\mathfrak{T}_g$ (comprising equivalence classes of marked Riemann surfaces of genus g) by the *mapping class group* Mod_g.

Here is the classical context for $\mathfrak{T}_g$ and $\mathfrak{M}_g = \mathfrak{T}_g/\mathsf{Mod}_g$: The fixed topology is a closed orientable surface Σ of genus g. A *marked Riemann surface of genus* g is a pair (M, f) where M is a Riemann surface and $\Sigma \xrightarrow{f} M$ is a diffeomorphism. The *Teichmüller space* is defined as the set of equivalence classes of marked Riemann surfaces of genus g, where two such marked Riemann surfaces (M, f), (M', f') are *equivalent* if and only if there is a biholomorphism $M \xrightarrow{\phi} M'$ such that $\phi \circ f$ is isotopic to ϕ'.

Exercise 7.1.1. Fix a Riemann surface M. The mapping class group

$$\mathsf{Mod}_g := \pi_0\big(\mathsf{Diff}(\Sigma)\big)$$

acts simply transitively on the set of equivalence classes of marked Riemann surfaces (M, f). Thus the Riemann moduli space $\mathfrak{M}_g$ is the quotient of the Teichmüller space $\mathfrak{T}_g$ by the mapping class group Mod_g.

7.1.2. Moduli of flat tori. Another common classification problem concerns flat tori. Recall (§5.5.1 a *flat torus* is a Euclidean manifold of the form $M^n := \mathbb{R}^n/\Lambda$, where $\Lambda \subset \mathbb{R}^n$ is a lattice. A *marking* of M is just a basis of Λ. Clearly the set of marked flat n-tori is the set of bases of $\mathbb{R}^n$, which is a torsor for the group $\mathsf{GL}(n, \mathbb{R})$. (The columns (respectively rows) of invertible $n \times n$ matrices are precisely bases of $\mathbb{R}^n$.)

Exercise 7.1.2. For $M = \mathbb{R}^n/\Lambda$ as above, compute the isometry group (respectively affine automorphism group) of M. Show that two invertible matrices $A, A' \in \mathsf{GL}(n, \mathbb{R})$ define isometric marked flat tori if and only if $A'A^{-1} \in \mathsf{O}(n)$. Show that all flat n-tori are affinely isomorphic.

The deformation space of marked flat tori identifies with the homogeneous space $\mathsf{GL}(n, \mathbb{R})/\mathsf{O}(n)$. The *mapping class group* $\mathsf{Mod}(\mathbb{T}^n)$ of the n-torus $\mathbb{T}^n$ identifies with $\mathsf{GL}(n, \mathbb{Z})$, which acts properly on the deformation space $\mathsf{GL}(n, \mathbb{R})/\mathsf{O}(n)$. The *moduli space of flat tori* in dimension n identifies with the biquotient $\mathsf{GL}(n, \mathbb{Z})\backslash\mathsf{GL}(n, \mathbb{R})/\mathsf{O}(n)$.

7.1.3. Marked geometric manifolds. Now we define the analogous construction for Ehresmann structures. As usual, we choose to work in the smooth category since (G, X)-manifolds carry natural smooth (in fact real analytic) structures, and the tools of differential topology are convenient. However, in general, there are many options, it may be more natural to consider homeomorphisms, or even homotopy equivalences, depending on the context. In dimension two these notions yield equivalent theories. Since our primary interest is in dimension two, we do not discuss the alternative context.

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Definition 7.1.3. Let Σ be a smooth manifold. A *marking* of an (G, X)-manifold M (with respect to Σ) is a diffeomorphism $\Sigma \xrightarrow{f} M$. A *marked* (G, X)*-manifold* is a pair (M, f) where f is a marking of M. Say that two marked (G, X)-manifolds (M, f) and (M', f') are *equivalent* if and only if a (G, X)-isomorphism $M \xrightarrow{\phi} M'$ exists such that $\phi \circ f$ is diffeotopic to ϕ'.

7.1.4. The infinitesimal approach. More useful for computations is another approach, where geometric structures are defined infinitesimally as structures on vector bundles associated to the tangent bundle. For example, a Euclidean manifold M can be alternatively described as a *Riemannian metric* on M with vanishing curvature tensor. Another example is defining a Riemann surface as a 2-manifold together with an *almost complex structure*, that is, a complex structure on its tangent bundle. A third example is defining an affine structure as a connection on the tangent bundle with vanishing curvature tensor. Projective structures and conformal structures can be defined in terms of *projective connections* and *conformal connections,* respectively.

In all of these cases, the underlying smooth structure is fixed, and the geometric structure is replaced by an infinitesimal object as above. The diffeomorphism group acts on this space, and the quotient by the full diffeomorphism group would serve as the moduli space. However, to avoid pathological quotient spaces, we prefer to quotient by the identity component of $\mathsf{Diff}(\Sigma)$. Alternatively define the deformation space of marked structures as the quotient of the space of the infinitesimal objects by the subgroup of $\mathsf{Diff}(\Sigma)$ consisting of diffeomorphisms isotopic to the identity.

The "infinitesimal objects" above are *Cartan connections,* for which we refer to Sharpe [**305**].

7.2. Deformation spaces of geometric structures

Fundamental in the deformation theory of locally homogeneous (Ehresmann) structures is the following principle, first observed in this generality by Thurston [**323**]:

Theorem 7.2.1. Let X be a manifold upon which a Lie group G acts transitively. Let M be a compact (G, X)-manifold with holonomy representation $\pi_1(M) \xrightarrow{\rho} G$.

(1) Suppose that ρ' is sufficiently near ρ in the representation variety $\mathsf{Hom}(\pi_1(M), G)$. Then there exists a (nearby) (G, X)-structure on M with holonomy representation ρ'.

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(2) If M' is a (G,X)-manifold near M having the same holonomy ρ, then M' is isomorphic to M by an isomorphism isotopic to the identity.

Here the topology on marked (G,X)-manifolds is defined in terms of the atlases of coordinate charts, or equivalently in terms of developing maps, or developing sections. (Compare §5.3.2.) In particular one can define a *deformation space* $\mathsf{Def}_{(G,X)}(\Sigma)$ whose points correspond to equivalence classes of marked (G,X)-structures on Σ.

We adopt the viewpoint of developing sections. The (G,X)-structure then consists of three parts: the (G,X)-bundle $\mathcal{E}_M$ over M, its transverse foliation $\mathcal{F}$ defined by the holonomy, its developing section $\mathcal{D}_M$. The *covering homotopy theorem* implies that the (G,X)-bundle $\mathcal{E}_M$ remains constant (up to isomorphism) under a deformation.

The foliation $\mathcal{F}$ is equivalent to a reduction of the structure group of the bundle from G with the classical topology to G with the discrete topology. This set of flat (G,X)-bundles over M identifies with the quotient of the $\mathbb{R}$-analytic set $\mathsf{Hom}(\pi_1(M),G)$ by the action of the group $\mathsf{Inn}(G)$ of inner automorphisms action by left-composition on homomorphisms $\pi_1(M)\to G$. The topology on developing sections is just the C^∞ topology on sections of $\mathcal{E}_M$, for which the $\mathcal{F}$-transversality condition defines an open subset. The equivalence relation of isotopy is sufficiently strong that the quotient space of isotopy classes of pairs $(\mathcal{F},\mathcal{D}_M)$ is "finite-dimensional," and indeed modeled on a real analytic set.

One might *like* to say the holonomy map

$$\mathsf{Def}_{(G,X)}(\Sigma) \xrightarrow{\mathsf{hol}} \mathsf{Hom}\big(\pi_1(\Sigma),G\big)/\mathsf{Inn}(G)$$

is a local homeomorphism, with respect to the quotient topology. Here the quotient topology on $\mathsf{Hom}\big(\pi_1(\Sigma),G\big)/\mathsf{Inn}(G)$ is induced from the classical topology on the $\mathbb{R}$-analytic set $\mathsf{Hom}\big(\pi_1(\Sigma),G\big)$. In many cases this is true (see below) but misstated in [**158**]. However, Kapovich [**209**] and Baues [**31**] observed that this is not quite true, because local isotropy groups acting on $\mathsf{Hom}\big(\pi_1(\Sigma),G\big)$ may not fix marked structures in the corresponding fibers.

In any case, these ideas have an important consequence:

Corollary 7.2.2. Let M be a closed manifold. The set of holonomy representations of (G,X)-structures on M is open in $\mathsf{Hom}(\pi_1(M),G)$ (with respect to the classical topology).

Proof of Theorem 7.2.1. The openness of the holonomy map can now be understood as follows. Deforming the holonomy representation amounts to deforming the foliation $\mathcal{F}$ in a C^1 way. Denote the universal covering of M

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as $\widetilde{M} \xrightarrow{\Pi} M$. Then the pullback $\Pi^*\mathcal{E}_M$ admits a smooth trivialization

$$\Pi^*\mathcal{E}_M \xrightarrow[\approx]{\Upsilon} \widetilde{M} \times X$$

relating the induced foliation $\Pi^*\mathcal{F}$ on $\widetilde{M}$ to the product foliation with leaves $\widetilde{M} \times \{x\}$, that is the foliation defined by the Cartesian projection $\widetilde{M} \times X \longrightarrow X$. Compare Figure 5.3.

The *covering homotopy theorem* (Steenrod [**317**],§11) asserts that, up to isomorphism, deformations of fiber bundles are trivial. Specifically, let $E \to M$ be a fibration, and

$$N \times [0,1] \xrightarrow{f} M$$

a homotopy. Then all $u \in [0,1]$, isomorphisms $f_u^* E \xrightarrow{F_u} f_0^E$ exist, where f_u denotes the map $p \mapsto f(p,u)$. More generally, if u ranges over a contractible parameter space U, all the induced fibrations are isomorphisms.

Exercise 7.2.3. Show that if $E \to M$ is a smooth fibration of smooth manifolds, that the above isomorphisms F_u can be chosen to vary smoothly in t.

Being an $\mathbb{R}$-analytic set, $\mathsf{Hom}(\pi_1(M), G)$ is local contractible. Find a contractible open neighborhood U of ρ and a now a family over U of smooth trivializations Υ_u of the flat bundle with holonomy u over $M \times \{u\}$:

$$\Pi^*\mathcal{E}_M \xrightarrow[\approx]{\Upsilon_W} (\widetilde{M} \times X) \times W.$$

With respect to this trivialization over W, the fibration $\mathcal{E}_M \to M$ remains constant, but the foliations $\mathcal{F}_w$ vary *smoothly* with $t \in W$.

The developing section $\mathcal{D}$ of $\mathcal{E}_M$ is transverse to $\mathcal{F}$. By openness of transversality (since M is compact), it remains transverse to $\mathcal{F}_w$ for w in a (possibly smaller) open neighborhood of ρ. Since transverse sections define geometric structures, this proves that the holonomy map is open.

Conversely, suppose M_1, M_2 are diffeomorphic (G,X)-manifolds which are nearby in the deformation space and have the same holonomy. As above, we can assume the tangent (G,X)-bundles $\mathcal{E}_{M_1}$ and $\mathcal{E}_{M_2}s$ are isomorphic to a fixed (G,X)-bundle $\mathcal{E}_M$. Furthermore since M_1 and M_2 have the same holonomy, their tangent (G,X)-bundles are isomorphic as flat bundles, that is, they have the same transverse foliation $\mathcal{F}$. Their developing sections $\mathcal{D}_{M1}, \mathcal{D}_{M2}$ are close in the C^1-topology and are each transverse to $\mathcal{F}$. In particular $\exists$ a tubular neighborhood of $\mathcal{D}_{M1}$ which is diffeomorphic to a product $M \times B_\epsilon$ by a diffeomorphism mapping the fibers to the sets $\{p\} \times B_\epsilon$ and the leaves to the sets $M \times \{x\}$. This local product structure defines an isotopy between $\mathcal{D}_{M1}, \mathcal{D}_{M2}$ by requiring that under the isotopy each point

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remains in the same leaf. as depicted in Figure 7.1. This proves local injectivity of hol.

The proof of Theorem 7.2.1 is complete. □

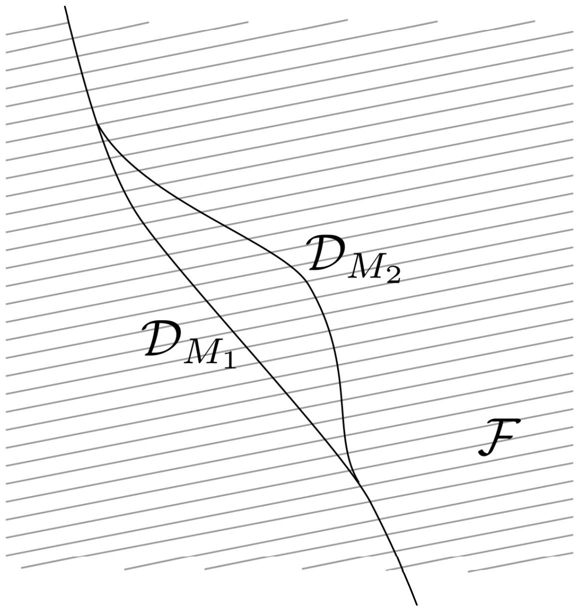

Figure 7.1. The isotopy between nearby $\mathcal{F}$-transverse sections

In the nicest cases, this means that under the natural topology on flat (G, X)-bundles $(X_\rho, \mathcal{F}_\rho)$ over M, the holonomy map hol is a local homeomorphism. Indeed, for many important cases such as hyperbolic geometry (or when the structures correspond to geodesically complete affine connections), hol is actually an embedding.

7.2.1. Historical remarks. Thurston's holonomy principle has a long and interesting history.

The first application is the theorem of Weil [**343**] that the set of *discrete embeddings* of the fundamental group $\pi = \pi_1(\Sigma)$ of a closed surface Σ in $G = \mathsf{PSL}(2, \mathbb{R})$ is open in the quotient space $\mathsf{Hom}(\pi, G)/G$. Indeed, a discrete embedding $\pi \hookrightarrow G$ is exactly a holonomy representation of a *hyperbolic structure* on Σ. The corresponding subset of $\mathsf{Hom}(\pi, G)/G$ is called the *Fricke space* $\mathfrak{F}(\Sigma)$ of Σ. Weil's results are clearly and carefully expounded in Raghunathan [**292**], (see Theorem 6.19), and extended in Bergeron–Gelander [**52**]. Fenchel and Nielsen proved that $\mathfrak{F}(\Sigma) \approx \mathbb{R}^{-\chi(\Sigma)}$; their approach is outlined in §7.4.

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For $\mathbb{C}\mathsf{P}^1$-structures, Theorem 7.2.1 is due to Hejhal [**187**, **188**]; see also Earle [**121**] and Hubbard [**196**]. This venerable subject originated with conformal mapping and the work of Schwarz, and closely relates to the theory of second order (Schwarzian) differential equations on Riemann surfaces. In this case, where $X = \mathbb{C}\mathsf{P}^1$ and $G = \mathsf{PSL}(2,\mathbb{C})$, we denote the deformation space $\mathsf{Def}_{(G,X)}(\Sigma)$ simply by $\mathbb{C}\mathsf{P}^1(\Sigma)$. See Dumas [**117**] and §14 below.

Thurston sketches the intuitive ideas for Theorem 7.2.1 in his unpublished notes [**323**], which contains the first explicit statement of this principle. The first detailed proofs of this fact are Lok [**246**] and Canary–Epstein–Green [**75**]. The proof here follows Goldman [**152**], which was worked out with M. Hirsch and was independently found by A. Haefliger. The ideas in these proofs may be traced to Ehresmann [**123**], although he didn't express them in terms of moduli of structures. Corollary 7.2.2 was noted by Koszul [**230**], Chapter IV, §3, Theorem 3; compare also the discussion in Kapovich [**210**], Theorem 7.2.

7.3. Representation varieties

As this theorem concerns the topology of the space of holonomy representations, we first discuss the space $\mathsf{Hom}(\pi, G)$ and its quotient $\mathsf{Rep}(\pi, G)$. Good general references for this theory are Kapovich [**210**], Lubotzky–Magid [**248**], Raghunathan [**292**] and Sikora [**307**]. For the special case of fundamental groups of compact Kähler manifolds (for example surfaces), see Goldman–Millson [**169**].

We shall assume that G is a (real) Lie group and π is finitely generated. Let $\{\gamma_1, \dots, \gamma_N\}$ be a set of generators.

Exercise 7.3.1. Consider the map

$$\begin{aligned} \mathsf{Hom}(\pi, G) &\longrightarrow G^N \\ \rho &\longmapsto \big(\rho(\gamma_1), \dots, \rho(\gamma_N)\big) \end{aligned}$$

- This map is injective.
- Its image is an analytic subset of G^N defined by

  $$R_\alpha(g_1, \dots, g_N) = 1,$$

  where the R_α are the *relations* among the generators $\gamma_1, \dots, \gamma_N$ of π, regarded as an an analytic map $G^N \xrightarrow{R_\alpha} G$.
- Furthermore the structure of this analytic variety is independent of the choice of generating set.
- The natural action of $\mathsf{Aut}(\pi) \times \mathsf{Aut}(G)$ on $\mathsf{Hom}(\pi, G)$ preserves the analytic structure.

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In many cases, G may be an *algebraic group,* that is a Zariski-closed subgroup of some $\mathsf{GL}(m,\mathbb{R})$. In that case $\mathsf{Hom}(\pi,G)$ has the structure of a *real algebraic subset* of $\mathsf{GL}(m,\mathbb{R})^N$, and this algebraic structure is preserved by the natural $\mathsf{Aut}(\pi)\times\mathsf{Aut}(G)$-action. Thus the map of Exercise 7.3.1 embeds $\mathsf{Hom}(\pi,G)$ an analytic or algebraic set.[1] Unless otherwise stated, we give this set the *classical topology* inherited from the topology of G as a Lie group. We shall use the fact that a real analytic set is locally contractible in the classical topology.

Exercise 7.3.2. Suppose that π is an n-generator free group. Let G be a reductive Lie group.

- Identify $\mathsf{Hom}(\pi,G)$ with the Cartesian power G^n. How does $\mathsf{Aut}(\pi)$ act on G^n?
- Let $\mathsf{Hom}(\pi,G)^-$ denote the subset comprising ρ such that the centralizer of $\rho(\pi)$ equals the center $\mathfrak{Z}(G)$ of G. Show that $\mathsf{Hom}(\pi,G)^-$ is $\mathsf{Aut}(G)$-invariant, open and dense in $\mathsf{Hom}(\pi,G)$.
- Show that $\mathsf{Inn}(G)$ acts freely and properly on $\mathsf{Hom}(\pi,G)^-$ and the quotient map is a smooth principal $\mathsf{Inn}(G)$-fibration. Deduce that the quotient space is a real analytic manifold having dimension $(n-1)\mathsf{dim}(G)+\mathsf{dim}\big(\mathfrak{Z}(G)\big)$.

As the holonomy homomorphism $\pi_1(M)\xrightarrow{h} G$ is only defined up to conjugation, it is natural to form the quotient of $\mathsf{Hom}(\pi,G)$ by the subgroup

$$\{1\}\times\mathsf{Inn}(G)\ <\ \mathsf{Aut}(\pi)\times\mathsf{Aut}(G)$$

where $\mathsf{Inn}(G)<\mathsf{Aut}(G)$ is the normal subgroup consisting of *inner automorphisms* of G. With this quotient topology inherited from the classical topology on $\mathsf{Hom}(\pi,G)$ as above, we denote this space by $\mathsf{Hom}(\pi,G)/\mathsf{Inn}(G)$ or simply $\mathsf{Hom}(\pi,G)/G$. This quotient space is the one arising in differential geometry/topology as the space of equivalence classes of flat connections, and is the quotient space upon which we concentrate.

Unfortunately this space is generally not well-behaved, and Murphy's law applies: *Everything that possibly could go wrong does go wrong.* In particular:

- Although the action of G by conjugation is algebraic/analytic, it is generally neither proper nor free. Thus $\mathsf{Hom}(\pi,G)/G$ is generally not a Hausdorff space.

[1]The Hilbert basis theorem implies that it is not necessary to assume that π has a finite presentation. An interesting question is how the defining ideal varies with the algebraic group when π is finitely generated but not finitely presentable.

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  - Even if the $\mathsf{Inn}(G)$-action is proper (for example if G is compact), then the action may not be free, and the quotient may not be a smooth manifold.

- Furthermore the analytic set $\mathsf{Hom}(\pi, G)$ is generally not smooth, and forming the quotient by G only makes matters worse.

Sometimes this can be repaired by forming the *algebra-geometric quotient* in the sense of *Geometric Invariant Theory.* Points in this quotient generally do *not* correspond to G-orbits themselves. This involves enlarging the equivalence relation as follows. For the ordinary quotient, equivalence classes are simply orbits. In general, however, orbits are not closed, so one is tempted to replace orbits by their closures. However, orbit closures are not disjoint. Say two points are equivalent if their orbit closures contain a common closed orbit. This represents a *maximal Hausdorff quotient space,* a "Hausdorffication" of the ordinary topological quotient. We denote this quotient $X//G$ and there is a natural map $X/G \longrightarrow X//G$.

Good references for this theory include Mumford–Fogarty–Kirwan [**276**] and Newstead [**281**]. For the specific case of representation varieties (and a careful scheme-theoretic treatment of the infinitesimal theory) see Sikora [**210**]. The infinitesimal theory, and its relation to cohomology, can be found in Raghunathan [**292**]. Explicit formulas using the *free differential calculus* of Fox [**131**] are described in Goldman [**161**].

In general, when π is the fundamental group of a closed oriented surface, the top stratum of $\mathsf{Hom}(\pi, G)/G$ enjoys the structure of a *symplectic manifold.* When G is compact, this construction is due to Narasimhan [**280**] and Atiyah–Bott [**11**], where the symplectic structure is defined as a *Marsden–Weinstein symplectic quotient* of the gauge group acting on the symplectic affine space of connections. More generally this construction applies whenever the adjoint representation of G on its Lie algebra $\mathfrak{g}$ preserves a nondegenerate symmetric bilinear form (Goldman [**149**]), for example if G is a semisimple Lie group. When $G = \mathsf{PSL}(2, \mathbb{R})$, this gives the Weil–Petersson Kähler form on the Fricke–Teichmüller space. It also gives $\mathsf{Mod}(\Sigma)$-invariant symplectic structures on the deformation spaces of $\mathbb{R}\mathsf{P}^2(\Sigma)$ and $\mathbb{C}\mathsf{P}^1(\Sigma)$ of real and complex projective structures, respectively.

When Σ is a compact, oriented surface with boundary, then Guruprasad–Huebschmann–Jeffrey–Weintstein [**183**] develop a Poisson geometry on the deformation space. This is a (possibly singular) foliation by symplectic manifolds obtained by constraining, for each component $\partial_i(\Sigma)$, the holonomy to lie in a fixed conjugacy class in G.

7.3.1. Example: $\mathsf{SL}(2)$-characters of $\mathbb{F}_2$. (This material is taken from [**157**], which includes proofs of the stated results.)

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A classical theorem of Vogt [**342**][2] asserts that, when $G = \mathsf{SL}(2, \mathbb{C})$ the algebro-geometric quotient of $\mathsf{Hom}(\mathbb{F}_2, G)$ by $\mathsf{Inn}(G)$ is $\mathbb{C}^3$. This generalizes the elementary fact that the quotient $G/\!/\mathsf{Inn}(G) \cong \mathbb{C}$, with coordinate given by the trace function

$$G \xrightarrow{\mathsf{tr}} \mathbb{C}$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto a + d.$$

Exercise 7.3.3. Compute the critical points and the critical values of tr. (Hint: First compute the tangent space $\mathsf{T}_g\mathsf{SL}(2, \mathbb{C})$, which lies in $\mathsf{T}_g\mathsf{Mat}_2(\mathbb{C}) \cong \mathsf{Mat}_2(\mathbb{C})$.)

Exercise 7.3.4. Recall that a function f on G is *regular* if $f(x)$ is a polynomial function of the matrix entries of $x \in G$.

- Show that for any $\mathsf{Inn}(G)$-invariant regular function f on G there exists a regular function $\mathbb{C} \xrightarrow{F} \mathbb{C}$ such that $f = F \circ \mathsf{tr}F$. (Hint: if $t := \mathsf{tr}(g) \neq \pm 2$, then g is conjugate to $\left[\begin{smallmatrix} 0 & -1 \\ 1 & t \end{smallmatrix}\right]$.)
- Extend the preceding result to the case when f is only assumed to be continuous.
- Show that $\mathsf{tr}^{-1}(t)$ consists of a single $\mathsf{Inn}(G)$-orbit when $t \neq \pm 2$. Describe $\mathsf{tr}^{-1}(t)$ when $t = \pm 2$.

It follows that tr defines an isomorphism $G/\!/\mathsf{Inn}(G) \cong \mathbb{C}$.

Exercise 7.3.5. Show that if $x \in G$, then $\mathsf{tr}(x) = \mathsf{tr}(x^{-1})$. Deduce that x and x^{-1} are conjugate in G. Is the same true when G is replaced by $\mathsf{SL}(2, \mathbb{R})$ or $\mathsf{GL}(2, \mathbb{R})$?

Now we consider 2-generator groups. Writing $\mathbb{F}_2 = \langle X, Y \rangle$ for a pair of free generators X, Y, the identification

$$\mathsf{Hom}(\mathbb{F}_2, G) \longleftrightarrow G \times G$$
$$\rho \longleftrightarrow \big(\rho(X), \rho(Y)\big)$$

is equivariant with respect to the action of $G \to \mathsf{Inn}(G)$ on $\mathsf{Hom}(\mathbb{F}_2, G)$ and the diagonal action of G on $G \times G$ given by:

$$g \cdot (x, y) := (gxg^{-1}, gyg^{-1}). \tag{7.1}$$

[2]This is often attributed to Fricke–Klein [**133**], although Vogt's paper [**342**] was published earlier.

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This action preserves the mapping

$$G \times G \longrightarrow \mathbb{C}^3 \qquad (7.2)$$

$$(x,y) \longmapsto \begin{bmatrix} \xi := \mathsf{tr}(x) \\ \eta := \mathsf{tr}(y) \\ \zeta := \mathsf{tr}(xy) \end{bmatrix}$$

which is an algebro-geometric quotient map, that is, defines an isomorphism

$$\mathsf{Hom}(\mathbb{F}_2, G)/\!/\mathsf{Inn}(G) \cong \mathbb{C}^3.$$

Theorem 7.3.6 (Vogt [**342**], Fricke–Klein [**133**])**.** Let

$$\mathsf{SL}(2,\mathbb{C}) \times \mathsf{SL}(2,\mathbb{C}) \xrightarrow{f} \mathbb{C}$$

be a regular function which is invariant under the diagonal action (7.1) of $\mathsf{SL}(2,\mathbb{C})$ by conjugation. There exists a polynomial function $F(\xi,\eta,\zeta) \in \mathbb{C}[\xi,\eta,\zeta]$ such that

$$f(x,y) = F(\mathsf{tr}(x), \mathsf{tr}(y), \mathsf{tr}(\xi\eta)).$$

Furthermore, for all $(\xi,\eta,\zeta) \in \mathbb{C}^3$, there exists $x, y \in \mathsf{SL}(2,\mathbb{C})$ such that

$$\xi = \mathsf{tr}(x), \eta = \mathsf{tr}(y), \zeta = \mathsf{tr}(xy).$$

Conversely, suppose $x, y, x', y' \in \mathsf{SL}(2,\mathbb{C})$ satisfy

$$\begin{bmatrix} \mathsf{tr}(x) \\ \mathsf{tr}(y) \\ \mathsf{tr}(xy) \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} \mathsf{tr}(x') \\ \mathsf{tr}(y') \\ \mathsf{tr}(x'y') \end{bmatrix}$$

where

$$\xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta \neq 4. \qquad (7.3)$$

Then $(x', y') = g \cdot (x, y)$ for some $g \in G$.

Condition (7.3) means that the matrix group $\langle x, y\rangle$ acts *irreducibly* on $\mathbb{C}^2$. That is, $\langle x, y\rangle$ preserves no proper nonzero linear subspace of $\mathbb{C}^2$. The irreducibility condition is crucial in several alternate descriptions of $\mathsf{SL}(2,\mathbb{C})$-representations of F_2. In particular, it is equivalent to the condition that the $\mathsf{SL}(2,\mathbb{C})$-orbit is closed in $\mathsf{Hom}(\mathsf{F}_2, \mathsf{SL}(2,\mathbb{C}))$. This condition is in turn equivalent to the orbit being *stable* in the sense of geometric invariant theory. In terms of hyperbolic geometry, it means that the representation fixes no point in $\mathbb{C}\mathsf{P}^1$.[3]

[3]This situation is remarkably clean; for a description of $\mathsf{SL}(3,\mathbb{C})$, see Lawton [**243**].

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7.3.1.1. *Coxeter Extension.* A more geometric description involves the action of the representation ρ with $\rho(X) = x$ and $\rho(Y) = y$ on hyperbolic 3-space H^3. The group $\mathsf{PSL}(2,\mathbb{C})$ acts by orientation-preserving isometries of H^3. An *involution,* that is, an element $g \in \mathsf{PSL}(2,\mathbb{C})$ having order two, is reflection in a unique geodesic $\mathsf{Fix}(g) \subset \mathsf{H}^3$. Denote the space of such involutions by Inv.

Theorem 7.3.7 (Coxeter extension)**.** Suppose that $x, y \in \mathsf{SL}(2,\mathbb{C})$ generate an irreducible representation and let $z = y^{-1}x^{-1}$ so that

$$xyz = \mathbb{I}.$$

Then there exists a unique triple of involutions

$$\iota_{xy}, \iota_{yz}, \iota_{zx} \in \mathsf{Inv}$$

such that the corresponding elements $\mathsf{P}(x), \mathsf{P}(y), \mathsf{P}(z) \in \mathsf{PSL}(2,\mathbb{C})$ satisfy:

$$\begin{aligned} \mathsf{P}(x) &= \iota_{zx}\iota_{xy} \\ \mathsf{P}(y) &= \iota_{xy}\iota_{yz} \\ \mathsf{P}(z) &= \iota_{yz}\iota_{zx}. \end{aligned}$$

For the proof see [**157**].

Exercise 7.3.8. Show that Inv identifies with the set of unoriented geodesics in H^3. Describe its topological type.

Exercise 7.3.9. Let $\widetilde{\mathsf{Inv}}$ denote the inverse image $\mathsf{P}^{-1}(\mathsf{Inv})$. Show that $\widetilde{\mathsf{Inv}} = \mathfrak{sl}(2,\mathbb{C}) \cap \mathsf{SL}(2,\mathbb{C})$.

We choose lifts $\widetilde{\iota}_{xy}, \widetilde{\iota}_{yz}, \widetilde{\iota}_{zx} \in \widetilde{\mathsf{Inv}}$ such that

$$\widetilde{\iota}_{xy}\widetilde{\iota}_{yz}\widetilde{\iota}_{zx} = \mathbb{I}.$$

These lifts will be used to parametrize hyperbolic structures on surfaces in terms of traces in $\mathsf{SL}(2,\mathbb{R})$.

7.3.1.2. *Hyperbolic three-holed spheres.* Theorem 7.3.7 implies the Fricke space of hyperbolic structures on the three-holed sphere Σ (sometimes called a “pair of pants” or a “trinion”) identifies with $(-\infty, -2]^3$ using trace coordinates. Namely, the three trace parameters correspond to the three boundary components of Σ. The Coxeter extension identifies a hyperbolic structure on Σ with (perhaps mildly degenerate) right-angled hexagon in the hyperbolic plane H^2. Right-angled hexagons are allowed to degenerate when some of the alternate edges covering boundary components degenerate to ideal points.

Suppose that $\xi, \eta, \zeta \le -2$. Then the corresponding elements $x, y, z \in \mathsf{SL}(2,\mathbb{C})$ have real representatives and are represented by hyperbolic or parabolic elements of $\mathsf{SL}(2,\mathbb{R})$. Furthermore if $\xi, \eta, \zeta < -2$, they are represented by hyperbolic elements of $\mathsf{SL}(2,\mathbb{R})$ whose axes pairwise do not intersect.

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The involutions $\iota_{xy}, \iota_{yz}, \iota_{zx}$ preserve $\mathsf{H}^2 \subset \mathsf{H}^3$ and their restrictions to H^2 act by (orientation-reversing) reflections in geodesics which are the common orthogonal geodesics to the invariant axes of x, y, z. Denote these gedoesics by

$$\mathsf{Fix}(\iota_{xy}), \mathsf{Fix}(\iota_{yz}), \mathsf{Fix}(\iota_{zx}) \subset \mathsf{H}^2$$

respectively. Theorem 7.3.7 implies that, for example, the invariant axis of x is the common orthogonal to the lines $\mathsf{Fix}(\iota_{xy}), \mathsf{Fix}(\iota_{zx})$. Their distance equals the distance between their closest points $\mathsf{Fix}(\iota_{xy}) \cap \mathsf{Axis}(x)$ and $\mathsf{Fix}(\iota_{zx}) \cap \mathsf{Axis}(x)$:

$$\mathsf{d}\big(\mathsf{Fix}(\iota_{xy}), \mathsf{Fix}(\iota_{zx})\big) \;=\; \mathsf{d}\big(\mathsf{Fix}(\iota_{xy}) \cap \mathsf{Axis}(x), \mathsf{Fix}(\iota_{zx}) \cap \mathsf{Axis}(x)\big)$$

Since $x = \iota_{zx}\iota_{xy}$, the hyperbolic isometry x is a transvection of displacement

$$\ell_x := 2\mathsf{d}(\mathsf{Fix}(\iota_{xy}), \mathsf{Fix}(\iota_{zx})$$

and the trace of the matrix x equals

$$\xi = -2\cosh(\ell_x/2).$$

For the detailed proof that $\xi, \eta, \zeta < 2$ implies that the six lines

$$\mathsf{Axis}(x),\ \mathsf{Fix}(\iota_{xy}),\ \mathsf{Axis}(y),\ \mathsf{Fix}(\iota_{yz}),\ \mathsf{Axis}(z),\ \mathsf{Fix}(\iota_{zx})$$

bound a convex right-angled hexagon, see §4.3 of [**157**]. This hexagon is a fundamental domain for the Coxeter group $\langle \iota_{xy}, \iota_{yz}, \iota_{zx}\rangle$. This Coxeter group contains $\langle x, y, z\rangle$ with index two. The union of two adjacent hexagons in the resulting tesselation is then a fundamental domain for $\langle x, y\rangle$. The quotient is a hyperbolic surface homeomorphic to a three-holed sphere, with three boundary components of length ℓ_x, ℓ_y, ℓ_z.

7.3.2. Twist flows and Fenchel–Nielsen earthquakes. Given a surface Σ and a simple closed curve $\mathcal{C} \subset \Sigma$, we define deformations of representations of $\pi_1(\Sigma)$ which are "supported" on $\mathcal{C}$. To this end, bordify the complement $\Sigma \setminus \mathcal{C}$ as a surface-with-boundary $\Sigma|_{\mathcal{C}}$ with boundary components $\mathcal{C}_i$ which are identified to form $\mathcal{C}$ in the quotient (which is Σ).

Suppose first that $\mathcal{C}$ separates Σ into two components Σ_1, Σ_2 so that Σ can be reconstructed from the disjoint union

$$\Sigma|_{\mathcal{C}} \;=\; \Sigma_1 \bigsqcup \Sigma_2$$

by a quotient map

$$\Sigma_1 \bigsqcup \Sigma_2 \xrightarrow{Q} \Sigma.$$

Write $Q^{-1}(\mathcal{C}) = \mathcal{C}_1 \sqcup \mathcal{C}_2$ where $\mathcal{C}_i \subset \Sigma_i$, so that Q identifies $\mathcal{C}_1$ and $\mathcal{C}_2$ to form $\mathcal{C}$.

Choose a basepoint $x_0 \in \mathcal{C} \subset \Sigma$ and let $Q^{-1}(x_0) = \{x_1, x_2\}$ where $x_i \in c_i \subset \Sigma_i$. Let $c \in \pi_1(\Sigma, x_0)$ be the element corresponding to $\mathcal{C}$. For

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$i = 1, 2$, let $c_i \in \pi_1(\Sigma_i, x_i)$, be the respective elements corresponding to c_i. By the van Kampen theorems, $\pi_1(\Sigma, x_0)$ may be reconstructed from $\pi_1(\Sigma_i, x_i)$ as an *amalgamated free product*

$$\pi_1(\Sigma, x_0) \cong \pi_1(\Sigma_1, x_1) \coprod_{\langle c \rangle} \pi_1(\Sigma_2, x_2).$$

Suppose that $\pi_1(\Sigma) \xrightarrow{\rho} G$ is a representation and $\mathfrak{z}_t$ is a parametrized family of elements of the centralizer of $\rho(c)$ in G. Then we can construct a parametrized family of representations ρ_t by the formula:

$$\rho_t(A) := \begin{cases} \rho(A) & \text{if } A \in \pi_1(\Sigma_1, x_1) \\ \mathfrak{z}_t\, \rho(A)\, \mathfrak{z}_t^{-1} & \text{if } A \in \pi_1(\Sigma_2, x_2) \end{cases} \tag{7.4}$$

Since $\pi_1(\Sigma_1, x_1)$ and $\pi_1(\Sigma_2, x_2)$ generate $\pi_1(\Sigma, x_0)$ and the only relations concern compatibility along c, that $\mathfrak{z}_t$ centralizes $\rho(c)$ implies that (7.4) defines a family of representations.

Exercise 7.3.10. Develop the analogous construction of a family of representations when $\mathcal{C}$ does not separate Σ, that is, when $\Sigma \setminus \mathcal{C}$ is connected.

When $\mathcal{C}$ is a simple closed geodesic on a complete hyperbolic surface M, then $\mathsf{dev}(\widetilde{\mathcal{C}})$ is a geodesic in H^2 and the holonomy $\rho(c)$ is a hyperbolic isometry stabilizing $\mathsf{dev}(\widetilde{c})$. The stabilizer $\mathsf{Stab}\big(\mathsf{dev}(\widetilde{\mathcal{C}})\big)$ is a one-parameter subgroup of $\mathsf{PSL}(2, \mathbb{R})$ consisting of transvections. The family of representations ρ_t corresponds to the following geometric operation: After cutting along $\mathcal{C}$, re-identify Σ from $\Sigma|_{\mathcal{C}}$ along the isometries corresponding to $\mathfrak{z}_t$. Thurston generalized this construction (originally due to Fenchel–Nielsen) to *earthquake flows* on Fricke space $\mathfrak{F}(\Sigma)$.

For a more general discussion of earthquakes and an important application, compare Kerckhoff [**213**]. Thurston's *bending deformations* of embeddings in $\mathsf{PSL}(2, \mathbb{C})$ and the higher-dimensional generalizations due to Johnson–Millson [**207**] are also special cases of this construction. McMullen [**264**] and Kamishima–Tan [**208**] consider a 2-parameter family of deformations in $\mathsf{PSL}(2, \mathbb{C})$ (*quakebend deformations*).

We describe a generalization to $\mathbb{R}\mathsf{P}^2$-structures in §13.1.2.

Exercise 7.3.11. If $\mathcal{C}_1, \dots \mathcal{C}_N \subset \Sigma$ are pairwise disjoint, with respective centralizing one-parameter subgroups then the corresponding flows on $\mathsf{Hom}(\pi, G)$ commute.

More generally, suppose $\mathcal{C} = \mathcal{C}_1 \sqcup \cdots \sqcup \mathcal{C}_N$ is a *multicurve,* that is, a disjoint union of simple closed curves. Then Exercise 7.3.11 implies the above operation can be performed along each of the curves $\mathcal{C}_i$ independently, obtaining an $\mathbb{R}^N$-action.

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Furthermore every $\mathsf{Inn}(G)$-invariant function $G \xrightarrow{f} \mathbb{R}$ defines an $\mathsf{Inn}(G)$-invariant function

$$\mathsf{Hom}(\pi, G) \xrightarrow{f_{\mathcal{C}}} \mathbb{R} \qquad (7.5)$$
$$\rho \longmapsto \sum_{i=1}^{N} f \circ \rho(c_i)$$

where $c_i \in \pi$ corresponds to $\mathcal{C}_i$.

Exercise 7.3.12. Show that the formula (7.5) is well-defined, that is, is independent of the elements c_i in the fundamental group.

7.4. Fenchel–Nielsen coordinates on Fricke space

See Abikoff [**5**], Hubbard [**197**] or Farb–Margalit [**128**] for more detailed accounts.

The Fenchel–Nielsen parametrization of $\mathfrak{F}(\Sigma)$ begins with the choice of a *pants decomposition,* that is, a decomposition into N three-holed spheres along a multicurve $\mathcal{C}$ as above.

Exercise 7.4.1. Show that if Σ is a closed orientable surface of genus $g > 1$, then $N = 3g - 3$.

In a sequence of papers, Wolpert [**350–353**] initiated the study of the symplectic geometry of $\mathfrak{F}(\Sigma)$. In particular he studied the twist flows and showed they were Hamiltonian flows for geodesic length functions. These were put into the more general context in Goldman[**149, 150**]. When applied to the geodesic length function of a pants decomposition $\mathcal{P}$ (as in (7.5), one obtains a map

$$\mathfrak{F}(\Sigma) \xrightarrow{\ell_{\mathcal{P}}} \left(\mathbb{R}^+\right)^N. \qquad (7.6)$$

Exercise 7.4.2. This map is a principal $\mathbb{R}^N$-fibration, where the fiber action is defined by the Fenchel–Nielsen earthquakes along the $\mathcal{C}_i$. Furthermore, giving $\mathfrak{F}(\Sigma)$ the Weil–Petersson symplectic structure, this $\mathbb{R}^N$-action is a Hamiltonian action and with momentum mapping (7.6).

It follows that the Fenchel–Nielsen earthquake flow defines a *completely integrable Hamiltonian system* ([**352**]).

More generally Wolpert [**353**] showed that the length functions $\ell_1, \dots, \ell_N$ are part of a *global Darboux coordinate system*

$$(\ell_1, \dots, \ell_N, \tau_1, \dots, \tau_N) \ \in \ (\mathbb{R}^+)^N \times \mathbb{R}^N$$

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on the symplectic manifold $\big(\mathfrak{F}(\Sigma), \omega_{WP}\big)$:

$$\omega_{WP} \;=\; \sum_{i=1}^{N} d\ell_i \wedge d\tau_i,$$

sometimes called *Wolpert's magic formula.* The choice of the *twist coordinates* τ_i is not as natural as the length coordinates $\ell_{\mathcal{P}}$: they involve a choice of section s of the mapping $\ell_{\mathcal{P}}$. This section corresponds to when all the $\tau_i = 0$.

The *Fenchel–Nielsen section* arises from the decomposition corresponding to $\mathcal{P}$as follows. We consider marked hyperbolic surfaces (M, f), where

$$\Sigma \xrightarrow{f} M$$

is a diffeomorphism. The hyperbolic surfaces M decompose into $2g-2$ pairs-of-pants P_j (where $j = 1, \dots, 2g-2$). Furthermore ∂P_j consists of closed geodesics

$$\partial P_j \;=\; \partial^1 P_j \sqcup \partial^2 P_j \sqcup \partial^3 P_j$$

where each boundary component is one of the $\mathcal{C}_i$ (for $i = 1, \dots, 3g-3$):

$$\partial^k P_j \;=\; \mathcal{C}_{i(j,k)}$$

for $k = 1, 2, 3$.

Each P_j decomposes into two right-angled hexagons $\hexagon_j^+ \cup \hexagon_j^-$; indeed each pants P_j is the *double* of a right-angled hexagon $\hexagon_j$.

Now fix a collection of lengths

$$\ell \;= (\ell_1, \dots, \ell_{3g-3}) \;\in\; (\mathbb{R}^+)^{3g-3}.$$

The Fenchel–Nielsen section is a *marked* hyperbolic surface with the given length parameters ℓ. Specifically, choose right-angled hexagons $\hexagon_1, \dots, \hexagon_{3g-3}$ with alternate triples of edge-lengths

$$\frac{\ell_i^{(1)}}{2}, \; \frac{\ell_i^{(2)}}{2}, \; \frac{\ell_i^{(3)}}{2}$$

for $i = 1, \dots, 3g-3$.

Exercise 7.4.3. Find other sections to $\ell_{\mathcal{P}}$.

Exercise 7.4.4. A *Dehn twist* about a simple closed curve $c \subset \Sigma$ is a homeomorphism $\Sigma \to \Sigma$ supported on a tubular neighborhood of c. Define a group of homeomorphisms $\mathbb{Z}^{3g-3}$ preserving $\mathcal{P}$ gener ated by Dehn twists and describe its action on Fenchel–Niesen coordinates. Describe the action of a Dehn twist about a curve *not* in $\mathcal{P}$ in Fenchel–Nielsen coordinates.

The following exercises are taken from [**150**] and will be used in §13.1.2. Choose an Ad-invariant nondegenerate symmetric bilinear form $\langle , \rangle$ on the Lie algebra $\mathfrak{g}$ of G. Choose an orientation on Σ as well.

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Exercise 7.4.5. Suppose that $G \xrightarrow{f} \mathbb{R}$ is a smooth $\mathsf{Inn}(G)$-invariant function. Define a function $G \xrightarrow{F} \mathfrak{g}$ by:

$$\langle F(x), Y \rangle \;=\; \frac{d}{dt}\Big|_{t=0} f\big((x \exp(tY)\big)$$

for all $Y \in \mathfrak{g}$.

- Show that F is G-equivariant with respect to the action Inn of inner automorphisms on G and the adjoint representation Ad of G on $\mathfrak{g}$.
- If $x \in G$, show that $F(x)$ lies in the infinitesimal centralizer of x. In particular the one-parameter subgroup $\exp\big(tF(x)\big)$ lies in the centralizer $\mathfrak{Z}_x < G$ of x.

We call F the *variation function* associated to f.

Now let $c \in \pi_1(M)$ and define a function f_c on $\mathsf{Hom}(\pi, G)$ as in (7.5) with $N = 1$. Generalizing Wolpert's theorem [**351**] that the Fenchel–Nielsen earthquake flow is the Hamiltonian flow for the geodesic length function is the following description of the Hamiltonian flow of f_c:

Exercise 7.4.6. Suppose that $\mathcal{C}$ is a simple closed curve on M and let c be an element of $\pi_1(M)$ corresponding to $\mathcal{C}$. Let

$$\mathfrak{z}_t := \exp\Big(tF\big(\rho(c)\big)\Big)$$

be the corresponding path in the centralizer of $\rho(c)$.

- Suppose first that $\mathcal{C}$ separates M into subsurfaces M_1, M_2 as in §7.3.2. Then (7.4) describes a flow on $\mathsf{Hom}(\pi, G)$ which leaves the function $f_{\mathcal{C}}$ invariant.
- Suppose that $M|_{\mathcal{C}}$ is connected. Describe the corresponding flow.

In the case of the Fricke component $\mathfrak{F}(\Sigma) \subset \mathsf{Hom}(\pi, G)/G$, every representation ρ with $[\rho] \in \mathfrak{F}(\Sigma)$ has the property that $\rho(c)$ is hyperbolic $\forall c \in \pi \setminus \{1\}$. Denote the open subset of hyperbolic elements of G by Hyp and use the invariant function ℓ defined by

$$\mathsf{tr}(A) = \pm 2\cosh(l/2)$$

on Hyp. It is uniquely determined by:

$$\mathsf{Hyp} \xrightarrow{\ell} \mathbb{R}^+ \tag{7.7}$$

$$\pm \begin{bmatrix} e^{l/2} & 0 \\ 0 & e^{-l/2} \end{bmatrix} \longmapsto l$$

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Exercise 7.4.7. Using the trace form on $\mathfrak{sl}(2,\mathbb{R})$ as the Ad-invariant inner product, show that the corresponding variation function for ℓ is the function

$$\pm \begin{bmatrix} e^{l/2} & 0 \\ 0 & e^{-l/2} \end{bmatrix} \xmapsto{L} \begin{bmatrix} l/2 & 0 \\ 0 & -l/2 \end{bmatrix}.$$

The corresponding one-parameter subgroups $\mathfrak{z}_t$ consists of transvections along $\mathsf{Axis}(A)$ displacing points on $\mathsf{Axis}(A)$ by distance t. In particular A itself equals $\mathfrak{z}_{\ell(A)}$.

When c corresponds to a simple closed curve $\mathcal{C} \subset \Sigma$, then ℓ_c is the function on $\mathfrak{F}(\Sigma)$ mapping a marked hyperbolic structure on M to the length of the unique closed geodesic homotopic to $\mathcal{C}$. The corresponding flow is the Fenchel–Nielsen earthquake flow along $\mathcal{C}$.

7.5. Open manifolds

The classification of (G, X)-structures on open manifolds is quite different than on closed manifolds. Indeed the classification is a relatively elementary special case of Gromov's *h-principle* [**174**], which extends the Smale–Hirsch theory of immersions. In particular the existence reduces to homotopy theory, and the effective classification uses a weaker equivalence relation. For a simple example, the (G, X)-structures on a disc D^n (for $n > 1$) correspond to immersions $D^n \looparrowright X$, and the quotient by isotopy is still an infinite-dimensional space. A more suggestive equivalence relation is modeled on *regular homotopy* whereby regular homotopy classes are classified by homotopy classes of sections of a natural fiber bundle. Without extra assumptions — the most notable being completeness — the developing maps are intractable and can by highly pathological, as illustrated in Figures 7.2 and 7.4.

Constructing incomplete geometric structures on noncompact manifolds M is easy. Take any immersion $M \xrightarrow{f} X$ which is not bijective; then f induces an (G, X)-structure on M. If M is parallelizable, then such an immersion always exists (Hirsch [**191**]).

More generally, let $\pi \xrightarrow{h} G$ be a representation. If the associated flat (G, X)-bundle $E \longrightarrow X$ possesses a section $M \xrightarrow{s} E$ whose normal bundle is isomorphic to $\mathsf{T}M$, then an (G, X)-structure exists having holonomy h. This follows from the extremely general *h-principle* of Gromov [**174**] (see Haefliger [**185**] or Eliashberg–Mishachev [**124**], for example).

Here is how it plays out in dimension two. First of all, every orientable noncompact surface admits an immersion into $\mathbb{R}^2$ and such an immersion determines an affine structure with trivial holonomy. Immersions can be

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classified up to crude relation of regular homotopy, although the isotopy classification of immersions of noncompact surfaces seems forbiddingly complicated. Furthermore suppose $\pi \xrightarrow{\mathsf{h}} \mathsf{Aff}(\mathsf{A})$ is a homomorphism such that the character

$$\pi \xrightarrow{\det \circ \mathsf{L} \circ \mathsf{h}} \mathbb{Z}/2$$

equals the first Stiefel–Whitney class. That is, suppose its kernel is the subgroup of π corresponding to the orientable double covering of M. Then M admits an affine structure with holonomy h. Classifying general geometric structures on noncompact manifolds without extra geometric hypotheses seems hopeless under anything but the crudest equivalence relations.

Constructing incomplete geometric structures on compact manifolds is much harder. Indeed for certain geometries (G, X), there exist closed manifolds for which *every* (G, X)-structure on M is complete. As a trivial example, if X is compact and M is a closed manifold with finite fundamental group, then Theorem 5.2.2 implies every (G, X)-structure is complete. As a less trivial example, if M is a closed manifold whose fundamental group contains a nilpotent subgroup of finite index and whose first Betti number equals one, then every affine structure on M is complete (see Fried–Goldman–Hirsch [**140**]). See §6.4.1 for a 3-dimensional example. Compare the discussion of Markus's question about the relation of parallel volume to completeness in §11.

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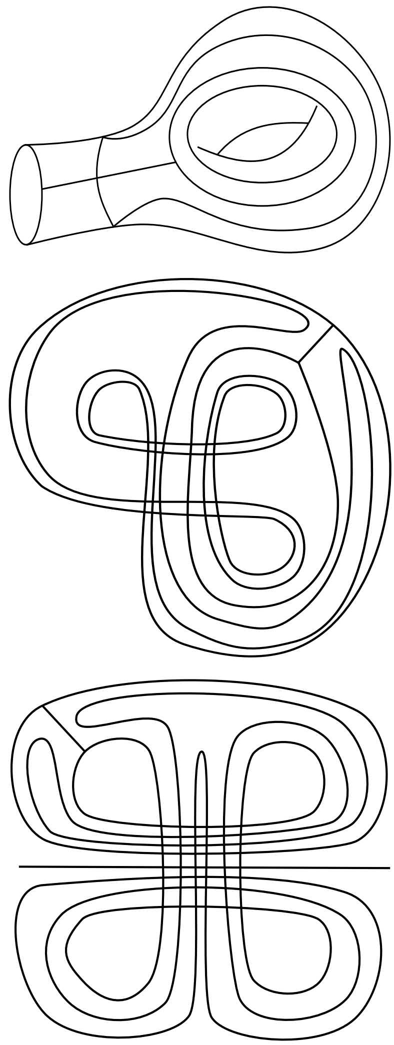

Figure 7.2. This sequence of drawings illustrate geometric structures on a one-holed torus M, beginning with structures with trivial holonomy. These correspond to immersions of $M \looparrowright X$ where X is either E^2 or $\mathbb{S}^2$. The developing map depicted in the middle picture defines a Euclidean structure on Σ which is highly incomplete. Points in the image have either one or two preimages. The bottom drawing depicts an immersion $\Sigma \looparrowright \mathbb{S}^2$ whose image contains the point at ∞. Think of these developing maps as overpasses and underpasses on a highway.

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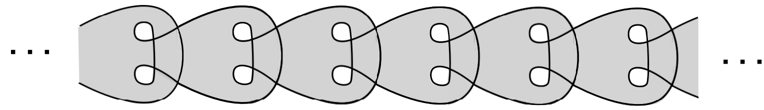

Figure 7.3. When the holonomy is nontrivial, the developing map is defined on a *nontrivial* covering space $\hat{M} \longrightarrow M$. This drawing and the next depict developing maps when the holonomy is cyclic. Here the holonomy is generated by a translation.

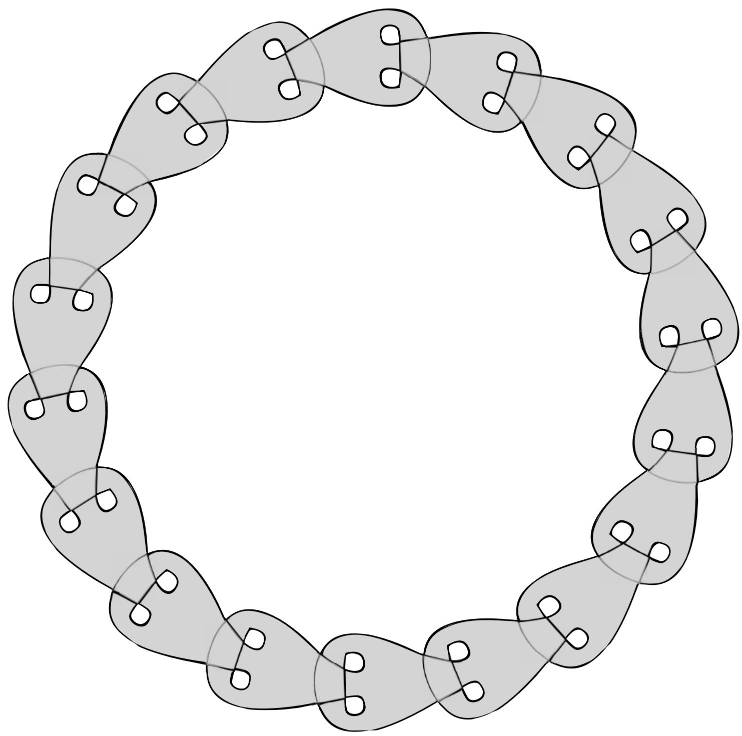

Figure 7.4. Here the holonomy is generated by a rotation of order 18. In each of these examples there is a loop on M with trivial holonomy. Deformations of the structure exist where the holonomy around this loop is nontrivial; indeed any action of $\pi_1(M)$ on $\mathbb{S}^2$ arises as the holonomy representation for a geometric structure modeled on $\mathbb{S}^2$.

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Chapter 8

Completeness

In many important cases the developing map is a diffeomorphism $\widetilde{M} \longrightarrow X$, or at least a covering map onto its image. In particular if $\pi_1(X) = \{e\}$, such structures are *quotient structures:*

$$M \cong \Gamma\backslash X$$

We also call such quotient structures *tame.* This chapter develops criteria for taming the developing map.

Many important geometric structures are modeled on *homogeneous Riemannian manifolds.* These structures determine Riemannian structures, which are locally homogeneous *metric spaces.* For these structures, completeness of the metric space will tame the developing map.

Although it is not completely necessary, this closely relates to *geodesic completeness* of the associated Levi–Civita connection. The key tool is the *Hopf–Rinow theorem:* Geodesic completeness (of the Levi–Civita connection) is equivalent to completeness of the associated metric space. In particular compact Riemannian manifolds are geodesically complete. Many Ehresmann structures have natural Riemannian structures whose completeness tames of the developing map. In particular such structures are quotient structures as above.

After giving some general remarks on the developing map, its relation to the exponential map (for affine connections), we describe all the complete affine structures on $\mathbb{T}^2$. The chapter ends with a discussion of incomplete affine structures on $\mathbb{T}^2$, and a general discussion of the most important incomplete examples — *Hopf manifolds,* which were introduced in §6.2.

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8.1. Locally homogeneous Riemannian manifolds

Suppose (G, X) is a *Riemannian homogeneous space*, that is, X possesses a G-invariant Riemannian metric g_X. Equivalently, $X = G/H$ where the isotropy group H is compact. Precisely, the image of the adjoint representation $\mathsf{Ad}(H) \subset \mathsf{GL}(\mathfrak{g})$ is compact.

Exercise 8.1.1. Prove that these two conditions on the homogeneous space (G, X) are equivalent.

If (G, X) is a Riemannian homogeneous space, then every (G, X)-manifold M inherits a Riemannian metric locally isometric to g_X. We say that M is *complete* if such a metric is geodesically complete.

Exercise 8.1.2. Prove that this notion of completeness is independent of the G-invariant Riemannian structure g_X on X.

8.1.1. Complete locally homogeneous Riemannian manifolds.

We use the following consequence of the *Hopf–Rinow theorem* from Riemannian geometry: *Geodesic completeness* of a Riemannian structure (the complete extendability of geodesics) is equivalent to the *completeness* of the corresponding metric space (convergence of Cauchy sequences). (Compare, for example, do Carmo [**113**], Kobayashi–Nomizu [**224**], Milnor [**269**], O'Neill [**283**], or Papadopoulos [**286**].) Our application to geometric structures is that *a local isometry from a complete Riemannian manifold is a covering space.*

Recall our standard notation from Chapter 5: M is a (G, X)-manifold with universal covering space $\widetilde{M} \xrightarrow{\Pi} M$; denote by π the associated fundamental group, and $(\mathsf{dev}, \mathsf{hol})$ a development pair.

Proposition 8.1.3. Let (G, X) be a Riemannian homogeneous space. Suppose that X is simply connected. Let M be a complete (G, X)-manifold. Then:

- $\widetilde{M} \xrightarrow{\mathsf{dev}} X$ is a diffeomorphism;
- $\pi \xrightarrow{\mathsf{hol}} G$ is an isomorphism of π onto a cocompact discrete subgroup $\Gamma \subset G$.

Corollary 8.1.4. Let (G, X) be a Riemannian homogeneous space, where X is simply connected, and let M be a compact (G, X)-manifold. Then the holonomy group $\Gamma \subset G$ is a discrete subgroup which acts properly and freely on X and M is isomorphic to the quotient X/Γ.

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Proof of Corollary 8.1.4 assuming Proposition 8.1.3.
Since (G, X) is a Riemannian homogeneous, M inherits a Riemannian structure locally isometric to X. Since M is compact, this Riemannian structure is complete. Now apply Proposition 8.1.3. □

Proof of Proposition 8.1.3. The Riemannian metric

$$\widetilde{\mathsf{g}} = \mathsf{dev}^*\mathsf{g}_X$$

on $\widetilde{M}$ is invariant under the group of deck transformations π of $\widetilde{M}$ and hence there is a Riemannian metric g_M on M such that $\Pi^*\mathsf{g}_M = \widetilde{\mathsf{g}}$. By assumption the metric g_M on M is complete and so is the metric $\widetilde{\mathsf{g}}$ on $\widetilde{M}$. By construction,

$$(\widetilde{M}, \widetilde{\mathsf{g}}) \xrightarrow{\mathsf{dev}} (X, \mathsf{g}_X)$$

is a local isometry. A local isometry from a complete Riemannian manifold into a Riemannian manifold is necessarily a covering map (Kobayashi–Nomizu [**224**]) so dev is a covering map of $\widetilde{M}$ onto X. Since X is simply connected and $\widetilde{M}$ is connected, dev is a diffeomorphism. Let $\Gamma \subset G$ denote the image of h. Since dev is equivariant respecting h, the action of π on X given by h is equivalent to the action of π by deck transformations on $\widetilde{M}$. Thus h is faithful and its image Γ is a discrete subgroup of G acting properly and freely on X. Furthermore dev induces a diffeomorphism

$$M = \widetilde{M}/\pi \longrightarrow X/\Gamma.$$

□

When M is compact, more is true: X/Γ is compact (and Hausdorff). Since the fibration $G \longrightarrow G/H = X$ is proper, the homogeneous space G/Γ is compact, that is, Γ is cocompact in G.

One may paraphrase the above observation abstractly as follows. Let (G, X) be a Riemannian homogeneous space. Then there is an equivalence of categories:

$$\Big\{\text{Compact } (G, X)\text{-manifolds}\Big\} \Longleftrightarrow$$

$$\Big\{\text{Discrete cocompact subgroups of } G \text{ acting freely on } X\Big\}$$

where the morphisms in the latter category are inclusions of subgroups composed with inner automorphisms of G).

8.1.2. Topological rigidity of complete structures. We say that a (G, X)-manifold M is *complete* if $\widetilde{M} \xrightarrow{\mathsf{dev}} X$ is a diffeomorphism.[1] A (G, X)-manifold M is complete if and only if its universal covering $\widetilde{M}$ is

[1]or a covering map if we don't insist that X be simply connected

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(G, X)-isomorphic to X, that is, if M is isomorphic to the quotient X/Γ (at least if X is simply connected). Note that if (G, X) is contained in (G', X') in the sense of §5.2.3 and $X \neq X'$, then a complete (G, X)-manifold is *never* complete as an (G', X')-manifold.

Here is an interesting characterization of completeness using elementary properties of developing maps.

Exercise 8.1.5. Let (G, X) be a (not necessarily Riemannian) homogeneous space and X be simply connected. Let M be a closed (G, X)-manifold with developing pair $(\mathsf{dev}, \mathsf{hol})$. Show that M is complete if and only if the holonomy representation $\pi \xrightarrow{\mathsf{hol}} G$ is an isomorphism of π onto a discrete subgroup of G which acts properly and freely on X. Find a counterexample when M is *not* assumed to be closed.

When G preserves an *indefinite* pseudo-Riemannian structure, then completeness is much trickier than in the Riemannian case. For example, when M is compact, the following conjecture is plausible:

Conjecture 8.1.6. A compact locally homogeneous pseudo-Riemannian manifold is geodesically complete.

This has been proved by Klingler [**217**] for *constant curvature* Lorentzian manifolds, following earlier work of Carriére [**78**] in the flat case.

However, when M is assumed to be compact and *homogeneous,* (that is, $\mathsf{Isom}(M)$ acts transitively on M), Marsden [**259**] proved that M is complete. However, without the assumption of compactness, §10.5.5.1 provides a homogeneous incomplete flat Lorentzian manifold. Its developing image is a halfplane. In contrast, every homogeneous *Riemannian* manifold is geodesically complete ([**224**],§IV, Theorem 4.5).

8.1.3. Euclidean manifolds.

Euclidean structures on closed manifolds provide an important example of this. Namely, E^n s a Riemannian homogeneous space whose isometry group $\mathsf{Isom}(\mathsf{E}^n)$ acts properly with isotropy group the orthogonal group $\mathsf{O}(n)$. As above, Euclidean structures on closed manifolds identify with lattices $\Gamma \subset \mathsf{Isom}(\mathsf{E}^n)$. This class of geometric structures forms the intersection of flat affine structures and locally homogeneous Riemannian structures.

Exercise 8.1.7. Let E be a Euclidean space with underlying vector space $\mathsf{V} = \mathsf{Trans}(\mathsf{E})$. Then every closed $\big(\mathsf{Trans}(\mathsf{E}), \mathsf{E}\big)$-manifold M is a quotient $\Lambda\backslash\mathsf{V}$, where $\Lambda < \mathsf{V}$ is a lattice, that is, M is a flat torus in the sense of §5.5.1.

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Since $\mathsf{Trans}(\mathsf{E}) < \mathsf{Isom}(\mathsf{E})$, every such structure is a Euclidean structure. Remarkably, every closed Euclidean manifold is finitely covered by a flat torus:

Theorem 8.1.8 ((Bieberbach)**.** Let M^n be a closed Euclidean manifold with affine holonomy group $\Gamma < \mathsf{Isom}(\mathsf{E}^n)$. Then $M^n \cong \Gamma\backslash\mathsf{E}^n$ is complete. Furthermore the translation subgroup $\Gamma \cap \mathbb{R}^n$ is a lattice in $\mathbb{R}^n$ and the quotient projection $\Lambda\backslash\mathsf{E}^n \twoheadrightarrow M$ is a finite covering space.

Euclidean structures identify with the more traditional notion of *flat Riemannian* structures.

8.2. Affine structures and connections

We have seen that a G-invariant metric on X is a powerful tool in classifying (G,X)-structures. However, without this extra structure, many pathological developing maps may arise, even on closed manifolds. In this section we discuss the notion of completeness for affine structures, for which the lack of an invariant metric leads to fascinating phenomena. The simplest example of a compact incomplete affine structure is a *Hopf manifold,* for which the 1-dimensional case was discussed in §5.5.2 and the general case in §6.2.

Just as Euclidean structures are flat Riemannian structures, general Ehresmann structures can be characterized in terms of more general differential-geometric objects. Affine structures are then *affine connections* ∇ which are locally equivalent to the affine connection ∇_{A} on a model affine space A. This is equivalent to the vanishing of both the curvature tensor and the torsion tensor of ∇. Thus affine structures are flat torsionfree affine connections. Such a connection is the Levi–Civita connection for a Euclidean structure, and the Euclidean structure can be recast as an affine structure with parallel Riemannian structure, as described in §1.4.1.

8.3. Completeness and convexity of affine connections

A more traditional proof of Proposition 8.1.3 uses the theory of *geodesics.* Geodesics are curves with zero acceleration, where *acceleration* of a smooth curve is defined in terms of an *affine connection,* which is just a connection on the tangent bundle of a smooth manifold. Connections appear twice in our applications: first, as Levi–Civita connections for Riemannian homogeneous spaces, and second, for flat affine structures. These contexts meet in the setting of Euclidean manifolds.

After we briefly review the standard theory of affine connections and the geodesic flow, we discuss the theorem of Auslander–Markus characterizing complete affine structures. Then we discuss the closely related notion of

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geodesic convexity and prove Koszul's theorem relating convexity to the developing map.

8.3.1. Review of affine connections. Suppose that M is a smooth manifold with an affine connection ∇. Let $p \in M$ be a point and $\mathbf{v} \in \mathsf{T}_pM$ a tangent vector. Then

$$\exists a, b \in \mathbb{R} \cup \{\pm\infty\}$$

such that

$$-\infty \leq a < 0 < b \leq \infty$$

and a geodesic $\gamma(t)$, defined for $a < t < b$, with $\gamma(0) = p$ and $\gamma'(0) = \mathbf{v}$. We call $(p, \mathbf{v})$ the *initial conditions.* Furthermore γ is unique in the sense that two such γ agree on their common interval of definition. We may choose the interval (a, b) to be maximal. When $b = \infty$ (respectively $a = -\infty$), the geodesic is *forwards complete* (respectively *backwards complete).* A geodesic is *complete* if and only if it is both forwards and backwards complete. In that case γ is defined on all of $\mathbb{R}$. We say (M, ∇) is *geodesically complete* if and only if every geodesic extends to a complete geodesic.

If γ is a geodesic with initial condition $(p, \mathbf{v}) \in \mathsf{T}M$, then we write

$$\gamma(t) = \mathsf{Exp}(t\mathbf{v})$$

in light of the uniqueness remarks above, and the remark that $t \longmapsto \gamma(st)$ is the geodesic with initial condition $(p, s\mathbf{v})$. For further clarification, we make the following definition:

Definition 8.3.1. The *exponential domain* $\mathcal{E} \subset \mathsf{T}M$ is the largest open subset of $\mathsf{T}M$ upon which Exp is defined. For $p \in M$, write $\mathcal{E}_p := \mathcal{E} \cap \mathsf{T}_pM$ and $\mathsf{Exp}_p := \mathsf{Exp}|_{\mathcal{E}_p}$.

- $\mathcal{E}$ contains the zero-section $\mathbf{0}_M$ of $\mathsf{T}M$.
- $\mathcal{E}_p$ is *star-shaped* about $\mathbf{0}_p$, that is, if $\mathbf{v} \in \mathcal{E}_p$ and $0 \leq t < 1$, then $t\mathbf{v} \in \mathcal{E}_p$.
- The set of all $t \in \mathbb{R}$ such that $t\mathbf{v} \in \mathcal{E}_p$ is an open interval

  $$(a_{\mathbf{v}}, b_{\mathbf{v}}) \subset \mathbb{R}$$

  containing 0, and

  $$\begin{aligned} (a_{\mathbf{v}}, b_{\mathbf{v}}) &\longrightarrow M \\ t &\longmapsto \mathsf{Exp}_p(t\mathbf{v}) \end{aligned}$$

  is a maximal geodesic.
- This maximal geodesic is complete if and only if

  $$(a_{\mathbf{v}}, b_{\mathbf{v}}) = (-\infty, +\infty).$$

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(M, ∇) is geodesically complete if and only if $\mathcal{E} = \mathsf{T}M$. Then

$$(p, \mathbf{v}) \xrightarrow{\Phi_t} \Big(\mathsf{Exp}_p(t\mathbf{v}), \frac{d}{dt}\mathsf{Exp}_p(t\mathbf{v})\Big)$$

defines a *flow* (that is, an additive $\mathbb{R}$-action) on $\mathsf{T}M$, called the *geodesic flow* of (M, ∇). The velocity vector

$$\frac{d}{dt}\mathsf{Exp}_p(t\mathbf{v})$$

is the image of $\mathbf{v}$ under parallel translation along the geodesic $\mathsf{Exp}_p|_{[0,t]}$.

Exercise 8.3.2. Suppose that M is connected, and ∇ is an affine connection on M. Let $p \in M$. Then (M, ∇) is complete if and only if $\mathcal{E}_p = \mathsf{T}_p M$.

Definition 8.3.3. Let (M, ∇) be a manifold with an affine connection, and let $x, y \in M$. Then y is *visible* from x if and only if a geodesic joins x to y. Equivalently, y lies in the image $\mathsf{Exp}_x(\mathcal{E}_x)$. Evidently y is visible from x if and only if x is visible from y. We say that y is *invisible* from x if and only if y is not visible from x.

The following idea will be used later in §11.4 and §12.2.

Exercise 8.3.4. Let M be an affine manifold and $p \in M$. Show that the set $M(p)$ of points in M visible from p is open in M. More generally, let M be a projective manifold and $p \in M$. Show that the union $M(p)$ of geodesic segments beginning at p is open in M.

Exercise 8.3.5. Let M be a manifold with an affine connection. For each $p \in M$, show that the function

$$\begin{aligned} \mathsf{T}_p M &\longrightarrow \mathbb{R}^+ \cup \{\infty\} \\ X_p &\longmapsto \sup\big\{t \in \mathbb{R} \,\big|\, tX_p \in \mathcal{E}_p\big\} \end{aligned}$$

is lower semicontinuous.[2]

When the affine connection is *flat,* that is, arises from an affine structure, the exponential map relates to the developing map as follows.

Proposition 8.3.6. Let M be a simply connected affine manifold with developing map $M \xrightarrow{\mathsf{dev}} \mathsf{A}$. Let $p \in M$. Then the composition

$$\mathcal{E}_p \xrightarrow{\mathsf{Exp}_p} M \xrightarrow{\mathsf{dev}} \mathsf{A}$$
$$\mathcal{E}_p \xrightarrow{\mathsf{dev} \circ \mathsf{Exp}_p} \mathsf{A}$$

[2]Semicontinuous functions are discussed in Appendix E.

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extends (uniquely) to an affine isomorphism $\mathsf{T}_p M \xrightarrow{\mathcal{A}_p} \mathsf{A}$: that is, the following diagram

(8.1)
$$\begin{array}{ccc} \mathcal{E}_p & \hookrightarrow & \mathsf{T}_x M \\ {\scriptstyle \mathsf{Exp}_p}\downarrow & & \vdots\,{\scriptstyle \mathcal{A}_p} \\ M & \xrightarrow[\mathsf{dev}]{} & \mathsf{A} \end{array}$$

commutes.

Exercise 8.3.7. Prove Proposition 8.3.6 .

Exercise 8.3.8. Relate the *parallel transport* along a path $x \overset{\gamma}{\rightsquigarrow} y$ to the composition

$$\mathsf{T}_x M \xrightarrow{\mathcal{A}_y^{-1} \circ \mathcal{A}_x} \mathsf{T}_y M.$$

8.3.2. Geodesic completeness and the developing map. Recall from Chapter 1 that Euclidean geodesics, that is, curves with zero acceleration, are motions along straight lines at constant Euclidean speed. Of course, in affine geometry, the speed doesn't make sense, which is why we prefer to characterize geodesics by acceleration. A fundamental result of Auslander–Markus [**12**] is that geodesic completeness of affine manifolds is equivalent to the bijectivity of the developing map.

Theorem 8.3.9 (Auslander–Markus [**12**])**.** Let M be an affine manifold, with a developing map $\widetilde{M} \xrightarrow{\mathsf{dev}} \mathsf{A}$. Then dev is an isomorphism if and only if M is geodesically complete.

That is, the following two conditions are equivalent:

- M is a quotient of affine space by a discrete subgroup $\Gamma \subset \mathsf{Aff}(\mathsf{A})$ acting properly on A;
- A particle on M moving at constant speed in a straight line will continue indefinitely.

Clearly if M is geodesically complete, so is its universal covering $\widetilde{M}$. Hence we may assume M is simply connected. Let $p \in M$. If M is complete, then Exp_p is defined on *all* of $\mathsf{T}_p M$ and

$$\begin{array}{ccc} \mathsf{T}_p M & \xrightarrow{(\mathsf{Ddev})_p} & \mathsf{T}_{\mathsf{dev}(p)}\mathsf{A} \\ {\scriptstyle \mathsf{Exp}}\downarrow & & \downarrow{\scriptstyle \mathsf{Exp}} \\ M & \xrightarrow[\mathsf{dev}]{} & \mathsf{A} \end{array}$$

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commutes. Since the vertical arrows and the top horizontal arrows are bijective, $M \xrightarrow{\mathsf{dev}} \mathsf{A}$ is bijective.

The other direction is a corollary of the following basic result (see also Kobayashi [**222**], Proposition 4.9, Shima [**306**], Theorem 8.1):

Theorem 8.3.10 (Koszul [**229**])**.** Let M be an affine manifold and $p \in M$. Suppose that the domain $\mathcal{E}_p \subset \mathsf{T}_pM$ of the exponential map Exp_p is convex. Then $\widetilde{M} \xrightarrow{\mathsf{dev}} \mathsf{A}$ is a diffeomorphism of $\widetilde{M}$ onto the open subset

$$\Omega_p := \mathsf{Exp}_p(\mathcal{E}_p) \subset \mathsf{A}$$

The proof is based on the following:

Lemma 8.3.11. The image $\mathsf{Exp}_p(\mathcal{E}_p) = M$.

Proof of Lemma 8.3.11. Clearly we may assume that M is simply connected, so that $M \xrightarrow{\mathsf{dev}} \mathsf{A}$ is defined. $\mathcal{E}_p \subset \mathsf{T}_pM$ is open and Exp_p is an open map, so the image $\mathsf{Exp}_p(\mathcal{E}_p)$ is open. Since M is assumed to be connected, we show that $\mathsf{Exp}_p(\mathcal{E}_p)$ is closed.

Let $q \in \overline{\mathsf{Exp}_p(\mathcal{E}_p)} \subset M$. Since M is simply connected, Exp_p maps $\mathcal{E}_p$ bijectively onto $\mathsf{Exp}_p(\mathcal{E}_p)$. Since $q \in \overline{\mathsf{Exp}_p(\mathcal{E}_p)}$, there exists $\mathbf{v} \in \mathsf{T}_pM$ such that

$$\lim_{t\to 1} \mathsf{Exp}_p(t\mathbf{v}) = q.$$

Since the star-shaped open subset $\mathcal{E}_p \subset \mathsf{T}_pM$ is convex, $t\mathbf{v} \in \mathcal{E}_p$ for $0 \le t < 1$. We want to show that $\mathbf{v} \in \mathcal{E}_p$ and $\mathsf{Exp}_p(\mathbf{v}) = q$.

Let $W \ni \mathbf{v}$ be a convex open neighborhood of $\mathbf{v}$ in T_pM, such that its parallel translate

$$W' := \mathbb{P}_{p,q}(W) \subset \mathsf{T}_qM$$

lies in $\mathcal{E}_q$. Then $\mathcal{E}_q \cap W'$ is nonempty. Furthermore, $\exists t_1 > 0$ so that $\mathsf{Exp}_q(\mathcal{E}_q \cap W')$ contains $\mathsf{Exp}_p(t\mathbf{v})$ for $t_1 \le t < 1$. Let $p_1 := \mathsf{Exp}_p(t_1\mathbf{v})$ and $\mathbf{v}_1 := \mathbb{P}_{p,p_1}(\mathbf{v})$. Then

$$\mathsf{Exp}_p(t\mathbf{v}) = \mathsf{Exp}_{p_1}\big((t - t_1)\mathbf{v}_1\big)$$

for $t_1 \le t < 1$ extends to $t = 1$. Thus $\mathbf{v} \in \mathcal{E}_p$ and $\mathsf{Exp}_p(\mathbf{v}) = q$ as desired. □

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Conclusion of proof of Theorem 8.3.10. By the commutativity of (8.1),

$$\begin{array}{ccc}
\mathsf{T}_p M & \xrightarrow[\cong]{(\mathsf{Ddev})_p} & \mathsf{T}_{\mathsf{dev}(p)}\mathsf{A} \\
\uparrow & & \uparrow \\
\mathcal{E}_p & \xrightarrow[\cong]{(\mathsf{Ddev})_p} & (\mathsf{Ddev})_p(\mathcal{E}_p) \\
\Big\downarrow{\scriptstyle \mathsf{Exp}_p} & & \Big\downarrow{\scriptstyle \mathcal{A}_p} \\
M & \xrightarrow[\mathsf{dev}]{} & \mathsf{A}
\end{array}$$

commutes, where the first vertical arrows are inclusions. By the previous argument (now applied to the subset $\mathcal{E}_p \subset \mathsf{T}_p M$) the developing map dev is injective. However, Lemma 8.3.11 implies that $\mathsf{Exp}_p(\mathcal{E}_p) = M$ and thus $\mathsf{dev}(M) = \Omega$. ☐

Exercise 8.3.12. Find a closed affine manifold M such that $\forall x \in M$, the restriction of dev to the closure $\overline{\mathsf{Exp}_x(\mathcal{E}_x)}$ is *not* injective.

More properties of the exponential map, including criteria for incompletenesss, are discussed in § 12.3.

8.4. Complete affine structures on the 2-torus

The compact complete affine 1-manifold $\mathbb{R}/\mathbb{Z}$ is unique up to affine isomorphism. Its Cartesian square $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ is a Euclidean structure on the two-torus, unique up to affine isomorphism. In this section we shall describe all other complete affine structures on the two-torus and show that they are parametrized by the plane $\mathbb{R}^2$.

These structures were first discussed by Kuiper [**234**]; compare also Baues [**32**, **33**] and Baues–Goldman [**34**].

Theorem 8.4.1 (Baues [**32**])**.** The deformation space of marked complete affine structures on $\mathbb{T}^2$ is homeomorphic to $\mathbb{R}^2$.

Recently P. Deligne has observed that this space naturally identifies with a cone over a twisted cubic curve in $\mathbb{R}\mathsf{P}^3$.

Changing the marking corresponds to the action of the *mapping class group* of $\mathbb{T}^2$, which is naturally isomorphic to $\mathsf{GL}(2,\mathbb{Z})$, on the deformation space. This action identifies with the usual *linear action* of $\mathsf{GL}(2,\mathbb{Z})$ on the vector space $\mathbb{R}^2$. The dynamics of this action is very complicated — it is ergodic with respect to Lebesgue measure (which is invariant but infinite) — but the union of its discrete orbits is dense. Its quotient $\mathsf{GL}(2,\mathbb{Z})\backslash\mathbb{R}^2$ is an intractable non-Hausdorff space. In contrast, $\mathsf{GL}(2,\mathbb{Z})$ acts *properly* on the deformation space of marked Euclidean structures on the torus, which identifies with the homogeneous space $\mathsf{GL}(2,\mathbb{R})/\mathsf{O}(2)$. (Compare §7.1.2.)

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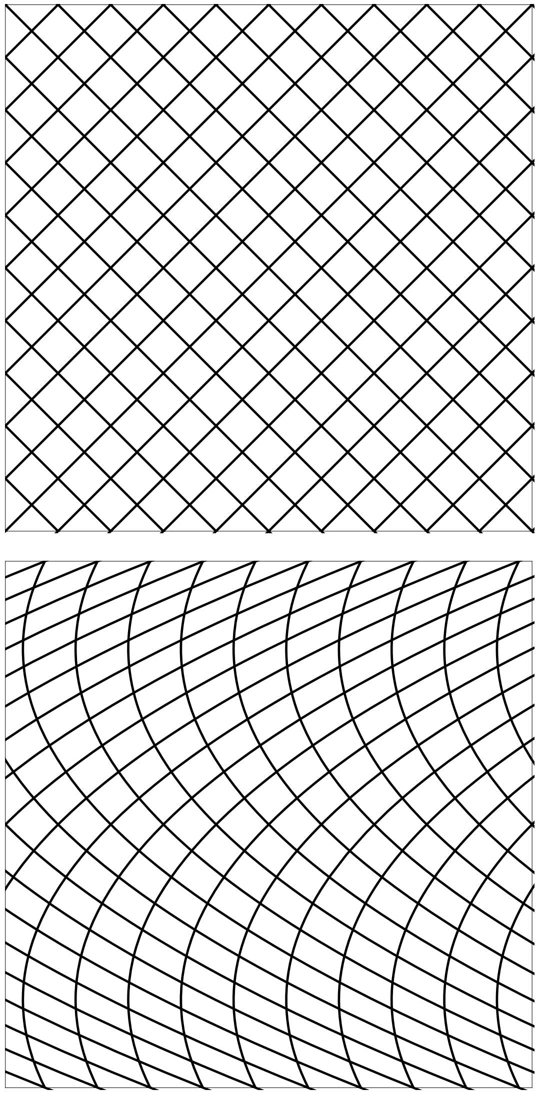

Figure 8.1. Tilings corresponding to some complete affine structures on the 2-torus. The second picture depicts a complete non-Riemannian deformation where the affine holonomy contains no nontrivial horizontal translation. The corresponding torus contains no closed geodesics.

We begin by considering the one-parameter family of (quadratic) diffeomorphisms of the affine plane A^2 defined by

$$\phi_r(x, y) = (x + ry^2, y)$$

Since $\phi_r \circ \phi_s = \phi_{r+s}$, the maps ϕ_r and ϕ_{-r} are mutually inverse. If $\mathbf{u} = (s, t) \in \mathbb{R}^2$ we denote translation by $\mathbf{v}$ as $\mathsf{A} \xrightarrow{\tau_{\mathbf{v}}} \mathsf{A}$. Conjugation of the

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translation $\tau_{\mathbf{u}}$ by ϕ_r yields the affine transformation

$$\alpha_r(\mathbf{u}) \;=\; \phi_r \circ \tau_{\mathbf{u}} \circ \phi_{-r} \;=\; \left[\begin{array}{cc|c} 1 & 2rt & s + rt^2 \\ 0 & 1 & t \end{array}\right]$$

and

$$\mathbb{R}^2 \xrightarrow{\alpha_r} \mathsf{Aff}(\mathsf{A})$$

defines a simply transitive affine action. (Compare [**139**], §1.19].) If $\Lambda \subset \mathbb{R}^2$ is a lattice, then $\mathsf{A}/\alpha_r(\Lambda)$ is a compact complete affine 2-manifold $M = M(r;\Lambda)$ diffeomorphic to a 2-torus.

The parallel 1-form dy defines a parallel 1-form η on M and its cohomology class

$$[\eta] \in H^1(M;\mathbb{R})$$

is a well-defined invariant of the affine structure up to scalar multiplication. In general, M will have no closed geodesics. If $\gamma \subset M$ is a closed geodesic, then it must be a trajectory of the vector field on M arising from the parallel vector field $\partial/\partial x$ on A; then γ is closed if and only if the intersection of the lattice $\Lambda \subset \mathbb{R}^2$ with the line $\mathbb{R} \oplus \{0\} \subset \mathbb{R}^2$ is nonzero.

To classify these manifolds, note that the normalizer of $G_r = \alpha_r(\mathbb{R}^2)$ equals

$$\left\{ \begin{bmatrix} \mu^2 & a \\ 0 & \mu \end{bmatrix} \;\middle|\; \mu \in \mathbb{R}^\times, a \in \mathbb{R} \right\} \cdot G_r$$

which acts on G_r conjugating

$$\alpha_r(s,t) \mapsto \alpha_r(\mu^2 s + at, \mu t)$$

Let

$$N = \left\{ \begin{bmatrix} \mu^2 & a \\ 0 & \mu \end{bmatrix} \;\middle|\; \mu \in \mathbb{R}^\times, a \in \mathbb{R} \right\};$$

then the space of affine isomorphism classes of these tori may be identified with the homogeneous space $\mathsf{GL}(2,\mathbb{R})/N$ which is topologically $\mathbb{R}^2 - \{0\}$. The groups G_r are all conjugate and as $r \longrightarrow 0$, each representation $\alpha_r|_\Lambda$ converges to an embedding of Λ as a lattice of translations $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$. It follows that the deformation space of complete affine structures on $\mathbb{T}^2$ form a space which is the union of $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ with a point O (representing the Euclidean structure) which is in the closure of every other structure.

These structures generalize to *left-invariant affine structures* on Lie groups, which form a rich and interesting algebraic theory, which will be discussed in §10. Many (but not all) closed affine 2-manifolds arise from invariant affine structures on $\mathbb{T}^2$ just as many (but not all) projective 1-manifolds arise from invariant projective structures on T^1 (see §5.5).

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We briefly summarize this more general point of view, referring to Exercise 10.1.1 and §10.3.1 for further details.

Exercise 8.4.2. Let $\mathfrak{a}$ be a 2-dimensional commutative associative $\mathbb{R}$-algebra and let $\Lambda < \mathfrak{a}$ be a lattice.

- Adjoin a (two-sided) identity element $\mathbf{1}$ to $\mathfrak{a}$ to define a 3-dimensional commutative associative $\mathbb{R}$-algebra with unit:
$$\mathfrak{a}' := \mathfrak{a} \oplus \mathbb{R}\mathbf{1}$$
Let G be the (commutative) group of invertible elements in the multiplicatively closed affine plane $\mathsf{A} := \mathfrak{a} \oplus \mathbf{1}$. Then G acts locally simply transitively (or étale) on A, and G inherits an invariant affine structure.
- If $\mathfrak{a}$ is nilpotent, then $\mathfrak{a}^3 = 0$. Then every element of the affine plane A is invertible and $G = \mathsf{A}$.
- Continue to assume that $\mathfrak{a}$ is nilpotent. The quotient Lie group $M := \Lambda\backslash G$ inherits a complete affine structure, and every orientable complete affine 2-manifold arises in this way.
- There are two isomorphism classes of nilpotent algebras $\mathfrak{a}$, depending on whether $\mathfrak{a}^2 = 0$ or $\mathfrak{a}^2 \neq 0$.

We call complete affine 2-tori M arising from a pair $(\mathfrak{a}, \Lambda)$ *Euclidean* if $\mathfrak{a}^2 = 0$ (in which case M is a flat (Euclidean) torus; otherwise we call M *non-Riemannian.*[3]

In general, a complete affine manifold will also have many incomplete affine structures. Indeed, completeness implies topological conditions (for example asphericity) which are not enjoyed by closed incomplete affine manifolds. Here is an amusing example of a 3-manifold which *only* admits complete affine structures (in fact, Euclidean structures).

Exercise 8.4.3. Let $\mathbb{S}^3_{\mathbb{Q}}$ be the rational homology 3-sphere constructed in Exercise 6.4.1. Prove that every affine structure on $\mathbb{S}^3_{\mathbb{Q}}$ is complete.

8.5. Unipotent holonomy

We give a general criterion for completeness of a closed affine manifold under the algebraic condition of *unipotence,* taken from Fried–Goldman–Hirsch [**140**]. Then an invariant parallel flag field exists (as guaranteed by Engel's theorem; see Appendix C), setting up a family of parallel foliations and vector fields. Completeness follows, iteratively, from the complete integrability of vector fields on closed manifolds.

[3]This terminology is meant to emphasize that this affine connection is not the Levi–Civita connection for any (pseudo-)Riemannian structure, although it is geodesically complete.

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Theorem 8.5.1 (Fried–Goldman–Hirsch [**140**], Theorem 8.4(a))**.** Let M be a closed affine manifold whose affine holonomy is unipotent. Then M is complete.

Conversely, if the affine holonomy group is a *nilpotent* group, then completeness is equivalent to unipotent holonomy ([**140**]).

Proof. Choose a basepoint $p_0 \in M$ and let $\widetilde{M} \xrightarrow{\Pi} M$ be the corresponding universal covering space; let $\widetilde{p}_0$ be a basepoint in $\widetilde{M}$ with $\Pi(\widetilde{p}_0) = p_0$. Choose an origin $\mathbf{0}$ to identify A^n with the vector space $\mathbb{R}^n$, and choose a developing map $\widetilde{M} \xrightarrow{\mathsf{dev}} \mathbb{R}^n$ such that $\mathsf{dev}(\widetilde{p}_0) = \mathbf{0}$. Let h be the corresponding affine holonomy representation and let $\Gamma := \mathsf{h}\big(\pi_1(M)\big)$ the affine holonomy group.

Suppose the linear holonomy group $\mathsf{L}(\Gamma) < \mathsf{GL}(n, \mathbb{R})$ is unipotent. Then $\mathsf{L}(\Gamma)$ is upper-triangular with respect to some basis of $\mathbb{R}^n$. That is, $\mathsf{L}(\Gamma)$ preserves a complete linear flag

$$0 = \mathsf{F}^0 \subset \mathsf{F}^1 \subset \cdots \subset \mathsf{F}^n = \mathbb{R}^n$$

where $\mathsf{dim}(\mathsf{F}^k) = k$. Furthermore the induced action on $\mathsf{F}^k/\mathsf{F}^{k-1}$ is trivial. Thus the restriction $\mathsf{L}(\Gamma)|_{\mathsf{F}^k}$ preserves a nonzero linear functional $\mathsf{F}^k \xrightarrow{l_k} \mathbb{R}$ with kernel F^{k-1}. (See, for example, Humphreys [**198**] and the discussion in §C.3.1.)

This invariant flag determines a family of parallel k-plane fields, $0 \leq k \leq n$. Evidently each k-plane field is integrable. We denote the corresponding foliation by $\mathfrak{F}^k$, and the original plane field by $\mathsf{T}\mathfrak{F}^k$. The leaves (maximal integral submanifolds) are totally geodesic affine submanifolds of M. A smooth path $\gamma(t)$ is $\mathfrak{F}^k$*-horizontal* (that is, an *integral curve of* $\mathfrak{F}^k$) if and only if it lies in a single leaf of $\mathfrak{F}^k$. Equivalently, its velocity $\gamma'(t) \in \mathsf{T}_{\gamma(t)}\mathsf{T}\mathfrak{F}^k$.

The linear functionals l_k on F^k determine parallel 1-forms ω_k on the leaves of $\mathfrak{F}^k$ vanishing on $\mathfrak{F}^{k-1}$. A partition of unity on M enables the construction of vector fields $\phi_k \in \mathsf{Vec}(M)$ such that:

- $(\phi_k)_p \in (\mathsf{T}\mathfrak{F}^k)_p \subset \mathsf{T}_pM$ for each $p \in M$,
- $\omega_k(\phi_k) = 1$ on each leaf of $\mathfrak{F}^k$.

Since M is closed, each ϕ_k integrates to a smooth flow on M. Since $(\phi_k)_p \in (\mathsf{T}\mathfrak{F}^k)_p$, the flow preserves the foliation $\mathfrak{F}^k$.

Lift each vector field ϕ_k to a vector field $\widetilde{\phi}_k \in \mathsf{Vec}(\widetilde{M})$. Then for each $\widetilde{p} \in \widetilde{M}$,

$$(\mathsf{Ddev})_{\widetilde{p}}\big(\widetilde{\phi}_k(\widetilde{p})\big) \in \mathsf{F}^k.$$

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Since ϕ_k integrates to a smooth flow on M, its lift $\widetilde{\phi}_k \in \mathsf{Vec}(\widetilde{M})$ integrates to a smooth flow $\widetilde{\Phi}_k$ on $\widetilde{M}$:

$$\begin{aligned} \mathbb{R} \times \widetilde{M} &\longrightarrow \widetilde{M} \\ (t, \widetilde{p}) &\longmapsto \widetilde{\Phi}_k(t)(\widetilde{p}) \end{aligned}$$

Furthermore, $\forall t \in \mathbb{R}, x \in \widetilde{M}$,

$$l_k\Big(\mathsf{dev}\big(\widetilde{\Phi}_k(t)x\big)\Big) \;=\; l_k\big(\mathsf{dev}(x)\big) + t \tag{8.2}$$

since $\omega_k(\phi_k) = 1$.

We first show that dev is surjective: for any $\mathbf{v} \in \mathsf{A}^n \longleftrightarrow \mathbb{R}^n$, we find $\widetilde{p}_n \in \widetilde{M}$ with $\mathsf{dev}(\widetilde{p}_n) = \mathbf{v}$.

Start with the basepoint $\widetilde{p}_0$ chosen above and the arbitrary tangent vector $\mathbf{v}$. Proceed, inductively, by finding a sequence $\widetilde{p}_0, \dots, \widetilde{p}_k$ (for $k \leq n$), beginning at $\widetilde{p}_0$. The point $\widetilde{p}_{k+1}$ after $\widetilde{p}_k$ is obtained by flowing $\widetilde{p}_k$ along $\widetilde{\Phi}_{n-k}$ for some time t_{n-k+1} to be determined. The leaf of $\mathfrak{F}^{n-k-1}$ containing $\widetilde{p}_{k+1}$ will contain the point $\widetilde{p}_n$ that we are seeking. The rest of the path (after $\widetilde{p}_{k+1}$) will lie in this leaf. The desired point $\widetilde{p}_n$ will be the endpoint of the last path.

The induction begins at $k = 0$: define

$$t_n := l_n(\mathbf{v}), \qquad \widetilde{p}_1 := \widetilde{\Phi}_n(t_n)(\widetilde{p}_0).$$

Then $\widetilde{\Phi}_n|_{[0,t_n]}(\widetilde{p}_0)$ is a geodesic path $\widetilde{p}_0 \rightsquigarrow \widetilde{p}_1$ in $\widetilde{M}$. The vector $\mathsf{dev}(\widetilde{p}_1) \in \mathbb{R}^n$ satisfies

$$l_n\big(\mathsf{dev}(\widetilde{p}_1)\big) = t_n = l_n(\mathbf{v})$$

and therefore lies in $\mathbf{v} + \mathsf{F}^{n-1}$.

Inductively suppose:

- $\widetilde{p}_k$ is a point $\widetilde{M}$ such that $\mathsf{dev}(\widetilde{p}_k)$ lies in $\mathbf{v} + \mathsf{F}^{n-k}$.

We construct an $\mathfrak{F}^{n-k}$-horizontal geodesic path of the form $\widetilde{\Phi}_{n-k}(t)(\widetilde{p}_k)$ ending at a point $\widetilde{p}_{k+1} \in \widetilde{M}$ where $\mathsf{dev}(\widetilde{p}_{k+1}) \in \mathbf{v} + \mathsf{F}^{n-k-1}$. Since $vv - \mathsf{dev}\widetilde{\Phi}_{n-k}(t)\widetilde{p}_k$ the linear functional $l_{n-k} \in (\mathsf{F}^{n-k})^*$ is defined on it. Define:

$$t_{n-k} := l_{n-k}\big(\mathbf{v} - \mathsf{dev}(\widetilde{p}_k)\big)$$

and

$$\widetilde{p}_{k+1} := \widetilde{\Phi}_{n-k}(t_{n-k})(\widetilde{p}_k).$$

By (8.2),

$$l_{n-k}\big(\mathsf{dev}(\widetilde{p}_{k+1})\big) = l_{n-k}(\mathbf{v})$$

and

$$\mathsf{dev}(\widetilde{p}_{k+1}) \in \mathbf{v} + \mathsf{F}^{n-k-1}$$

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as desired, completing the induction. The induction ends at $k = n-1$, when

$$\mathsf{dev}(\widetilde{p}_n) \in \mathbf{v} + \mathsf{F}^0 = \{\mathbf{v}\}$$

so $\mathsf{dev}(\widetilde{p}_n) = \mathbf{v}$, as claimed.

Next we prove injectivity.

Suppose $p, q \in \widetilde{M}$ satisfy $\mathsf{dev}(p) = \mathsf{dev}(q)$; we prove that $p = q$. Start with a smooth path $\widetilde{\gamma}_0 : p \rightsquigarrow q$. Inductively, as $k = 1, \dots, n$, we find successive paths $\widetilde{\gamma}_k : p \rightsquigarrow q$, such that each $\widetilde{\gamma}_k$ is $\mathfrak{F}^{n-k}$-horizontal, that is, all $\widetilde{\gamma}_k(t)$ lie in a single leaf of $\mathfrak{F}^{n-k}$. Since the leaves of $\mathfrak{F}^0$ are points, $\widetilde{\gamma}_n$ is constant so $p = q$.

To set up the argument, assume $\mathsf{dev}(p) = \mathbf{0}$. Join p to q by a path $[a, b] \xrightarrow{\widetilde{\gamma}} \widetilde{M}$ such that $\widetilde{\gamma}(a) = p$ and $\widetilde{\gamma}(b) = q$.

Inductively, suppose that $\widetilde{\gamma}_k$ is a $\mathfrak{F}^{n-k}$-horizontal path $p \rightsquigarrow q$. For any $s \in \mathbb{R}$, the path $\widetilde{\Phi}_k(s) \circ \widetilde{\gamma}_k$ is also $\mathfrak{F}^{n-k}$-horizontal. Thus the velocity

$$\big(\widetilde{\Phi}_k(s) \circ \widetilde{\gamma}_k\big)'(t) \in \mathsf{T}\mathfrak{F}^{n-k},$$

and

$$l_{n-k}\Big(\big(\mathsf{dev} \circ \widetilde{\Phi}_k(s) \circ \widetilde{\gamma}_k\big)'(t)\Big)$$

is defined.

Apply the flow $\widetilde{\Phi}_k(s)$ to the path $\widetilde{\gamma}_k(t)$ and evaluate the functional l_{n-k}, obtaining

$$l_{n-k}\Big(\big(\mathsf{dev} \circ \widetilde{\Phi}_k(s) \circ \widetilde{\gamma}_k\big)'(t)\Big) = l_{n-k}\big((\mathsf{dev} \circ \widetilde{\gamma}_k)'(t)\big) + s \tag{8.3}$$

by (8.2). Define

$$\sigma(t) := -\int_0^t l_{n-k}\big((\mathsf{dev} \circ \widetilde{\gamma}_k)'(s)\big) ds,$$

so

$$\widetilde{\gamma}_{k+1}(t) := \widetilde{\Phi}_k\big(\sigma(t)\big)\big(\widetilde{\gamma}_k(t)\big),$$

is a path $p \rightsquigarrow q$ which is $\mathfrak{F}^{n-k-1}$-horizontal by (8.3) and the chain rule. At the final stage ($k = n$),

$$(\widetilde{\gamma}_n)'(t) \in \mathsf{T}\mathfrak{F}^0 = 0,$$

so $\widetilde{\gamma}_n$ is constant, as desired. This completes the proof of Theorem 8.5.1. □

Fried [**137**] proves the following generalization of Theorem 8.5.1:

Theorem 8.5.2. Suppose that M is a closed affine manifold whose linear holonomy preserves a flag

$$0 = \mathsf{F}^0 \subset \mathsf{F}^1 \subset \cdots \subset \mathsf{F}^r = \mathbb{R}^n$$

and acts orthogonally on each quotient $\mathsf{F}^s/\mathsf{F}^{s-1}$. Then M is complete.

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Indeed, M is finitely covered by a complete affine nilmanifold. The hypothesis is equivalent to the *distality* of the affine holonomy.

8.6. Complete affine manifolds

This section describes the general theory of complete affine structures; compare §5.5.1 and § 8.4 for the specific cases of $\mathbb{S}^1$ and $\mathbb{T}^2$ respectively.

The model for the classification is Bieberbach's theorem that every closed *Euclidean* manifold M is finitely covered by a flat torus: that is, M is a quotient of A^n by a lattice of translations.[4]

For complete affine structures on closed manifolds, the conjectural picture replaces the simply transitive group of translations by a more general simply transitive group G of affine transformations, such as the group

$$G = \alpha_r(\mathbb{R}^2) \subset \mathsf{Aff}(\mathsf{A}^2)$$

of §8.4. This statement had been claimed by Auslander [**14**] but the his proof was flawed. The ideas were clarified in Milnor's wonderful paper [**272**] and Fried–Goldman [**139**], which classifies complete affine structures on closed 3-manifolds. (Compare §15.3.) Milnor observed that Auslander's claim was equivalent to the *amenability* of the fundamental group. He asked whether the fundamental group of any complete manifold (possibly noncompact) must be amenable; this is equivalent by Tits's theorem [**328**] to whether a two-generator free group can act properly and affinely.

In the late 1970's, Margulis proved that such group actions do exist; compare §15.4.

8.6.1. The Bieberbach theorems. In 1911–1912 Bieberbach found a general group-theoretic criterion for such groups in arbitrary dimension. In modern parlance, Γ is a *lattice* in $\mathsf{Isom}(\mathsf{E}^n)$, that is, a discrete cocompact subgroup. Furthermore $\mathsf{Isom}(\mathsf{E}^n)$ decomposes as a semidirect product $\mathbb{R}^n \rtimes \mathsf{O}(n)$ where $\mathbb{R}^n$ is the vector space of *translations.* In particular every isometry g is a composition of a translation $x \longmapsto x + \mathbf{b}$ by a vector $\mathbf{b} \in \mathbb{R}^n$, with an orthogonal linear map $A \in \mathsf{O}(n)$:

$$x \xmapsto{g} \mathbf{A}x + \mathbf{b} \tag{8.4}$$

where $\mathbf{A} = \mathsf{L}(g)$ the linear part of g and $\mathbf{b} = u(g)$ is the translational part of g. If $\mathbf{A}$ is only required to be linear, then g is *affine.* An affine automorphism is a Euclidean isometry if and only if its linear part lies in $\mathsf{O}(n)$.

Bieberbach showed:

- $\Gamma \cap \mathbb{R}^n$ is a lattice $\Lambda \subset \mathbb{R}^n$;

[4] An excellent general reference for this classification of Euclidean manifolds and the algebraic theory of their fundamental groups is Charlap [**84**].

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- The quotient Γ/Λ is a finite group, which L maps isomorphically into $\mathsf{O}(n)$.
- Any isomorphism $\Gamma_1 \longrightarrow \Gamma_2$ between Euclidean crystallographic groups $\Gamma_1, \Gamma_2 \subset \mathsf{Isom}(\mathsf{E}^n)$ is induced by an *affine automorphism* $\mathsf{E}^n \longrightarrow \mathsf{E}^n$.
- There are only finitely many affine isomorphism classes of crystallographic subgroups of $\mathsf{Isom}(\mathsf{E}^n)$.

A *Euclidean manifold* is a flat Riemannian manifold, that is, a Riemannian manifold of zero curvature. A Euclidean manifold is *complete* if the underlying metric space is complete, which by the Hopf–Rinow theorem, is equivalent to the condition of geodesic completeness.

A torsionfree Euclidean crystallographic group $\Gamma \subset \mathsf{Isom}(\mathsf{E}^n)$ acts freely on E^n and the quotient E^n/Γ is a complete Euclidean manifold. Conversely every complete Euclidean manifold is a quotient of E^n by a crystallographic group. The geometric version of Bieberbach's theorems is:

- Every compact complete Euclidean manifold is a quotient of a flat torus E^n/Λ (where $\Lambda \subset \mathbb{R}^n$ is a lattice of translations by a finite group of isometries acting freely on E^n/Λ).
- Any homotopy equivalence $M_1 \longrightarrow M_2$ of compact complete Euclidean manifolds is homotopic to an affine diffeomorphism.
- There are only finitely many affine isomorphism classes of compact complete Euclidean manifolds in each dimension n.

8.6.2. Complete affine solvmanifolds. Bieberbach's theorems give a very satisfactory qualitative picture of closed Euclidean manifolds, or, (essentially) equivalently Euclidean crystallographic groups. Does a similar picture hold for *affine crystallographic groups,* that is, for discrete subgroups $\Gamma \subset \mathsf{Aff}(\mathsf{A}^n)$ which act properly on A^n?

For abelian groups, embedding a finitely generated torsionfree group Γ in a Lie group is just taking the span of a discrete subgroup in a vector space, succinctly expressed as tensor product $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$, and containing Γ as a lattice. In particular every element of Γ has a *normal form*

$$\gamma = (b_1)^{n_1} \dots (b_r)^{n_r} \tag{8.5}$$

with exponents $n_j \in \mathbb{Z}$. The connected abelian Lie group $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ is obtained by extending the exponents n_j to $\mathbb{R}$.

Malcev [**250**] gives a beautiful construction for finitely generated torsionfree *nilpotent* groups. In particular he finds a generating set $b_1, \dots, b_r$ so that every $\gamma \in \Gamma$ has normal form (8.5).

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(However unlike the abelian case, the order of the factors is important.) Then, as in the abelian case, he extends to the exponents from integers to real numbers. In particular, this abstract construction embeds every finitely generated torsionfree nilpotent group as a lattice in a 1-connected nilpotent Lie group, called the *Malcev completion* of Γ. (Compare also §2 of Raghunathan [**292**].)

Exercise 8.6.1. Show that if Γ is a discrete subgroup of $\mathsf{GL}(n,\mathbb{R})$ consisting of unipotent matrices, then the Zariski closure of Γ in $\mathsf{GL}(n,\mathbb{R})$ is isomorphic as a Lie group to the Malcev completion.

8.6.2.1. *The Heisenberg group.* The simplest example of a nonabelian nilpotent group is the 3-dimensional *Heisenberg group* $\mathsf{Heis}_{\mathbb{Z}}$, which is a lattice in the 3-dimensional nilpotent Lie group, which we denote $\mathsf{Heis}_{\mathbb{R}}$ (or simply Heis when the context is clear). $\mathsf{Heis}_{\mathbb{R}}$ is the group of upper triangular unipotent matrices

$$(a;b,c) := \begin{bmatrix} 1 & b & a \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \tag{8.6}$$

where $a,b,c \in \mathbb{R}$. Its *integer lattice* $\mathsf{Heis}_{\mathbb{Z}} := \mathsf{Heis}_{\mathbb{R}} \cap \mathsf{GL}(3,\mathbb{Z})$, is obtained by taking $a,b,c \in \mathbb{Z}$.

Exercise 8.6.2. Here is Malcev's general theory in this special case.

- Find a presentation for $\mathsf{Heis}_{\mathbb{Z}}$ of the form:
$$\langle A,B,C \mid AB = BA,\ AC = CA,\ BC = CBA\rangle$$
and find a normal form for elements in $\mathsf{Heis}_{\mathbb{Z}}$, with generators A,B,C of the above form.
- Show that $\mathsf{center}(\mathsf{Heis}_{\mathbb{Z}}) \cong \mathbb{Z}$ with $\mathsf{Heis}_{Z}/\mathsf{center}(\mathsf{Heis}_{\mathbb{Z}}) \cong \mathbb{Z}^2$, making $\mathsf{Heis}_{\mathbb{Z}}$ a *nontrivial* central extension:
$$\mathbb{Z} \rightarrowtail \mathsf{Heis}_{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}^2$$
- Alternatively, express $\mathsf{Heis}_{\mathbb{Z}}$ as a semidirect product $\mathbb{Z}^2 \rtimes \mathbb{Z}$, making $\mathsf{Heis}_{\mathbb{Z}}$ a *split* extension:
$$\mathbb{Z}^2 \rightarrowtail \mathsf{Heis}_{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}.$$
Identify the automorphism of $\mathbb{Z}^2$ defining the semidirect product.
- Find infinitely many non-isomorphic lattices $\Gamma < \mathsf{Heis}_{\mathbb{R}}$.
  - Identify the quotient 3-manifolds $M := \Gamma\backslash\mathsf{Heis}_{\mathbb{R}}$ as oriented $\mathbb{S}^1$-bundles over and alternatively as $\mathbb{T}^2$-bundles over $\mathbb{S}^1$. Identify the homeomorphism of $\mathbb{T}^2$ for which M is the mapping torus.

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8.6.2.2. *Auslander–Markus examples.* In [**13**] Auslander and Markus constructed examples of affine crystallographic groups Γ in dimension 3, for which all three Bieberbach theorems directly fail. Furthermore their examples consist of Lorentzian isometries of $\mathsf{E}^{2,1}$, so in their examples, the quotients

$$M^3 \;=\; \Gamma\backslash\mathsf{A}^3$$

are *complete flat Lorentzian* 3*-manifolds.* Topologically these are all $\mathbb{T}^2$-torus bundles over $\mathbb{S}^1$; conversely every torus bundle over the circle admits such a structure. These constitute three of the 3-dimensional *Thurston geometries* (see [**323**],§3.8):

- Euclidean geometry;
- *Nilgeometry,* modeled on the 3-dimension Heisenberg group $\mathsf{Heis}_{\mathbb{R}}$ with a left-invariant *Riemannian* structure;
- *Solvgeometry,* modeled on the 3-dimensional Lie group $\mathsf{Sol}_{\mathbb{R}}$ with a left-invariant Riemannian structure (as a concrete model, Sol is isomorphic to the isometry group $\mathsf{Isom}^0(\mathsf{E}^{1,1}) \cong \mathbb{R} \rtimes \mathbb{R}^2$ where $\mathbb{R}$ acts on $\mathbb{R}^2$ by
$$t \longmapsto \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}.$$
These include *hyperbolic torus bundles,* (mapping tori of hyperbolic automorphisms of $\mathbb{T}^2$) and, analogous to the above discussion of nilmanifolds, are coset spaces of Sol.

Affine structures arising from affine Lie groups isomorphic to $\mathbb{R}^3$, Heis and Sol are discussed in Chapter 10 and §15.3, §15.3. We next describe an example of complete affine structures on Heisenberg nilmanifolds.

8.6.2.3. *Cartan's example of affine structure on* Heis. In his 1927 paper [**80**][5], É. Cartan describes a bi-invariant complete affine structure on Heis. In the notation of (8.6), the group element $(a; b, c)$ corresponds to $\mathbf{1} + N$, where N is an element of the matrix algebra

$$\mathfrak{a}_{\mathsf{Heis}} := \left\{ \begin{bmatrix} 0 & b & a \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}. \tag{8.7}$$

Unlike the preceding examples, this algebra is *not* commutative, although it is associative and nilpotent. The coordinate basis A, B, C for which

$$(a; b, c) = \mathbf{1} + aA + bB + cC$$

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has multiplication described in Table 8.1.

Table 8.1. This algebra of nilpotent upper triangular matrices is not commutative, and describes Cartan's bi-invariant complete affine structure on $\mathsf{Heis}_{\mathbb{R}}$.

	A	B	C
A	0	0	0
B	0	0	A
C	0	0	0

The procedure described in Exercise 8.4.2 applies to the *group* $\mathsf{Heis}_R = \mathfrak{a}_{\mathsf{Heis}} \oplus \mathbf{1}$, despite the noncommutativity of $\mathfrak{a}_{\mathsf{Heis}}$. Both left-multiplication and right-multiplication by $\mathsf{Heis}_{\mathbb{R}}$ on $\mathfrak{a}_{\mathsf{Heis}}$ are affine and lead to simply transitive 3-dimensional affine actions. Although these are different actions (and lead to different affine structures) they are affinely isomorphic.

8.6.2.4. *Hulls in solvable groups.* These examples arise from a more general construction: namely, Γ embeds as a lattice in a closed Lie subgroup $G \subset \mathsf{Aff}(\mathsf{A})$ whose identity component G^0 acts *simply transitively* on A.

Furthermore $\Gamma^0 := \Gamma \cap G^0$ has finite index in Γ, so the flat Lorentz manifold M^3 is finitely covered by the homogeneous space G^0/Γ^0. Necessarily G^0 is simply connected solvable. The group G^0 plays the role of the translation group $\mathbb{R}^n$ and G is called the *crystallographic hull* in Fried–Goldman [**139**].

Exercise 8.6.3. If G is a Lie group which admits a simply transitive affine action, then G is solvable.

A weaker version of this construction is the *syndetic hull,* defined in [**139**], but known to H. Zassenhaus, H. C. Wang, and L. Auslander. A good general reference for this theory is Raghunathan [**292**] and Grunewald–Segal [**178**]. In particular Grunewald–Segal [**178**] correct an error in the discussion of syndetic hulls in [**139**]. If $\Gamma \subset \mathsf{GL}(n)$ is a solvable group, then a *syndetic hull* for Γ is a subgroup G such that:

- $\Gamma \subset G \subset \mathbb{A}(\Gamma)$, where $\mathbb{A}(\Gamma) \subset \mathsf{GL}(n)$ is the Zariski closure (algebraic hull) of Γ in $\mathsf{GL}(n)$;
- G is a closed subgroup having finitely many connected components;
- G/Γ is compact (although not necessarily Hausdorff).

The last condition is somewhat called *syndetic,* since "cocompact" sometimes refers to a subgroup whose coset is space is compact *and Hausdorff.* (This terminology is due to Gottschalk and Hedlund [**171**].) Equivalently,

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$\Gamma \subset G$ is syndetic if and only if $\exists K \subset G$ which is compact and meets every left coset $g\Gamma$, for $g \in G$. (Compare §A.2.) In general, syndetic hulls fail to be unique; in fact:

Exercise 8.6.4. Find an example of an affine crystallographic group with infinitely many syndetic hulls.

If $M = \Gamma\backslash\mathsf{A}$ is a complete affine manifold, then $\Gamma \subset \mathsf{Aff}(\mathsf{A})$ is a discrete subgroup acting properly and freely on A.

Exercise 8.6.5. Find an affine transformation g of A generating a cyclic group $\langle g\rangle$ which is discrete but doesn't act properly on A.

A proper action of a discrete group is the usual notion of a *properly discontinuous action.* If the action is also free (that is, no fixed points), then the quotient is a (Hausdorff) smooth manifold, and the quotient map $\mathsf{A} \longrightarrow \Gamma\backslash\mathsf{A}$ is a covering space. A properly discontinuous action whose quotient is compact as well as Hausdorff is said to be *crystallographic,* in analogy with the classical notion of a *crystallographic group:* A *Euclidean crystallographic group* is a discrete cocompact group of Euclidean isometries. Its quotient space is a Euclidean orbifold. Since such groups act isometrically on metric spaces, discreteness here does imply properness; this dramatically fails for more general discrete groups of *affine transformations.*

L. Auslander [**14**] claimed to prove that the Euler characteristic vanishes for a compact complete affine manifold, but his proof was flawed. It rested upon the following question, which in [**139**], was demoted to a "conjecture," and is now (perhaps erroneously) known as the "Auslander Conjecture":

Conjecture 8.6.6. Let M be a compact complete affine manifold. Then $\pi_1(M)$ is virtually polycyclic.

In that case the affine holonomy group $\Gamma \cong \pi_1(M)$ embeds in a closed Lie subgroup $G \subset \mathsf{Aff}(\mathsf{A})$ satisfying:

- G has finitely many connected components;
- The identity component G^0 acts simply transitively on A.

Then $M = \Gamma\backslash\mathsf{A}$ admits a finite covering space $M^0 := \Gamma^0\backslash\mathsf{A}$ where

$$\Gamma^0 := \Gamma \cap G^0.$$

The simply transitive action of G^0 define a complete *left-invariant affine structure* on G^0 and the developing map is just the evaluation map of this action. Necessarily G^0 is a simply connected solvable Lie group and M^0 is affinely isomorphic to the *complete affine solvmanifold* $\Gamma^0\backslash G^0$. In particular $\chi(M^0) = 0$ and thus $\chi(M) = 0$.

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The existence of syndetic hulls iss the natural extension of Bieberbach's theorems describing the structure of flat Riemannian (or Euclidean) manifolds; see Milnor [**271**] for an exposition of this theory and its historical importance. Every flat Riemannian manifold is finitely covered by a *flat torus,* the quotient of A by a lattice of translations. In the more general case, G^0 plays the role of the group of translations of an affine space and the solvmanifold M^0 plays the role of the flat torus. The importance of Conjecture 8.6.6 is that it would provide a detailed and computable structure theory for compact complete affine manifolds.

Fried–Goldman [**139**] established Conjecture 8.6.6 in dimension 3. The proof involves classifying the possible Zariski closures $\mathbb{A}\big(\mathsf{L}(\Gamma)\big)$ of the linear holonomy group inside $\mathsf{GL}(\mathsf{A})$. Goldman–Kamishima [**165**] prove Conjecture 8.6.6 for flat Lorentz manifolds. Grunewald–Margulis [**177**] establish Conjecture 8.6.6 when the Levi component of $\mathsf{L}(\Gamma)$ lies in a real rank-one subgroup of $\mathsf{GL}(\mathsf{A})$. See Tomanov [**330**], Soifer [**314**] and Abels–Margulis–Soifer [**3**, **4**] for further results. The conjecture is now known in all dimensions ≤ 6 (Abels–Margulis–Soifer [**2**]).

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Part 3

Affine and projective structures

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Chapter 9

Affine structures on surfaces and the Euler characteristic

One of our first goals is to classify affine structures on closed 2-manifolds. As noted in §7.5, classification of structures on noncompact manifolds is much different, and reduces to a homotopy-theoretic problem since the equivalence relation is much bigger.

The classification of closed affine 2-manifolds splits into two steps: first is the basic result of Benzécri that a closed surface admits an affine structure if and only if its Euler characteristic vanishes. This was conjectured by Kuiper [**234**] who proved it in the convex case, using the asymptotics of holonomy sequences (see §2.6, which goes back to Myrberg [**278**]). In particular the affine holonomy group of a closed affine 2-manifold is abelian. The second step uses simple algebraic methods to classify affine structures. This chapter deals with the first step and its generalizations.

9.1. Benzécri's theorem on affine 2-manifolds

The following result was first proved in [**45**]. Shortly afterwards, Milnor [**268**] gave a more general proof, clarifying its homotopic-theoretic nature. For generalizations of Milnor's result, see Benzécri [**47**], Gromov [**173**], Sullivan [**319**] and Smillie [**309**]. For an interpretation of this inequality in terms of hyperbolic geometry, see [**146**]. More recent developments are surveyed in [**159**].

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Theorem 9.1.1 (Benzécri 1955)**.** Let M be a closed 2-dimensional affine manifold. Then $\chi(M) = 0$.

Proof. Replace M by its orientable double covering to assume that M is orientable. By the classification of surfaces, M is diffeomorphic to a closed surface of genus $g \geq 0$. Since a simply connected closed manifold admits no affine structure, (§5.2.4), M cannot be a 2-sphere and hence $g \neq 0$. We assume that $g > 1$ and will obtain a contradiction.

9.1.1. The surface as an identification space. Begin with the topological model for M. There exists a decomposition of M along $2g$ simple closed curves $a_1, b_1, \dots, a_g, b_g$ which intersect in a single point $x_0 \in M$. (Compare Fig. 9.1.) The complement

$$M \setminus \bigcup_{i=1}^{g} (a_i \cup b_i)$$

is the interior of a $4g$-gon F with edges

$$a_1^+, a_1^-, b_1^+, b_1^-, \dots, a_g^+, a_g^-, b_g^+, b_g^-.$$

(Compare Fig. 9.2.) There exist maps

$$A_1, B_1, \dots, A_g, B_g \in \pi$$

defining identifications:

$$A_i(b_i^+) = b_i^-,$$
$$B_i(a_i^+) = a_i^-$$

for a quotient map $F \longrightarrow M$. A universal covering space is the quotient space of the product $\pi \times F$ by identifications defined by the generators $A_1, B_1, \dots, A_g, B_g$.

9.1.2. The turning number. Let $[t_0, t_1]$ be a closed interval. If $[t_0, t_1] \xrightarrow{f} \mathbb{R}^2$ is a smooth immersion, then its *turning number* $\tau(f)$ is defined as the total angular displacement of its tangent vector (normalized by dividing by 2π). Explicitly, if $f(t) = (x(t), y(t))$, then

$$\tau(f) = \frac{1}{2\pi}\int_{t_1}^{t_2} d\tan^{-1}(y'(t)/x'(t)) = \frac{1}{2\pi}\int_{t_1}^{t_2} \frac{x'(t)y''(t) - x''(t)y'(t)}{x'(t)^2 + y'(t)^2} dt.$$

Extend τ to piecewise smooth immersions as follows. Suppose that $[t_0, t_N] \xrightarrow{f} \mathbb{R}^2$ is an immersion which is smooth on subintervals $[a_i, a_{i+1}]$ where

$$t_0 < t_1 < \cdots < t_{N-1} < t_N.$$

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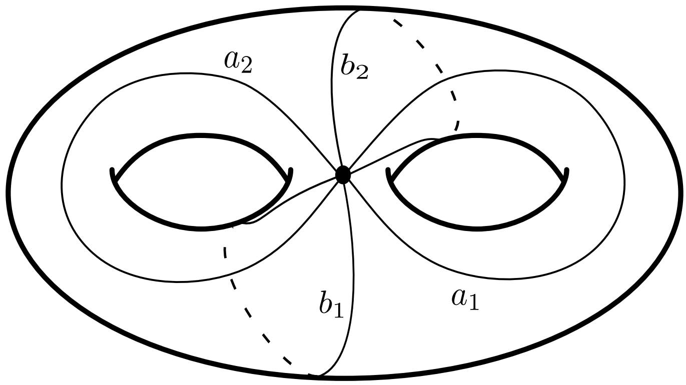

Figure 9.1. Decomposing a genus $g = 2$ surface along $2g$ curves into a $4g$-gon. The single common intersection of the curves is a single point which decomposes into the $4g$ vertices of the polygon.

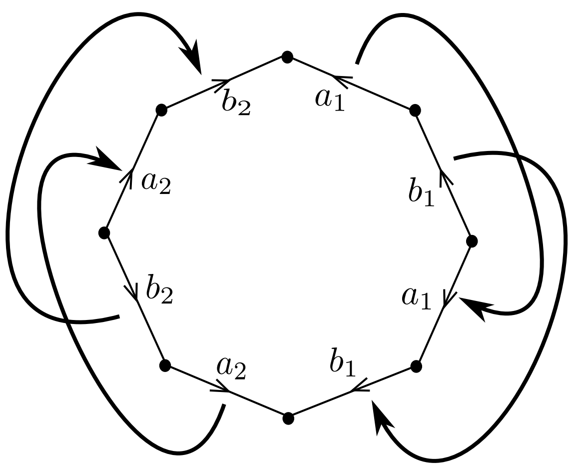

Figure 9.2. Identifying the edges of a $4g$-gon into a closed surface of genus g. The sides are paired into $2g$ curves, which meet at the single vertex.

Let $f'_+(t_i) = \lim_{t\to t_i+} f'(t)$ and $f'_-(t_i) = \lim_{t\to t_i-} f'(t)$ be the two tangent vectors to f at t_i; define the total turning number of f by:

$$\tau(f) := \tau^{cont}(f) + \tau^{disc}(f)$$

where the *continuous contribution* is:

$$\tau^{cont}(f) := \sum_{i=0}^{N-1} (\tau(f|_{[t_i,t_{i+1}]})$$

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and the *discrete contribution* is:

$$\tau^{disc}(f) := \frac{1}{2\pi}\sum_{i=0}^{N-2} \angle(f'_-(t_{i+1}), f'_+(t_{i+1}))$$

where $\angle(v_1, v_2)$ represents the positively measured angle between the vectors v_1, v_2.

Here are some other elementary properties of τ:

- Denote $-f$ the immersion obtained by reversing the orientation on t:
$$(-f)(t) := f(t_0 + t_N - t)$$
Then $\tau(-f) = -\tau(f)$.
- If $g \in \mathsf{Isom}(\mathsf{E}^2)$ is an orientation-preserving Euclidean isometry, then $\tau(f) = \tau(g \circ f)$.
- If f is an immersion of $\mathbb{S}^1$, then $\tau(f) \in \mathbb{Z}$.
- Furthermore, if an immersion $\partial D^2 \xrightarrow{f} \mathsf{E}^2$ extends to an orientation-preserving immersion $D^2 \longrightarrow \mathsf{E}^2$, then $\tau(f) = 1$.

The *Whitney–Graustein theorem* asserts that immersions $\mathbb{S}^1 \xrightarrow{f_i} \mathbb{R}^2$ ($i = 1, 2$) are regularly homotopic if and only if $\tau(f_1) = \tau(f_2)$, which implies the last remark.

Exercise 9.1.2. Suppose that S is a compact oriented surface with boundary components $\partial_1 S, \ldots, \partial_k S$. Suppose that $S \xrightarrow{f} \mathsf{E}^2$ is an orientation-preserving immersion. Then

$$\sum_{i=1}^{k} \tau(f|_{\partial_i S}) = \chi(S).$$

An elementary property relating turning number to affine transformations is the following:

Lemma 9.1.3. Suppose that $[a, b] \xrightarrow{f} \mathbb{R}^2$ is a smooth immersion and $\phi \in \mathsf{Aff}^+(\mathbb{R}^2)$ is an orientation-preserving affine automorphism. Then

$$|\tau(f) - \tau(\phi \circ f)| < \frac{1}{2}.$$

Proof. If ψ is an orientation-preserving Euclidean isometry, then $\tau(f) = \tau(\psi \circ f)$; by composing ϕ with an isometry we may assume that

$$f(a) = (\phi \circ f)(a)$$
$$f'(a) = \lambda(\phi \circ f)'(a)$$

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for $\lambda > 0$. That is,

$$\mathsf{L}(\phi)(f'(a)) = \lambda f'(a). \tag{9.1}$$

Suppose that $|\tau(f) - \tau(\phi \circ f)| \geq 1/2$. Since for $a \leq t \leq b$, the function

$$|\tau(f|_{[a,t]}) - \tau(\phi \circ f|_{[a,t]})|$$

is a continuous function of t and equals 0 for $t = a$ and is $\geq 1/2$ for $t = b$. The intermediate value theorem implies that there exists $0 < t_0 \leq b$ such that

$$|\tau(f|_{[a,t_0]}) - \tau(\phi \circ f|_{[a,t_0]})| = 1/2.$$

Then the tangent vectors $f'(t_0)$ and $(\phi \circ f)'(t_0)$ have opposite direction, that is, there exists $\mu < 0$ such that

$$\mathsf{L}(\phi)(f'(t_0)) = (\phi \circ f)'(t_0) = \mu f'(t_0). \tag{9.2}$$

Combining (9.1) with (9.2), the linear part $\mathsf{L}(\phi)$ has eigenvalues λ, μ with $\lambda > 0 > \mu$. However ϕ preserves orientation, contradicting $\mathsf{Det}\big(\mathsf{L}(\phi)\big) = \lambda\mu < 0$. □

We apply these ideas to the restriction of the developing map dev to ∂F. Since $f := \mathsf{dev}|_{\partial F}$ is the restriction of the immersion $\mathsf{dev}|_F$ of the 2-disc,

$$1 = \tau(f) = \tau^{disc}(f) + \tau^{cont}(f)$$

where

$$\begin{aligned}\tau^{cont}(f) &= +\sum_{i=1}^{g} \tau(\mathsf{dev}|_{a_i^+}) + \tau(\mathsf{dev}|_{a_i^-}) + \tau(\mathsf{dev}|_{b_i^+}) + \tau(\mathsf{dev}|_{b_i^-}) \\ &= \sum_{i=1}^{g} \tau(\mathsf{dev}|_{a_i^+}) - \tau(h(B_i) \circ \mathsf{dev}|_{a_i^+}) \\ &\qquad\qquad + \tau(\mathsf{dev}|_{b_i^+}) - \tau(h(A_i) \circ \mathsf{dev}|_{b_i^+})\end{aligned}$$

since $h(B_i)$ identifies $\mathsf{dev}_{a_i^+}$ with $-\mathsf{dev}_{a_i^-}$ and $h(A_i)$ identifies $\mathsf{dev}_{b_i^+}$ with $-\mathsf{dev}_{b_i^-}$. By Lemma 9.1.3, each

$$\begin{aligned}|\tau(\mathsf{dev}|_{a_i^+}) - \tau(h(B_i) \circ \mathsf{dev}|_{a_i^+})| &< \frac{1}{2} \\ |\tau(\mathsf{dev}|_{b_i^+}) - \tau(h(A_i) \circ \mathsf{dev}|_{b_i^+})| &< \frac{1}{2}\end{aligned}$$

and thus

$$|\tau^{cont}(f)| < \sum_{i=1}^{g} \frac{1}{2} + \frac{1}{2} = g \tag{9.3}$$

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Now we estimate the discrete contribution. The j-th vertex of ∂F contributes $1/2\pi$ of the angle

$$\angle\big(f'_-(t_j), f'_+(t_j)\big),$$

which is supplementary to the i-th *interior angle* α_j of the polygon ∂F, as measured in the Euclidean metric $\mathsf{dev}^*\mathsf{g}_{\mathsf{E}^2}$.

Let $m_0 \in M$ be the point corresponding to the $4g$ vertices of F. The total angle around m_0 (as measured in the metric $\mathsf{dev}^*\mathfrak{g}_{\mathsf{E}^2}$ restricted to the lift of a coordinate patch equals 2π and we would like to identify this as the sum $\sum_{j=1}^{4g} \alpha_j = 2\pi$. However, the interior angle of the side of ∂F may not equal the corresponding angle in the tangent space of m_0, since they are related by an element of the holonomy group, which is an affine transformation. Angles are generally *not* preserved by affine transformations, unless they are multiples of a straight angles π radians. Thus, we assume that the edges emanating from each vertex meet at an angle α_j, which is a multiple of π. (Benzécri considers the case when all of the angles are 0 except one, which equals 2π, as in Figure 9.3.)

Then the total cone angle at m_0 (as measured in this local Euclidean metric) equals 2π, that is,

$$\sum_{j=1}^{4g} \alpha_j = 2\pi$$

as desired, and

$$\begin{aligned}\tau^{disc}(f) &= \frac{1}{2\pi}\sum_{j=1}^{4g} \angle\big(f'_-(t_j), f'_+(t_j)\big) \\ &= \frac{1}{2\pi}\sum_{j=1}^{4g}(\pi - \alpha_j) = 2g - 1.\end{aligned}$$

Now

$$\tau^{cont}(f) = \tau(f) - \tau^{disc} = 1 - (2g-1) = 2 - 2g$$

but (9.3) implies $2g - 2 < g$, that is, $g < 2$ as desired. □

Benzécri's original proof uses a decomposition where all the sides of F have the same tangent direction at x_0; thus all the α_j equal 0 except for one which equals 2π (as in Figure 9.4).

9.1.3. The Milnor–Wood inequality.

Shortly after Benzécri proved the above theorem, Milnor observed that this result follows from a more general theorem on flat vector bundles. Let E be the 2-dimensional oriented vector bundle over M whose total space is the quotient of $\widetilde{M} \times \mathbb{R}^2$ by the diagonal action of π by deck transformations on $\widetilde{M}$

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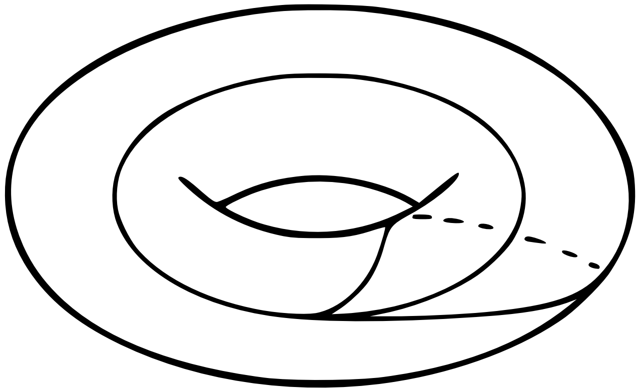

Figure 9.3. Cell-division of a torus where all but one angle at the vertex is 0.

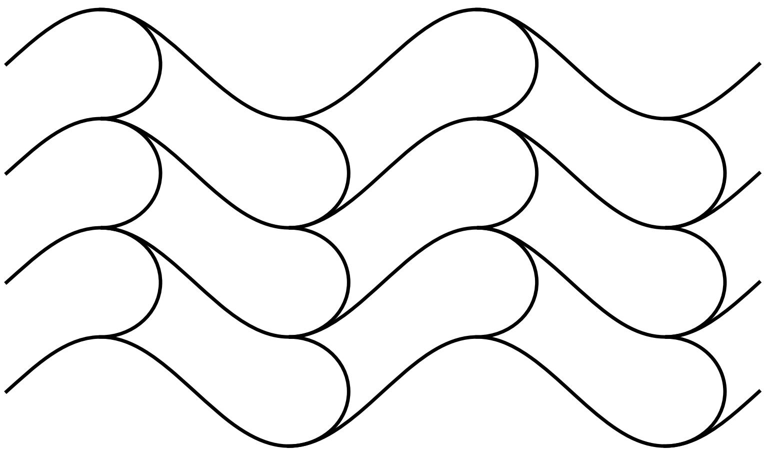

Figure 9.4. Doubly periodic tiling of the Euclidean plane with all but one angle at vertex 0.

and via $\mathsf{L} \circ h$ on $\mathbb{R}^2$, (that is, the *flat vector bundle over M associated to the linear holonomy representation.)* This bundle has a natural flat structure, since the coordinate changes for this bundle are (locally) constant linear maps. Now an oriented $\mathbb{R}^2$-bundle ξ over a space M is classified by its *Euler class*

$$\mathsf{Euler}(\xi) \in H^2(M; \mathbb{Z}).$$

For M a closed oriented surface, the orientation defines an isomorphism $H^2(M; \mathbb{Z}) \cong \mathbb{Z}$, and we henceforth identify these groups when the context is clear. If ξ is an oriented $\mathbb{R}^2$-bundle over M which admits a flat structure, Milnor [**268**] showed that

$$|\mathsf{Euler}(\xi)| < g.$$

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Furthermore every integer in this range is realized by a flat oriented 2-plane bundle. If M is an affine manifold, then the bundle E is isomorphic to the tangent bundle $\mathsf{T}M$ of M and hence has Euler number

$$\mathsf{Euler}(\mathsf{T}M) = 2 - 2g.$$

Thus the only closed orientable surface whose tangent bundle has a flat structure is a torus. Furthermore Milnor showed that any $\mathbb{R}^2$-bundle whose Euler number satisfies the above inequality has a flat connection.

Extensions of the Milnor–Wood inequality to higher dimensions have been proved by Benzécri [**47**], Smillie [**309**], Sullivan [**319**], Burger–Iozzi–Wienhard [**71**] and Bucher–Gelander [**67**].

9.2. The Euler Characteristic in higher dimensions

In the early 1950's Chern suggested that in general the Euler characteristic of a compact affine manifold must vanish. Based on the Chern–Weil theory of representing characteristic classes by curvature, several special cases of this conjecture can be solved: if M is a compact *complex* affine manifold, then the Euler characteristic is the top Chern number and hence can be expressed in terms of curvature of the complex linear connection (which is zero). However, in general, for a real vector bundle, only the Pontryagin classes are polynomials in the curvature — indeed Milnor's examples show that the Euler class *cannot* be expressed as a polynomial in the curvature of a linear connection (although it can be expressed as a polynomial in the curvature of an *orthogonal* connection).

This question has been an extremely important impetus for research in this subject.

Deligne–Sullivan [**110**] proved a strong vanishing theorem for flat *complex* vector bundles. Namely, every flat complex vector bundle ξ over a finite complex M is *virtually trivial:* that is, there exists a finite covering space $\hat{M} \xrightarrow{f} M$ such that $f^*\xi$ is trivial. This immediately implies that $\mathsf{Euler}(\xi) = 0$. Hirsch and Thurston [**193**] gave a very general criterion for vanishing of the Euler class of flat bundle with amenable holonomy; compare Goldman–Hirsch [**162**] for an elementary proof in the case of flat vector bundles.

Using an ingenious argument, Sullivan [**318**] proved that the Euler number of a flat vector bundle with *integral* holonomy vanishes.

Exercise 9.2.1. Prove Sullivan's theorem: Suppose that M is a closed affine manifold with linear holonomy in $\mathsf{SL}(n, \mathbb{Z})$. Show that $\chi(M) = 0$. (Hint: Since the linear holonomy group preserves the integral lattice $\mathbb{Z}^n < \mathbb{R}^n$, the

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tangent bundle passes down to a $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$-bundle $\mathcal{T}_M$ over M. The 2-torsion in $\mathbb{T}^n$ defines a closed submanifold $N \subset \mathcal{T}_M$ disjoint from the image of the zero-section $\mathbf{0}_M$ in $\mathcal{T}_M$.)

9.2.1. The Chern–Gauss–Bonnet Theorem. Most of the known special cases of the Chern–Sullivan conjecture follow from the Chern's intrinsic generalization of the Gauss–Bonnet theorem [**87**] and the Chern–Weil theory of characteristic classes. This includes flat pseudo-Riemannian manifolds, flat complex manifolds, and complete affine manifolds (Kostant–Sullivan). Notable exceptions are Benzécri's theorem for surfaces and the Kobayashi–Vey theorem for hyperbolic affine structures.

Chern's theorem concerns an oriented orthogonal rank n vector bundle $\xi \longrightarrow M$ over an oriented closed n-dimensional manifold M. That is, ξ is a smooth $\mathbb{R}^n$-bundle over M with an orthogonal connection ∇ and an orientation on the fibers. Let

$$\mathsf{Euler}(\xi) \in H^n(M, \mathbb{Z})$$

denote the Euler class of the oriented $\mathbb{R}^n$-bundle ξ. (Compare Milnor–Stasheff [**273**] and Steenrod [**317**].) The orthogonal connection ∇ determines an exterior n-form $\mathsf{Euler}(\nabla)$, the *Euler form* of ∇ on M, such that

$$\int_M \mathsf{Euler}(\nabla) = \mathsf{Euler}(\xi) \cdot [M]$$

where $[M] \in H_n(M, \mathbb{Z})$ denotes the fundamental class of M arising from the orientation. The Euler form is a polynomial expression in the curvature of ∇ and vanishes if ∇ is flat. When ξ is the tangent bundle of M, then

$$\mathsf{Euler}(\xi) \cdot [M] = \chi(M),$$

the Euler characteristic of M.

Milnor [**268**] showed that, over a closed oriented surface of genus $g > 1$, every oriented $\mathbb{R}^2$-bundle E with $|\mathsf{Euler}(\xi)| < g$ admits a flat structure. That is, ξ admits a flat *linear connection* ∇, but if $\mathsf{Euler}(\xi) \neq 0$, then ∇ cannot be orthogonal.

We summarize some of the ideas in Chern's theorem.[1] A key point in this proof is the use of the associated principal $\mathsf{SO}(n)$-bundle over M, which is the bundle of positively oriented orthonormal frames. When $n = 2$, this is the unit tangent bundle of M and is an $\mathbb{S}^1$-bundle over M. As discussed in Steenrod [**317**] and Milnor–Stasheff [**273**], the Euler class is really an invariant of the associated oriented $\mathbb{S}^{n-1}$-bundle. The quotient of the total space by the antipodal map on the fiber is an $\mathbb{R}\mathsf{P}^{n-1}$-bundle, which when $n = 2$, identifies with a sphere bundle itself. (Compare Exercise 9.2.2.)

[1] We refer to Poor [**291**] (§3.56–3.73, pp. 138–49) for detailed proofs and discussion. According to Poor, this geometric approach is due to Gromoll.

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Let ξ be a smooth oriented real vector bundle of even rank $n = 2m$ over an oriented smooth n-manifold M with an orthogonal connection ∇. Let $\mathfrak{so}(\xi)$ be the vector bundle to ξ associated to the adjoint representation

$$\mathsf{SO}(2m) \longrightarrow \mathsf{Aut}\big(\mathfrak{so}(2m)\big).$$

The *curvature tensor* $R(\nabla)$ is an $\mathfrak{so}(\xi)$-valued exterior 2-form on M. The *Pfaffian* is an Ad-invariant polynomial mapping $\mathfrak{so}(2m,\mathbb{R}) \xrightarrow{\mathsf{Pfaff}} \mathbb{R}$ such that

$$\mathsf{Det}(A) = \mathsf{Pfaff}(A)^2$$

and Pfaff is a polynomial of degree m. For example, for $m = 1$, and

$$A = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix},$$

$\mathsf{Det}(A) = a^2$ and $\mathsf{Pfaff}(A) = a$. Applying the Pfaffian to $R(\nabla)$ yields an exterior $2m$-form $\mathsf{Pfaff}\big(R(\nabla)\big)$ on M^{2m}.

The Euler number of ξ (relative to the orientations of ξ and A) can be computed by the *Poincaré–Hopf theorem:* Namely, let η be a section of ξ with isolated zeroes $p_1, \dots, p_k$. Find an open ball B_i containing p_i and a trivialization

$$E_{B_i} \xrightarrow{\approx} B_i \times \mathbb{R}^{2m}.$$

With respect to this trivialization, the restriction of η to B_i is the graph of a map $B_i \xrightarrow{f_i} \mathbb{R}^{2m}$ where $f_i(x) \neq 0$ if $x \neq p_i$. The degree of the smooth map

$$\mathbb{S}^{2m-1} \approx \partial B_i \longrightarrow \mathbb{S}^{2m-1}$$
$$x \longmapsto \frac{f_i(x)}{\|f_i(x)\|}$$

is independent of the trivialization, and is called the *Poincaré–Hopf index* $\mathsf{Ind}(\xi, p_i)$ of η at p_i. The *Euler number* of E, defined as

$$\mathsf{Euler}(\xi, \eta) := \sum_{i=1}^{k} \mathsf{Ind}(\eta, p_i) \in \mathbb{Z}$$

is independent of ξ and the local trivializations of ξ. Compare Bott–Tu [**62**], Theorem 11.17, p.125.)

The *intrinsic Gauss–Bonnet theorem,* due to Chern [**87**], states that there is a constant $c_m \in \mathbb{R}$ such that

$$\mathsf{Euler}(\xi) = c_m \int_M \mathsf{Pfaff}\big(R(\nabla)\big).$$

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In particular if $\xi = \mathsf{T}M$, and $R(\nabla) = 0$, then

$$\chi(M) = \mathsf{Euler}(\mathsf{T}M) = 0.$$

Gromoll's proof (Poor [**291**], §3.56–3.73, pp. 138–49) begins with a smooth vector field η on M which is nonzero in the complement of a finite subset $Z \subset M$. From an orthogonal connection and η determine a $2m-1$-form whose exterior derivative equals $\mathsf{Pfaff}\big(R(\nabla)\big)$ on M^{2m}. Applying Stokes's theorem on the complement of a small neighborhood of $Z \subset M$ implies Chern's theorem.

9.2.2. Smillie's examples of flat tangent bundles. Two oriented 2-plane bundles over a space M are isomorphic if their Euler classes in $H^2(M;\mathbb{Z})$ are equal. (Compare Milnor–Stasheff [**273**].) Milnor [**268**] showed that an oriented 2-plane bundle ξ over an oriented surface of genus $g \geq 0$ admits a flat structure if and only if

$$|\mathsf{Euler}(\xi)| < g.$$

Suppose that ξ is such a bundle which is nontrivial, that is, $\mathsf{Euler}(\xi) \neq 0$. Then ξ admits a connection ∇ with curvature zero.

Exercise 9.2.2. Show that the complexification of such a bundle is trivial.

Exercise 9.2.3. Suppose that $F \longrightarrow \Sigma$ is an oriented $\mathbb{S}^1$-bundle which admits a free proper action of the cyclic group $\mathbb{Z}/m\mathbb{Z}$ on the fibers so that $F' := F/(\mathbb{Z}/m\mathbb{Z})$ is an oriented $\mathbb{S}^1$-bundle over Σ. Show that

$$\mathsf{Euler}(F') = \mathsf{Euler}(F)/m$$

and

$$m|\mathsf{Euler}(F).$$

Deduce that the Euler number of a flat $\mathbb{R}^2$-bundle over Σ is always even.

Exercise 9.2.4. Show that the 3-sphere $\mathbb{S}^3$ admits a flat affine connection (that is, a connection on its tangent bundle $\mathsf{T}M$ with vanishing curvature), but no flat affine connection with vanishing torsion.

Thus, in general, a manifold can have a flat tangent bundle even if it fails to have a flat affine structure. In this direction, Smillie [**311**] showed that Chern's conjecture is false if one only requires that the curvature vanishes. We outline his (elementary) argument below.

First, he considers the class of *stably parallelizable manifolds,* that is, manifolds with stably trivial tangent bundles. Recall that a vector bundle $E \to M$ is *stably trivial* if the Whitney sum $\mathsf{E} \oplus \mathbb{R}_M$ is trivial, where $\mathbb{R}_M \to M$ denotes the trivial line bundle over M. (Smillie uses the terminology "almost" instead of "stably" although we think that "stable" is more standard.)

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Exercise 9.2.5. If $\xi \to M$ is a stably trivial vector bundle, then $\xi = f^*\mathsf{T}\mathbb{S}^n$, for some map $M \xrightarrow{f} \mathbb{S}^n$. Furthermore two stably trivial vector bundles ξ, ξ' are isomorphic if and only if

$$\mathsf{Euler}(\xi) = \mathsf{Euler}(\xi') \in H^n(M; \mathbb{Z}).$$

An oriented 2-plane bundle ξ over a closed oriented surface is stably trivial if and only its Euler number $\mathsf{Euler}(\xi)$ is even (equivalently, if its second Stiefel–Whitney class $w_2(\xi) = 0$).

Exercise 9.2.6. Let M be an orientable n-manifold. Show that the following conditions are equivalent:

- M is stably parallelizable;
- M immerses in $\mathbb{R}^{n+1}$;
- For any point, the complement $M \backslash \{x\}$ is parallelizable.

Deduce that the connected sum of two stably parallelizable manifolds is parallelizable.

Smillie constructs a 4-manifold N^4 with $\chi(N) = 4$, and a 6-manifold Q^6 with $\chi(Q) = 8$ such that both $\mathsf{T}N$ and $\mathsf{T}Q$ have flat structures. He begins with a closed orientable surface Σ_3 of genus 3 and the flat $\mathsf{SL}(2, \mathbb{R})$-bundle ξ over Σ_3 with $\mathsf{Euler}(\xi) = -2$. (This bundle arises by lifting a Fuchsian representation

$$\pi_1(\Sigma_3) \longrightarrow \mathsf{PSL}(2, \mathbb{R})$$

from $\mathsf{PSL}(2, \mathbb{R})$ to $\mathsf{SL}(2, \mathbb{R})$.) Then ξ is stably trivial and admits a flat structure.

The product 4-plane bundle $\xi \times \xi$ over $\Sigma_3 \times \Sigma_3$ is also stably trivial and admits a flat structure. Furthermore its Euler number

$$\mathsf{Euler}(\xi \times \xi) = 2 + 2 = 4.$$

Let P^4 be a parallelizable 4-manifold and let N be the connected sum of six copies of P with $\Sigma_3 \times \Sigma_3$.

Exercise 9.2.7. Prove that $\mathsf{T}N \cong f^*(\xi \times \xi)$ for some degree one map

$$N \xrightarrow{f} \Sigma_3 \times \Sigma_3.$$

Deduce that $\mathsf{T}N$ admits a flat structure and that $\chi(N) = 4$. Find a similar construction for a 6-manifold Q^6 with flat tangent bundle but $\chi(Q) = 8$. Find, for any even $n \geq 8$, an n-dimensional manifold with flat tangent bundle and positive Euler characteristic.

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9.2.3. The Kostant–Sullivan Theorem. In 1960, L. Auslander published a false proof that the Euler characteristic of a closed *complete* affine manifold is zero [**14**]. Of course, the difficulty is that the Euler characteristic can only be computed as a curvature integral for an *orthogonal connection,* but *not* for a linear connection (as Milnor's examples show).

This difficulty was overcome by a clever trick by Kostant and Sullivan [**225**] who showed that the Euler characteristic of a compact *complete* affine manifold vanishes.

Proposition 9.2.8 (Kostant–Sullivan [**225**])**.** Let M^{2n} be a compact affine manifold whose affine holonomy group acts freely on A^{2n}. Then $\chi(M) = 0$.

Corollary 9.2.9. The Euler characteristic of a compact complete affine manifold vanishes.

The main lemma is the following elementary fact, which the authors attribute to Hirsch:

Lemma 9.2.10. Let $\Gamma \subset \mathsf{Aff}(\mathsf{A})$ be a group of affine transformations which acts freely on A. Let $\mathbb{A}\big(\mathsf{L}(\Gamma)\big)$ in $\mathsf{GL}(\mathsf{V})$ denote the Zariski closure of the linear part $\mathsf{L}(\Gamma)$ in $\mathsf{GL}(\mathsf{V})$. Then every element of $\mathbb{A}\big(\mathsf{L}(\Gamma)\big)$ has 1 as an eigenvalue.

Proof. First we show that the linear part $\mathsf{L}(\gamma)$ has 1 as an eigenvalue for every $\gamma \in \Gamma$. This condition is equivalent to the noninvertibility of $\mathsf{L}(\gamma) - \mathbb{I}$. Suppose otherwise; then $\mathsf{L}(\gamma) - \mathbb{I}$ is invertible. Writing

$$x \stackrel{g}{\longmapsto} \mathsf{L}(g)x + u(g),$$

where the vector $u(g)$ is the translational part $u(g) = g(0)$ of g. The point

$$p := -\big(\mathsf{L}(g) - \mathbb{I}\big)^{-1} u(g)$$

is fixed by γ, contradicting our hypothesis that g acts freely.

Noninvertibility of $\mathsf{L}(\gamma) - \mathbb{I}$ is equivalent to

$$\mathsf{Det}\big(\mathsf{L}(\gamma) - \mathbb{I}\big) = 0,$$

evidently a polynomial condition on γ. Thus $\mathsf{L}(g) - \mathbb{I}$ is noninvertible for every $g \in G$, as desired. ☐

Proof of Proposition 9.2.8. To show that $\chi(M) = 0$, we find an orthogonal connection ∇ for which the Gauss–Bonnet integrand $\mathsf{Pfaff}\big(R(\nabla)\big) = 0$. To this end, observe first that the tangent bundle $\mathsf{T}M$ is associated to the linear holonomy $\mathsf{L}(\Gamma)$. representation of M, and hence its structure group reduces from $\mathsf{Aff}(\mathsf{A})$ to the algebraic hull G of $\mathsf{L}(\Gamma)$. Since M is complete, its affine holonomy group Γ acts freely and Lemma 9.2.10 implies that every element of G has 1 as an eigenvalue.

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Let $K \subset G$ be a maximal compact subgroup of G. (One can take K to be the intersection of G with a suitable conjugate of the orthogonal group $\mathsf{O}(2m) \subset \mathsf{GL}(2m, \mathbb{R})$.) A section of the G/K-bundle associated to the G-bundle corresponding to the tangent bundle always exists (since G/K is contractible), and corresponds to a Riemannian metric on M. Let ∇ be the corresponding Levi–Civita connection. Its curvature $R(\nabla)$ lies in the Lie algebra $\mathfrak{k} \subset \mathfrak{so}(2m)$.

Since every element of G has 1 as an eigenvalue, every element of K has 1 as an eigenvalue, and every element of $\mathfrak{k}$ has 0 as eigenvalue. That is, $\mathsf{Det}(k) = 0$ for every $k \in \mathfrak{k}$. Since

$$\mathsf{Pfaff}(k)^2 = \mathsf{Det}(k) = 0,$$

the Gauss–Bonnet integrand $\mathsf{Pfaff}\big(R(\nabla)\big) = 0$. Applying Chern's intrinsic Gauss–Bonnet theorem (§9.2.1), $\chi(M) = 0$. □

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Chapter 10

Affine Lie groups

Many geometric structures on closed manifolds arise from *homogeneous structures* — structures invariant under a transitive Lie group action. In particular, left-invariant structures on Lie groups themselves furnish many examples, as we have already seen in dimensions 1 and 2. A large class of affine structures on the 2-torus $\mathbb{T}^2$ arises from invariant[1] affine structures on the Lie group $\mathbb{T}^2$ — see Baues's survey [**33**] for an account of this. For example, the heart of Auslander's approach to classifying affine crystallographic groups is the conjecture that every compact complete affine manifold is a biquotient of a left-invariant complete affine structure on a solvable Lie group G. Similarly, Dupont's classification [**120**] of affine structures on 3-dimensional hyperbolic torus bundles reduces to left-invariant affine structures on the solvable unimodular exponential non-nilpotent Lie group $\mathsf{Sol} := \mathsf{Isom}^0(\mathsf{E}^{1,1})$.

We begin with a brief summary in dimensions 1 and 2 of *compact* affine Lie groups. This exercise is a warm-up for the general theory developed in the rest of this chapter. After the general theory is developed, the 2-dimensional *noncommutative theory* is described. Although no compact quotients exst, the theory is quite rich and leads to compact 3-dimensional examples given later in this chapter. The chapter ends with several more low-dimensional examples.

We emphasize the interplay between Lie group coordinates and affine coordinates. In particular we find many important additional geometric structures in these affine structures.

[1]Because the Lie group is abelian, left-invariance equals right-invariance.

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Left-invariant objects on a Lie group G of course reduce to algebraic constructions on its Lie algebra $\mathfrak{g}$. Left-invariant affine structures on G correspond to *left-invariant structures* on $\mathfrak{g}$, where the bi-invariant structures correspond to compatible *associative* structures on $\mathfrak{g}$. Specifically, covariant differentiation of vector fields preserves the Lie subalgebra of left-invariant vector fields, leading to a bilinear operation

$$\begin{aligned} \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (X, Y) &\longmapsto XY := \nabla_X(Y) \end{aligned} \tag{10.1}$$

Conversely, any bilinear mapping $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ defines a left-invariant connection on G. The connection is flat and torsionfree if and only if the corresponding algebra is *left-symmetric* (In this chapter an *algebra* is a finite-dimensional $\mathbb{R}$-vector space $\mathfrak{a}$ together with bilinear map $\mathfrak{a} \times \mathfrak{a} \longrightarrow \mathfrak{a}$.) Denote the commutator operation by:

$$[x, y] := xy - yx. \tag{10.2}$$

In general, however, a left-invariant structure will *not* be right-invariant, but instead will satisfy the *left-symmetry* condition, that the *associator operation*

$$[x, y, z] := (xy)z - x(yz) \tag{10.3}$$

is symmetric in its first two variables:

$$[x, y, z] = [y, x, z]. \tag{10.4}$$

Such algebras arise in many mathematical contexts, and we call them *left-symmetric algebras.* (They also go by other names : *pre-Lie algebras, Koszul algebras, Vinberg algebras,* or *Koszul–Vinberg algebras.*

The literature is vast. We particularly recommend Burde's survey article [**70**], as well as works by Vinberg [**340**], Helmstetter[**189**] , and Vey [**337**], Medina [**266**], Medina–Saldarriaga–Giraldo [**265**], Segal [**303**], Kim [**215**], Valencia [**332**], and the references cited therein for more information. See also Rothaus [**295**], Dorfmeister [**114**], Vinberg [**340**], Koszul [**226**], and Matsushima [**262**].)

We do not attempt to be comprehensive, but mainly give a glimpse into this fascinating algebraic theory, which provides a rich class of interesting geometric examples.

10.1. Affine Lie tori

By means of an introduction, we describe the six examples of 2-dimensional abelian Lie groups. Alternatively, we describe the examples of *compact* affine Lie groups of dimension 2, based on our earlier analysis of compact affine 1-manifolds in §5.5.1 and §5.5.2. Up to isomorphism, only one complete

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structure exists, namely $\mathbb{R}/\mathbb{Z}$. The incomplete structures are *Hopf circles,* $\mathbb{R}^+/\langle\lambda\rangle$, where $\lambda > 0$. The complete structure is a degeneration as the parameter $\lambda \searrow 0$.

In a similar way, the compact 2-dimensional examples can be understood as quotients of the vector group $\mathbb{R}^2$ with an invariant affine structure by a lattice $\Lambda < \mathbb{R}^2$. Products of 1-dimensional affine Lie groups give three examples: $\mathbb{R} \times \mathbb{R}$, $\mathbb{R} \times \mathbb{R}^+$, and $\mathbb{R}^+ \times \mathbb{R}^+$. (These structures are depicted in the top picture in Figure 5.10, Figure 5.13, and Figure 5.11, respectively.)

However there are three other group structures other than the products, which have the same developing images: the non-Euclidean *complete structure* (depicted in the bottom picture in Figure 5.10), a radiant structure on the halfplane $\mathbb{R} \times \mathbb{R}^+$ which does not decompose as a product of affine Lie groups, (depicted in Figure 5.14), and the radiant structure on the universal covering of $\mathbb{C}^\times \cong \mathbb{R}^2 \setminus \{\mathbf{0}\}$ (depicted in Figure 5.10). These structures are elegantly described as the 2-dimensional commutative case of the following general construction with associative algebras.

Exercise 10.1.1. Let $\mathfrak{a}$ be an associative algebra and adjoin a two-sided identity element (denoted "$\mathbf{1}$") to form a new associative algebra $\mathfrak{a} \oplus \mathbb{R}\mathbf{1}$ with identity.

- The invertible elements[2] in $\mathfrak{a}\oplus\mathbb{R}\mathbf{1}$ form an open subset closed under multiplication. The element $e := 0 \oplus \mathbf{1}$ is the two-sided identity element.
- The universal covering group G of the group of invertible elements of the form
$$a \oplus \mathbf{1} \ \in \ \mathfrak{a} \oplus \mathbb{R}\mathbf{1}$$
acts locally simply transitively and affinely on the affine space
$$\mathsf{A} = \mathfrak{a} \oplus \{\mathbf{1}\}.$$
The Lie algebra of G naturally identifies with the algebra $\mathfrak{a}$ and there is an exponential map $\mathfrak{a} \xrightarrow{\exp} G$ defined by the usual power series (in $\mathfrak{a}$). The corresponding evaluation map at e defines a developing map for an bi-invariant affine structure on G.

Multiplication tables for covariant differentiation of left-invariant vector fields conveniently describe left-invariant affine structures on Lie groups. Table 10.1 describes the two complete affine Lie groups isomorphic to $\mathbb{R}^2$. Table 10.2 describes the four incomplete affine Lie groups isomorphic to $\mathbb{R}^2$.

[2]Here it suffices to assume only left-invertibilty. A pleasant exercise is to show that for a finite-dimensional associative algebra with two-sided identity, that left-inverses and right-inverses agree.

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Table 10.1. Multiplication tables for complete (Euclidean and non-Riemannian) structures on the vector group $\mathbb{R}^2$

	X	Y
X	0	0
Y	0	0

	X	Y
X	0	0
Y	0	X

Table 10.2. Multiplication tables for incomplete (nonradiant halfplane, radiant halfplane, hyperbolic, and $\mathbb{C}$-affine) structures on the vector group $\mathbb{R}^2$. The first is a product $\mathbb{R} \times \mathbb{R}^+$ and the others are radiant, with radiant vector fields as follows: In the second, $\mathsf{Rad} = Y$, the third $\mathsf{Rad} = X + Y$, and in the fourth $\mathsf{Rad} = X$.

	X	Y
X	0	0
Y	0	Y

	X	Y
X	0	X
Y	X	Y

	X	Y
X	X	0
Y	0	Y

	X	Y
X	X	Y
Y	Y	$-X$

These structures will be revisited in §10.4 after the general theory is developed.

10.2. Étale representations and the developing map

If G is a Lie group and $a \in G$, then define the *left-* and *right-multiplication operations* $\mathcal{L}_a, \mathcal{R}_a$, respectively:

$$\mathcal{L}_a(b) := ab$$
$$\mathcal{R}_a(b) := ba.$$

Suppose that G is a Lie group with an affine structure. The affine structure is *left-invariant* (respectively *right-invariant*) if and only if the operations $G \xrightarrow{\mathcal{L}_a} G$ (respectively $G \xrightarrow{\mathcal{R}_a} G$) are affine. An affine structure is *bi-invariant* if and only if it is both left-invariant and right-invariant. In this section we describe the relationship between left-invariant affine structures and étale (locally simply transitive) affine representations.

10.2.1. Reduction to simply-connected groups. Suppose that G is a Lie group with a left-invariant (respectively right-invariant, bi-invariant) affine structure. Let $\widetilde{G}$ be its universal covering group and

$$\pi_1(G) \hookrightarrow \widetilde{G} \longrightarrow G$$

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the corresponding central extension. Then the induced affine structure on $\widetilde{G}$ is also left-invariant (respectively right-invariant, bi-invariant). Conversely, since $\pi_1(G)$ is central in $\widetilde{G}$, every left-invariant (respectively right-invariant, bi-invariant) affine structure on $\widetilde{G}$ determines a left-invariant (respectively right-invariant, bi-invariant) affine structure on G. Thus there is a bijection between left-invariant (respectively right-invariant, bi-invariant) affine structures on a Lie group and left-invariant (respectively right-invariant, bi-invariant) affine structures on any covering group. For this reason we shall mainly only consider simply connected Lie groups.

10.2.2. The translational part of the étale representation. Suppose that G is a simply connected Lie group with a left-invariant affine structure. Let $G \xrightarrow{\mathsf{dev}} \mathsf{A}$ be a developing map. Corresponding to the affine action of G on itself by left-multiplication is a homomorphism $G \xrightarrow{\alpha} \mathsf{Aff}(\mathsf{A})$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\mathsf{dev}} & \mathsf{A} \\ {\scriptstyle \mathcal{L}_g}\big\downarrow & & \big\downarrow{\scriptstyle \alpha(g)} \\ G & \xrightarrow[\mathsf{dev}]{} & \mathsf{A} \end{array} \tag{10.5}$$

commutes for each $g \in G$. We may assume that dev maps the identity element $e \in G$ to an origin $p_0 \in \mathsf{A}$. Then (10.5) implies that the developing map is the translational part of the affine representation:

$$\mathsf{dev}(g) = (\mathsf{dev} \circ \mathcal{L}_g)(e) = \big(\alpha(g) \circ \mathsf{dev}\big)(e) = \alpha(g)(p_0)$$

Furthermore since dev is open, it follows that the orbit $\alpha(G)(p_0)$ equals the developing image and is open. Indeed the translational part, which is the differential of the evaluation map

$$\mathsf{T}_e G = \mathfrak{g} \longrightarrow \mathsf{V} = \mathsf{T}_{p_0}\mathsf{A}$$

is a linear isomorphism. Such an action will be called *locally simply transitive* or simply an *étale* representation.

Conversely suppose that $G \xrightarrow{\alpha} \mathsf{Aff}(\mathsf{A})$ is an étale affine representation with an open orbit $\mathcal{O} \subset \mathsf{A}$. Then for any point $x_0 \in \mathcal{O}$, the evaluation map

$$g \mapsto \alpha(g)(x_0)$$

defines a developing map dev for an affine structure on G. Since

$$\begin{aligned} \mathsf{dev}(\mathcal{L}_g h) &= \alpha(gh)(x_0) \\ &= \alpha(g)\alpha(h)(x_0) \\ &= \alpha(g)\ \mathsf{dev}(h) \end{aligned}$$

for $g, h \in G$, this affine structure is left-invariant.

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Thus a left-invariant affine structure on a Lie group G precisely corresponds to a pair $(\alpha, \mathcal{O})$, where $G \xrightarrow{\alpha} \mathsf{Aff}(\mathsf{A})$ is an étale affine representation and $\mathcal{O} \subset \mathsf{A}$ is an open orbit. Pull back the connection on $\mathcal{O} \subset \mathsf{A}$ by dev to obtain a flat torsionfree affine connection ∇ on G.

The affine representation $G \xrightarrow{\alpha} \mathsf{Aff}(\mathsf{A})$ corresponds to left-multiplication. Therefore the associated Lie algebra representation $\mathfrak{g} \xrightarrow{\alpha} \mathsf{aff}(\mathsf{A})$ maps $\mathfrak{g}$ into affine vector fields corresponding to the infinitesimal generators of left-multiplications, that is, *right-invariant vector fields.*.

Thus with respect to a left-invariant affine structure on a Lie group G, the right-invariant vector fields are affine.

10.3. Left-invariant connections and left-symmetric algebras

Let G be a Lie group with Lie algebra $\mathfrak{g}$, which we realize as *left-invariant vector fields.* Suppose that ∇ is a left-invariant connection, that is, each left-multiplication $\mathcal{L}_g$ preserves ∇. If $X, Y \in \mathfrak{g}$ are left-invariant vector fields, then left-invariance of ∇ implies that the *covariant derivative* $\nabla_X Y$ is also a left-invariant vector field. Thus the operation defined in (10.1) turns $\mathfrak{g}$ into a finite-dimensional algebra over $\mathbb{R}$.

Now we describe the algebraic consequences of the vanishing of the torsion and curvature of ∇. The condition that $\mathsf{Tor}_\nabla = 0$ is precisely the commutator property (10.2). Using (10.2), the condition that the curvature $\mathsf{Riem}_\nabla = 0$ is:

$$(XY - YX)Z = [X, Y]Z = X(YZ) - Y(XZ)$$

which rearranges to:

$$(XY)Z - X(YZ) \;=\; (YX)Z - Y(XZ)$$

This is the *left-symmetric property* (10.4), that is, the *associator* defined in (10.3) is is symmetric in its first two variables:

$$[X, Y, Z] = [Y, X, Z].$$

An algebra satisfying this condition will be called *left-symmetric.*

Every left-symmetric algebra determines a Lie algebra. This generalizes the well-known fact that underlying every associative algebra is a Lie algebra. We shall sometimes call a left-symmetric algebra with underlying Lie algebra $\mathfrak{g}$ an *affine structure* on the Lie algebra $\mathfrak{g}$.

Exercise 10.3.1. Let $\mathfrak{a}$ be an algebra with commutator operation $[X, Y] := XY - YX$, and define a trilinear alternating map

$$\begin{aligned} &\mathfrak{a} \times \mathfrak{a} \times \mathfrak{a} \xrightarrow{\mathsf{Jacobi}} \mathfrak{a}. \\ &(X, Y, Z) \longmapsto [[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] \end{aligned}$$

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- Show that

$$\mathsf{Jacobi}(X,Y,Z) = [X,Y,Z] + [Y,Z,X] + [Z,X,Y] \\ - [Z,Y,X] - [X,Z,Y] - [Y,X,Z]$$

  where $[X,Y,Z]$ denotes the *associator* defined in (10.3). Deduce that underlying every left-symmetric algebra is a Lie algebra.
- Find an algebra $\mathfrak{a}$ which is *not* left-symmetric but its commutator nonetheless satisfies the Jacobi identity.
- Suppose that $\mathfrak{a}$ itself is a Lie algebra. If the Lie operation in $\mathfrak{a}$ is a also associative, show that $\mathfrak{a}$ is a 2-step nilpotent Lie algebra. If the Lie operation in $\mathfrak{a}$ is left-symmetric, must $\mathfrak{a}$ be 2-step nilpotent?

In terms of left-multiplication and the commutator operation defined in (10.2), a condition equivalent to (10.4) is:

$$\mathcal{L}_{[X,Y]} \;=\; [\mathcal{L}_X, \mathcal{L}_Y] \tag{10.6}$$

that is, that $\mathfrak{g} \xrightarrow{\mathcal{L}} \mathsf{End}(\mathfrak{a})$ is a Lie algebra homomorphism. We denote by $\mathfrak{a}_{\mathcal{L}}$ the corresponding $\mathfrak{g}$-module. Furthermore the identity map $\mathfrak{g} \xrightarrow{\mathbb{I}} \mathfrak{a}_{\mathcal{L}}$ defines a cocycle of the Lie algebra $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module $\mathfrak{a}_{\mathcal{L}}$:

$$\mathcal{L}_X Y - \mathcal{L}_Y X \;=\; [X,Y]. \tag{10.7}$$

Let A denote an affine space with associated vector space $\mathfrak{a}$. Then (10.6) and (10.7) imply the map $\mathfrak{g} \xrightarrow{\alpha} \mathsf{aff}(\mathsf{A})$ defined by

$$Y \xmapsto{\alpha(X)} \mathcal{L}_X Y + X \tag{10.8}$$

is a Lie algebra homomorphism.

Theorem 10.3.2. An isomorphism of categories exists between left-symmetric algebras and simply connected Lie groups with left-invariant affine structure. Under this isomorphism associative algebras correspond to bi-invariant affine structures.

We proved the first assertion, and now prove the second assertion.

10.3.1. Bi-invariance and associativity. Under the correspondence between left-invariant affine structures on G and left-symmetric algebras $\mathfrak{a}$, *bi-invariant* affine structures corresponds to *associative* algebras $\mathfrak{a}$.

Proposition 10.3.3. Let $\mathfrak{a}$ be the left-symmetric algebra corresponding to a left-invariant affine structure on G. Then $\mathfrak{a}$ is associative if and only if the left-invariant affine structure is bi-invariant.

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Proof. Let ∇ be the affine connection corresponding to a left-invariant affine structure on G. Then ∇ is left-invariant and defines the structure of a left-symmetric algebra $\mathfrak{a}$ on the Lie algebra of left-invariant vector fields on G.

Suppose that the affine structure is bi-invariant; then ∇ is also right-invariant. Therefore right-multiplications on G are affine maps with respect to the affine structure on G. It follows that the infinitesimal right-multiplications — the left-invariant vector fields — are affine vector fields as well. For a flat torsionfree affine connection a vector field Z is affine if and only if the second covariant differential $\nabla\nabla Z$ vanishes. (Compare Exercise 1.6.7.) Now $\nabla\nabla Z$ is the tensor field which associates to a pair of vector fields X, Y the vector field

$$\begin{aligned}\nabla\nabla Z(X,Y) &:= \nabla_X\big(\nabla Z(Y)\big) - \nabla Z(\nabla_X Y)\\ &= \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y}(Z).\end{aligned}$$

If X, Y, Z are left-invariant vector fields, then

$$\nabla\nabla Z(X,Y) = X(YZ) - (XY)Z = [X, Y, Z]$$

in $\mathfrak{a}$, so $\mathfrak{a}$ is an associative algebra, as desired.

Conversely, suppose $\mathfrak{a}$ is an associative algebra, We construct from $\mathfrak{a}$ a Lie group $G = G(\mathfrak{a})$ with a bi-invariant structure, such that the corresponding left-symmetric algebra equals $\mathfrak{a}$.

Denote by $\mathbb{R}\mathbf{1}$ a 1-dimensional algebra (isomorphic to $\mathbb{R}$) generated by a two-sided identity element $\mathbf{1}$. The direct sum $\mathfrak{a} \oplus \mathbb{R}\mathbf{1}$ admits an associative algebra structure where $\mathbf{1}$ is a two-sided identity element:

$$(a \oplus a_0\mathbf{1})(b \oplus b_0\mathbf{1}) := (ab + a_0 b + a b_0) \oplus a_0 b_0 \mathbf{1},$$

that is, "adjoint to $\mathfrak{a}$ a two-sided identity element." The affine hyperplane $\mathfrak{a} \oplus \mathbf{1}$ is multiplicatively closed, with the *Jacobson product*

$$(a \oplus \mathbf{1})(b \oplus \mathbf{1}) = (a + b + ab) \oplus \mathbf{1}.$$

In particular left-multiplications and right-multiplications are affine maps.

Let $G = G(\mathfrak{a})$ be the set of all $a \oplus \mathbf{1}$ which have left-inverses (necessarily also in $\mathfrak{a} \oplus \{\mathbf{1}\}$). Associativity implies $a \oplus \mathbf{1}$ is left-invertible if and only if it is right-invertible as well. Evidently G is an open subset of $\mathfrak{a} \oplus \{\mathbf{1}\}$ and forms a group. Furthermore, associativity implies both the actions of G by left- and right- multiplication, respectively, on $\mathsf{A} = \mathfrak{a} \oplus \mathbf{1}$ are affine, obtaining a bi-invariant affine structure on G.

The proof concludes with the following exercise. ☐

Exercise 10.3.4. Show that the corresponding left-symmetric algebra on the Lie algebra of G is $\mathfrak{a}$.

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When G is commutative, left-invariance and right-invariance coincide. Thus every left-invariant affine structure is bi-invariant. It follows that every commutative left-symmetric algebra is associative. This purely algebraic fact has a purely algebraic proof, using the following relationship between commutators and associators.

Exercise 10.3.5. Suppose that $\mathfrak{a}$ is an $\mathbb{R}$-*algebra.* If $X, Y, Z \in \mathfrak{a}$, show that

$$[X, Y, Z] - [X, Z, Y] + [Z, X, Y] = [XY, Z] + X[Z, Y] + [Z, X]Y.$$

However, even in dimension two, commutativity alone does not imply associativity (or, equivalently, left-symmetry):

Exercise 10.3.6. Show that Table 10.3 describes a commutative algebra which is not associative. (Hint: compute $[X, X, Y]$.)

Table 10.3. A commutative nonassociative 2-dimensional algebra

	X	Y
X	Y	0
Y	0	Y

10.3.2. The characteristic polynomial. Recall that a Lie group is *unimodular* if and only if its left and right Haar measures agree. The group $\mathsf{Aff}_+(1, \mathbb{R})$ is *not* unimodular. Among 2-dimensional 1-connected Lie groups nonunimodularity characterizes $\mathsf{Aff}_+(1, \mathbb{R})$. Unimodularity is obstructed by the *modular character,* the homomorphism

$$\begin{aligned} G &\xrightarrow{\Delta} \mathbb{R}^+ \\ g &\longmapsto \mathsf{Det}\ \mathsf{Ad}(g) \end{aligned}$$

which relates the left-invariant and right-invariant Haar measures:

$$\mu_{\mathfrak{R}} \ = \ \Delta \ \cdot \mu_{\mathfrak{L}},$$

that is, if $g \in G$, then

$$S \xmapsto{g_* \mu_{\mathfrak{R}}} \mu_{\mathfrak{R}}(g^{-1}S) = \Delta(g) \cdot \mu_{\mathfrak{R}}(S)$$

for $S \subset G$.

If G contains a *lattice* $\Gamma < G$, that is, a discrete subgroup for which $\mu_{\mathfrak{L}}(\Gamma\backslash G) < \infty$, then G is necessarily unimodular. (Consider the action of G by right-multiplications on $\Gamma\backslash G$.) The group $\mathsf{Aff}_+(1, \mathbb{R})$ discussed in §10.5

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is not *unimodular* and hence contains no lattices. However an important 3-dimensional extension

$$\mathsf{Sol} \cong \mathsf{Aff}_+(1,\mathbb{R}) \rtimes \mathbb{R}$$

does contain lattices and has quotients closed 3-manifolds (for example hyperbolic torus bundles) with many interesting affine structures.

Here are the right- and left-invariant Haar measures, and the modular function, for the structure $\mathfrak{a}_{\mathfrak{L}}$ above:

$$\mu_{\mathfrak{R}} = y\,|\partial_x \wedge \partial_y|$$
$$\mu_{\mathfrak{L}} = y^2\,|\partial_x \wedge \partial_y|$$

and therefore

$$\Delta\Big(\begin{bmatrix} y & \Big| & x \end{bmatrix} \Big) = \mu_{\mathfrak{R}}/\mu_{\mathfrak{L}} = y^{-1}.$$

The *characteristic polynomial* (defined in §10.3.5) is thus y.

10.3.3. Completeness and right-nilpotence. One can translate geometric properties of a left-invariant affine structure on a Lie group G into algebraic properties of the corresponding left-symmetric algebra $\mathfrak{a}$. For example, the following theorem of Helmstetter [**189**] and Segal [**303**].indicates a kind of infinitesimal version of Markus's conjecture relating geodesic completeness to parallel volume. (See also Goldman–Hirsch [**164**].)

Theorem 10.3.7. Let G be a simply connected Lie group with left-invariant affine structure. Let $G \xrightarrow{\alpha} \mathsf{Aff}(\mathsf{A})$ be the corresponding locally simply transitive affine action and $\mathfrak{a}$ the corresponding left-symmetric algebra. The following conditions are equivalent:

- G is a complete affine manifold;
- α is simply transitive;
- A volume form on G is parallel if and only if it is right-invariant;
- For each $g \in G$,

  $$\det \mathsf{L}\big(\alpha(g)\big) = \det \mathsf{Ad}(g)^{-1},$$

  that is, the distortion of parallel volume by α equals the modular function of G;
- $\mathfrak{a}$ is *right-nilpotent:* The subalgebra of $\mathsf{End}(\mathfrak{a})$ generated by right-multiplications $x \xmapsto{\mathfrak{R}_a} xa$ is nilpotent.

The original equivalence of completeness and right-nilpotence is due to Helmstett er [**189**] and was refined by Segal [**303**]. Goldman–Hirsch [**164**]

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explain the characterization of completeness by right-invariant parallel volume as an "infinitesimal version" of the Markus conjecture (using the radiance obstruction). However, left-nilpotence of a left-symmetric algebra is a much more restrictive condition; indeed it implies right-nilpotence:

Theorem 10.3.8 (Kim [**214**]). The following conditions are equivalent:

- The left-multiplications $x \xmapsto{\mathcal{L}_a} xa$ generate a nilpotent subalgebra of $\mathsf{End}(\mathfrak{a})$;
- G is nilpotent and the affine structure is complete;
- $\mathfrak{g}$ is a nilpotent Lie algebra and $\mathfrak{a}$ is right-nilpotent.

10.3.4. Radiant vector fields. In a different direction, we may say that a left-invariant affine structure is *radiant* if and only if the affine representation corresponding to left-multiplication has a fixed point, that is, is conjugate to a representation $G \longrightarrow \mathsf{GL}(\mathsf{V})$. Equivalently, $\alpha(G)$ preserves a radiant vector field Rad on A. In other words, G admits a left-invariant radiant vector field. A left-invariant affine structure on G is radiant if and only if the corresponding left-symmetric algebra has a right-identity, that is, an element $e \in \mathfrak{a}$ satisfying $ae = a$ for all $a \in \mathfrak{a}$.

10.3.5. Volume forms and the characteristic polynomial. Choose a basis $X_1, \dots, X_n$ of right-invariant vector fields. In affine coordinates $(x^1, \dots, x^n)$ the exterior product

$$\alpha(X_1) \wedge \cdots \wedge \alpha(X_n)$$

is a nonzero exterior n-form. Furthermore since the $\alpha(X_i)$ are polynomial vector fields of degree ≤ 1,

$$\alpha(X_1) \wedge \cdots \wedge \alpha(X_n) \;= f(x)\; \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}$$

for a nonzero polynomial $f(x_1, \dots, x_n) \in \mathbb{R}[x^1, \dots, x^n]$ of degree $\leq n$. It defines the right-invariant Haar measure on G. The dual exterior n-form

$$f(x)^{-1}\; dx^1 \wedge \cdots \wedge dx^n$$

corresponds to a right-invariant volume form. This polynomial is called the *characteristic polynomial* of the left-invariant affine structure, and is reminiscent of Vinberg's characteristic function (§4.4.2), agreeing with it in the homogeneous case. By Helmstetter [**189**] and Goldman–Hirsch [**164**], the developing map is a covering space, mapping G onto a connected the component of complement of $f^{-1}(0)$.

In particular the nonvanishing of f is equivalent to completeness of the affine structure.

The following is due to Goldman–Hirsch [**164**]:

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Exercise 10.3.9 (Infinitesimal Markus conjecture)**.** A left-invariant affine structure on a Lie group is complete if and only if right-invariant volumes forms are parallel. (Hint: first reduce to working over $\mathbb{C}$. Then the characteristic function f is a nonzero polynomial $\mathbb{C}^n \to \mathbb{C}$. Unless it is nonconstant, it vanishes somewhere, and the étale representation is not transitive.)

10.4. Affine structures on $\mathbb{R}^2$

As discussed above in §8.4 and §10.1, commutative associative 2-dimensional algebras provide many examples of affine structures on closed 2-manifolds.

Now let $\Lambda \subset \mathfrak{a}$ be a lattice. The quotient $\mathfrak{a}/\Lambda$ is a torus with an invariant affine structure. The complete structures were discussed in §8.4 in terms of a polynomial deformation. Now we discuss all the structures on $\mathbb{R}^2$, in terms of commutative associative algebras. (Recall that for commutative algebras, associativity and left-symmetry are equivalent; see §10.3.1. This represents a vast simplification of the theory.) We summarize the classification of 2-dimensional commutative associative algebras in terms of a basis $X, Y \in \mathfrak{a}$:

- *Euclidean case:* $\mathfrak{a}^2 = 0$ (all products are zero). The corresponding affine representation is the action of $\mathbb{R}^2$ on the plane by translation and the corresponding affine structures on the torus are the Euclidean structures. These structures are depicted in the first picture in Figure 5.10 and described algebraically the first multiplication table in Table 10.1.
- *Non-Riemannian complete case:* $\mathsf{dim}(\mathfrak{a}^2) = 1$ and $\mathfrak{a}$ is nilpotent ($\mathfrak{a}^3 = 0$). We may take X to be a generator of $\mathfrak{a}^2$ and $Y \in \mathfrak{a}$ to be an element with $Y^2 = X$. The corresponding affine representation is the simply transitive action discussed in §8.4 as a polynomial deformation of the Euclidean case. These are the two isomorphism classes described in §8.4.

  The corresponding affine structures are complete but non-Riemannian. These structures deform to the first one, where $\mathfrak{a} = \mathbb{R}[X, Y]$ where $X^2 = \lambda Y$ and $X^3 = 0$. These are all equivalent when $\lambda \neq 0$, and deforms to the Euclidean structure as $\lambda \to 0$. These structures are depicted in the second picture in Figure 5.10 and described algebraically by the second multiplication table in Table 10.1.
- $\mathfrak{a}$ is a direct sum of 1-dimensional algebras, one with zero multiplication corresponding to the complete 1-dimensional structure), and one with nonzero multiplication (corresponding to the 1-dimensional radiant structure). We can choose $X^2 = X$ for the radiant summand and this is the only nonzero basic product ($XY = YX =$

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$Y^2 = 0$). For various choices of Λ one obtains parallel suspensions of Hopf circles. In these cases the developing image is a halfplane. Figure 5.13 depicts these structures. The first multiplication table in Table 10.2 describes them algebraically.

- The next structure is a radiant suspension of the Euclidean 1-dimensional structure. As in the previous one, the developing image is a halfplane. Take X to be the radiant vector field, so that it is an identity element in $\mathfrak{a}$. For various choices of Λ one obtains radiant suspensions of the complete affine 1-manifold $\mathbb{R}/\mathbb{Z}$. The developing image is a halfplane. Figure 5.14 depicts these structures. The second multiplication table in Table 10.2 describes them algebraically.
- $\mathfrak{a}$ is a direct sum of nonzero 1-dimensional algebras. Taking X, Y to be the generators of these summands, we can assume they are *idempotent,* that is, $X^2 = X, Y^2 = Y$ and $XY = YX = 0$. This structure is radiant since $X + Y$ is an identity element, that is, a radiant vector field. Products of Hopf circles, and, more generally, radiant suspensions of Hopf circles are examples of these affine manifolds. The developing image is a quadrant in $\mathbb{R}^2$. These are *hyperbolic affine structures,* in the sense of Chapter 12, where the developing map is a homeomorphism onto a sharp convex cone (§4). Figure 5.11 and Figure 5.12 depict these structures. The third multiplication table in Table 10.2 describes them algebraically.
- Finally $\mathfrak{a} \cong \mathbb{C}$ is the field of complex structures, regarded as an $\mathbb{R}$-algebra. In this case we obtain the complex affine 1-manifolds, in particular the (usual) 2-dimensional Hopf manifolds are all obtained from this algebra. Clearly $X \leftrightarrow 1 \in \mathbb{C}$ is the identity, and $Y \leftrightarrow \sqrt{-1} \in \mathbb{C}$. These structures are radiant. The developing image is the complement of a point in the plane. The complex exponential map lifts to a biholomorphism $\mathbb{C} \to \widetilde{\mathbb{C}^\times}$ which is a developing map. Figure 5.15 depicts these structures and fourth multiplication table in Table 10.2 describes them algebraically.

Baues [**33**], surveys the theory of affine structures on the 2-torus. In particular he describes how the homogeneous structures (*affine Lie groups*) deform one into another. Kuiper [**234**] classified convex affine structures on $\mathbb{T}^2$, including the complete case (see §8.4). Nagano–Yagi [**279**] and Arrowsmith–Furness [**141**],[**10**] completed the classification. The complex-affine structures can be understood easily in terms of nonzero abelian differentials on the underlying elliptic curve (see, for example, Gunning [**181**, **182**] and the discussion in §14. Projective structures on $\mathbb{T}^2$ were classified by Goldman [**145**] and on higher-dimensional tori by Benoist [**39**].

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10.5. Affine structures on $\mathsf{Aff}_+(1,\mathbb{R})$

Two-dimensional Lie algebras $\mathfrak{g}$ fall into two isomorphism types:

- $\mathfrak{g} \cong \mathbb{R}^2$ (abelian);
- $\mathfrak{g} \cong \mathsf{aff}(1,\mathbb{R})$ the Lie algebra of affine vector fields on A^1.

The latter is the unique nontrivial semidirect sum $\mathbb{R} \rtimes \mathbb{R}$.

As we have just treated the abelian case, we turn now to the nonabelian case, using our model for $\mathsf{aff}(1,\mathbb{R})$. The classification of affine structures on $\mathsf{aff}(1,\mathbb{R})$ is due to Burde [**69**], Proposition 4.1. The deformation space is much more complicated than for $\mathbb{R}^2$, where the deformation space consists of 6 points. After discussing the structure of the group and its Lie algebra, we mention the two bi-invariant structure. We jump into the discussion of the complete structure, since it deforms to each of the bi-invariant structures. Then we describe the other structures as deformations of the two bi-invariant structures.

10.5.1. The Lie group $\mathsf{Aff}_+(1,\mathbb{R})$. The corresponding 1-connected Lie group is the group $G^0 = \mathsf{Aff}_+(1,\mathbb{R})$ of affine transformations of the line A^1 with positive linear part. Thus G^0 is the open subset of A^2 with coordinates (x, y). Its standard matrix representation (§1.8) is:

$$\begin{bmatrix} y & \Big| & x \end{bmatrix} \longmapsto \begin{bmatrix} y & x \\ 0 & 0 \end{bmatrix}$$

for the Lie algebra ($x, y \in \mathbb{R}$) and

$$\begin{bmatrix} y & \Big| & x \end{bmatrix} \longmapsto \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}$$

for the Lie group ($x \in \mathbb{R}, y \in \mathbb{R}^+$). Indeed, the space of affine maps $\mathsf{A}^1 \longrightarrow \mathsf{A}^1$ is a 2-dimensional associative algebra $\mathfrak{a}^{(2)}$ whose units comprise the two halfplanes defined by $y \neq 0$. The exponential map is:

$$\mathsf{aff}(1,\mathbb{R}) \xrightarrow{\exp} \mathsf{Aff}_+(1,\mathbb{R})$$
$$\begin{bmatrix} y & \Big| & x \end{bmatrix} \longmapsto \begin{bmatrix} e^y & \Big| & \varepsilon_y(x) \end{bmatrix}$$

where ε_y denotes the continuous function $\mathbb{R} \to \mathbb{R}$ defined by:

$$\varepsilon_y(t) := \begin{cases} \frac{e^{yt}-1}{y} & \text{if } y \neq 0 \\ t & \text{if } y = 0 \end{cases} \tag{10.9}$$

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Under this identification $G^0 \leftrightarrow \mathbb{R} \times \mathbb{R}^+ \subset \mathsf{A}^2$, both left- and right-multiplications extend to affine transformations of A^2, and thus define a *bi-invariant* affine structure on G^0.

Specifically, an element of $\mathsf{Aff}_+(1,\mathbb{R})$ is the transformation

$$\mathsf{A}^1 \xrightarrow{\left[y \,\middle|\, x\right]} \mathsf{A}^1$$
$$\xi \longmapsto y\xi + x.$$

which is the restriction of the usual linear representation

$$\begin{bmatrix} \xi \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix} = \begin{bmatrix} y\xi + x \\ 1 \end{bmatrix}$$

on $\mathbb{R}^2$ to the affine line $\mathsf{A}^1 \leftrightarrow \mathbb{R} \oplus \{1\} \subset \mathbb{R}^2$, where the affine coordinate ξ corresponds to $\xi \oplus 1 = \left[\begin{smallmatrix} \xi \\ 1 \end{smallmatrix}\right]$.

The identity element is $\left[1 \,\middle|\, 0\right]$, inversion is:

$$\left[y \,\middle|\, x\right] \longmapsto \left[y^{-1} \,\middle|\, -y^{-1}x\right]$$

and the group operation is:

$$\left[y_1 \,\middle|\, x_1\right] \left[y_2 \,\middle|\, x_2\right] = \left[y_1 y_2 \,\middle|\, x_1 + y_1 x_2\right]. \tag{10.10}$$

In particular left-multiplication by $g = \left[\eta \,\middle|\, \xi\right]$ extends from G^0 to the affine transformation

$$\mathsf{A}^2 \xrightarrow{\mathcal{L}_g} \mathsf{A}^2$$
$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} \eta x + \xi \\ \eta y \end{bmatrix} = \left[\begin{array}{cc|c} \eta & 0 & \xi \\ 0 & \eta & 0 \end{array}\right] \begin{bmatrix} x \\ y \end{bmatrix}$$

$\big($taking $y_1 = \eta,\ x_1 = \xi,\ y_2 = y,\ x_2 = x$ in (10.10)$\big)$.

This associative algebra is noncommutative. This provides *two* examples of left-symmetric algebras, which we denote $\mathfrak{a}_{\mathcal{L}}$ and $\mathfrak{a}_{\mathfrak{R}}$ which are opposites of each other. The corresponding affine Lie groups $\big($isomorphic to $\mathsf{Aff}_+(1,\mathbb{R})\big)$ have bi-invariant affine structures. Their algebraic structures are tabulated in Table 10.5.

Exercise 10.5.1. These two left-symmetric algebras are not isomorphic.

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10.5.2. A complete structure on $\mathsf{Aff}_+(1,\mathbb{R})$.

Exercise 10.5.2. We begin by describing the unique *complete* affine structure on $\mathsf{Aff}_+(1,\mathbb{R})$.

- Let $\lambda \in \mathbb{R}$ be a real parameter. If $\lambda \neq 0$,

$$\mathsf{Aff}_+(1,\mathbb{R}) \longrightarrow \mathsf{Aff}(2,\mathbb{R})$$
$$\begin{bmatrix} y & \big| & x \end{bmatrix} \longmapsto \left[\begin{array}{cc|c} y^\lambda & 0 & x \\ 0 & 1 & y \end{array}\right]$$

  defines a simply transitive affine representation, and hence a left-invariant complete affine structure on $\mathsf{Aff}_+(1,\mathbb{R})$.
- When $\lambda \to 0$, these representations converge to the simply transitive action of $\mathbb{R}^2$ by translations.
- All the representations where $\lambda \neq 0$ are affinely equivalent.
- Show that

$$X_{\mathcal{R}}^{\mathsf{complete}} := \partial_x, \qquad Y_{\mathcal{R}}^{\mathsf{complete}} := \lambda x\, \partial_x + \partial_y$$

  base the right-invariant vector fields and

$$X_{\mathcal{L}}^{\mathsf{complete}} := y^\lambda \partial_x, \qquad Y_{\mathcal{L}}^{\mathsf{complete}} := \partial_y$$

  base the left-invariant vector fields.[3] Table 10.4 describes the corresponding left-symmetric algebra.
- Show the corresponding affine Lie group is the only *complete* affine Lie group isomorphic to $\mathsf{Aff}_+(1,\mathbb{R})$. The left-invariant area form is $e^{-y} dx \wedge dy$ and a right-invariant area form is the parallel form $dx \wedge dy$. Note that in the basis $X = X_{\mathcal{L}}^{\mathsf{complete}}, Y = Y_{\mathcal{L}}^{\mathsf{complete}}$ of $\mathfrak{a}$, right-multiplications are:

$$\mathcal{R}_X \leftrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \mathcal{R}_Y \leftrightarrow \mathbf{0}$$

  and generate an nilpotent algebra, as in Theorem 10.3.7. (However, left-multiplication $\mathcal{L}_Y \leftrightarrow \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is *not* nilpotent.)

[3] Our convention for vector fields denoted X, Y is that the value of X at the basepoint is the coordinate vector ∂_x and the value of Y at the basepoint is the coordinate vector ∂_y. The computation for $[X, Y]$ may differ for the various examples, but $[X, Y]$ will always be a multiple of X in our examples.

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Table 10.4. The complete structure on $\mathsf{aff}(1,\mathbb{R})$. The algebra depends on a parameter $\lambda \in \mathbb{R}$, which we assume is nonzero. The left-symmetric algebras for nonzero λ are all isomorphic. However, for $\lambda = 0$, the structure is the trivial structure on $\mathbb{R}^2$. To simplify notation in the table, we write $X = X_{\mathfrak{L}}^{\text{complete}}$ and $Y = Y_{\mathfrak{L}}^{\text{complete}}$.

	X	Y
X	0	0
Y	λX	0

Table 10.5. Left-invariant vector fields on $\mathsf{Aff}_+(1,\mathbb{R})$ form an associative algebra $\mathfrak{a}_{\mathfrak{L}}$ and the right-invariant vector fields form the *opposite* algebra $\mathfrak{a}_{\mathfrak{R}}$. The multiplication tables of opposite algebras are transposes of each other and the commutation relations differ by -1. Namely $[X_{\mathfrak{L}}, Y_{\mathfrak{L}}] = -X_{\mathfrak{L}}$ and $[X_{\mathfrak{R}}, Y_{\mathfrak{R}}] = X_{\mathfrak{R}}$.

	$X_{\mathfrak{L}}$	$Y_{\mathfrak{L}}$
$X_{\mathfrak{L}}$	0	0
$Y_{\mathfrak{L}}$	$X_{\mathfrak{L}}$	$Y_{\mathfrak{L}}$

	$X_{\mathfrak{R}}$	$Y_{\mathfrak{R}}$
$X_{\mathfrak{R}}$	0	$X_{\mathfrak{R}}$
$\tilde{Y}_{\mathfrak{R}}$	0	$Y_{\mathfrak{R}}$

10.5.3. Bi-invariant structures on $\mathsf{Aff}_+(1,\mathbb{R})$. We return to the associative algebra $\mathfrak{a}$. If $\mathfrak{a}$ is an algebra, then define its *opposite* as the algebra $\mathfrak{a}^o$ with same underlying vector space but multiplication defined by:

$$\begin{aligned} \mathfrak{a} \times \mathfrak{a} &\longrightarrow \mathfrak{a} \\ (A, B) &\longmapsto BA \end{aligned}$$

Exercise 10.5.3. Suppose that $\mathfrak{a}$ is an associative algebra.

- Show that the opposite $\mathfrak{a}^o$ of $\mathfrak{a}$ is also associative.
- Find an isomorphism between the corresponding Lie groups of invertible elements in $\mathfrak{a}$ and $\mathfrak{a}^o$ respectively.
- Define a notion of *right-symmetric* algebra so that the opposite of a left-symmetric algebra is right-symmetric, and vice versa.
- Let $\mathfrak{a}_{\mathfrak{L}} := \mathfrak{a}^{(2)}$ be the associative algebra defined above in §10.5 and $\mathfrak{a}_{\mathfrak{R}}$ its opposite. Show $\mathfrak{a}_{\mathfrak{L}}$ and $\mathfrak{a}_{\mathfrak{R}}$ are not isomorphic to each other (although the corresponding Lie algebras are isomorphic).
- Show that an associative algebra whose underlying Lie algebra equals $\mathfrak{g} = \mathsf{aff}(1,\mathbb{R})$ is isomorphic to either $\mathfrak{a}_{\mathfrak{L}}$ or $\mathfrak{a}_{\mathfrak{R}}$.

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We will describe other left-invariant affine structures on G^0 in terms of the algebra of affine maps of A^1, first using an étale representation corresponding to left-multiplication. Left-multiplication by one-parameter subgroups define flows whose infinitesimal generators are *right-invariant* vector fields. By describing the developing maps in terms of one-parameter subgroups, we find the left-invariant vector fields and compute the left-symmetric algebra.

Here is the procedure applied to this first example $\mathfrak{a}_{\mathcal{L}}$.

One-parameter groups of positive homotheties $\left[e^t \mid 0\right]$ and translations $\left[1 \mid s\right]$ generate G^0:

$$\left[e^t \mid s\right] = \left[1 \mid s\right] \left[e^t \mid 0\right]$$

and we use $s, t \in \mathbb{R}^2$ as coordinates on the group. Left multiplication by $\left[e^t \mid s\right]$ then corresponds to the affine representation

$$\mathcal{L}_{\left[e^t \mid s\right]} = \left[\begin{array}{cc|c} e^t & 0 & s \\ 0 & e^t & 0 \end{array}\right]$$

on points

$$p := \begin{bmatrix} x \\ y \end{bmatrix} \longleftrightarrow \left[y \mid x\right]. \tag{10.11}$$

The developing map requires a choice of basepoint p_0 which lies in an open orbit of this affine representation of G^0 on A^2. For the choice

$$p_0 := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathsf{A}^2, \tag{10.12}$$

the developing map is evaluation at p_0:

$$G^0 \xrightarrow{\mathsf{dev}} \mathsf{A}^2$$

$$\left[e^t \mid s\right] \longmapsto \left[\begin{array}{cc|c} e^t & 0 & s \\ 0 & e^t & 0 \end{array}\right] p_0 = \begin{bmatrix} s \\ e^t \end{bmatrix}$$

which will be used to compute *affine coordinates* on G^0.

Affine coordinates relate to *group coordinates* as follows. Use the identification (10.11) above to solve the equation

$$\left[e^t \mid s\right] p_0 = p \tag{10.13}$$

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for s, t, given p. That is, we want to solve the system

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s \\ e^t \end{bmatrix} \tag{10.14}$$

for the group coordinates (s, t) in terms of affine coordinates (x, y), which can be done as long as $y > 0$. Suppose that $(x, y) \in \mathbb{R} \times \mathbb{R}^+$. Then the left-multiplication which maps p_0 to p is the affine transformation

$$\left[\begin{array}{cc|c} e^t & 0 & s \\ 0 & e^t & e^t \end{array}\right] = \left[\begin{array}{cc|c} y & 0 & x \\ 0 & y & y \end{array}\right] \tag{10.15}$$

The developing map takes the identity element $e \in G^0$ to the basepoint p_0. Moreover dev maps G^0 diffeomorphically onto the halfplane $\mathbb{R} \times \mathbb{R}^+ \subset \mathsf{A}^2$. A left-invariant vector field on G^0 is determined by its value on any point, for example e. Let $g \in G^0$. For any tangent vector $\mathbf{v} \in \mathsf{T}_e(G^0)$, the value at g of the left-invariant vector field on G^0 extending $\mathbf{v}$ equals the image $(\mathsf{D}\mathfrak{L}_g)_e(\mathbf{v})$ of $\mathbf{v}$ under the differential of left-multiplication

$$G^0 \xrightarrow{\mathfrak{L}_g} G^0$$
$$e \longmapsto g.$$

Since the differential of an affine transformation (in affine coordinates) is its linear part, the columns of the linear part form a basis for left-invariant vector fields. In this example, the first column and second column, respectively determine left-invariant vector fields which we denote $X_{\mathfrak{L}}, Y_{\mathfrak{L}}$:

$$X_{\mathfrak{L}} := y\partial_x \leftrightarrow \begin{bmatrix} y \\ 0 \end{bmatrix}, \qquad Y_{\mathfrak{L}} := y\partial_y \leftrightarrow \begin{bmatrix} 0 \\ y \end{bmatrix}. \tag{10.16}$$

(These left-invariant vector fields are affine because the structure is bi-invariant.)

The dual basis of left-invariant 1-forms (corresponding to the *rows* of the transpose inverse of the linear part)

$$y^{-1}dx, \qquad y^{-1}dy.$$

The sum of their squares is the *left-invariant Poincaré metric* on G^0, regarded as the *upper half-plane* $y > 0$*:*

$$y^{-2}\big(dx^2 + dy^2\big)$$

for which $\mathfrak{X}_{\mathfrak{L}}, Y_{\mathfrak{R}}$ form an *orthonormal* basis. For a discussion of the corresponding Riemannian connection, see Exercise B.6.1 in Appendix B.6.

Computing covariant derivatives yields the multiplication table (Table 10.5) for the corresponding left-symmetric structure, which is evidently

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associative. In particular $[Y_{\mathfrak{L}}, X_{\mathfrak{L}}] = X_{\mathfrak{L}}$ which defines the Lie algebra $\mathfrak{g}$ of G^0 up to isomorphism. Similarly we find a basis of left-invariant vector fields $\mathfrak{X}_{\mathfrak{R}}, Y_{\mathfrak{R}}$ for the opposite structure[4], obtaining the commutator relation $[Y_{\mathfrak{R}}, X_{\mathfrak{R}}] = -X_{\mathfrak{R}}$ for the opposite structure.

10.5.4. The opposite structure. Now we apply the procedure to $\mathfrak{a}_{\mathfrak{R}}$.

Since the affine structure on $\mathfrak{a}_{\mathfrak{L}}$ is also invariant under right-multiplications, the left-invariant vector fields generate flows of right-multiplication by one-parameter subgroups, obtaining:

Right multiplication by $\left[e^t \;\middle|\; s\right]$ then corresponds to the affine representation

$$\mathfrak{R}_{\left[e^t \,\middle|\, s\right]} = \left[\begin{array}{cc|c} 1 & s & 0 \\ 0 & e^t & 0 \end{array}\right] \tag{10.17}$$

since

$$\left[y \;\middle|\; x\right] \left[e^t \;\middle|\; s\right] = \left[e^t y \;\middle|\; x + sy\right]$$

whence

$$\mathfrak{R}_{\left[e^t \,\middle|\, s\right]} = \left[\begin{array}{cc|c} 1 & s & 0 \\ 0 & e^t & 0 \end{array}\right]$$

and evaluation at the same basepoint (as in (10.12)) gives a developing map in group coordinates:

$$G^0 \xrightarrow{\mathsf{dev}} \mathsf{A}^2$$

$$\left[e^t \;\middle|\; s\right] \longmapsto \left[\begin{array}{cc|c} 1 & s & 0 \\ 0 & e^t & 0 \end{array}\right] p_0 = \begin{bmatrix} s \\ e^t \end{bmatrix}.$$

Converting to affine coordinates:

$$\left[\begin{array}{cc|c} 1 & s & s \\ 0 & e^t & e^t \end{array}\right] = \left[\begin{array}{cc|c} 1 & x & x \\ 0 & y & y \end{array}\right].$$

The columns of the linear part then give a basis of the left-invariant vector fields for the opposite structure $\mathfrak{a}_{\mathfrak{R}}$:

$$X_{\mathfrak{R}} := \partial_x \leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad Y_{\mathfrak{R}} := x\partial_x + y\partial_y \leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix}. \tag{10.18}$$

Thus the structure admits both a parallel vector field $X_{\mathfrak{R}}$ and a radiant vector field $Y_{\mathfrak{R}} = \mathsf{Rad}$. Their covariant derivatives are recorded in Table 10.5.

[4]keeping in mind our convention that $X = \partial_x$ and $Y = \partial_y$ at the basepoint

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10.5.5. The first halfplane family: deformations of $\mathfrak{a}_{\mathfrak{L}}$. We embed the algebra $\mathfrak{a}_{\mathfrak{L}}$ in a family $\mathfrak{a}_{\mathfrak{L}}^{\beta}$ as follows, where $\beta \in \mathbb{R}$ is a real parameter. When $\beta = 1$ this structure agrees with the bi-invariant structure $\mathfrak{a}_{\mathfrak{L}}$ depicted in Table 10.5 and for all $\beta \neq 0$ the developing images are halfplanes. $\beta = 0$ corresponds to the complete structure described in Table 10.4.

To begin, assume that $\beta \neq 0$. The affine vector fields

$$X_{\mathfrak{R}}^{\beta} := \partial_x, \qquad Y_{\mathfrak{R}}^{\beta} := x\,\partial_x + \beta y\,\partial_y$$

generate an affine action of G^0 which agrees with the first action when $\beta = 1$. The product of the corresponding one-parameter subgroups acts by:

$$\left[\begin{array}{cc|c} 1 & 0 & s \\ 0 & 1 & 0 \end{array}\right] \left[\begin{array}{cc|c} e^t & 0 & 0 \\ 0 & e^{\beta t} & 0 \end{array}\right] = \left[\begin{array}{cc|c} e^t & 0 & s \\ 0 & e^{\beta t} & 0 \end{array}\right] \tag{10.19}$$

which maps the basepoint p_0 defined in (10.12) to $\begin{bmatrix} s \\ e^{\beta t} \end{bmatrix}$. Writing

$$p = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s \\ e^{\beta t} \end{bmatrix}$$

obtains the étale representation in affine coordinates:

$$\left[\begin{array}{cc|c} y^{1/\beta} & 0 & x \\ 0 & y & y \end{array}\right]$$

and the columns of the linear part give left-invariant vector fields

$$X_{\mathfrak{L}}^{\beta} \;:=\; y^{1/\beta}\,\partial_x, \qquad Y_{\mathfrak{R}}^{\beta} \;:=\; y\,\partial_y.$$

Although the second vector field is affine, the first vector field is affine if and only if $\beta = 1$, that is, only for the bi-invariant structure. Taking covariant derivatives, Table 10.6 describes the left-symmetric structure with respect to this basis. Note that

$$[X_{\mathfrak{L}}^{\beta}, Y_{\mathfrak{L}}^{\beta}] \;=\; -1/\beta\, X_{\mathfrak{L}}^{\beta}.$$

When $\beta = 0$, the original affine representation (10.19) has no open orbits. However, a simple modification extends this structure to $\beta = 0$, by including a *complete affine structure* at this parameter value.

To this end, replace the second one-parameter subgroup by:

$$\left[\begin{array}{cc|c} e^t & 0 & 0 \\ 0 & e^{\beta t} & \varepsilon_\beta(t) \end{array}\right] = \exp \left[\begin{array}{cc|c} t & 0 & 0 \\ 0 & \beta t & t \end{array}\right]$$

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Table 10.6. Left-invariant vector fields of deformation of bi-invariant structure. To simplify notation in the table, we write $X = X^\beta_{\mathfrak{L}}$ and $Y = Y^\beta_{\mathfrak{L}}$.

	X	Y
X	0	0
Y	$1/\beta\, X$	Y

where ε_β denotes the continuous function defined in (10.9).

Note that when $\beta \neq 0$, the orbit of p_0 is the halfplane $y > -1/\beta$, and the new action is conjugate to the original action by the translation $\mathsf{Trans}_{(0,-1/\beta)}$. Thus, the developing halfplane now varies with the parameter β while retaining the same basepoint p_0. As $\beta \to 0$, these halfplanes converge to all of A^2.

The étale representation of G^0 is given by

$$\left[\begin{array}{cc|c} 1 & 0 & s \\ 0 & 1 & 0 \end{array}\right] \left[\begin{array}{cc|c} e^t & 0 & 0 \\ 0 & e^{\beta t} & \varepsilon_\beta(t) \end{array}\right] = \left[\begin{array}{cc|c} e^t & 0 & s \\ 0 & e^{\beta t} & \varepsilon_\beta(t) \end{array}\right].$$

Evaluating at p_0 yields the étale representation in affine coordinates:

$$\left[\begin{array}{cc|c} (1+\beta y)^{1/\beta} & 0 & x \\ 0 & (1+\beta y) & y \end{array}\right]$$

and the columns of the linear part yields a basis of left-invariant vector fields:

$$X^\beta_{\mathfrak{L}} = \begin{cases} (1+\beta y)^{1/\beta}\partial_x & \text{if } \beta \neq 0 \\ e^y \partial_x & \text{if } \beta = 0 \end{cases}$$
$$Y^\beta_{\mathfrak{L}} = (1+\beta y)\partial_y$$

whose covariant derivatives are recorded in Table 10.7. Now that $\beta = 1$ corresponds to the bi-invariant (associative) structure as before but now $\beta = 0$ corresponds to a complete structure discussed in §10.5.2:

10.5.5.1. *Lorentzian structure.* The case $\beta = -1$ is also interesting. Then the affine structure arises from an invariant flat Lorentzian structure. Explicitly, if X, Y base the left-invariant vector fields:

$$X = y^{-1}\,\partial_x, \qquad Y = y\,\partial_y$$

and X^*, Y^* is the dual basis of left-invariant 1-forms:

$$X^* = y\,dx, \qquad Y^* = y^{-1}\,dy,$$

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Table 10.7. Left-invariant vector fields on the halfplane family structures are deformations of $\mathfrak{a}_{\mathcal{L}}$ depending on a parameter $\beta \in \mathbb{R}$. This family includes the complete structure ($\beta = 0$, depicted in Table 10.4), the bi-invariant (associative) structure ($\beta = 1$, depicted in Table 10.6) and the Lorentzian structure ($\beta = -1$). To simplify notation in the table, we write $X = X^{\beta}_{\mathcal{L}}$ and $Y = Y^{\beta}_{\mathcal{L}}$.

	X	Y
X	0	0
Y	X	βY

then the symmetric product $X^* \cdot Y^* = dx \cdot dy$ is a parallel Lorentzian structure, defining a left-invariant flat Lorentzian structure on G.

The developing image of the corresponding flat structure is a halfplane. Therefore this structure is an *incomplete flat Lorentzian structure,* despite its *homogeneity.* This contrasts Marsden's theorem [**259**] on the completeness of a *compact* homogeneous pseudo-Riemannian manifold, indicating that the compactness hypothesis is necessary. Compare the discussion in Conjecture 8.1.6.

10.5.6. Parabolic deformations. The case $\beta = 2$ is interesting for several reasons. Deformations exist whose developing images are *parabolic subdomains* of A^2, the components of the complement of a parabola in A^2. (Recall that the complement of a parabola has *two* connected components, one which is convex and the other concave.)

After describing this structure as a deformation, we mention its surprising role as the first *simple* (in the sense of Burde [**69**]) left-symmetric algebra whose underlying Lie algebra is solvable. Then we describe it as a *clan* in the sense of Vinberg [**340**], and briefly describe Vinberg's theory of convex homogeneous domains, which was one of the historical origins of the theory of left-symmetric algebras.

To this end, consider a parameter $\delta \in \mathbb{R}$. when $\delta = 0$, these examples are just the $\mathfrak{a}^2_{\mathcal{L}}$ as before, but when $\delta \neq 0$, these examples are all affinely conjugate, but the halfplane deforms to a parabolic subdoman. We denote these structures by $\mathfrak{a}^{2,\delta}$. Exercise 10.5.4 asserts that there is one affine isomorphism type for all $\delta > 0$ (respectively $\delta < 0$) and that $\mathfrak{a}^{2,\delta}$ are polynomially conjugate.

10.5.6.1. *Parabolic domains.* Let

$$\mathcal{Q}_\delta(x,y) := y - \delta\, x^2/2; \tag{10.20}$$

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Table 10.8. Left-invariant vector fields on the parabolic deformation $\mathfrak{a}_{\mathfrak{L}}^{2,\delta}$. When $\delta = 0$, this is just $\mathfrak{a}_{\mathfrak{L}}^{2}$ as depicted in Table 10.7 with $\beta = 2$. To simplify notation in the table, we write $X = X^{2,\delta}$ and $Y = Y^{2,\delta}$.

	X	Y
X	$\delta/2\ Y$	0
Y	X	$2Y$

the parabolic subdomain

$$\Omega_\delta := \{(x, y) \in \mathsf{A}^2 \mid \mathcal{Q}_\delta(x, y) > 0\}$$

is a halfplane if $\delta = 0$, strictly convex if $\delta > 0$, and strictly concave if if $\delta < 0$. The one-parameter subgroups

$$\exp\left[\begin{array}{cc|c} 0 & 0 & s \\ \delta s & 0 & 0 \end{array}\right] \exp\begin{bmatrix} t & 0 \\ 0 & 2t \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & s \\ \delta s & 1 & \delta\, s^2/2 \end{array}\right] \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$$
$$= \left[\begin{array}{cc|c} e^t & 0 & s \\ \delta e^t s & e^{2t} & \delta\, s^2/2 \end{array}\right]$$

generate its affine automorphism group. The image of p_0 is:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s \\ e^{2t} + \delta\, s^2/2 \end{bmatrix}$$

relating group coordinates (s, t) to affine coordinates (x, y) by:

$$s = x$$
$$t = \frac{1}{2} \log\Big(y - \delta\, x^2/2\Big) \ = \ \frac{1}{2} \log \mathcal{Q}_\delta(x, y)$$

where $\mathcal{Q}_\delta$ is defined in (10.20). The left-invariant vector fields are based by the first two columns of the matrix

$$\left[\begin{array}{cc|c} \sqrt{y - \delta\, x^2/2} & 0 & x \\ \delta\sqrt{y - \delta\, x^2/2}\, x & \big(y - \delta\, x^2/2\big) & y \end{array}\right] = \left[\begin{array}{cc|c} \sqrt{\mathcal{Q}_\delta(x,y)} & 0 & x \\ \delta\sqrt{\mathcal{Q}_\delta(x,y)}\, x & \mathcal{Q}_\delta(x,y) & y \end{array}\right]$$

which are:

$$X^{2,\delta} := \sqrt{\mathcal{Q}_\delta(x,y)}\ \big(\partial_x + \delta x\, \partial_y\big), \qquad Y^{2,\delta} := \mathcal{Q}_\delta(x,y)\, \partial_y.$$

Table 10.8 describes the corresponding left-symmetric algebras.

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Exercise 10.5.4.

- Show that, as δ varies, these structures are related by the polynomial diffeomorphism:

$$\begin{aligned}\mathsf{A}^2 &\longrightarrow \mathsf{A}^2\\ (x,y) &\longmapsto \big(x, y-\delta x^2/2\big) = \big(x, \mathcal{Q}_\delta(x,y)\big)\end{aligned}$$

- Show that for $\delta \neq 0$, these structures are all isomorphic.

10.5.6.2. *Simplicity.* The algebra $\mathfrak{a}^2_{\mathfrak{L}}(\delta)$, where $\delta \neq 0$, also has special *algebraic* significance. Define a left-symmetric algebra to be *simple* if and only if it contains no nonzero proper two-sided ideals. Somewhat surprisingly, the Lie algebra underlying a simple left-symmetric algebra can even be solvable. Indeed, Burde [**69**] proved that, over $\mathbb{C}$, the complexification of the above example is the only *simple* left-symmetric algebra of dimension two. Over $\mathbb{R}$, this example, and the field $\mathbb{C}$ itself (regarded as an $\mathbb{R}$-algebra), are the only simple 2-dimensional left-symmetric algebras. The classification of simple left-symmetric algebras in general is a difficult unsolved algebraic problem; see [**69**, **70**] for more details.

10.5.6.3. *Clans and homogeneous cones.* Vinberg [**340**] classifies convex homgeneous domains in terms of special left-symmetric algebras, which he calls *clans*. The example above is the first nontrivial example of such a clan, and can be approached in several different ways.

Let $\Omega \subset \mathsf{Mat}_2(\mathbb{R})$ denote the convex cone comprising *positive definite symmetric* 2×2 real matrices. It is an open subset of the 3-dimensional linear subspace of $\mathsf{Mat}_2(\mathbb{R})$ consisting of *symmetric* matrices. Thus $\Omega \subset \mathsf{W}$ is an open convex cone.

$\Omega \subset \mathsf{W}$ is *homogeneous:* Namely $\mathsf{Aut}(\mathbb{R}^2) = \mathsf{GL}(2,\mathbb{R})$ acts on W by the induced action on symmetric bilinear forms:

$$\begin{aligned}\mathsf{GL}(2,\mathbb{R})\times\mathsf{W} &\longrightarrow \mathsf{W}.\\ (A,w) &\longmapsto A^\dagger w A\end{aligned}$$

where $A^\dagger$ denotes the transpose of A. Under the linear correspondence

$$\begin{aligned}\mathsf{W} &\longrightarrow \mathbb{R}^3\\ \begin{bmatrix} x & y\\ y & z\end{bmatrix} &\longmapsto \begin{bmatrix} x\\ y\\ z\end{bmatrix}\end{aligned}$$

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defines an action of $\mathsf{GL}(2,\mathbb{R})$ on $\mathbb{R}^3$ preserving the open subset $\Omega \subset \mathsf{W}$ defined by the positivity of determinant of the matrix

$$\Delta(x,y,z) = xz - y^2,$$

which appears as the *characteristic polynomial* of the corresponding left-invariant structure (or left-symmetric algebra). Furthermore, by the Gram–Schmidt orthonormalization process, $\mathsf{GL}(2,\mathbb{R})$ acts transitively on Ω.

10.5.7. The second halfplane family: deformations of $\mathfrak{a}_{\mathfrak{R}}$. The other associative structure $\mathfrak{a}_{\mathfrak{R}}$ has two kinds of deformations.

In the first deformation all the structures are radiant. The representation replaces the diagonal one-parameter subgroup with one eigenvalue 1 by one with two eigenvalues of varying strength. This deformation is parametrized by a nonzero parameter $\mu \in \mathbb{R}$, corresponding to the relative strength of the eigenvalues. The original bi-invariant structure $\mathfrak{a}_{\mathfrak{R}}$ occurs at $\mu = 1$ corresponding to the bi-invariant structure. At $\mu = -1$ we find a structure which is both radiant and area-preserving, contrasting affine structures on *closed* manifolds (Theorem 6.5.9).

A modification similar to the modification of the first halfplane family (§10.5.5) gives the complete structure (§10.5.2) as the limit as $\mu \to 0$. (The complete structure is of course nonradiant.)

The second deformation of $\mathfrak{a}_{\mathfrak{R}}$ is nonradiant, and uses a left-invariant parallel vector field on $\mathfrak{a}_{\mathfrak{R}}$ to construct a deformation, depending a parameter $\eta \in \mathbb{R}$. All the structures with $\eta \neq 0$ are equivalent.

10.5.7.1. *The radiant deformations of* $\mathfrak{a}_{\mathfrak{R}}$. The étale affine representation

$$\left[\begin{array}{c|c} e^t & s \end{array}\right] \longmapsto \left[\begin{array}{cc|c} e^{\mu t} & e^{\mu t}s & 0 \\ 0 & e^t & 0 \end{array}\right] \tag{10.21}$$

deforms the representation (10.17) when $\mu \neq 0$. The original étale representation for $\mathfrak{a}_{\mathfrak{R}}$ occurs at $\mu = 1$, and when $\mu = 0$, the resulting group is abelian. Evaluation at p_0 gives a developing map:

$$\left[\begin{array}{cc|c} e^{\mu t} & e^{\mu t}s & 0 \\ 0 & e^t & 0 \end{array}\right] p_0 = \begin{bmatrix} e^{\mu t}s \\ e^t \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

so group coordinates relate to the affine coordinates by:

$$s := y^{-1/\mu}x$$
$$t := \log(y)$$

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Table 10.9. Radiant deformation of $\mathfrak{a}_{\mathfrak{R}}$ depending on a nonzero eigenvalue parameter $\mu \in \mathbb{R}$. When $\mu = 1$ this structure agrees with the bi-invariant structure $\mathfrak{a}_{\mathfrak{R}}$ depicted in Table 10.5 When $\mu = 0$ this algebra is commutative. To simplify notation in the table, we write $X = X^{\mu}_{\mathfrak{R}}$ and $Y = Y^{\mu}_{\mathfrak{R}}$.

	X	Y
X	0	X
Y	μX	Y

and in affine coordinates the étale representation is:

$$\left[\begin{array}{cc|c} y^{\mu} & x & x \\ 0 & y & y \end{array}\right] \tag{10.22}$$

The columns of the linear part determine left-invariant vector fields:

$$X^{\mu}_{\mathfrak{R}} := y^{\mu}\partial_x, \qquad Y^{\mu}_{\mathfrak{R}} = x\partial_x + y\partial_y = \mathsf{Rad}.$$

Table 10.9 tabulates their covariant derivatives.

As above, we can reparametrize this family to include the complete structure when $\mu = 1$. It will be convenient for the exposition to temporarily include a new parameter δ (which we will eventually take to be $\mu - 1$, so that $\delta \to 0$ as $\mu \to 1$).

Exercise 10.5.5. For $\mu \neq 1$ and $\delta \neq 0$, reparametrize and conjugate the étale affine representation of (10.21) to the étale affine representation

$$\begin{aligned} \left[\begin{array}{cc|c} e^{\mu t} & \delta\, e^{\mu t}\, s & e^{\mu t} s \\ 0 & e^{\delta t} & \varepsilon_\delta(t) \end{array}\right] &= \exp\left(t \left[\begin{array}{cc|c} \mu & 0 & 0 \\ 0 & \delta & 1 \end{array}\right]\right). \\ &\exp\left(s \left[\begin{array}{cc|c} 0 & \delta & 1 \\ 0 & 0 & 0 \end{array}\right]\right). \end{aligned} \tag{10.23}$$

Suppose that $\delta = \mu - 1$. The two one-parameter subgroups above correspond to affine vector fields

$$\begin{aligned} S_\delta &:= (1 + \delta y)\partial_x, \\ T_\delta &:= (1+\delta)x\partial_x + (1+\delta y)\partial_y \\ &= x\partial_x + \partial_y + \delta\mathsf{Rad} \end{aligned}$$

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which, as $\delta \to 0$, converge to the affine vector fields

$$X_{\mathfrak{R}}^{\mathsf{complete}} = x\partial_x + \partial_y,$$
$$Y_{\mathfrak{R}}^{\mathsf{complete}} = \partial_x$$

generating the complete affine structure of §10.5.2.

The columns of the étale affine representation in affine coordinates:

$$\left[\begin{array}{cc|c} (1+\delta y)^{1+1/\delta} & \delta x & x \\ 0 & 1+\delta y) & y \end{array}\right]$$

define a basis of left-invariant vector fields:

$$X := (1+\delta y)^{1+1/\delta}\partial_x,$$
$$Y := \partial_y + \delta(x\partial_x + y\partial_y)$$

with covariant derivatives tabulated in Table 10.10.

Table 10.10. Deformations of $\mathfrak{a}_{\mathfrak{R}}$ depending on an eigenvalue parameter $\delta \in \mathbb{R}$ containing the complete structure at $\delta = 0$. When $\delta \neq 0$, this structure is isomorphic to the radiant deformation $\mathfrak{a}_{\mathfrak{R}}^{\mu}$ where $\mu = 1 + \delta$.

	X	Y
X	0	δX
Y	$(\delta+1)X$	δY

10.5.7.2. *Radiance and parallel volume.* When $\mu = -1$, the structure has parallel volume — that is, area forms are parallel if and only if they are left-invariant. In that case the vector fields

$$x\partial_x + y\partial_y,\ y^{-1}\partial_x$$

base the space of left-invariant vector fields and the 1-forms

$$y^{-1}dy,\ y\,dx - x\,dy$$

base the space of left-invariant 1-forms. In contrast to affine structures on closed manifolds (Theorem 6.5.9), this left-invariant affine structure is both radiant and has parallel volume.

10.5.7.3. *Nonradiant deformation.* The étale representation

$$\left[\begin{array}{cc|c} 1 & s & \eta t \\ 0 & e^t & 0 \end{array}\right] \tag{10.24}$$

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deforms the representation (10.17) along the parallel vector field $\partial/\partial x$ with strength η. The developing map is:

$$p_0 := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} s + \eta t \\ e^t \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Group coordinates relate to the affine coordinates by:

$$\begin{aligned} s &:= x - \eta \log(y) \\ t &:= \log(y) \end{aligned}$$

In terms of affine coordinates, the étale affine representation is:

$$\left[\begin{array}{cc|c} 1 & x - \eta \log(y) & x \\ 0 & y & y \end{array}\right]$$

The columns of the linear part determine left-invariant vector fields:

$$\begin{aligned} X := \partial_x, \qquad Y_\eta &= \big(x - \eta \log(y)\big)\partial_x + y\partial_y \\ &= \mathsf{Rad} - \eta \log(y)\partial_x \end{aligned}$$

Their covariant derivatives are tabulated in Table 10.11. When $\eta \to 0$, they specialize to the left-invariant vector fields on $\mathfrak{a}_{\mathfrak{R}}$ as in (10.18) basing the algebra depicted in the first multiplication table in Table 10.5.

Table 10.11. This a nonradiant deformation of $\mathfrak{a}_{\mathfrak{R}}$, depending on a parameter $\eta \neq 0$. All the structures with nonzero η are isomorphic, and converge to $\mathfrak{a}_{\mathfrak{R}}$ as $\eta \to 0$.

	X	Y
X	0	X
Y	0	$Y - \eta X$

10.6. Complete affine structures on 3-manifolds

Fried–Goldman [**139**] classified closed complete affine 3-manifolds, based on the positive resolution of the Auslander–Milnor question in dimension 3. Namely a compact complete affine 3-manifold admits a finite covering space of the form $\Gamma\backslash G$, where G is a Lie group with a left-invariant complete affine structure and $\Gamma < G$ is a lattice. Furthermore G is isomorphic to $\mathbb{R}^3$, the 3-dimensional Heisenberg group $\mathsf{Heis}_{\mathbb{R}}$ introduced in §8.6.2.1 or the group Sol introduced in §8.6.2.2. Thus we classify the simply transitive affine actions of these three Lie groups. We begin with the nilpotent case, namely when $G \cong \mathbb{R}^3$ or $\mathsf{Heis}_{\mathbb{R}}$.

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In the nilpotent case, simple transitivity is equivalent to unipotence of the linear part:

Proposition 10.6.1 (Scheuneman [**299**]). Let G be a nilpotent group with with left-invariant affine structure. Let $G \xrightarrow{\rho} \mathsf{Aff}(\mathsf{A})$ be the étale representation corresponding to left-multiplication. Then the affine structure is complete (that is, ρ is simply transitive) if and only if $\mathsf{L} \circ \rho$ is unipotent.

Proof. This follows immediately from Helmstetter's characterization [**189**] of completeness as right-nilpotence of the corresponding left-symmetric algebra (Theorem 10.3.7). We sketch the original proof.

Suppose first that $\mathsf{L} \circ \rho$ is unipotent. Let O be an open orbit corresponding to the developing image. By Rosenlicht [**294**] (and independently, Kostant (unpublished)), every orbit of a connected unipotent group is Zariski-closed. Thus O is both open and closed (in the classical topology). Since A is connected, $O = \mathsf{A}$, that is, G acts transitively.

Since $\mathsf{dim}(\mathsf{A}) = \mathsf{dim}(G)$, every isotropy group is discrete. Since G is unipotent, every isotropy group is torsionfree and Zariski closed, which implies that G acts freely. Thus G acts simply transitively as desired.[5]

Conversely, suppose that G acts simply transitively. By the structure theorem for affine representations of nilpotent groups (Corollary C.3.2) there exists a maximal invariant affine subspace A_0 upon which ρ acts unipotently. Transitivity implies that $\mathsf{A}_0 = \mathsf{A}$, so ρ is unipotent. ☐

10.6.1. Central translations. The classification in dimension 3 is simplified by the existence of *central translations* in a simply transitive group of unipotent affine transformations. This was conjectured by Auslander [**15**] and erroneously claimed by Scheuneman [**299**]. Fried [**137**] produced a 4-dimensional counterexample, which is discussed in Appendix D; see Kim [**214**] for further developments in this basic question.

In terms of left-symmetric algebras, translations in an étale affine representation correspond to *right-invariant parallel vector fields.* A translation is central if and only if the corresponding right-invariant parallel vector field is also *left-invariant.* Left-invariant parallel vector fields correspond to elements $P \in \mathfrak{a}$ of the left-symmetric algebra $\mathfrak{a}$ such that $\mathfrak{a} \cdot P = 0$; being central means that $[\mathfrak{a}, P] = 0$. Thus the central translations correspond to elements $P \in \mathfrak{a}$ such that $P\mathfrak{a} = \mathfrak{a}P = 0$. Such elements P clearly form a two-sided ideal $\mathfrak{Z} = \mathfrak{Z}(\mathfrak{a})$ and the quotient $\mathfrak{a}/\mathfrak{Z}$ is a left-symmetric algebra.

Exercise 10.6.2. If $\mathfrak{a}$ corresponds to a complete affine structure, then $\mathfrak{a}/\mathfrak{Z}$ corresponds to a complete affine structure.

[5]When G admits a cocompact lattice, then completeness follows from Theorem 8.5.1. However, most nilpotent Lie groups do not admit cocompact lattices.

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Table 10.12. Multiplication tables for $\mathsf{dim}(\mathfrak{z}) = 3$ and $\mathsf{dim}(\mathfrak{z}) = 2$, respectively. The first example is $\mathbb{R}^3$, the Euclidean structure $\big(\mathsf{dim}(\mathfrak{z}) = 3\big)$. The second example is a product of $\mathbb{R}$ with the non-Riemannian structure on $\mathbb{R}^2$ $\big(\mathsf{dim}(\mathfrak{z}) = 2\big)$.

	X	Y	Z
X	0	0	0
Y	0	0	0
Z	0	0	0

	X	Y	Z
X	0	0	0
Y	0	0	0
Z	0	0	Y

Table 10.13. When $\mathsf{dim}(\mathfrak{z}) = 1$ and $G/\mathfrak{z}$ Euclidean, the structure is defined by a nondegenerate bilinear form $G/\mathfrak{z} \times G/\mathfrak{z} \xrightarrow{\mathbf{a}} \mathfrak{z}$. The affine Lie group is abelian if and only if **a** is symmetric; otherwise it is isomorphic to Heis. The Cartan bi-invariant affine structure on Heis in §8.6.2.3 arises when **a** is skew-symmetric and nonzero.

	X	Y	Z
X	0	0	0
Y	0	$a_{11}X$	$a_{12}X$
Z	0	$a_{21}X$	$a_{22}X$

10.6.2. 3-dimensional nilpotent algebras. Three-dimensional nilpotent left-symmetric algebras correspond to unipotent simply transitive affine actions on A^3. These are, in turn, equivalent to complete left-invariant affine structures on nilpotent Lie groups (compare §11.2.) If a 1-connected 3-dimensional nilpotent Lie group is not abelian, then it is isomorphic to the *Heisenberg group* $\mathsf{Heis}_{\mathbb{R}}$, discussed in §8.6.2.1.

Exercise 10.6.3. The 3-dimensional nilpotent associative algebras are tabulated in Tables 10.12,10.13, and 10.14. (Compare [**139**])

G is abelian when $\mathsf{dim}(\mathfrak{z}) \geq 2$, and in these cases G is a product of a 1-dimensional structure and a 2-dimensional structure. These cases appear as limits in the generic situation when $\mathsf{dim}(\mathfrak{z}) = 1$.

When $\mathsf{dim}(\mathfrak{z}) = 1$ and the structure on $G/\mathfrak{z}$ is Euclidean, then the induced product

$$\mathfrak{a}/\mathfrak{z} \times \mathfrak{a}/\mathfrak{z} \longrightarrow \mathfrak{a}$$
$$(A + \mathfrak{z}, B + \mathfrak{z}) \longmapsto AB$$

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Table 10.14. When $\mathsf{dim}(\mathfrak{Z}) = 1$ and $G/\mathfrak{Z}$ non-Riemannian, the structure is defined by a nonzero pair $(b, c) \in \mathbb{R}^2$.

	X	Y	Z
X	0	0	0
Y	0	0	bX
Z	0	cX	Y

defines a bilinear form $\mathbb{R}^2 \times \mathbb{R}^2 \xrightarrow{a} \mathfrak{Z} \cong \mathbb{R}$. Table 10.13 arises by taking X to be a generator of $\mathfrak{Z}$. The structure is abelian if and only if $\mathbf{a}$ is symmetric. In this case $\mathfrak{a}$ is Euclidean if $\mathbf{a} = 0$ and a product ($\mathsf{dim}(\mathfrak{Z}) = 2$) when $\mathbf{a}$ is nonzero and degenerate.

When $\mathsf{dim}(\mathfrak{Z}) = 1$ and the structure on $G/\mathfrak{Z}$ is non-Riemannian, let X be generate $\mathfrak{Z}$, and extend to a basis $\{X, Y, Z\}$ so that

$$(Z + \mathfrak{Z})^2 = Y$$

in the non-Riemannian quotient. The results are tabulated in Table 10.14. The structure is abelian if and only if $b = c$, and corresponds to the product structure ($\mathsf{dim}(\mathfrak{Z}) = 2$) if and only if $b = c = 0$.

10.6.3. Structures on Heis. We single out some interesting examples of affine structures on Heis, including an incomplete (radiant) structure.

The first examples are bi-invariant structures complete structures due to Caran, mentioned above. They correspond to the associative algebra with basis X, Y, Z with the only nonzero basic relations:

$$YZ \;=\; -ZY \;= X$$

Exercise 10.6.4. Show that this structure is isomorphic to its opposite.

The second structure on Heis which we mention is the incomplete *radiant suspension* of the Euclidean structure on $\mathbb{R}^2$ given by the étale affine representation (depending on a parameter $\lambda \neq 0$):

$$(a; b, c) \longmapsto e^{\lambda b} \begin{bmatrix} 1 & b & a \\ & 1 & c \\ & & 1 \end{bmatrix},$$

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which (using the basepoint p_0 above) gives left-multiplication in affine coordinates:

$$\left[\begin{array}{ccc|c} z & z\log(z) & x & x \\ 0 & z & y & y \\ 0 & 0 & z & z \end{array}\right]$$

and a basis of left-invariant vector fields:

$$X := z\partial_x, \quad Y := z\big(\log(z)\partial_x + \partial_y\big), \quad Z = \mathsf{Rad} = x\partial_x + y\partial_y + z\partial_z$$

whose covariant derivatives tabulated in Table 10.15.

Table 10.15. A radiant structure on Heis. It can be regarded as an infinitesimal version of the radiant suspension of a twist flow of the Euclidean structure.

	X	Y	Z
X	0	0	X
Y	0	0	Y
Z	X	$X+Y$	Z

10.7. Solvable 3-dimensional algebras

The solvable case reduces to the nilpotent case by the useful fact that the unipotent radical the Zariski closure (sometimes called the *nil-shadow* of G) of a simply transitive subgroup itself acts simply transitively.

Thus, underlying every simply transitive affine action is a *unipotent* simply transitive affine action. Every solvable group with complete left-invariant affine structure has an underlying structure as a *nilpotent* Lie group with complete left-invariant affine structure.

If M^3 is a closed manifold with complete affine structure, then M is affinely isomorphic to a finite quotient of a *complete affine solvmanifold,* that is, a homogeneous space $\Gamma\backslash G$, where G is a Lie group with a complete affine structure and $\Gamma < G$ is a lattice. Equivalently, G admits a simply transitive affine action (corresponding to left-multiplication). Necessarily G is solvable and we may assume that G is simply-connected. Since G admits a lattice, it is unimodular.

We have already discussed the cases when G is nilpotent. There are two isomorphism classes of simply connected solvable unimodular non-nilpotent Lie groups:

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- The universal covering $\widetilde{\mathsf{Isom}^0(\mathsf{E}^2)}$ of orientation-preserving isometries of the group $\mathsf{Isom}^0(\mathsf{E}^2)$ of orientation-preserving isometries of the Euclidean plane E^2;
- The identity component of the group $\mathsf{Sol} = \mathsf{Isom}^0(\mathsf{E}^{1,1})$ of orientation-preserving isometries of 2-dimensional Minkowski space.

Exercise 10.7.1. Prove that every lattice $\Gamma < \widetilde{\mathsf{Isom}^0(\mathsf{E}^2)}$ contains a free abelian subgroup of finite index. In particular, Γ is a finite extension of $\mathbb{Z}^3$ and is a 3-dimensional Bieberbach group.

Thus the only interesting remaining case for classifying complete affine 3-manifolds occurs for the group $\mathsf{Isom}^0(\mathsf{E}^{1,1}) \cong \mathsf{Sol}$.

There are two isomorphism classes of complete left-invariant affine structures on this group: the Auslander–Markus example [**13**] is a group of isometries of a parallel Lorentzian metric is discussed in §8.6.2.2; the other, due to Auslander [**15**], and corresponds to a simply transitive action containing no translations (compare Fried–Goldman [**139**], Theorem 4.1):

$$(s,t,u) \xmapsto{\rho_\lambda} \left[\begin{array}{ccc|c} 1 & \lambda e^s u & \lambda e^{-s} t & s + \lambda t u \\ 0 & e^s & 0 & t \\ 0 & 0 & e^{-s} & u \end{array}\right].$$

We denote the image of this affine representation by G_λ.

The corresponding left-invariant vector fields are:

$$\begin{aligned} X &:= \partial_x \\ Y &:= e^{x-\lambda yz}\Big(\lambda z\ \partial_x + \partial_y\Big) \\ Z &:= e^{-x+\lambda yz}\Big(\lambda y\ \partial_x + \partial_z\Big) \end{aligned}$$

with multiplication given in Table 10.16.

The case $\lambda = 0$ is the original Auslander–Markus flat Lorentzian structure, and contains a one-parameter group of translations. When $\lambda \neq 0$, these actions are all affinely conjugate. However, even when $\lambda = 0$, these actions are *polynomially conjugate,* as follows.

The algebraic hull $\mathbb{A}(G_0)$ of G_0 is the product $U \cdot D$, where the *unipotent radical* U is just the group of translations of A^3 and the *Levi subgroup* D is

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the one-parameter group of diagonal matrices

$$\delta_s := \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-s} \end{bmatrix}$$

(U is the nil-shadow of G_0, the nilpotent group underlying G).

Exercise 10.7.2. The polynomial diffeomorphism

$$\mathsf{A}^3 \xrightarrow{\Phi_\lambda} \mathsf{A}^3$$
$$(x, y, z) \longmapsto (x + \lambda yz, y, z)$$

conjugates G_0 to G_λ, that is,

$$\Phi_\lambda \circ \rho_0 \circ \Phi_\lambda^{-1} \;=\; \rho_\lambda.$$

- Show that

  $$U_\lambda := \Phi_\lambda \circ U_0 \circ \Phi_\lambda^{-1}$$

  defines the affine Lie group (isomorphic to $\mathbb{R}^3$) corresponding to

  $$\mathbf{a} = \begin{bmatrix} 0 & \lambda \\ \lambda & 0 \end{bmatrix}$$

  in Table 10.13. This nilpotent affine Lie group underlies G_λ.
- $\Phi_\lambda \circ D \circ \Phi_\lambda^{-1} = D$ so $\mathbb{A}(G_\lambda) = U_\lambda \cdot D$.
- $\Phi_\lambda \circ \rho_0 \circ \Phi_\lambda^{-1} = \rho_\lambda$, that is, ρ_λ is obtained by equating the parameter s in δ_s with the parameter s in $\alpha_\lambda(s, t, u)$.

Table 10.16. Complete affine structures on Sol depending on a parameter $\lambda \in \mathbb{R}$. The original Auslander–Markus example of a complete flat Lorentzian manifold arises for $\lambda = 0$. For $\lambda \neq 0$, these structures are affinely equivalently, but the corresponding simply transitive affine action contains no translations.

	X	Y	Z
X	0	Y	$-Z$
Y	0	0	λX
Z	0	λX	0

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10.8. Parabolic cylinders

We can extend this structure to structures on a 3-dimensional solvable Lie group G, which admits compact quotients. These provide examples of compact convex incomplete affine 3-manifolds which are not *properly convex,* and nonradiant. Therefore Vey's result that compact hyperbolic affine manifolds are radiant is sharp.

Further examples from the same group action give *concave* affine structures on these same 3-manifolds.

The function:

$$\begin{aligned} \mathsf{A}^3 &\xrightarrow{f} \mathbb{R} \\ (x,y,z) &\longmapsto x - y^2/2 \end{aligned}$$

is invariant under the affine $\mathbb{R}^2$-action defined by:

$$\begin{aligned} \mathbb{R}^2 &\xrightarrow{\mathfrak{U}} \mathsf{Aff}(\mathsf{A}^3) \\ (t,u) &\longmapsto \exp \left[\begin{array}{ccc|c} 0 & t & 0 & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & u \end{array}\right] = \left[\begin{array}{ccc|c} 1 & t & 0 & t^2/2 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & u \end{array}\right]. \end{aligned}$$

Under the 1-parameter group of dilations

$$\begin{aligned} \mathbb{R} &\xrightarrow{\delta} \mathsf{Aff}(\mathsf{A}^3) \\ s &\longmapsto \exp \begin{bmatrix} 2s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & -s \end{bmatrix} = \begin{bmatrix} e^{2s} & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-s} \end{bmatrix} \end{aligned}$$

the function f scales as:

$$f \circ \delta(s) = e^{2s} f.$$

The group $G \subset \mathsf{Aff}(\mathsf{A}^3)$ generated by $\mathfrak{U}(t,u)\delta(s)$ (for $s,t,u \in \mathbb{R}$) acts simply transitively on the open convex parabolic cylinder defined by $f(x,y,z) > 0$ as well as on the open concave parabolic cylinder defined by $f(x,y,z) < 0$. The corresponding left-invariant affine structure on the Lie group G has a basis of left-invariant vector fields

$$\begin{aligned} X &:= f(x,y,z)\partial_x \\ Y &:= f(x,y,z)^{1/2}\big(y\partial_x + \partial_y\big) \\ Z &:= f(x,y,z)^{-1/2}\partial_z \end{aligned}$$

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Table 10.17. Algebra corresponding to parabolic 3-dimensional halfspaces

	X	Y	Z
X	X	$Y/2$	$-Z/2$
Y	0	X	0
Z	0	0	0

with multiplication recorded in Table 10.17. The dual basis of left-invariant 1-forms is:

$$\begin{aligned} X^* &:= f(x,y,z)^{-1}(dx - y dy) = -d\log(f) \\ Y^* &= f(x,y,z)^{-1/2}dy \\ Z^* &= f(x,y,z)^{1/2}dz \end{aligned}$$

with *bi-invariant* volume form

$$X^* \wedge Y^* \wedge Z^* = f(x,y,z)^{-1} dx \wedge dy \wedge dz.$$

This example is due to Goldman [**147**] providing examples of non-conical convex domains covering compact affine manifolds.

10.8.1. Nonradiant deformations of radiant halfspace quotients.

Another example arises from radiant suspensions. Namely, consider the radiant affine representation

$$\begin{aligned} \mathbb{R} \ltimes \mathbb{R}^2 &\xrightarrow{\rho_\delta} \mathsf{Aff}(\mathsf{A}^3) \\ (s;t,u) &\longmapsto e^{\delta s} \exp \begin{bmatrix} s & 0 & t \\ 0 & -s & u \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} e^{(\alpha+1)s} & 0 & e^{\alpha s}t \\ 0 & e^{(\alpha-1)s} & e^{\alpha s}u \\ 0 & 0 & e^{\alpha s} \end{bmatrix} \end{aligned}$$

depending on a parameter $\alpha \in \mathbb{R}$. When $\alpha \neq 0$, the action is locally simply transitive; the open orbits are the two halfspaces defined by $z > 0$ and $z < 0$

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Table 10.18. The complete structure on $\mathsf{aff}(1,\mathbb{R})$

	X	Y	Z
X	0	0	X
Y	0	0	Y
Z	$\big((\alpha+1)/\alpha\big)X$	$\big((\alpha-1)/\alpha\big)Y$	Z

respectively. The vector fields

$$\begin{aligned} X &:= z^{(\alpha+1)/\alpha}\partial_x \\ Y &:= z^{(\alpha-1)/\alpha}\partial_y \\ Z &:= R = x\partial_x + y\partial_y + z\partial_z \end{aligned}$$

correspond to a basis of left-invariant vector fields, with multiplication recorded in Table 10.18.

When $\alpha = \pm 1$, then this action admits *nonradiant* deformations. Namely let $\beta \in \mathbb{R}$ be another parameter, and consider the case when $\alpha = 1$. The affine representation

$$\begin{aligned} \mathbb{R} \ltimes \mathbb{R}^2 &\xrightarrow{\rho^\beta} \mathsf{Aff}(\mathsf{A}^3) \\ (s;t,u) &\longmapsto \exp\left[\begin{array}{ccc|c} 2s & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta s \\ 0 & 0 & s & 0 \end{array}\right] \cdot \exp\begin{bmatrix} 0 & 0 & t \\ 0 & 0 & u \\ 0 & 0 & 0 \end{bmatrix} \\ &= \left[\begin{array}{ccc|c} e^{2s} & 0 & e^{2s}t & 0 \\ 0 & 1 & u & \beta s \\ 0 & 0 & e^s & 0 \end{array}\right] \end{aligned}$$

maps

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xmapsto{\rho^\beta} \begin{bmatrix} e^{2s}t \\ u + \beta s \\ e^s \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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Table 10.19. Nonradiant Deformation

	X	Y	Z
X	0	0	X
Y	0	0	Y
Z	$2X$	0	$Z-\beta Y$

so the group element with coordinates (s,t,u) corresponds to the point with coordinates

$$\begin{aligned} x &= e^{2s}t \\ y &= u+\beta s \\ z &= e^{s} \end{aligned}$$

with inverse transformation:

$$\begin{aligned} s &= \log(z) \\ t &= z^{-2}x \\ u &= y-\beta\log(z). \end{aligned}$$

Then the linear part $\mathsf{L}\rho^{\beta}(s,t,u)$ corresponds to the matrix

$$\begin{bmatrix} z^2 & 0 & x \\ 0 & 1 & y-\beta\log(z) \\ 0 & 0 & z \end{bmatrix}$$

whose columns base the left-invariant vector fields:

$$\begin{aligned} X &:= z^2\partial_x \\ Y &:= \partial_y \\ Z &:= x\partial_x + \big(y-\beta\log(z)\big)\partial_y + z\partial_z \end{aligned}$$

Table 10.19 records their covariant derivatives.

10.9. Structures on $\mathsf{gl}(2,\mathbb{R})$

The algebra $\mathfrak{a}$ of 2×2 matrices has several structures. As $\mathsf{End}(\mathbb{R}^2)$ it is an associative algebra, and hence corresponds to bi-invariant affine structures on $\mathsf{GL}(2,\mathbb{R})$. This structure is radiant and defines bi-invariant $\mathbb{R}\mathsf{P}^3$-structures on $\mathsf{PSL}(2,\mathbb{R})$.

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Write

$$m := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longleftrightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \quad p := \begin{bmatrix} x & y \\ z & w \end{bmatrix} \longleftrightarrow \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathsf{A}^4.$$

Then left-multiplication by m maps p to

$$\mathcal{L}_m(p) = \begin{bmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{bmatrix} \longleftrightarrow \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}.$$

Choose for the basepoint

$$p_0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

the identity matrix; then evaluation at the basepoint defines affine coordinates

$$m(p) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longleftrightarrow \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}.$$

Corresponding to the coordinates a, b, c, d is a basis for right-invariant vector fields:

$$A := x\partial_x + y\partial_y, \quad B := z\partial_x + w\partial_y, \quad C := x\partial_z + y\partial_w, \quad D := z\partial_z + w\partial_w.$$

The affine representation corresponding to $\mathcal{L}_g$, using p_0 as the origin is:

$$\left[\begin{array}{cccc|c} x & 0 & y & 0 & x \\ 0 & x & 0 & y & y \\ z & 0 & w & 0 & z \\ 0 & z & 0 & w & x \end{array}\right]$$

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which implies that the vector fields

$$x\partial_x + z\partial_z, \qquad x\partial_y + z\partial_w, \qquad y\partial_x + w\partial_z, \qquad y\partial_y + w\partial_w$$

form a basis left-invariant vector fields. The characteristic function is given by the determinant of the linear part

$$\begin{vmatrix} x & 0 & y & 0 \\ 0 & x & 0 & y \\ z & 0 & w & 0 \\ 0 & z & 0 & w \end{vmatrix} = (xw - yz)^2$$

Exercise 10.9.1. Show that *right-multiplication* is given by:

$$\mathcal{R}_m = m^\dagger \oplus m^\dagger = \begin{bmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{bmatrix}$$

Exercise 10.9.2. Find an isomorphism between this associative algebra and its opposite.

10.9.1. The third symmetric power. Another structure is given by the irreducible 4-dimension irreducible representation of the algebra $\mathsf{gl}(2,\mathbb{R})$. This corresponds to the action of $\mathsf{gl}(2,\mathbb{R}) = \mathsf{End}(\mathbb{R}^2)$ on the third symmetric power $\mathsf{Sym}^3(\mathbb{R}^2)$.

Let $e, f \in \mathbb{R}^2$ be a basis. Then the monomials[6]

$$e^3, \qquad \sqrt{3}\, e^2 f, \qquad \sqrt{3}\, e f^2, \qquad f^3$$

base $\mathsf{Sym}^3(\mathbb{R}^2)$ and the induced action of the matrix $m \in \mathsf{gl}(2,\mathbb{R})$ above is:

$$\begin{bmatrix} a^3 & \sqrt{3}a^2 b & \sqrt{3}ab^2 & b^3 \\ \sqrt{3}a^2 c & 2abc + a^2 d & b^2 c + 2abd & \sqrt{3}b^2 d \\ \sqrt{3}ac^2 & bc^2 + 2acd & 2bcd + ad^2 & \sqrt{3}bd^2 \\ c^3 & \sqrt{3}c^2 d & \sqrt{3}cd^2 & d^3 \end{bmatrix}.$$

Exercise 10.9.3. The vector fields

$$3x\partial_x + 2y\partial_y + z\partial_z, \qquad \sqrt{3}y\partial_x + 2z\partial_y + \sqrt{3}w\partial_z,$$
$$\sqrt{3}x\partial_y + 2y\partial_z + \sqrt{3}z\partial_w, \qquad y\partial_y + 2z\partial_z + 3w\partial_w$$

base the Lie algebra of right-invariant vector fields.

[6]The factor $\sqrt{3}$ is there to obtain a nicer form for the invariant symplectic structure.

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- Find a basis for the left-invariant vector fields.
- Compute the multiplication table for the corresponding left-symmetric algebra.
- Show that the *characteristic function* is:

$$\phi(x,y,z,w) := \frac{9}{2}x^2w^2 - 9xyzw \\ - \frac{3}{2}y^2z^2 + 2\sqrt{3}(y^3w + xz^3)$$

and relate it to the *discriminant* of the cubic polynomial

$$f(t) \;:=\; xt^3 + yt^2 + zt + w.$$

This gives interesting examples of *contact* $\mathbb{R}\mathsf{P}^3$*-manifolds* as follows. (Compare Exercise 6.6.1. The left-invariant affine structure on $\mathsf{GL}(2,\mathbb{R})$ induces a left-invariant projective structure on $\mathsf{PGL}(2,\mathbb{R})$. Furthermore an $\mathsf{SL}(2,\mathbb{R})$-invariant symplectic structure on $\mathbb{R}^2$ induces an invariant symplectic structure on $\mathsf{Sym}^3(\mathbb{R}^2)$ given by the exterior 2-form

$$dx \wedge dw + dy \wedge dz$$

and induces a left-invariant contact $\mathbb{R}\mathsf{P}^3$-structure on $\mathsf{PGL}(2,\mathbb{R})$. Passing to the quotient by a lattice $\Gamma_0 < \mathsf{PGL}(2,\mathbb{R})$ defines contact projective structures on the 3-manifolds

$$M_0 \;:=\; \Gamma_0\backslash\mathsf{PGL}(2,\mathbb{R}).$$

Deforming the holonomy representation

$$\pi_1(M_0) \longrightarrow \Gamma_0 \hookrightarrow \mathsf{PGL}(2,\mathbb{R}) \longrightarrow \mathsf{PSp}(4,\mathbb{R})$$

in

$$\mathsf{Hom}\big(\pi_1(M_0), \mathsf{PSp}(4,\mathbb{R})\big)$$

gives the *Hitchin representations* studied in Guichard–Wienhard [**180**]. The deformed contact $\mathbb{R}\mathsf{P}^3$-manifolds have additional structure, which Guichard and Wienhard call *convex foliated.*

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Chapter 11

Parallel volume and completeness

A particularly tantalizing open problem about closed affine manifolds is whether geodesic completeness (a geometric 1-dimensional property) is equivalent to parallel volume (an algebraic n-dimensional property).[1]

Question. Let M be a closed affine manifold. Then M is geodesically complete if and only if M has parallel volume.

An affine manifold M has *parallel volume* if and only if it satisfies any of the following equivalent conditions:

- The orientable double-covering of M admits a parallel volume form (in the sense of §1.4.3);
- M admits a coordinate atlas where the coordinate changes are volume-preserving;
- M admits a refined $(\mathsf{SAff}(\mathsf{A}), \mathsf{A})$-structure, where $\mathsf{SAff}(\mathsf{A})$ denotes the subgroup $\mathsf{L}^{-1}\big(\mathsf{SL}_{\pm}(\mathbb{R}^n)\big)$ of volume-preserving linear transformations;
- For each $\phi \in \pi_1(M)$, the linear holonomy $\mathsf{L} \circ \mathsf{h}(\phi)$ has determinant ± 1.

[1] This question was raised in 1963 by L. Markus [**254**] as a "Research Problem" in unpublished mimeographed lecture notes from the University of Minnesota (Problem 8, §6, p.58) and has been called the *Markus conjecture.*

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11.1. The volume obstruction

Exercise 11.1.1. Prove the equivalence of the conditions stated in the introduction to Chapter 11.

The last condition suggests a topological interpretation. The composition of the linear holonomy representation $\mathsf{L} \circ \mathsf{h}$ with the logarithm of the absolute value of the determinant

$$\pi_1(M) \xrightarrow{\mathsf{L}\circ\mathsf{h}} \mathsf{GL}(\mathsf{V}) \xrightarrow{|\det|} \mathbb{R}^+ \xrightarrow{\log} \mathbb{R}$$

defines an additive homomorphism $\nu_M \in \mathsf{Hom}\big(\pi_1(M), \mathbb{R}\big) \cong H^1(M;\mathbb{R})$ which we call the *volume obstruction.* M has parallel volume if and only if $\nu_M = 0$.

Exercise 11.1.2. Suppose that M is a manifold with zero first Betti number. Then every affine structure on M has parallel volume.

One amusing corollary of this is that the *only* affine structures on the $\mathbb{Q}$-homology 3-sphere $\mathcal{S}^3_{\mathbb{Q}}$ defined in §6.4.1 are *complete affine* structures. Since $\mathcal{S}^3_{\mathbb{Q}}$ is covered by a 3-torus, the Markus conjecture for abelian holonomy (Smillie [**310**], Fried–Goldman–Hirsch [**140**], see §11.2) implies that $\mathcal{S}^3_{\mathbb{Q}}$ must be complete, and must be covered by a complete affine nilmanifold. Furthermore, Benoist's classification of projective structures on tori [**36**] implies all projective structures on $\mathcal{S}^3_{\mathbb{Q}}$ are affine, and hence complete affine. Helmstetter's theorem [**189**] states that a left-invariant affine structure on a Lie group is complete $\Longleftrightarrow$ right-invariant volume forms are parallel is an "infinitesimal version" of Markus's conjecture. (Compare also Goldman–Hirsch [**163**].)

In some cases, a natural closed 1-form whose de Rham cohomology class represents the volume obstruction exists. For example, when M is *Koszul hyperbolic,* that is, the quotient of a sharp convex cone Ω/Γ, the *Koszul 1-form* (as defined in §4.4.4) defines a closed 1-form representing the volume obstruction. Similarly, when G is an affine Lie group, the logarithmic differential of the characteristic polynomial $\det(\mathcal{R}_{X\oplus 1})$ as in §10.3.5 defines such a 1-form.

The plausibility of Markus's question is a major barrier in constructing examples of affine manifolds. A purely topological consequence of this conjecture is that a compact affine manifold M with zero first Betti number $\beta_1(M)$ is covered by Euclidean space: in particular all of its higher homotopy groups vanish. Thus, if $\beta_1(M) = 0$ there should be no such structure on a nontrivial connected sum in dimensions greater than two.[2]

[2]In fact no affine structure — or projective structure — is presently known on a nontrivial connected sum.

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Jo and Kim [**206**] resolve this question for *convex* affine manifolds.

11.2. Nilpotent holonomy

One of the first results on Markus's question is its resolution in the case the affine holonomy group is nilpotent.

The structure theory of affine structures on closed manifolds with nilpotent holonomy is relatively well understood, due to the work of Smillie [**310**], Fried–Goldman–Hirsch [**140**] and Benoist [**36**, **39**]. Smillie's thesis develops the basic theory for affine structures with abelian holonomy. Subsequently Fried–Goldman–Hirsch extended Smillie's theory to nilpotent holonomy, which Benoist extended to projective structures. The transition from nilpotent to solvable is much larger than the transition from abelian to nilpotent, and §15.2 discusses the few results in the solvable non-nilpotent case, due to Serge Dupont [**119**],[**120**]. The classification of closed similarity manifolds (see §11.4)

The key technique in the discussion of nilpotent holonomy is the structure theory of linear representations of nilpotent groups, described in Appendix C. The guiding principle is that nilpotence ensures a nontrivial center, producing lots of commuting transformations. Specifically, elements of a nilpotent linear group have strongly compatible Jordan decompositions, which leads to invariant structures for manifolds with nilpotent holonomy. In another context this was used by Goldman [**148**] to give the first examples of 3-manifolds *without* flat conformal structures — for a geometric approach to these algebraic facts, see Thurston [**324**], Corollary 4.1.17 and Ratcliffe [**293**]. (Compare the discussion in §15.6.)

The structure theory developed here "reduces" the classification of affine manifolds with nilpotent holonomy to finite-sheeted covering space of a deformation an iterated fibration over a complete nilmanifold, with fibers affine manifolds whose holonomy are diagonal matrices. The initial work of Smillie [**310**] and Sullivan–Thurston [**320**] (see also [**145**]) showed the richness of these structures, which was later analyzed by Smillie [**308**] and Benoist [**36**, **39**]. Compare the discussion in §13.4.

11.2.1. Affine representations of nilpotent groups.

Now we extend the preceding theory of *linear* representations to affine representations. The linearization of affine representations §1.8 implies that the $n+1$-dimensional linear representation of an n-dimensional affine representation always has 1 as a weight, so there is nontrivial generalized eigenspace upon which the

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group acts unipotently. Theorem C.2.1 has the following geometric consequence:

Corollary 11.2.1. Let A be an affine space over $\mathbb{C}$ and $\Gamma < \mathsf{Aff}(\mathsf{A})$. Then $\exists!$ a maximal Γ-invariant affine subspace $\mathsf{A}_1 < \mathsf{A}$ such that the restriction of Γ to A_1 is unipotent.

A_1 is called the *Fitting subspace* in [**140**].

Since Γ preserves the affine subspace A_1, it induces an affine action on the quotient space A/A_1. Denote by V the vector space underlying A. Since the affine action on A/A_1 is radiant (it preserves the coset $\mathsf{A}_1 < \mathsf{A}$), we may describe A as an *affine direct sum:*

$$\mathsf{A} \;=\; \mathsf{A}_1 \oplus \mathsf{V}_1$$

where $\mathsf{V}_1 \subset \mathsf{V}$ is an $\mathsf{L}(\Gamma)$-invariant linear subspace. Compare Fried–Goldman–Hirsch [**140**]] where this is proved cohomologically, using techniques from Hirsch [**192**].

Theorem 11.2.2 (Smillie [**310**], Fried–Goldman–Hirsch [**140**])**.** Let M be a closed manifold with an affine structure whose affine holonomy group Γ is nilpotent. Let A_1 be its Fitting subspace and V_1 its linear complement as above. Then $\exists\gamma \in \Gamma$ such that the restriction $\gamma|_{\mathsf{V}_1}$ is a linear expansion.

The last assertion follows from the following characterization of linear expansions, combined with the structure of affine representations of nilpotent groups (simultaneous Jordan normal form):

Exercise 11.2.3. Suppose that $\gamma \in \mathsf{GL}(\mathsf{V})$ is a linear map. Then either γ is a linear expansion or there exists a proper γ-invariant *open* continuous function $\mathsf{V} \setminus \{\mathbf{0}\} \longrightarrow \mathbb{R}^+$.

As in [**140, 310**], Markus's conjecture for nilpotent holonomy follows:

Corollary 11.2.4. Let M be a closed affine manifold whose affine holonomy group is nilpotent. Then M is complete if and only if it has parallel volume.

Proof. Suppose M has parallel volume. Theorem 11.2.2 guarantees an element $\gamma \in \Gamma$ whose linear part $\mathsf{L}(\gamma)$ is an expansion on V_1. Thus $\mathsf{V}_1 = 0$, and $\mathsf{A} = \mathsf{A}_1$, that is, Γ is unipotent. Apply Theorem 8.5.1 to deduce that M is complete.

Conversely suppose M is complete, that is, the developing map $\widetilde{M} \xrightarrow{\mathsf{dev}} \mathsf{A}$ is a diffeomorphism. Then

$$M_1 \;:=\; \big(\mathsf{dev}^{-1}(\mathsf{A}_1)\big)/\Gamma \;\subset\; \mathsf{A}/\Gamma \cong M$$

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is a closed affine submanifold and $M_1 \hookrightarrow M$ is a homotopy-equivalence. Since $\widetilde{M} \approx \mathsf{A}$ and $\widetilde{M_1} \approx \mathsf{A}_1$, both M_1 and M are aspherical. Thus[3]

$$\mathsf{dim}(\mathsf{A}_1) = \mathsf{dim}(M_1) = \mathsf{cd}(\Gamma) = \mathsf{dim}(M) = \mathsf{dim}(\mathsf{A}),$$

$\mathsf{A}_1 = \mathsf{A}$ and $\mathsf{L}(\Gamma)$ is unipotent, and hence volume-preserving. Thus M has parallel volume. □

The geometric consequence of Theorem 11.2.2 is that the Γ-invariant decomposition

$$\mathsf{A}_1 \hookrightarrow \mathsf{A} \twoheadrightarrow \mathsf{V}_1$$

defines two transverse affine foliations of M. The affine subspaces parallel to A_1 define the leaves of a foliation $\mathcal{F}^u$ of M. The leaves of $\mathcal{F}^u$ are affine submanifolds of M with unipotent holonomy. The affine subspaces parallel to V_1 define the leaves of a foliation $\mathcal{F}^{\mathsf{Rad}}$ of M. The leaves of $\mathcal{F}^{\mathsf{Rad}}$ are affine submanifolds of M with radiant affine structure.

Exercise 11.2.5. Suppose that M is closed. Show that each leaf of $\mathcal{F}^u$ is a complete affine manifold.

11.3. Smillie's nonexistence theorem

Theorem 11.3.1 (Smillie [**312**])**.** Let M be a closed affine manifold with parallel volume. Then the affine holonomy homomorphism cannot factor through a free group.

This theorem can be generalized much further — see Smillie [**312**] and Goldman–Hirsch [**163**].

Corollary 11.3.2 (Smillie [**312**])**.** Let M be a closed manifold whose fundamental group is a free product of finite groups (for example, a connected sum of manifolds with finite fundamental group). Then M admits no affine structure.

Proof of Corollary 11.3.2 assuming Theorem 11.3.1.

Suppose M has an affine structure. Since $\pi_1(M)$ is a free product of finite groups, the first Betti number of M is zero. Thus M has parallel volume. Furthermore if $\pi_1(M)$ is a free product of finite groups, there exists a free subgroup $\Gamma \subset \pi_1(M)$ of finite index. Let $\hat{M}$ be the covering space with $\pi_1(\hat{M}) = \Gamma$. Then the induced affine structure on $\hat{M}$ also has parallel volume contradicting Theorem 11.3.1. □

[3] Here $\mathsf{cd}(\Gamma)$ denotes the *cohomological dimension* of Γ. We are using the standard fact that for an aspherical manifold X with $\pi_1(X) \cong \Gamma$, $\mathsf{dim}(X) \geq \mathsf{cd}(\Gamma)$ and $\mathsf{dim}(X) = \mathsf{cd}(\Gamma)$ if X is compact.

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Proof of Theorem 11.3.1. Let M be a closed affine manifold modeled on an affine space E, $\widetilde{M} \xrightarrow{\Pi} M$ a universal covering, and

$$\left(\widetilde{M} \xrightarrow{\mathsf{dev}} E,\ \pi \xrightarrow{\mathsf{hol}} \mathsf{Aff}(E)\right)$$

a development pair. Suppose that M has parallel volume and that there is a free group Π through which the affine holonomy homomorphism h factors:

$$\pi \xrightarrow{\phi} \Pi \xrightarrow{\bar{\mathsf{h}}} \mathsf{Aff}(E)$$

Choose a graph G with fundamental group Π; then there exists a map $j : M \longrightarrow G$ inducing the homomorphism

$$\pi = \pi_1(M) \xrightarrow{\phi} \pi_1(G) = \Pi.$$

By general position, there exist $s_1, \dots, s_k \in G$ such that j is transverse to s_i and the complement $G - \{s_1, \dots, s_k\}$ is connected and simply connected. Let H_i denote the inverse image $f^{-1}(s_i)$ and let $H = \cup_i H_i$ denote their disjoint union. Then H is an oriented closed smooth hypersurface such that the complement $M - H \subset M$ has trivial holonomy. Let $M|H$ denote the manifold with boundary obtained by *splitting* M along H; that is, $M|H$ has two open and closed subsets H_i^+, H_i^- for each H_i with diffeomorphisms $H_i^+ \xrightarrow{g_i} H_i^-$ (generating a groupoid containing Π) such that M is the quotient of $M|H$ by the identifications g_i. There is a canonical diffeomorphism of $M - H$ with $\mathsf{int}(M|H)$.

Let ω_E be a parallel volume form on E; then there exists a parallel volume form ω_M on M such that

$$\Pi^* \omega_M = \mathsf{dev}^* \omega_E.$$

Since $H^n(E) = 0$, $\exists$ an $(n-1)$-form η on E such that $d\eta = \omega_E$. For any immersion $S \xrightarrow{f} E$ of an oriented closed $(n-1)$-manifold S, the integral

$$\int_S f^* \eta$$

is independent of the choice of η satisfying $d\eta = \omega_E$. Indeed, $H^{n-1}(E) = 0$, and any other η' must satisfy $\eta' = \eta + d\theta$. Thus

$$\int_S f^* \eta' - \int_S f^* \eta = \int_S d(f^* \theta) = 0.$$

Since $M \setminus H$ has trivial holonomy there is a developing map

$$M - H \xrightarrow{\mathsf{dev}} E$$

and its restriction to $M \setminus H$ extends to a developing map $M|H \xrightarrow{\mathsf{dev}} E$ such that

$$\mathsf{dev}|_{H_i^+} = \bar{h}(g_i) \circ \mathsf{dev}|_{H_i^-}$$

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and the normal orientations of H_i^+, H_i^- induced from $M|H$ are opposite. Since $\mathsf{h}(g_i)$ preserves the volume form ω_E,

$$d(h(g_i)^*\eta) = d(\eta) = \omega$$

and

$$\int_{H_i^+} \mathsf{dev}^*\eta = \int_{H_i^+} \mathsf{dev}^* h(g_i)^*\eta = -\int_{H_i^-} \mathsf{dev}^*\eta$$

since the normal orientations of $H_i^\pm$ are opposite. We now compute the ω_M-volume of M:

$$\begin{aligned} \mathsf{vol}(M) &= \int_M \omega_M = \int_{M|H} \mathsf{dev}^*\omega_E \\ &= \int_{\partial(M|H)} \eta \;= \sum_{i=1}^k \left(\int_{H_i^+} \eta + \int_{H_i^-} \eta \right) = 0, \end{aligned}$$

a contradiction. □

One basic method of finding a primitive η for ω_E involves a radiant vector field ρ. Since ρ expands volume, specifically,

$$d\iota_\rho \omega_E = n\omega_E,$$

the $(n-1)$-form

$$\eta := \frac{1}{n}\iota_\rho \omega_E$$

is a primitive for ω_E. Recall that an affine manifold is *radiant* if and only if it possesses a radiant vector field if and only if the affine structure comes from an $\big(\mathsf{GL}(\mathsf{V}), \mathsf{V}\big)$-structure if and only if its affine holonomy has a fixed point in A (§6.5). The following result generalizes the above theorem:

Theorem 11.3.3 (Smillie)**.** Let M be a closed affine manifold with a parallel exterior differential k-form which has nontrivial de Rham cohomology class. Suppose $\mathcal{U}$ is an open covering of M such that for each $U \in \mathcal{U}$, the affine structure induced on U is radiant. Then $\mathsf{dim}(\mathcal{U}) \geq k$; that is, there exist $k+1$ distinct open sets

$$U_1, \dots, U_{k+1} \in \mathcal{U}$$

such that the intersection

$$U_1 \cap \cdots \cap U_{k+1} \neq \emptyset.$$

(Equivalently the nerve of $\mathcal{U}$ has dimension at least k.)

A published proof of this theorem can be found in Goldman–Hirsch [**163**].

Using these ideas, Carrière, Dal'bo, and Meignez [**79**] have proved that a nontrivial Seifert 3-manifold with hyperbolic base cannot have an affine

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structure with parallel volume. This implies that the 3-dimensional Brieskorn manifolds $M(p,q,r)$ with

$$p^{-1} + q^{-1} + r^{-1} < 1$$

admit no affine structure whatsoever. (Compare Milnor [**270**].)

There is a large class of discrete groups Γ for which every affine representation $\Gamma \longrightarrow \mathsf{Aff}(\mathsf{A})$ is conjugate to a representation factoring through $\mathsf{SL}(\mathsf{V})$, that is,

$$\Gamma \longrightarrow \mathsf{SL}(\mathsf{V}) \subset \mathsf{Aff}(\mathsf{A}).$$

For example finite groups have this property, and the above theorem gives an alternate proof that the holonomy of a compact affine manifold must be infinite. Another class of groups having this property are the *Margulis-superrigid groups,* that is, irreducible lattices Γ in semisimple Lie groups G of $\mathbb{R}$-rank greater than one (for example, $\mathsf{SL}(n,\mathbb{Z})$ for $n > 2$). Margulis proved [**252**] that every unbounded finite-dimensional linear representation of Γ extends to a representation of G. It then follows that the affine holonomy of a compact affine manifold cannot factor through a Margulis-superrigid group. However, since $\mathsf{SL}(n;\mathbb{R})$ does admit a left-invariant $\mathbb{R}\mathsf{P}^{n^2-1}$-structure, it follows that if $\Gamma \subset \mathsf{SL}(n;\mathbb{R})$ is a torsion-free cocompact lattice, then there exists a compact affine manifold with holonomy group $\Gamma \times \mathbb{Z}$ although Γ itself is not the holonomy group of an affine structure.

11.4. Fried's classification of closed similarity manifolds

Fried [**135**] gives a sharp classification of closed similarity manifolds; this was announced earlier by Kuiper [**231**], although the proof contains a gap. Using a completely different set of ideas. Reischer and Vaisman [**331**] independently proved this as well. Miner [**274**] extended this theorem to manifolds modeled on the Heisenberg group and its group of similarity transformations. Recently this has been extended to the boundary geometry of any rank-one symmetric space by Raphaël Alexandre [**7**]. In §15.6, we relate the ideas in Fried's proof to Thurston's parametrization of $\mathbb{C}\mathsf{P}^1$-structures and the Kulkarni–Pinkall theory of flat conformal structures [**237, 238**].

11.4.1. Completeness versus radiance. This theorem is a prototype of a theorem about geometric structures on *closed manifolds.* Here $X = \mathsf{E}^n$ and $G = \mathsf{Sim}(\mathsf{E}^n)$. A (G,X)-structure on a closed manifold M must reduce to one of two special types, corresponding to *subgeometries* $(G',X') \rightsquigarrow (G,X)$. Specifically, a closed similarity manifold must be one of the following two types:

- A *Euclidean manifold* where $X' = X = \mathsf{E}^n$ and

$$G' = \mathsf{Isom}(\mathsf{E}^n) \hookrightarrow \mathsf{Sim}(\mathsf{E}^n).$$

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This is precisely the case when the underlying affine structure on M is complete;

- A finite quotient of a *Hopf manifold* where $X' = \mathsf{E}^n \backslash \{\mathbf{0}\}$ and $G' = \mathsf{Sim}_0(\mathsf{E}^n) \hookrightarrow \mathsf{Sim}(\mathsf{E}^n)$, the group of linear similarity (or conformal) transformations of E^n. The is precisely the case when the underlying affine structures is incomplete.

The complete case is easy to handle, since in that case Γ acts freely, and any similarity transformation which is not isometric must fix a point. When M is incomplete, very little can be said in general, and the compactness hypothesis must be crucially used.

Exercise 11.4.1. Prove that a complete similarity manifold is a Euclidean manifold, and diffeomorphic to a finite quotient of a product $\mathbb{T}^r \times \mathsf{E}^{n-r}$.

The other extreme — the case of radiant similarity manifolds — was discussed in Exercise 6.3.12.

The recurrence of an incomplete geodesic on a compact manifold guarantees a divergent sequence in the affine holonomy group Γ.

This *holonomy sequence* converges to a singular projective transformation ϕ as in §2.6. The condition that $\Gamma \subset \mathsf{Sim}(\mathsf{E}^n)$ strongly restricts ϕ; in particular it has *rank one* or its limits are *proximal,* in that most points approach a single point, which must lie in E^n. From this follows radiance, proving the structure is modeled on $\mathbb{R}^n \backslash \{\mathbf{0}\}$ and $\mathsf{Sim}_0(\mathsf{E}^n) \cong \mathbb{R}^+ \times \mathsf{O}(n)$. In particular just from compactness and incompleteness both the radiant vector field and the canonical Riemannian metric are constructed.

11.4.2. Canonical metrics and incompleteness. Choose a Euclidean metric $\mathsf{g_E}$ on E^n; the pullback $\mathsf{dev}^*\mathsf{g_E}$ is a Euclidean metric on $\widetilde{M}$. Unless M is a Euclidean manifold, this metric is *not* invariant under π. Rather, it transforms by the *scale factor homomorphism:* $\pi \xrightarrow{\lambda \circ \mathsf{h}} \mathbb{R}^+$:

$$\gamma^*\big(\mathsf{dev}^*\mathsf{g_E}\big) \;=\; \big(\lambda \circ \mathsf{h}\big)(\gamma) \cdot \mathsf{dev}^*\mathsf{g_E}. \tag{11.1}$$

defined in §1.4.1.

Exercise 11.4.2. Relate the scale factor $\lambda \circ \mathsf{h}$ to the volume obstruction ν_M.

Unless M is Euclidean, $\mathsf{dev}^*\mathsf{g_E}$ is incomplete. Thus we assume that $(\widetilde{M}, \mathsf{dev}^*\mathsf{g_E})$ is an *incomplete Euclidean manifold* with distance function

$$\widetilde{M} \times \widetilde{M} \xrightarrow{\widetilde{\mathsf{d}}_{\mathsf{E}}} \mathbb{R},$$

non-bijective developing map $\widetilde{M} \xrightarrow{\mathsf{dev}} \mathsf{E}$ and *nontrivial* scale factor homomorphism $\pi_1(M) \xrightarrow{\lambda \circ \mathsf{h}} \mathbb{R}^+$.

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We begin with some general facts about an incomplete Euclidean manifold N with trivial holonomy. We apply these facts to the case when N is the universal covering $\widetilde{M}$ of a compact incomplete similarity manifold M.

Exercise 11.4.3. Let N be a Euclidean manifold with trivial holonomy. Choose a developing map $N \xrightarrow{\mathsf{dev}} \mathsf{E}$. Let $B \subset N$ be an open subset. The following conditions are equivalent:

- B is an open ball in N, that is, $\exists c \in N, r > 0$ such that $B = \mathsf{B}_r(c)$.
- B develops to an open ball in E, that is, $\exists c \in \mathsf{E}, r > 0$ such that the restriction $\mathsf{dev}|_B$ is a diffeomorphism $B \longrightarrow \mathsf{B}_r(c) \subset \mathsf{E}$;
- B is the exponential image of a metric ball in the tangent space $\mathsf{T}_c N$, that is, $\exists c \in N, r > 0$ such that the restriction $\mathsf{Exp}|_{\mathsf{B}_r(\mathbf{0}_c)}$ is a diffeomorphism $\mathsf{B}_r(\mathbf{0}_c) \longrightarrow B$.

Under these conditions, the maps

$$\mathsf{B}_r(\mathbf{0}_c) \xrightarrow{\mathsf{Exp}_c} B \xrightarrow{\mathsf{dev}} \mathsf{B}_r\big(\mathsf{dev}(c)\big)$$

are isometries with respect to the restrictions of the Euclidean metrics on $\mathsf{T}_c N$, N and E, respectively.

Definition 11.4.4. A *maximal ball* in N is an open ball which is maximal among open balls with respect to inclusion.

Exercise 11.4.5. A Euclidean manifold with trivial holonomy is complete (that is, is isomorphic to Euclidean space) if and only if no ball is maximal.

Exercise 11.4.6. Suppose N is an incomplete Euclidean manifold with trivial holonomy.

- Let $B \subset N$ be a maximal ball and let c be its center. Then B is maximal among open balls centered at c.
- Every open ball lies in a maximal ball.
- Every $x \in N$ is the center of a unique maximal ball $\mathcal{B}(x)$.
- For each x, not every point on $\partial\mathcal{B}(x)$ is visible from x.

Definition 11.4.7. Let N be an incomplete Euclidean manifold with trivial holonomy. For each $x \in N$, let $R(x) < \infty$ be the radius of the maximal ball $\mathcal{B}(x) \subset N$ centered at x, so that

$$\mathcal{B}(x) = \mathsf{B}_{R(x)}(x).$$

Exercise 11.4.8. $R(x)$ is the supremum of r such that $\mathsf{B}_r(\mathbf{0}_x) \subset \mathcal{E}_x$, where $\mathcal{E}_x \subset \mathsf{T}_x N$ denotes the domain of Exp_x defined in §8.3 of Chapter 8.

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Lemma 11.4.9. The function R is Lipschitz:

$$|R(x) - R(y)| \leq \widetilde{\mathsf{d}}(x, y) \tag{11.2}$$

if $x, y \in N$ are sufficiently close. In particular R is continuous.

Proof. Suppose that $x \in N$ and ϵ such that

$$\epsilon < \sup\big(R(x), R(y)\big).$$

Choose $r < R(x)$ so that $\mathsf{B}_r(\mathbf{0}_x) \subset \mathcal{E}_x$. First we show that if $\widetilde{\mathsf{d}}(x, y) < \epsilon$, then

$$r < \widetilde{\mathsf{d}}(x, y) + R(y) \tag{11.3}$$

Choose $u \in \mathcal{B}(x)$ such that $\widetilde{\mathsf{d}}(x, u) = r$. Suppose that $\widetilde{\mathsf{d}}(x, y) < \epsilon$. Then closed ball $\overline{\mathsf{B}_r(x)}$ lies in the convex set $\mathcal{B}(x)$ which also contains y. Thus $u \in \partial\mathsf{B}_r(x)$ is visible from y, whence

$$\widetilde{\mathsf{d}}(y, u) < R(y).$$

Thus

$$r = \widetilde{\mathsf{d}}(x, u) \leq \widetilde{\mathsf{d}}(x, y) + \widetilde{\mathsf{d}}(y, u) < \widetilde{\mathsf{d}}(x, y) + R(y),$$

proving (11.3). Taking the supremum over r yields:

$$R(x) < \widetilde{\mathsf{d}}(x, y) + R(y),$$

so $R(x) - R(y) < \widetilde{\mathsf{d}}(x, y)$ if $\widetilde{\mathsf{d}}(x, y) < \epsilon$. Similarly, symmetry of $\widetilde{\mathsf{d}}$ implies that $R(x) - R(y) < \widetilde{\mathsf{d}}(x, y)$ if $\widetilde{\mathsf{d}}(x, y) < \epsilon$ which implies (11.2). □

We return to the case that M is a compact incomplete similarity manifold. Choose a universal covering $N \xrightarrow{\Pi} M$, a developing map $N \xrightarrow{\mathsf{dev}} \mathsf{E}$, and a holonomy representation $\pi_1(M) \xrightarrow{\mathsf{h}} \mathsf{Sim}(\mathsf{E}^n)$. If $\phi \in \pi_1(M)$ is a deck transformation (also denoted $N \xrightarrow{\phi} N$), then $\phi\big(B_{R(\widetilde{p})}\big)$ is a maximal ball at $\phi(\widetilde{p})$, so:

$$R\big(\phi\widetilde{p}\big) = \lambda \circ \mathsf{h}(\phi) R(\widetilde{p}) \tag{11.4}$$

This leads to a natural conformal Riemannian structure on N which descends to a conformal Riemannian structure on M.

This will be the canonical Riemannian structure on a radiant similarity manifold. If M is closed and incomplete, then M is finitely covered by a Hopf manifold M' homeomorphic to $S^{n-1} \times S^1$. The induced Riemannian structure on M' is the Cartesian product of a spherical metric on S^{n-1} with a Euclidean metric on S^1. By (11.4), the Riemannian metric $\widetilde{\mathsf{g}}$ on $\widetilde{M}$ defined by:

$$\widetilde{\mathsf{g}}(\widetilde{p}) := R(\widetilde{p})^{-1} \mathsf{dev}^* \mathsf{g}_{\mathsf{E}} \tag{11.5}$$

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is $\pi_1(M)$-invariant. Therefore $\widetilde{\mathsf{g}}$ passes down to a Riemannian metric g_M on M, that is, $\Pi^*\mathsf{g}_M = \widetilde{\mathsf{g}}$.

The Riemannian structure g_M has the property that its unit ball is maximal inside the domain $\mathcal{E}$ of the exponential map Exp. When M is closed, even more is true:

Proposition 11.4.10. Let M be a compact incomplete similarity manifold with universal covering $N \xrightarrow{\Pi} M$. Then $\exists \xi \in \mathsf{Vec}(M)$ which is Π-related to a vector field $\widetilde{\xi} \in \mathsf{Vec}(N)$ such that:

- $\|\xi\|_{\mathsf{g}_M} = \|\widetilde{\xi}\|_{\widetilde{\mathsf{g}}} = 1$;
- The halfspace
$$\mathcal{H}_x := \{\mathbf{v} \in \mathsf{T}_x N \mid \widetilde{\mathsf{g}}(\mathbf{v}, \xi) < 1\}$$
lies in $\mathcal{E}_x$ for all $x \in N$.

In particular
$$H_x := \mathsf{Exp}_x(\mathcal{H}_x) \subset N$$
is a natural halfspace neighborhood of x.

By analyzing H_x, we shall prove that ξ is a radiant vector field and M is (covered by) a Hopf manifold. A key ingredient is a *holonomy sequence* $\mathsf{h}(\phi_{ij}) \in \mathsf{Sim}(\mathsf{E}^n)$, where $\phi_{ij} \in \pi_1(M)$, which contracts to $\mathbf{0}$ as $j \nearrow \infty$ (Proposition 11.4.13). The proof of Proposition 11.4.10 will be given in §11.4.4, following several preliminary lemmas needed in the proof.

11.4.3. Incomplete geodesics recur. Fried makes a detailed analysis of an incomplete geodesic $[0,1) \xrightarrow{\gamma} M$. That is, $\gamma(t) = \mathsf{Exp}(t\mathbf{v})$, where $\mathbf{v} \in \mathsf{T}_x M$ but $t\mathbf{v} \in \mathcal{E}_x \iff t < 1$.

Since M is compact (and $[0,1)$ isn't compact), the path $\gamma(t)$, *accumulates*. That is, for some sequence $t_n \nearrow 1$, the sequence $\gamma(t_n) \in M$ converges in M as $n \nearrow +\infty$. Denote

$$p := \lim_{n \to +\infty} \gamma(t_n) \tag{11.6}$$

Next, we use the recurrence of $\gamma(t)$ to obtain a *holonomy sequence* converging to a singular projective transformation as in §2.6. To that end, we pass to a specific universal covering space and a developing map. Employ $p \in M$ as a basepoint to define a universal covering space $\widetilde{M} \xrightarrow{\Pi} M$. The total space $\widetilde{M}$ comprises relative homotopy classes of paths $[0,T] \xrightarrow{\gamma} M$ with $\gamma(0) = p$ and the projection is:

$$\begin{aligned} \widetilde{M} &\xrightarrow{\Pi} M \\ [\gamma] &\longrightarrow \gamma(T), \end{aligned}$$

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The constant path defines a basepoint $\widetilde{p} \in \widetilde{M}$ with $\Pi(\widetilde{p}) = p$. The group of deck transformations is $\pi_1(M, p)$ consisting of relative homotopy classes of loops in M based at p.

Lift the incomplete geodesic $[0,1) \xrightarrow{\gamma} M$ to an incomplete geodesic $[0,1) \xrightarrow{\widetilde{\gamma}} N$ so that

$$\lim_{n\to+\infty} \widetilde{\gamma}(t_n) = \widetilde{p}$$

and let $\widetilde{x} := \widetilde{\gamma}(0)$ be the initial endpoint of $\widetilde{\gamma}$ and $\widetilde{\mathbf{v}} := \widetilde{\gamma}'(0) \in \mathsf{T}_{\widetilde{x}}N$ the initial velocity.

Choose a developing map $\widetilde{M} \xrightarrow{\mathsf{dev}} \mathsf{E}^n$. A Euclidean metric tensor g_E on E induces a Euclidean metric tensor $\mathsf{dev}^*\mathsf{g}$ on N. By rescaling, we may assume that $\mathsf{dev}^*\mathsf{g}(\widetilde{\mathbf{v}}, \widetilde{\mathbf{v}}) = 1$. Denote the corresponding distance function by $N \times N \xrightarrow{\mathsf{d}} \mathbb{R}$. Choose a coordinate patch $U \ni p$ such that the restriction $\mathsf{dev}|_{\widetilde{U}}$ is injective, where $\widetilde{U} \subset \widetilde{M}$ is the component of $\Pi^{-1}(U)$ containing $\widetilde{p}$. Choose $\epsilon > 0$ such that:

- $\mathcal{B}_\epsilon := \mathsf{B}_\epsilon(\mathbf{0}_{\widetilde{p}}) \subset \mathcal{E}_{\widetilde{p}}$;
- The ball $B := \mathsf{Exp}_{\widetilde{p}}(\mathcal{B}_\epsilon)$ lies in $\widetilde{U}$;
- $\epsilon < \frac{1}{2}$.

In particular, the restriction $\mathsf{dev}|_B$ is injective.

Lemma 11.4.11. Let $N \xrightarrow{R} \mathbb{R}^+$ be the radius function (defined in 11.4.7). Then

$$R\big(\widetilde{\gamma}(t)\big) = 1 - t.$$

Exercise 11.4.12. Prove Lemma 11.4.11.

After possibly passing to a subsequence, (11.6) implies that $\gamma(t_i) \in B$. Let

$$s_i := d\big(\gamma(t_i), p\big)$$

and $[0, s_i] \xrightarrow{\eta_i} M$ the unit-speed geodesic in B with $\eta_i(0) = p$ and $\eta_i(s_i) = \gamma(t_i)$.

Lift η_i to

$$[0, s_i] \xrightarrow{\widetilde{\eta_i}} \widetilde{M}$$

with $\widetilde{\eta_i}(s_i) = \widetilde{\gamma}(t_i)$. Let $\widetilde{p_i} := \widetilde{\eta_i}(0)$. For $i < j$ define

$$\phi_{ij} := \big[\eta_j^{-1} \star \gamma|_{[t_i,t_j]} \star \eta_i\big] \in \pi_1(M, p).$$

A crucial fact is that the scale factors decrease to 0 along the incomplete geodesic $\gamma(t)$. In particular the limit of the holonomy sequence is a singular projective transformation whose image is a single point.

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Proposition 11.4.13. Fix $i \in \mathbb{N}$ and $\delta > 0$. Then $\exists J(i)$ such that the scale factor $\lambda \circ \mathsf{h}(\phi_{ij}) < \delta$ for $j \geq J(i)$.

Proof. Denote the $\tilde{\mathsf{g}}$-length of $\widetilde{\eta_i}$ by

$$l_i := \widetilde{\mathsf{d}}\big(\widetilde{p_i}, \mathsf{dev}\,\widetilde{\gamma}(t_i)\big).$$

First, we claim that:

$$\epsilon > \frac{l_i}{l_i + R\big(\widetilde{\gamma}(t_i)\big)}. \tag{11.7}$$

If $0 \leq s \leq s_i$, then $\widetilde{\mathsf{d}}\big(\widetilde{\gamma}(t_i), \widetilde{\eta_i}(s)\big) \leq l_i$. Lemma 11.4.9 implies:

$$R\big(\widetilde{\eta_i}(s)\big) \leq R\big(\widetilde{\gamma}(t_i)\big) + l_i$$

so the $\tilde{\mathsf{g}}$-length of $\widetilde{\eta_i}$ equals:

$$\begin{aligned}\widetilde{\mathsf{d}}\big(\widetilde{p_i}, \widetilde{\gamma}(t_i)\big) &= \int_0^{s_i} \frac{(\widetilde{\eta_i})^* ds}{R\big(\tilde{\eta}_i(s)\big)} \\ &\geq \frac{l_i}{l_i + R\big(\widetilde{\gamma}(t_i)\big)}.\end{aligned}$$

Thus,

$$\frac{l_i}{l_i + R\big(\widetilde{\gamma}(t_i)\big)} \leq \widetilde{\mathsf{d}}\big(\widetilde{p_i}, \widetilde{\gamma}(t_i)\big) = \mathsf{d}\big(x, \gamma_i(t_i)\big) < \epsilon$$

as desired, proving (11.7). Next we prove:

$$l_i < 2\epsilon(1 - t_i). \tag{11.8}$$

In general, $\epsilon > l/(l + R)$ implies that $l < R\epsilon/(1 - \epsilon)$. Furthermore $\epsilon < 1/2$ implies that $\epsilon/(1 - \epsilon)R < 2\epsilon R$ so

$$l_i < 2\epsilon R\big(\widetilde{\gamma}(t_i)\big) = 2\epsilon R$$

by Lemma 11.4.11, thereby establishing (11.8).

Lemma 11.4.14.

$$1 - 2\epsilon < \frac{R(\widetilde{p_i})}{1 - t_i} < 1 + 2\epsilon.$$

Proof. Lemma 11.4.9 implies

$$|R(\widetilde{p_i}) - R\big(\widetilde{\gamma}(t_i)\big)| \leq \mathsf{d}\big(\mathsf{dev}(\widetilde{p_i}), \mathsf{dev}\big(\widetilde{\gamma}(t_i)\big) = l_i.$$

Lemma 11.4.11 and (11.8) together imply

$$|R(\widetilde{p_i}) - (1 - t_i)| < 2\epsilon(1 - t_i).$$

Now divide by $1 - t_i$. □

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Lemma 11.4.15. For $i < j$,

$$\frac{R(\widetilde{p_j})}{R(\widetilde{p_i})} < \frac{1+2\epsilon}{1-2\epsilon}\,\frac{1-t_j}{1-t_i}$$

Proof. Apply Lemma 11.4.14 to obtain:

$$R(\widetilde{p_j}) < (1+2\epsilon)(1-t_j) \tag{11.9}$$

and

$$(1-2\epsilon)(1-t_i) < R(\widetilde{p_i})$$

that is,

$$\frac{1}{R(\widetilde{p_i})} < \frac{1}{(1-2\epsilon)(1-t_i)}. \tag{11.10}$$

Multiplying (11.9) and (11.10) implies Lemma 11.4.15. ☐

Since $t_j \nearrow 1$, for any fixed i, there exists $J(i)$ such that $j > J(i)$ implies

$$\frac{1+2\epsilon}{1-2\epsilon}\,\frac{1-t_j}{1-t_i} \;<\; \delta.$$

Since $\widetilde{p_j} = \phi_{ij}\,\widetilde{p_i}$, (11.4) and (11.1) imply:

$$\frac{R(\widetilde{p_j})}{R(\widetilde{p_i})} \;=\; \big(\lambda\circ h\big)(\phi_{ij}).$$

Now apply Lemma 11.4.15, completing the proof of Proposition 11.4.13. ☐

Recall from Exercise 2.6.8 that a sequence of similarity transformations accumulates to either:

- The *zero* affine transformation (undefined at the ideal hyperplane, otherwise constant;
- A singular projective transformation of rank one, taking values at an ideal point.

The contraction of scale factors (Proposition 11.4.13) implies that the second case cannot occur.

11.4.4. Existence of halfspace neighborhood. The contraction of scale factors (Proposition 11.4.13) easily implies the existence of halfspace neighborhoods (Proposition 11.4.10).

The proof uses the following elementary fact in Euclidean geometry:

Exercise 11.4.16. Let $y \in \mathsf{E}^n$ be a point and $\mathbf{v} \in \mathsf{T}_y\mathsf{E}^n \longleftrightarrow \mathbb{R}^n$ be a tangent vector. Choosing coordinates, we may assume that $\mathsf{Exp}_y(\mathbf{v}) = \mathbf{0}$ is

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the origin. Let $\mathcal{S}_t$ be a one-parameter family of homotheties approaching zero:

$$\begin{aligned} \mathsf{E}^n &\xrightarrow{\mathcal{S}_t} \mathsf{E}^n \\ p &\longmapsto e^{-t}p \end{aligned}$$

Let B be the ball centered at y of radius $R = \|\mathbf{v}\|$:

$$B := \{x \in \mathsf{E}^n \mid \mathsf{d}(y,x) < R\}$$

If $t_n \nearrow \infty$, then the union of $\mathcal{S}_{t_n}(B)$ is the halfspace H containing y and orthogonal to the line segment $\overleftrightarrow{y\mathbf{0}}$:

$$H(y,\mathbf{0}) := \{y + \mathbf{w} \mid \mathbf{w}\cdot\mathbf{v} < R\}.$$

Let $\mathcal{A}_p$ denote the extension of $\mathsf{dev}\circ\mathsf{Exp}_p$ to $\mathsf{T}_pN \longrightarrow \mathsf{A}$ described in Proposition 8.3.6.

Conclusion of proof of Proposition 11.4.10. Let $x \in M$ and lift to $\widetilde{x} \in N$. Since M is incomplete, a maximal ball $\mathcal{B}(\widetilde{x}) \subset \mathcal{E}_{\widetilde{x}}$ exists, and denote its radius by $R := R(\widetilde{x})$. Furthermore $\partial\mathcal{B}(\widetilde{x})$ contains a vector $\widetilde{\mathbf{v}}$ of length R such that $\widetilde{\mathbf{v}} \notin \mathcal{E}_{\widetilde{x}}$ but $t\widetilde{\mathbf{v}} \in \mathcal{E}_{\widetilde{x}}$ for $|t| < 1$. (Soon we shall see that $\widetilde{\mathbf{v}}$ is unique and equals $\xi(\widetilde{x})$.)

Apply this construction to go from the incomplete geodesic

$$\gamma(t) := \mathsf{Exp}_x(t\mathbf{v}),\ (0 \le t < R),$$

in M to a holonomy sequence $\gamma_{ij} \in \pi_1(M,p)$ satisfying Proposition 11.4.13.

Define the halfspace

$$\begin{aligned} \mathcal{H}_{\widetilde{x}} &:= \{Y \in \mathsf{T}_{\widetilde{x}}N \mid \mathsf{dev}^*\mathsf{g}_\mathsf{E}(\widetilde{\mathbf{v}},Y) < R\} \\ &= \{Y \in \mathsf{T}_{\widetilde{x}}N \mid \mathsf{g}_N(\widetilde{\mathbf{v}},Y) < 1\}. \end{aligned}$$

We show $\mathcal{H}_{\widetilde{x}} \subset \mathcal{E}_{\widetilde{x}}$. Suppose that Y is a tangent vector at $\widetilde{x}$ such that

$$\mathsf{dev}^*\mathsf{g}_\mathsf{E}(\widetilde{\mathbf{v}},Y) < R.$$

By Proposition 11.4.13, for $j \gg 1$ and fixed $i < j$, the holonomy $\mathsf{h}(\gamma_{ij})$ is a very sharp contraction with rotational component close to the identity — that is, it's very close to a strong homothety about $\mathbf{0}$. By Exercise 11.4.16 above, $(\mathsf{D}\gamma_{ij})_{\widetilde{x}}$ maps Y to a vector in $\mathcal{B}_{\gamma_{ij}(\widetilde{x})} \subset \mathcal{E}_{\gamma_{ij}(\widetilde{x})}$ for $j \gg i$. Thus $(\mathsf{D}\gamma_{ij})_{\widetilde{x}}(Y)$ is visible from $\gamma_{ij}(\widetilde{x})$. It follows Y is visible from $\widetilde{x}$, as claimed.

In particular every point in $\partial\mathcal{B}(\widetilde{x}) \setminus \{\widetilde{\mathbf{v}}\}$ lies in $\mathcal{H}(\widetilde{x})$, and therefore $\widetilde{\mathbf{v}}$ uniquely determines X, so we write $\widetilde{\mathbf{v}} =: \xi(\widetilde{x})$.

□

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11.4.5. Visible points on the boundary of the halfspace. Bounding the halfspace $\mathcal{H}_{\widetilde{x}}$ is the affine hyperplane

$$\partial\mathcal{H}_{\widetilde{x}} := \{Y \in \mathsf{T}_{\widetilde{x}} \mid \tilde{\mathsf{g}}(\widetilde{\mathbf{v}}, Y) = 1\},$$

which decomposes as the disjoint union of two subsets:

- The *visible set* $\partial\mathcal{H}_{\widetilde{x}} \cap \mathcal{E}_{\widetilde{x}}$;
- The *invisible set* $\partial\mathcal{H}_{\widetilde{x}} \setminus \mathcal{E}_{\widetilde{x}}$.

The invisible set is nonempty, since it contains $\xi(\widetilde{x})$.

Lemma 11.4.17. The *visible set* $\partial\mathcal{H}_{\widetilde{x}} \cap \mathcal{E}_{\widetilde{x}}$ is nonempty.

Proof. Suppose every point of $\partial\mathcal{H}_{\widetilde{x}}$ is invisible. Then $\mathcal{H}_{\widetilde{x}}$ is a closed subset of $\mathcal{E}_{\widetilde{x}}$. By construction, $\mathcal{H}_{\widetilde{x}}$ is open and connected, so $\mathcal{H}_{\widetilde{x}} = \mathcal{E}_{\widetilde{x}}$.

Thus the corresponding subset $H_{\widetilde{x}} = \mathsf{Exp}(\mathcal{H}\langle_t x)$ of N equals all of N, so M is a quotient of a halfspace. The contradiction now follows from the following exercise. ☐

Exercise 11.4.18. Let $H \subset \mathsf{E}^n$ be an open halfspace, and let $\Gamma < \mathsf{Sim}(\mathsf{E}^n)$ be a discrete subgroup stabilizing H and acting properly on H. Then the quotient $\Gamma\backslash H$ is not compact.

11.4.6. The invisible set is affine and locally constant. This uses the variation of $H_{\widetilde{x}}$ as $\widetilde{x}$ varies. The key is Lemma 1 of Fried [**135**], which says that if $y = \mathsf{Exp}_{\widetilde{x}}(Y)$, where $Y \in \partial\mathcal{H}_{\widetilde{x}} \cap \mathcal{E}_{\widetilde{x}}$ is a visible vector, then $\xi(\widetilde{x})$ "is also invisible" from y. That is, the vector in $\mathsf{T}_y N$ whose parallel transport to $\widetilde{x}$ equals $X \in \mathsf{T}_{\widetilde{x}}(N)$ is invisible from y.

Now apply the strong contraction γ_{ij} to show that the halfspace $H(y)$ is moved closer and closer to $\mathbf{0}$. If it contains $\mathbf{0}$, then $X = \xi(\widetilde{x})$ is visible, a contradiction. If $\mathbf{0} \notin \overline{H(y)}$, then for $j \gg i$, the maximal ball $\mathcal{B}(y)$ is contained in a visible halfspace from $\widetilde{x}$, contradicting maximality. Thus $\xi(\widetilde{x})$ is also invisible from y.

This shows that the map $\widetilde{x} \longmapsto \xi(\widetilde{x})$ is locally constant, as claimed.

Exercise 11.4.19. Prove this map is affine and deduce that the invisible subspace $\partial\mathcal{H}_{\widetilde{x}} \setminus \mathcal{E}_{\widetilde{x}}$ is an affine subspace.

Now we return to the case that M is a closed affine manifold with holonomy covering space N.

Fried concludes the proof by observing now that the vector field corresponds to an affine projection from the ambient affine space to the invisible subspace $\partial\mathcal{H}_{\widetilde{x}} \setminus \mathcal{E}_{\widetilde{x}}$. Since it is invariant under deck transformations it descends to a vector field on the compact quotient M.

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Recall the *divergence* of a vector field (see §1.7.2) which measures the infinitesimal distortion of volume:

Exercise 11.4.20. The divergence of the vector field ξ equals $\mathsf{dim}\partial\mathcal{H}_{\widetilde{x}} \setminus \mathcal{E}_{\widetilde{x}}$. Since M is closed, the vector field has divergence zero.

Thus $\partial\mathcal{H}_{\widetilde{x}} \cap \mathcal{E}_{\widetilde{x}}$ is a single point, and M is radiant, as desired.

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Chapter 12

Hyperbolicity

Opposite to geodesic completeness is *hyperbolicity* in the sense of Vey [**336–338**] and Kobayashi [**220–222**]. This condition is equivalent to the following notion: An affine manifold M is *completely incomplete* if and only if every affine map $\mathbb{R} \longrightarrow M$ is constant, that is, M admits *no* complete geodesic. As noted by the author (see Kobayashi [**222**]), the combined results of Kobayashi, Vey and Wu [**354**] imply:

Theorem. Let M be a closed affine manifold. Suppose that M is completely incomplete. Then M is a quotient of a properly convex cone.

Kobayashi defines a pseudometric $\mathsf{d}^{\mathsf{Kob}}$ and calls an affine (respectively projective) manifold *hyperbolic* if and only if $\mathsf{d}^{\mathsf{Kob}}$ defines a metric, that is, if $\mathsf{d}^{\mathsf{Kob}}(x, y) > 0$ for $x \neq y$. Thus an affine manifold M is hyperbolic if and only if if it is a quotient of a properly convex cone; a compact projective manifold is hyperbolic if and only if it is a quotient of a properly convex domain in projective space. The tameness of developing maps of hyperbolic affine and projective structures suggests, when the pseudometric $\mathsf{d}^{\mathsf{Kob}}$ fails to be a metric, that $\mathsf{d}^{\mathsf{Kob}}$ and its infinitesimal form may provide useful tools to understand pathological developing maps.

For the two extreme cases of geodesic behavior for closed affine manifolds, the developing map is an embedding. That is, the developing map for a *complete* affine manifold is a diffeomorphism, whereas the developing map for a *completely incomplete* affine manifold embeds the universal covering as a sharp convex cone.

In particular a completely incomplete affine manifold M is radiant. Because the Koszul 1-form is closed and everywhere nonzero, Furthermore M fibers over $\mathbb{S}^1$ as a radiant suspension of an automorphism of a projective

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manifold of codimension one. Topological consequences are that the Euler characteristic vanishes and the first Betti number is positive.

Along the way we will also show that complete incompleteness is equivalent to the nonexistence of nonconstant *projective* maps $\mathbb{R} \longrightarrow M$. Their constructions were in turn inspired by the intrinsic metrics of Carathéodory and Kobayashi for holomorphic mappings between complex manifolds.

We begin by discussing Kobayashi's pseudometric for domains in projective space, and then extend this construction to projective manifolds.

12.1. The Kobayashi metric

To motivate Kobayashi's construction, consider the basic case of intervals in P^1. (Compare the discussion in Exercise 2.5.8 on the cross-ratio.)

There are several natural choices to take, for example, the interval of positive real numbers $\mathbb{R}^+ = (0, \infty)$ or the open unit interval $\mathbf{I} = (-1, 1)$. They relate via the projective transformation $\mathbf{I} \xrightarrow{\tau} \mathbb{R}^+$

$$x = \tau(u) = \frac{1+u}{1-u}$$

mapping $-1 < u < 1$ to $0 < x < \infty$ with $\tau(0) = 1$. The corresponding Hilbert metrics are given by

$$d_{\mathbb{R}^+}(x_1, x_2) = \log\left|\frac{x_1}{x_2}\right|$$

$$d_{\mathbf{I}}(u_1, u_2) = 2\left|\tanh^{-1}(u_1) - \tanh^{-1}(u_2)\right|. \tag{12.1}$$

This follows from the fact that τ pulls back the parametrization corresponding to Haar measure on $\mathbb{R}^+$:

$$\frac{|dx|}{x} = |d \log x|$$

to the *Poincaré metric* on $\mathbf{I}$:

$$ds_{\mathbf{I}} = \frac{2\,|du|}{1-u^2} = 2\,|d \tanh^{-1} u|.$$

A slight generalization of this will be useful in §12.2 in the proof of the projective Brody Lemma 12.2.19:

Exercise 12.1.1. Let $r > 0$ and denote by $\mathbf{I}(r)$ the open interval $(-r, r) \subset \mathbb{R}$. Show that the diffeomorphism

$$\mathbf{I} \longrightarrow \mathbf{I}(r)$$
$$u \longmapsto v = ru$$

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maps $0 \mapsto 0$, takes $\frac{d}{du}$ to $r\,\frac{d}{dv}$, and, dually,

$$ds_{\mathbf{I}} \;=\; \frac{2\,|du|}{1-u^2} \longleftarrow \frac{2r\,|dv|}{r^2-v^2}.$$

Show that the infinitesimal form of the Hilbert metric on $\mathbf{I}(r)$ is:

$$ds_{\mathbf{I}(r)} \;=\; \frac{2\,r\;|dv|}{r^2-v^2} \;=\; 2\,d\tanh^{-1}(v/r)$$

for $v \in \mathbf{I}(r)$ and $v = r\tanh(s/2)$ where s is the arc length parameter.

Exercise 12.1.2. Let $x_-, x_+ \in \mathbb{R}\backslash\{0\}$ be distinct. Show that the projective map mapping

$$\begin{aligned} -1 &\longmapsto x_- \\ 0 &\longmapsto 0 \\ 1 &\longmapsto x_+ \end{aligned}$$

is given by:

$$t \longmapsto \frac{2(x_-x_+)\;t}{(t+1)x_- \;+\; (t-1)x_+},$$

and a projective automorphism of $\mathbf{I}$ by:

$$t \longmapsto \frac{\cosh(s)t+\sinh(s)}{\sinh(s)t+\cosh(s)} = \frac{t+\tanh(s)}{1+\tanh(s)t}$$

for $s \in \mathbb{R}$.

Let $\Omega \subset \mathbb{R}\mathsf{P}^n$ be convex domain bounded by a quadric. In terms of the Poincaré metric on $\mathbf{I}$ the Hilbert distance $\mathsf{d}(x,y)$ can be characterized as an infimum over all projective maps $\mathbf{I} \longrightarrow \Omega$:

$$\mathsf{d}(x,y) = \inf\left\{ \mathsf{d}_{\mathbf{I}}(a,b) \;\middle|\; f \in \mathsf{Proj}(\mathbf{I},\Omega),\; a,b \in \mathbf{I},\; f(a)=x,\; f(b)=y \right\}.$$

We now define the Kobayashi pseudometric for any domain Ω and, more generally, any manifold with a projective structure. This proceeds by a general universal construction forcing two properties:

- The *triangle inequality*: $\mathsf{d}(a,c) \le \mathsf{d}(a,b) + \mathsf{d}(b,c)$;
- The *projective Schwarz lemma:* Projective maps do not increase distance.

However, the resulting pseudometric may not be positive; indeed for many domains it is identically zero.

Let $\Omega \subset \mathsf{P}$ be a domain and $x, y \in \Omega$. A (projective) *chain* from x to y is a sequence C of projective maps $f_1, \dots, f_m \in \mathsf{Proj}(\mathbf{I},\Omega)$ and pairs $a_i, b_i \in \mathbf{I}$,

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for $i = 1, \dots, m$ such that:

$$f_1(a_1) = x,\ f_1(b_1) = f_2(a_2),\ \dots,$$
$$f_{m-1}(b_{m-1}) = f_m(a_m),\ f_m(b_m) = y.$$

Denote the set of all projective chains from x to y by $\mathsf{Chain}(x \rightsquigarrow y)$. Define *length* of a projective chain by:

$$\ell(C) = \sum_{i=1}^{m} \mathsf{d}_{\mathbf{I}}(a_i, b_i).$$

and the *Kobayashi pseudodistance* $\mathsf{d}^{\mathsf{Kob}}(x, y)$:

$$\mathsf{d}^{\mathsf{Kob}}(x, y) = \inf\left\{ \ell(C) \;\middle|\; C \in \mathsf{Chain}(x \rightsquigarrow y) \right\}.$$

The resulting function enjoys the following obvious properties:

- $\mathsf{d}^{\mathsf{Kob}}(x, y) \geq 0$;
- $\mathsf{d}^{\mathsf{Kob}}(x, x) = 0$;
- $\mathsf{d}^{\mathsf{Kob}}(x, y) = \mathsf{d}^{\mathsf{Kob}}(y, x)$;
- (Triangle inequality) $\mathsf{d}^{\mathsf{Kob}}(x, y) \leq \mathsf{d}^{\mathsf{Kob}}(y, z) + \mathsf{d}^{\mathsf{Kob}}(z, x)$.
- (Projective Schwarz lemma) If Ω, Ω' are two domains in projective spaces with Kobayashi pseudometrics d, d' respectively and
$$\Omega \xrightarrow{f} \Omega'$$
is a projective map, then
$$\mathsf{d}'(f(x), f(y)) \leq \mathsf{d}(x, y).$$
- The Kobayashi pseudometric on any interval in $\mathbb{R}\mathsf{P}^1$ equals its Hilbert metric.
- $\mathsf{d}^{\mathsf{Kob}}$ is invariant under the group $\mathsf{Aut}(\Omega)$ consisting of all collineations of P preserving Ω.

Proposition 12.1.3 (Kobayashi [**222**])**.** If $\Omega \subset \mathsf{P}$ is a properly convex domain, then $\mathsf{d}_\Omega^{\mathsf{Hilb}} = \mathsf{d}_\Omega^{\mathsf{Kob}}$.

Corollary 12.1.4. The function $\Omega \times \Omega \xrightarrow{\mathsf{d}^{\mathsf{Hilb}}} \mathbb{R}$ is a complete metric on Ω.

Proof of Proposition 12.1.3. Let $x, y \in \Omega$ be distinct points and $\ell = \overleftrightarrow{x\ y}$ be the line incident to them. Now

$$\mathsf{d}_\Omega^{\mathsf{Hilb}}(x, y) \;=\; \mathsf{d}_{\ell\cap\Omega}^{\mathsf{Hilb}}(x, y) \;=\; \mathsf{d}_{\ell\cap\Omega}^{\mathsf{Kob}}(x, y) \;\leq\; \mathsf{d}_\Omega^{\mathsf{Kob}}(x, y)$$

by the projective Schwarz lemma applied to the projective map $\ell \cap \Omega \hookrightarrow \Omega$. For the opposite inequality, let x_∞, y_∞ be intersections of ℓ with $\partial\Omega$. Let S

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be the intersection of a supporting hyperplane to $\partial\Omega$ at x_∞ and a supporting hyperplane to $\partial\Omega$ at y_∞. Projection from S to ℓ defines a projective map

$$\Pi_{S,\ell}\Omega \xrightarrow{\Pi_{S,\ell}} \ell \cap \Omega$$

retracting Ω onto $\ell \cap \Omega$. Thus, again using the projective Schwarz lemma,

$$\mathsf{d}_\Omega^{\mathsf{Kob}}(x,y) \;\leq\; \mathsf{d}_{\ell\cap\Omega}^{\mathsf{Kob}}(x,y) \;=\; \mathsf{d}_\Omega^{\mathsf{Hilb}}(x,y)$$

as desired. □

Corollary 12.1.5. Line segments in Ω are geodesics. If $\Omega \subset \mathsf{P}$ is properly convex, $x, y \in \Omega$, then the chain consisting of a single projective isomorphism

$$\mathbf{I} \longrightarrow \overleftrightarrow{xy} \cap \Omega$$

minimizes the length among all chains in $\mathsf{Chain}(x \rightsquigarrow y)$.

Exercise 12.1.6. Prove that straight line segments in Ω are geodesics with respect to this metric. Find an example of a properly convex domain Ω for which there are also geodesic paths which are not segments of straight lines.

Kobayashi [**222**],§3, compares his construction with that of a metric analogous to the Carathéodory pseudometric on complex domains. The *projective Carathéodory pseudometric* $\mathsf{d}^{\mathsf{Car}}$ is defined using maps $\Omega \longrightarrow \mathbf{I}$ rather than maps $\mathbf{I} \longrightarrow \Omega$. It satisfies a universal property with respect to projective maps, and $\mathsf{d}^{\mathsf{Car}} \leq \mathsf{d}^{\mathsf{Kob}}$. Furthermore both agree with the HIlbert metric when Ω is a properly convex domain.

12.2. Kobayashi hyperbolicity

Now we discuss intrinsic metrics on affine and projective manifolds.

Recall from §12.1 the open unit interval $\mathbf{I} = (-1, 1)$ with Poincaré metric

$$\mathsf{g}_\mathbf{I} := \frac{4\ du^2}{(1-u^2)^2} = \Big(ds_\mathbf{I}\Big)^2,$$

where

$$ds_\mathbf{I} \;=\; \sqrt{\mathsf{g}_\mathbf{I}} \;:=\; \frac{2\ |du|}{1-u^2} \;=\; \Big|d\big(2\tanh^{-1}(u)\big)\Big|$$

defines the associated norm on the tangent spaces. As in §12.1, the natural parameter s for arc length on $\mathbf{I}$ relates to the Euclidean coordinate u on $\mathbf{I} \subset \mathbb{R}$ by:

$$u \;=\; \tanh(s/2).$$

For projective manifolds M, one defines a "universal" pseudometric

$$M \times M \xrightarrow{\mathsf{d}_M^{\mathsf{Kob}}} \mathbb{R}$$

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such that affine (respectively projective) maps $\mathbf{I} \to M$ are distance non-increasing with respect to $ds_{\mathbf{I}}$. This generalizes the Kobayashi metric for projective domains discussed in §12.1.

The definition of $\mathsf{d}_M^{\mathsf{Kob}}$ for an arbitary $\mathbb{R}\mathsf{P}^n$-manifold M enforces the triangle inequality and Schwarz lemma by taking the infimum of $g_{\mathbf{I}}$-distances over *chains* in M, as in §12.1. Recall that if $x, y \in M$, a (projective) *chain* from x to y is a sequence of projective maps $f_1, \dots, f_m \in \mathsf{Proj}(\mathbf{I}, M)$ and pairs $a_i, b_i \in \mathbf{I}$, for $i = 1, \dots, m$ such that:

$$f_1(a_1) = x,\ f_1(b_1) = f_2(a_2),\ \dots,\\ f_{m-1}(b_{m-1}) = f_m(a_m),\ f_m(b_m) = y.$$

Denote the set of all chains from x to y by $\mathsf{Chain}(x \rightsquigarrow y)$ and define *length:*

$$\begin{aligned} \mathsf{Chain}(x \rightsquigarrow y) &\xrightarrow{\ \ell\ } \mathbb{R}_{\geq 0} \\ \Big(\big((a_1, b_1), f_1\big), \dots, \big((a_m, b_m), f_m\big)\Big) &\longmapsto \sum_{i=1}^{m} \mathsf{d}_{\mathbf{I}}(a_i, b_i), \end{aligned}$$

where $\mathsf{d}_{\mathbf{I}}$ is the distance function on the Riemannian 1-manifold $(\mathbf{I}, \mathsf{g}_{\mathbf{I}})$.

Now define the *Kobayashi pseudodistance* $\mathsf{d}^{\mathsf{Kob}}(x, y)$ as:

$$\begin{aligned} M \times M &\xrightarrow{\mathsf{d}^{\mathsf{Kob}}} \mathbb{R}_{\geq 0} \\ (x, y) &\longmapsto \inf\left\{ \ell(C) \;\middle|\; C \in \mathsf{Chain}(x \rightsquigarrow y) \right\}. \end{aligned}$$

just as in as in §12.1. Just as in the case of domains, $\mathsf{d}^{\mathsf{Kob}}$ satisfies the triangle inequality and the *projective Schwarz lemma:*

Lemma 12.2.1 (Projective Schwarz Lemma)**.** Projective maps do not increase pseudodistances: If $x, y \in N$, and $f \in \mathsf{Proj}(N, M)$, then

$$\mathsf{d}_M^{\mathsf{Kob}}\big(f(x), f(y)\big) \leq \mathsf{d}_N^{\mathsf{Kob}}\big(x, y\big).$$

Definition 12.2.2. Let M be an $\mathbb{R}\mathsf{P}^n$-manifold. Then M is *(projectively) hyperbolic* if and only if $\mathsf{d}_M^{\mathsf{Kob}} > 0$, that is, if $\big(M, \mathsf{d}_M^{\mathsf{Kob}}\big)$ is a metric space. Say that M is *complete hyperbolic* if the metric space $\big(M, \mathsf{d}_M^{\mathsf{Kob}}\big)$ is complete.

12.2.1. Complete hyperbolicity and convexity.

The following convexity theorem is due to Kobayashi [**222**] and Vey [**336, 337**], independently. but from somewhat different viewpoints. We closely follow Kobayashi [**222**]; see also [**220, 221**].

Proposition 12.2.3. Let M be a complete hyperbolic projective manifold.

Then M is properly convex, that is, M is isomorphic to a quotient of a properly convex domain by a discrete group of collineations.

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The proof will be based on the following fundamental compactness property of projective maps (compare Vey [**337**], Proposition IV, Chapitre II):

Lemma 12.2.4. Suppose that M, N are projective manifolds, where M is complete hyperbolic. Let $p \in N$ and K a compact subset of M. Then

$$\mathsf{Proj}_{p,K}(N, M) \;:=\; \big\{ f \in \mathsf{Proj}(N, M) \;\big|\; f(p) \in K \big\}$$

is compact.

The proof uses the following general lemma on pseudometric spaces; we include the proof for the reader's convenience.

Lemma 12.2.5 (Kobayashi [**223**], Chapter V, Theorem 3.1)**.** Let (N, d_N) and (M, d_M) be connected locally compact pseudometric spaces. Suppose that (N, d_N) is separable (that is, it has a countable dense subset), and (M, d_M) is a complete metric space.

Then the subset of $\mathsf{Map}(N, M)$ comprising distance-nonincreasing maps

$$N \xrightarrow{f} M$$

is locally compact. In particular, if $p \in N$ and $K \subset\subset M$, the subset of all such maps with $f(p) \in K$ is compact.

We briefly sketch the proof; see Kobayashi [**223**] for further details.

Proof of Lemma 12.2.5. Let f_n be a sequence of distance-nonincreasing projective maps taking $p \in N$ to $K \subset\subset M$. Choose a countable dense subset $\{p_1, \dots\} \subset N$; then

$$K_i \;:= \overline{\mathsf{B}}_{\mathsf{d}_M(K, p_i)}(K) \;\subset\subset M.$$

Since the f_n are distance-nonincreasing,

$$f_n(p_i) \in K_i.$$

Passing to a subsequence, assume that the sequence $f_n(p_i)$ converges, for each i. Then for each $q \in N$, the sequence $f_n(q)$ is Cauchy. Since N is complete, $f_n(q)$ converges. Finally, this convergence is uniform on compact subsets of N. □

Proof. Apply the Projective Schwarz Lemma 12.2.1 to Lemma 12.2.5 (Theorem 3.1 of Chapter V of Kobayashi [**223**]). □

Proof of Proposition 12.2.3. We show that M is *geodesically convex,* that is, if $\forall p, q \in M$, every path $p \rightsquigarrow q$ is relatively homotopic to a geodesic path from p to q. We may assume that M is simply connected.

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For $p \in M$, let $M(p) \subset M$ be the union of geodesic segments in M beginning at p. Exercise 8.3.4 implies $M(p)$ is open. Since M is connected, it suffices to show $M(p)$ is closed.

Suppose that $q_n \in M(p)$ for $n = 1, 2, \ldots$ be a convergent sequence in M with $q = \lim_{n\to\infty} q_n$. We show that $q \in M(p)$.

Let $0 < a < 1$ and $f_n \in \mathsf{Proj}(\mathbf{I}, M)$ with $f_n(0) = p$ and $f_n(a) = q_n$. Lemma 12.2.4 guarantees a subsequence of f_n converging to a projective map $f \in \mathsf{Proj}(\mathbf{I}, M)$ with $f(0) = p$. Then

$$q \;=\; \lim_{n\to\infty} q_n \;=\; \lim_{n\to\infty} f_n(a) \;=\; f(a) \in M(p)$$

and $M(p)$ is closed, as desired. □

12.2.2. The infinitesimal form. Kobayashi's pseudometric $\mathsf{d}^{\mathsf{Kob}}$ has an infinitesimal form Φ^{Kob} defined by a function $TM \xrightarrow{\Phi^{\mathsf{Kob}}} \mathbb{R}$. That is, $\mathsf{d}^{\mathsf{Kob}}(p, q)$ is the infimum of the *pseudolengths*

$$\ell(\gamma) := \int_\gamma \Phi^{\mathsf{Kob}}(\gamma')$$

over piecewise C^1 paths $p \overset{\gamma}{\rightsquigarrow} q$. For $x \in M$ and $\xi \in \mathsf{T}_x M$, define:

$$\Phi^{\mathsf{Kob}}(\xi) := \inf\left\{ |ds_{\mathbf{I}}(\mathbf{v})| \;\middle|\; f \in \mathsf{Proj}(\mathbf{I}, M),\ f(u) = x,\ (\mathsf{D}f)_u(\mathbf{v}) = \xi \right\} \tag{12.2}$$

where $u \in \mathbf{I}$ and $ds_{\mathbf{I}}(\mathbf{v})$ denotes the norm of $\mathbf{v} \in \mathsf{T}_u\mathbf{I}$ with respect to the Poincaré metric $(ds_{\mathbf{I}})^2$ on $\mathbf{I}$.

Exercise 12.2.6. For affine manifolds, completeness is equivalent to $\Phi^{\mathsf{Kob}} \equiv 0$. For a Hopf manifold, $\mathsf{d}^{\mathsf{Kob}} \equiv 0$ but $\Phi^{\mathsf{Kob}} \neq 0$. Indeed $\Phi^{\mathsf{Kob}}(\mathsf{Rad}) = 1$ where Rad is the radiant vector field.

Exercise 12.2.7. Show that Φ^{Kob} is *homogeneous of degree one,* that is,

$$\Phi^{\mathsf{Kob}}(r\xi) = r\,\Phi^{\mathsf{Kob}}(\xi)$$

for $r \geq 0$. Deduce that $\ell(\gamma)$ is independent of the parametrization of γ.

Proposition 12.2.8. Φ^{Kob} is upper semicontinuous.

Recall that a function $X \xrightarrow{f} \mathbb{R}$ is *upper semicontinuous* at $x \in X$ if and only if $\forall \epsilon > 0$,

$$f(y) < f(x) + \epsilon$$

for y in an open neighborhood of x. That is, the values of f cannot "jump down" in limits.

$$\lim_{n\to\infty} f(\xi_n) \leq f\big(\lim_{n\to\infty} \xi_n\big)$$

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for convergent sequences ξ_n. Equivalently, f is a continuous mapping from X to $\mathbb{R}$, where $\mathbb{R}$ is given the topology whose open sets are intervals $(-\infty, a)$ where $a \in \mathbb{R}$. The indicator function of a closed set is upper semicontinuous. Semicontinuous functions are further discussed in Appendix E.

Proof of Proposition 12.2.8. Let $x \in M$ and $\xi \in \mathsf{T}_x M$ and write

$$\Phi^{\mathsf{Kob}}(\xi) = k.$$

Let $\epsilon > 0$. Then $\exists f \in \mathsf{Proj}(\mathbf{I}, M)$ with $f(u) = x$, and $\mathbf{v} \in \mathsf{T}_u\mathbf{I}$ with $(\mathsf{D}f)_u(\mathbf{v}) = \xi$ and

$$\|\mathbf{v}\| < k + \epsilon/2.$$

Lift f to $\widetilde{f} \in \mathsf{Proj}(\mathbf{I}, \widetilde{M})$ and extend $\widetilde{f}$ to $\widetilde{F} \in \mathsf{Proj}(\mathbb{B}, \widetilde{M})$. Let $\|\|_{\mathbb{B}}$ the corresponding norm for the intrinsic metric on $\mathbb{B}$ defined in §3.3. We may assume that $\|\xi\|_{\mathbb{B}} = \|\xi\|$. Then an open neighborhood $\mathcal{N}$ of $\xi \in \mathsf{T}\widetilde{M}$ exists so that if $\xi' \in \mathcal{N}$, then:

- $\xi' \in \mathsf{T}_{x'}\widetilde{M}$ where x' lies in the image $\widetilde{F}(\mathbb{B}) \subset \widetilde{M}$;
- $\|(\mathsf{D}\widetilde{F})^{-1}(\xi')\|_{\mathbb{B}} < k + \epsilon$.

Since projective maps do not increase distance, $\|\xi'\|_{\widetilde{M}} < k+\epsilon$, as desired. ☐

Corollary 12.2.9 (Kobayashi [**222**],Proposition 5.16)**.** If M is a complete hyperbolic projective manifold, then $\mathsf{T}M \xrightarrow{\Phi^{\mathsf{Kob}}} \mathbb{R}$ is continuous.

The proof uses the following fact, stating that the infimum in the definition of Φ^{Kob} is actually achieved.

Lemma 12.2.10 (Kobayashi [**222**],Lemma 5.17)**.** Suppose M is complete hyperbolic and $\xi \in \mathsf{T}_p M$. Then $\exists f \in \mathsf{Proj}(\mathbf{I}, M)$ and $\mathbf{v} \in \mathsf{T}_0\mathbf{I}$ with $f(0) = p$ and $(\mathsf{D}f)_0(\mathbf{v}) = \xi$ such that $\Phi^{\mathsf{Kob}}(\xi) = \|\mathbf{v}\|$.

Proof. Lemma 12.2.4 implies that the subset of $\mathsf{Proj}(\mathbf{I}, M)$ comprising projective maps $\mathbf{I} \xrightarrow{f} M$ with $f(0) = p$ is compact. Thus the set of $\|(\mathsf{D}f)^{-1}(\xi)\|$ is a compact subset of $\mathbb{R}^+$, so its infimum $\Phi^{\mathsf{Kob}}(\xi)$ is positive. ☐

Proof of Corollary 12.2.9. Suppose, for $k = 1, 2, \dots$, that $\xi_k \in \mathsf{T}_{p_k} M$, defines a sequence converging to $\xi_\infty \in \mathsf{T}_{p_\infty} M$.

We show that $\lim_{k\to\infty} \Phi^{\mathsf{Kob}}(\xi_k) = \Phi^{\mathsf{Kob}}(\xi_\infty)$. Proposition 12.2.8 (semicontinuity of Φ^{Kob}) implies that

$$\Phi^{\mathsf{Kob}}(\xi_\infty) \geq \lim_{k\to\infty} \Phi^{\mathsf{Kob}}(\xi_k), \tag{12.3}$$

so it suffices to show that $\Phi^{\mathsf{Kob}}(\xi_\infty) \leq \lim_{k\to\infty} \Phi^{\mathsf{Kob}}(\xi_k)$.

The above lemma guarantees $f_k \in \mathsf{Proj}(\mathbf{I}, M)$ with $f_k(0) = p_k$ and

$$\mathbf{v}_k := (\mathsf{D}f_k)^{-1}(\xi_k) \in \mathsf{T}_0\mathbf{I}$$

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such that $\Phi^{\mathsf{Kob}}(\xi_k) = \|\mathbf{v}_k\|$. By (12.3), $\mathbf{v}_k$ contains a convergent subsequence. By extracting such a subsequence, let

$$\mathbf{v}_\infty := \lim_{k\to\infty} \mathbf{v}_k.$$

Lemma 12.2.4 guarantees that by passing to a further subsequence, we may assume that f_k converges to $f_\infty \in \mathsf{Proj}(\mathbf{I}, M)$ with $f_\infty(0) = p_\infty$ and $\mathsf{D}f_\infty(\mathbf{v}_\infty) = \xi_\infty$. By the definition of Φ^{Kob},

$$\Phi^{\mathsf{Kob}}(\xi_\infty) \le \|\mathbf{v}_\infty\| = \lim_{k\to\infty} \|\mathbf{v}_k\| = \lim_{k\to\infty} \Phi^{\mathsf{Kob}}(\xi_k)$$

as desired. ☐

Exercise 12.2.11. Find an example of a projectively hyperbolic domain for which Φ^{Kob} is not continuous.

Exercise 12.2.12. Find an example of a domain Ω for which:

- $\Phi^{\mathsf{Kob}}(\xi) < \infty$ for all nonzero $\xi \in \mathsf{T}\Omega$;
- Ω contains no complete geodesic rays.

Theorem 12.2.13. Φ^{Kob} is the *infinitesimal form* of the Kobayashi pseudometric $\mathsf{d}_M^{\mathsf{Kob}}$, that is,

$$\mathsf{d}_M^{\mathsf{Kob}}(x,y) \;=\; \mathsf{inf}\left\{ \int_a^b \Phi^{\mathsf{Kob}}\big(\gamma'(t)\big)dt \;\middle|\; \gamma \in \mathsf{Path}(x \rightsquigarrow y)\right\}$$

where $\mathsf{Path}(x \rightsquigarrow y)$ denotes the set of piecewise C^1 paths $[a,b] \xrightarrow{\gamma} M$ with $\gamma(a) = x,\ \gamma(b) = y$.

Since positive upper semicontinuous functions are bounded on compact sets (Exercise E.2.3) and measurable (Exercise E.1.2), the above integral is well-defined.

Proof of Theorem 12.2.13. Define

$$\delta^{\mathsf{Kob}}(p,q) := \mathsf{inf}\left\{ \int_a^b \Phi^{\mathsf{Kob}}\big(\gamma'(t)\big)dt \;\middle|\; \gamma \in \mathsf{Path}(p \rightsquigarrow q)\right\}$$

We must prove that $\delta^{\mathsf{Kob}} = \mathsf{d}^{\mathsf{Kob}}$.

To prove $\delta^{\mathsf{Kob}} \le \mathsf{d}^{\mathsf{Kob}}$, note that every chain $C \in \mathsf{Chain}(x \rightsquigarrow y)$ determine a piecewise C^1 path $\gamma_C \in \mathsf{Path}(p \rightsquigarrow q)$. Since $\mathsf{g}_{\mathbf{I}}$ is the infinitesimal form of $\mathsf{d}_{\mathbf{I}}$, the path γ_C has shorter length than the chain C, that is, their lengths satisfy $\ell(\gamma_C) \le \ell(C)$. Taking infima implies $\delta^{\mathsf{Kob}} \le \mathsf{d}^{\mathsf{Kob}}$ as desired.

We prove $\delta^{\mathsf{Kob}} \ge \mathsf{d}^{\mathsf{Kob}}$. Suppose that $\gamma \in \mathsf{Path}(p \rightsquigarrow q)$ as above. Suppose $\epsilon > 0$. We seek a chain $C \in \mathsf{Chain}(p \rightsquigarrow q)$ such that

$$\ell(C) \le \ell(\gamma) + \epsilon. \tag{12.4}$$

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Proposition 12.2.8 implies that the function

$$\begin{aligned} [a,b] &\xrightarrow{\phi} \mathbb{R} \\ u &\longmapsto \Phi^{\mathsf{Kob}}\big(\gamma'(t)\big) \end{aligned}$$

is upper semicontinuous. Exercise E.2.3 implies ϕ is bounded from above. Apply Proposition E.2.4 to conclude that ϕ is the limit of a monotonically decreasing sequence of nonnegative continuous functions.

Apply Lebesgue's monotone convergence theorem (Rudin [**297**],1.26) to find a continuous function $[a,b] \xrightarrow{h} \mathbb{R}$ such that:

$$\phi(t) < h(t) \text{ for } a \le t \le b \tag{12.5}$$

$$\int_{[a,b]} h < \ell(\gamma) + \epsilon \tag{12.6}$$

for $a \le t \le b$.

We claim that for each $s \in [a,b]$,

$$\int_s^t \phi(u)\,du \le (1+\epsilon)\,h(s)\,|s-t| \tag{12.7}$$

for t in an interval I_s centered at s.

To this end, first assume that γ is C^1. We choose the open neighborhood I_s of s in three steps:

First, choose a convex ball W_s containing $\gamma(s)$, so that $\gamma(t) \in W_s$ for $t \in I_s$. Let $f_t \in \mathsf{Proj}(\mathbf{I}, M)$ extend the geodesic in W_s joining $\gamma(s)$ to $\gamma(t)$. Then (12.5) implies that $\|(\mathsf{D}f_t)^{-1}\big(\gamma'(s)\big)\| < h(s)$. Since γ is C^1 and $t \mapsto f_t$ is C^0,

$$\|(\mathsf{D}f_t)^{-1}\big(\gamma'(t)\big)\| < h(s) \tag{12.8}$$

for t sufficiently near s.

Next, choose I_s so that (12.8) holds for $t \in I_s$. Since

$$h(u) \le (1+\epsilon)h(s)$$

for all $u \in I_s$,

$$\int_s^t h(u)du \le (1+\epsilon)\,h(s)\,|s-t| \tag{12.9}$$

for t sufficiently near s.

Finally, choose I_s so that (12.9) holds for $t \in I_s$.

Combining (12.9) with (12.5) implies

$$\int_s^t \phi(u)du < \int_s^t h(u)du \le (1+\epsilon)\,h(s)\,|s-t|,$$

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establishing the claim when γ is C^1. Extending (12.9) to the case that γ is only *piecewise* C^1 is a routine exercise.

Continuing to follow Wu [**354**], we pick up the argument of Royden [**296**]. Let $\eta > 0$ be a *Lebesgue number* for the open cover $\{I_s \mid s \in \mathbf{I}\}$ of $\mathbf{I}$: *every closed interval of length* $< \eta$ *lies in some* I_s.[1]

Thus subdivision $a = t_0 < t_1 < \cdots < t_k = b$ exists with $t_i - t_{i-1} < \eta$; let s_i be such that $[t_{i-1}, t_i] \subset I_{s_i}$. Continuity of h and (12.6) imply

$$\sum_{i=1}^{k} h(s_i)(t_i - t_{i-1}) \; < \; \int_a^b h(u) < \ell(\gamma) + \epsilon. \tag{12.10}$$

By (12.7),

$$\begin{aligned} \mathsf{d}^{\mathsf{Kob}}\big(\gamma(s_i), \gamma(s_{i-1})\big) \le \quad & \mathsf{d}^{\mathsf{Kob}}\big(\gamma(s_i), \gamma(s_{-1})\big) \quad + \; \mathsf{d}^{\mathsf{Kob}}\big(\gamma(s_i), \gamma(s_{-1})\big) \\ & \le (1+\epsilon)\,\big(\,h(s_i)(s_i - t_i) \quad + \; h(s_{i-1})(t_i - s_{i-1})\,\big). \end{aligned}$$

Apply (12.9) and (12.10), obtaining:

$$\begin{aligned} \mathsf{d}^{\mathsf{Kob}}(p,q) &\le \sum_{i=1}^{k} \mathsf{d}^{\mathsf{Kob}}\big(\gamma(s_i), \gamma(s_{i-1})\big) \\ &\le \sum_{i=1}^{k} h(s_i)(t_i - t_{i-1}) \\ &< \ell(\gamma) + \epsilon. \end{aligned}$$

Now the chain $C \in \mathsf{Chain}(p \rightsquigarrow q)$ defined by:

$$C \; := \; \Big(\big((s_1, s_2, f_1\big), \ldots, (s_{k-1}, s_k), f_k)\Big)$$

has length

$$\ell(C) = \sum_{i=1}^{k-1} \mathsf{d}^{\mathsf{Kob}}\big(\gamma(s_i, s_{i+1})\big)$$

and (12.4) follows. Since $\epsilon > 0$ is arbitrary, $\mathsf{d}^{\mathsf{Kob}}(p,q) \le \ell(\gamma) \le \delta$ as desired. □

Wu [**354**] actually proves a much stronger statement, valid for affine connections which are *not necessarily* flat. His proof is based on the analog for the Kobayashi pseudometric for complex manifolds, due to Royden [**296**].

Closely related is the universal property of Φ^{Kob} among infinitesimal pseudometrics for which projective maps are infiniteimally nonincreasing (Kobayashi [**222**], Proposition 5.5):

[1]For a proof of the existence of a Lebesgue number, see Burago–Burago–Invanov [**68**], Theorem 1.6.11.

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Exercise 12.2.14. Let M be an $\mathbb{R}\mathsf{P}^n$-manifold and $M \xrightarrow{\Phi} \mathbb{R}_{\geq 0}$ a function such that $\forall f \in \mathsf{Proj}(\mathbf{I}, M)$

$$\Phi\big((\mathsf{D}f)_a(\xi)\big) \leq \|\xi\|_a,$$

where $-1 < a < 1$ and $\xi \in \mathsf{T}_a\mathbf{I}$. Then $\Phi \leq \Phi^{\mathsf{Kob}}$.

12.2.3. Completely incomplete manifolds.

Exercise 12.2.15. Let M be a affine manifold. Suppose that every geodesic ray is incomplete. Then M is noncompact.

Theorem 12.2.16. Suppose that M is an $\mathbb{R}\mathsf{P}^n$-manifold. Then $\mathsf{d}_M^{\mathsf{Kob}} > 0$ if and only if every projective map $\mathbb{R} \longrightarrow M$ is constant.

More specifically, we prove (following Kobayashi [**222**]):

Proposition 12.2.17. Let M be a projective manifold, $p \in M$ and $\xi \in \mathsf{T}_pM$. Suppose $\Phi^{\mathsf{Kob}}(\xi) = 0$. Then $\exists f \in \mathsf{Proj}(\mathbb{R}, M)$ with $f(0) = p$ and $f'(0) = \xi$.

Since $\Phi^{\mathsf{Kob}}(p, \xi) = 0$, there is a sequence $j_m \in \mathsf{Proj}(\mathbf{I}, M)$ with $j_m(0) = p$ and a sequence $a_m > 0$ with a_m decreasing, $a_m \searrow 0$ such that the differential $(\mathsf{D}j_m)_0$ of j_m at $0 \in \mathbf{I}$ maps:

$$\begin{aligned} \mathsf{T}_0\mathbf{I} &\xrightarrow{(\mathsf{D}j_m)_0} \mathsf{T}_pM \\ a_m\left(\frac{d}{du}\right)_0 &\longmapsto \xi. \end{aligned}$$

Let $r_m = 1/a_m$ so that

$$\left(\frac{d}{du}\right)_0 \xmapsto{(\mathsf{D}j_m)_0} r_m\xi \tag{12.11}$$

with $r_m \nearrow +\infty$ monotonically.

As in Exercise 12.1.1, let $\mathbf{I}(r)$ denotes the open interval $(-r, r) \subset \mathbb{R}$. Then

$$\mathbf{I}(r_1) \subset \mathbf{I}(r_2) \subset \cdots \subset \mathbf{I}(r_m) \subset \cdots \subset \mathbb{R}$$

and $\bigcup_{m=1}^{\infty} \mathbf{I}(r_m) \;=\; \mathbb{R}$.

The strategy of the proof is to reparametrize the maps j_m to obtain a subsequence of projective maps

$$h_m \in \mathsf{Proj}\big(\mathbf{I}(r_m), M\big),$$

such that the restriction of h_m to h_l equals h_l for $l \leq m$.

First renormalize j_m to a projective map $f_m \in \mathsf{Proj}(\mathbf{I}(r_m), M)$:

$$\begin{aligned} \mathbf{I}(r_m) &\xrightarrow{f_m} M \\ u &\longmapsto j_m(u/r_m) \end{aligned}$$

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Lemma 12.2.18. The differential $(\mathsf{D}f_m)_0$ of f_m at 0 maps the tangent vector $\big(\frac{d}{du}\big)_0 \in \mathsf{T}_0\big(\mathbf{I}(r_m)\big)$ to ξ.

Proof. f_m equals the composition of $\mathbf{I} \xrightarrow{j_m} M$ with the contraction

$$\begin{aligned} \mathbf{I}(r_m) &\longrightarrow \mathbf{I} \\ u &\longmapsto u/r_m \\ 0 &\longmapsto 0 \end{aligned}$$

whose differential at 0 is multiplication by $(r_m)^{-1}$. Now apply the chain rule and (12.11). □

Next reparametrize the maps f_m using the following analog of Brody's reparametrization lemma for holomorphic mappings (Brody [**66**]), whose proof is given later.

As in Exercise 12.1.1, the infinitesimal norm for $\mathbf{I}(r)$ equals:

$$ds_{\mathbf{I}(r)} \;=\; \frac{2r|du|}{r^2 - u^2},$$

a function on $\mathsf{T}\mathbf{I}(r)$. As in Kobayashi [**222**][2] choose a Riemannian metric g on M such that $\mathsf{g}(\xi) = 1$. The corresponding norm on $\mathsf{T}M$ is $\sqrt{\mathsf{g}}$ and pulls back to a norm $f^*\sqrt{\mathsf{g}}$ on $\mathsf{T}_u\mathbf{I}(r)$. Then there is a continuous function

$$\mathbf{I}(r) \xrightarrow{\mathcal{W}_f} \mathbb{R}^+$$

such that

$$f^*\sqrt{\mathsf{g}}(u) \;=\; \mathcal{W}_f(u)\, ds_{\mathbf{I}(r)}(u). \tag{12.12}$$

Lemma 12.2.19 (Reparametrization Lemma)**.** Let M be an projective manifold, $p \in M$ and $\xi \in \mathsf{T}_pM$ nonzero. Suppose that $f \in \mathsf{Proj}(\mathbf{I}(r), M)$ with $f(0) = p$ and $f'(0) = \xi$. Choose c such that $\mathcal{W}_f(0) > c > 0$. Then $\exists a, b$ with $0 < a < 1$ and $b \in \mathsf{Aut}\big(\mathbf{I}(r)\big)$ such that $h \in \mathsf{Proj}(\mathbf{I}(r), M)$ defined by:

$$h(u) := \; f\big(ab(u)\big)$$

satisfies

- $\mathcal{W}_h(u) \leq c$;
- $\mathcal{W}_h(0) = c$,

where $\mathcal{W}_h$ is defined in (12.12)

[2] Remark 5.27 after Theorem 5.22, pp.145–146

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Conclusion of proof of Proposition 12.2.17 assuming Lemma 12.2.19. Applying Lemma 12.2.19 to f_m, $\exists h_m \in \mathsf{Proj}(\mathbf{I}(r_m), M)$ and $c_m > 0$ such that

$$\mathcal{W}_{h_m}(u) \leq c_m \text{ and } \mathcal{W}_{h_m}(0) = c \tag{12.13}$$

and the image of f_m contains the image of h_m.

Denote the restriction of h_m to $\mathbf{I}(r_)$ by $h_{l,m}$. Equation (12.13) implies that, for each $l \in \mathbb{N}$, the family

$$\mathcal{F}_l \ := \ \{h_{l,m} \mid m \geq l\}$$

is equicontinuous.

We construct the projective map $h \in \mathsf{Proj}(\mathbb{R}, M)$ by consecutive extensions $h_l \in \mathsf{Proj}(\mathbf{I}(r_l), M)$ to $\mathbf{I}(r_l) \supset \mathbf{I}(r_{l-1})$ as follows.

Beginning with $l = 1$, the Arzelà–Ascoli theorem guarantees a convergent subsequence $h_{1,m}$ in $\mathsf{Proj}\big(\mathbf{I}(r_1), M\big)$. Write

$$h_1 = \lim_{m\to\infty} h_{1,m} \in \mathsf{Proj}(\mathbf{I}(r_1), M).$$

Suppose inductively that $h_l \in \mathsf{Proj}\big(\mathbf{I}(r_l), M\big)$ has been defined such that h_l extends h_k for all $k \leq l$. Since $\mathcal{F}_l$ is equicontinuous, the Arzelà–Ascoli theorem guarantees a convergent subsequence of $h_{l,m}$. Define

$$h_l := \lim_{m\to\infty} h_{l,m}.$$

The value of $h_m^*\sqrt{\mathsf{g}}$ at $u = 0$ equals $2c\ du \ \neq\ 0$. Since this is the value of $h^*\sqrt{\mathsf{g}} \ = \ h_l^*\sqrt{\mathsf{g}}$ at $u = 0$, the map h is nonconstant. This concludes the proof of Proposition 12.2.17 assuming Lemma 12.2.19. □

Proof of Lemma 12.2.19. For $0 \leq t \leq 1$, consider the projective map

$$\begin{aligned} \mathbf{I}(r) &\xrightarrow{f_t} M \\ u &\longmapsto f(tu). \end{aligned}$$

Then the corresponding function $\mathcal{W}_t := \mathcal{W}_{f_t}$ $\big($defined as in (12.12)$\big)$

$$\mathbf{I}(r) \xrightarrow{\mathcal{W}_t} \mathbb{R}^+$$

satisfies the following elementary properties, whose proofs are left as exercises:

$$\mathcal{W}_t(u) \ = \ \mathcal{W}_f(tu)\,\frac{t(r^2-u^2)}{r^2-t^2u^2}. \tag{12.14}$$

$$\mathcal{W}_t(u) \geq 0 \text{ and } \mathcal{W}_t(u) = 0 \iff t = 0. \tag{12.15}$$

$$\lim_{u\to\pm r} \mathcal{W}_t(u) \ = \ 0. \tag{12.16}$$

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The function

$$A(t) \ := \ \sup_{u\in\mathbf{I}(r)} \mathcal{W}_t(u)$$

of $t \in [0,1]$ satisfies the following elementary properties, whose proofs are also left as exercises:

- $A(t) < \infty$.
- $[0,1] \xrightarrow{A} \mathbb{R}^+$ is continuous.
- A is increasing.

Furthermore $A(0) = 0$ and $A(1) = c$, so the Intermediate Value Theorem guarantees $\exists a \in [0,1]$ such that $A(a) = c$. Thus,

$$c \ = \ \sup_{u\in\mathbf{I}(r)} \mathcal{W}_a(u).$$

By (12.16), $\mathcal{W}_a$ assumes its maximum on $u_0 \in \mathbf{I}(r)$; let $b \in \mathsf{Aut}\big(\mathbf{I}(r)\big)$ take 0 to u_0. The proof of Lemma 12.2.19 is complete.

□

12.3. Hessian manifolds

When M is hyperbolic and affine, then Corollary 4.3.2 implies that M is a quotient of a properly convex cone Ω by a discrete group of collineations acting properly on Ω. In addition to the Hilbert metric, Ω enjoys the natural Riemannian metric introduced by Vinberg [**340**], Koszul [**226**, **227**, **229**, **230**], and Vesentini [**335**]. (Compare §4.4.) In particular Koszul and Vinberg observe that this Riemannian structure is the covariant differential $\nabla\omega$ of a closed 1-form ω. In particular ω is everywhere nonzero, so by Tischler [**327**], M fibers over $\mathbb{S}^1$.

This implies Koszul's beautiful theorem [**230**] that the holonomy mapping hol (described in §7.2) embeds the space of convex structures onto an open subset of the representation variety. This has recently been extended to noncompact manifolds by Cooper–Long–Tillmann [**101**].

Hyperbolic affine manifolds closely relate to *Hessian manifolds.* If ω is a closed 1-form, then its covariant differential $\nabla\omega$ is a symmetric 2-form. Since closed forms are locally exact, $\omega = df$ for some function; in that case $\nabla\omega$ equals the *Hessian* $d^2 f$. Koszul [**230**] showed that hyperbolicity is equivalent to the existence of a closed 1-form ω whose covariant differential $\nabla\omega$ is positive definite, that is, a Riemannian metric. More generally, Shima [**306**] considered Riemannian metrics on an affine manifold which are locally Hessians of functions, and proved that such a closed *Hessian* manifold is a quotient of a convex domain, thus generalizing Koszul's result.

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We briefly sketch some of the ideas in Koszul's paper.[3]

Let (M, ∇) be an affine manifold with connection ∇. Let $x \in M$. Denote by $\mathcal{E}_x \subset \mathsf{T}_x M$ the domain of exponential map as in §8.3.1. For $\xi \in \mathsf{T}_x M$, let

$$\lambda(\xi) := \sup\{t \in \mathbb{R} \mid t\xi \in \mathcal{E}_x\} \in (0, \infty].$$

Then the line $\mathbb{R}\xi$ intersects $\mathcal{E}_x$ in

$$\mathbb{R}\xi \cap \mathcal{E}_x \;=\; \big(-\lambda(-\xi), \lambda(\xi)\big)\,\xi.$$

Suppose that ω is a closed 1-form as above, such that the covariant differential $\nabla\omega > 0$. Koszul's theory is based on the two lemmas below. For notational simplicity, write

$$\begin{aligned} \big(-\lambda(-\xi), \lambda(\xi)\big) &\xrightarrow{\gamma} M \\ t &\longmapsto \mathsf{Exp}_x(t\xi) \end{aligned}$$

for the maximal geodesic with velocity $\xi = \gamma'(0)$ at time $t = 0$. Observe that for any $-\lambda(-\xi) < t < \lambda(\xi)$, the velocity vector at time t is

$$\gamma'(t) = \mathbb{P}_{\gamma_t}(\xi)$$

where

$$s \xrightarrow{\gamma_t} \gamma(s),$$

$0 \le s \le t$, is the restriction of γ to $[0, t]$ and

$$\mathsf{T}_x M \xrightarrow{\mathbb{P}_{\gamma_t}} \mathsf{T}_{\gamma(t)} M$$

denotes parallel transport along γ_t. The two basic lemmas are:

Lemma 12.3.1. If $\omega(\xi) > 0$, then $\lambda(\xi) < \infty$.

Lemma 12.3.2. If $\lambda(\xi) < \infty$, then

$$\int_0^{\lambda(\xi)} \omega\big(\gamma'(t)\big) dt = +\infty.$$

The role of positivity is apparent from the simple 1-dimensional example when $M = \mathbb{R}^+ \subset \mathbb{R}$. To develop intuition for these conditions, we work out these lemmas in the basic example when $M = \mathbb{R}^+ \subset \mathbb{R}$.

Proof of Lemma 12.3.1. We show that $\omega(\xi) > 0$ implies the geodesic ray γ is incomplete. The cone Ω^* dual to M consists of all $\psi > 0$ and the characteristic function is

$$\begin{aligned} M &\xrightarrow{f} \mathbb{R} \\ x &\longmapsto \int_{\Omega^*} e^{-\psi x} d\psi = \int_0^\infty e^{-\psi x} d\psi = \frac{1}{x} \end{aligned}$$

[3]Shima's book [**306**] contains a very accessible and comprehensive exposition of these and related ideas.

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and the logarithmic differential equals

$$\omega = d\log f = -\frac{dx}{x}.$$

If $\xi = y_0\partial_x \in \mathsf{T}_{x_0}M$ and $\xi \neq 0$, then $\gamma(t) = x_0 + ty_0$ and

$$\lambda(\xi) = \begin{cases} -x_0/y_0 & \text{if } y_0 < 0 \\ \infty & \text{if } y_0 > 0. \end{cases}$$

Similarly

$$\omega\big(\gamma'(t)\big) = \big(-dx/(x_0+ty_0)\big)\big(y_0\partial_x\big|_{x=x_0+ty_0}\big) = -y_0/(x_0+ty_0),$$

so if $\omega\big(\gamma'(t)\big) > 0$, then $y_0 < 0$ and $\lambda(\xi) < \infty$ as desired. □

Proof of Lemma 12.3.2. We show that along an incomplete geodesic ray, the integral of $\omega\big(\gamma'(t)\big)$ diverges. Suppose $\lambda(\xi) < \infty$. Then $y_0 < 0$ and

$$\int_0^{\lambda(\xi)} \omega\big(\gamma'(t)\big)dt = \log \frac{x_0}{x_0+ty_0}\Bigg|_{t=0}^{-x_0/y_0} = \infty$$

as desired. □

12.4. Functional characterization of hyperbolic affine manifolds

Vey [**337**] proves the following elegant characterization of hyperbolic affine manifolds using the space $\mathsf{Aff}(N,M)$ of affine maps $N \to M$ of affine manifolds M, N:

Theorem 12.4.1 (Vey [**337**], Chapitre II)**.** An affine manifold M is hyperbolic if and only if $\forall$ connected affine manifolds N, the natural map:

$$\begin{aligned} \mathsf{Aff}(N,M)\times N &\longrightarrow M\times N \\ (f,x) &\longmapsto \big(f(x),x\big) \end{aligned}$$

is proper.

Exercise 12.4.2. Prove Theorem 12.4.1 using the techniques developed in this chapter. (Hint: reduce to the case that N is a bounded interval in $\mathbb{R}$.)

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Chapter 13

Projective structures on surfaces

$\mathbb{R}\mathsf{P}^2$-manifolds are relatively well understood, due to intense activity in recent years. Rather than give an detailed description of this theory, we only summarize the results, and refer to the literature. In particular we recommend the recent book by Casella–Tate–Tillmann [**82**].

Aside from the two structures with finite fundamental group ($\mathbb{R}\mathsf{P}^2$ itself, and its double cover S^2), this class of geometric structures includes affine structures on surfaces, some new $\mathbb{R}\mathsf{P}^2$-structures on tori (first analyzed by Sullivan–Thurston [**320**], Smillie [**310**] and the author [**145**] in 1976–1977). as well as convex structures (which are *hyperbolic* in the sense of Kobayashi and Vey; see §12.2). Strikingly the answer is much more satisfactory for surfaces with $\chi < 0$, aside from $\mathbb{R}\mathsf{P}^2$-structures on tori and Klein bottles which are *not* affine, this chapter concentrates on surfaces of $\chi < 0$. For these, the convex structures play a fundamental role.

13.1. Classification in higer genus

The deformation space $\mathbb{R}\mathsf{P}^2_{\mathsf{convex}}(\Sigma)$ of convex $\mathbb{R}\mathsf{P}^2$ structures was calculated by the author [**153**] in 1985, using the analog of Fenchel–Nielsen coordinates. Shortly thereafter, in his doctoral thesis, Suhyoung Choi proved his *Convex Decomposition Theorem* [**88, 89**], expressing that on a closed surface of $\chi < 0$, *every* $\mathbb{R}\mathsf{P}^2$ is obtained from a convex surface by grafting annuli. We summarize their classification as follows (compare Choi–Goldman [**91**]):

Theorem. Let Σ be a closed orientable surface of genus $g > 1$.

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- The deformation space $\mathbb{RP}^2_{\mathsf{convex}}(\Sigma)$ of marked convex $\mathbb{RP}^2$-structures on Σ is homeomorphic to $\mathbb{R}^{16g-16}$, upon which $\mathsf{Mod}(\Sigma)$ acts properly.
- The holonomy map hol embeds $\mathbb{RP}^2_{\mathsf{convex}}(\Sigma)$ as a connected component of $\mathsf{Hom}(\pi, G)/\mathsf{Inn}(G)$ where $G = \mathsf{PGL}(3,\mathbb{R})$).
- The deformation space of marked $\mathbb{RP}^2$-structures on Σ is homeomorphic to $\mathbb{R}^{16g-16} \times \mathbb{N}$, upon which $\mathsf{Mod}(\Sigma)$ acts properly.

The first proof that $\mathbb{RP}^2_{\mathsf{convex}}(\Sigma)$ is a cell of dimension $16g-16$ (in [**153**]) involves a more general statement, valid when $\partial\Sigma \neq \emptyset$ but with some boundary conditions. The proof introduces an extension of the Fenchel–Nielsen coordinates on the Fricke space — the deformation space of hyperbolic structures on Σ — as described in §4. A particularly tractable and suggestive set of coordinates is due to Fock and Goncharov in [**129, 130**], and based on parametrizations of hyperbolic structures due to Thurston (see Bonahon [**55**]) and Penner [**288, 289**]. Bonahon–Kim [**58**] describe the relationship between the author's original extended Fenchel–Nielsen coordinates and Fock–Goncharov's extended Penner–Thurston coordinates. While the former uses decompositions of Σ into three-holed spheres (*pants decompositions*), the latter uses ideal triangulations.

The symplectic geometry of $\mathbb{RP}^2_{\mathsf{convex}}(\Sigma)$ in the coordinates developed in [**153**] involves a more general statement, described in Choi–Jung–Kim [**94**]. Another approach using affine connections is discussed in Goldman [**154**]; compare the discussion in §B.4.

In general if Ω/Γ is a convex $\mathbb{RP}^2$-manifold which is a *closed* surface S with $\chi(S) < 0$ then either $\partial\Omega$ is a conic, or $\partial\Omega$ is a C^1 convex curve (Benzécri [**45**]) which is not C^2 (Kuiper [**235**]). The key general point is that, if Σ is a closed surface of $\chi < 0$, then the dynamics of the holonomy group $\Gamma \cong \pi_1(\Sigma)$ is standard: If

$$1 \neq \gamma \in \Gamma < \mathsf{SL}(3,\mathbb{R}),$$

then γ has one repelling fixed point p_+ on $\partial\Omega$ and one attracting fixed point p_- on $\partial\Omega$. Furthermore, since $\partial\Omega$ is C^1, the projective lines $l_\pm$ tangent to $\partial\Omega$ at $p_\pm$ respectively are also γ-invariant. It follows that the intersection $p_0 := l_+ \cap l_-$ is a third fixed point for γ, which has saddle point dynamics. This implies that γ is represented by a 3×3 diagonal matrix with real and distinct eigenvalues. (Compare Figure 13.1.)

Such examples seem to be conjectured *not* to exist in the original articles of Ehresmann [**122**] and Benzécri [**45**]. They are analogous to *quasi-Fuchsian surface groups* in $\mathsf{PSL}(2,\mathbb{C})$. However $\partial\Omega$ is considerably more regular hree than in the classical quasi-Fuchsian case, where the limit circle is not even rectifiable.

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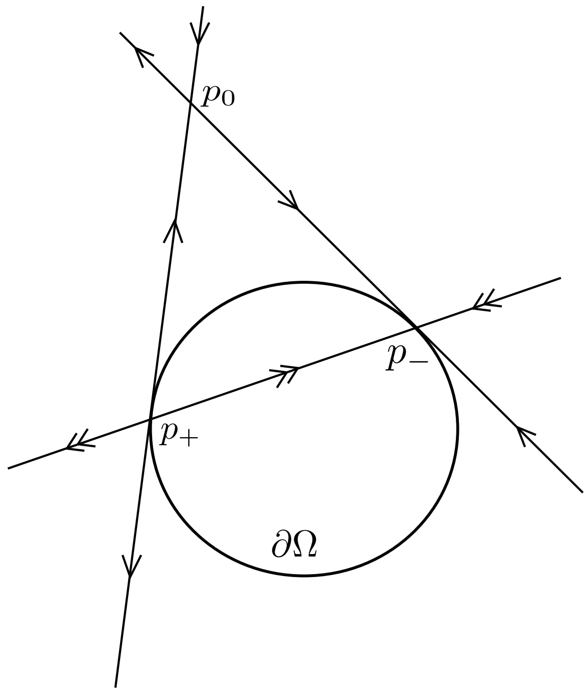

Figure 13.1. A projective transformation leaving invariant a closed convex curve.

In fact the derivative of $\partial\Omega$ is Hölder continuous with Hölder exponent strictly between 1 and 2. The Hölder exponent of the limit circle is a fascinating invariant, which for reasons of space, we do not discuss. We refer to Guichard [**179**] for some of the first work on this subject. (Compare also Exercise 13.1.2.) Recently this invariant has been related to the *entropy* of the Hilbert geodesic flow associated to the $\mathbb{R}\mathsf{P}^2$-structure. This has been vastly extended by Bridgeman–Canary–Labourie–Sambarino [**64**] to Hitchin representations, and more generally *Anosov representations* in the sense of Labourie [**242**].

13.1.1. Triangle groups. Figure 3.3 depicts the first example of a convex $\mathbb{R}\mathsf{P}^2$-manifold Ω/Γ (actually an orbifold) which is *not* homogeneous.[1] It was discovered by Kac–Vinberg [**341**]). It arises from a $(3, 3, 4)$-triangle tessellation, and Γ is the Weyl group of a Kac–Moody Lie algebra of hyperbolic

[1]It also appears on the cover of the November 2002 *Notices* of the American Mathematical Society).

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type as follows. Namely the Cartan matrix

$$C = \begin{bmatrix} 2 & -1 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

determines a *Coxeter group,* that is, a group generated by reflections.

For $i = 1, 2, 3$ let E_{ii} denote the elementary matrix having entry 1 in the i-th diagonal slot. Then, for $i = 1, 2, 3$, the reflections

$$\rho_i = I - E_{ii} C$$

generate a discrete subgroup $\Gamma < \mathsf{SL}(3, \mathbb{Z})$ which acts properly on the convex domain depicted in Figure 3.3

Recall from Exercise 2.3.3, that a reflection ρ in $\mathbb{R}\mathsf{P}^2$ is determined by its fixed set $\mathsf{Fix}(\rho)$, which is a disjoint union $\{f\} \sqcup \ell$, where f is a point and ℓ is a line not containing f. Write f_i for the isolated fixed point of ρ_i and ℓ_i for the fixed line of ρ_i. Exercise 2.5.9 imply the relations in the Coxeter group imply conditions on cross ratios of lines passing through f_i.

Exercise 13.1.1. Let $p, q, r \in \mathbb{N} \cup \{\infty\}$ and define

$$\begin{aligned} \Gamma(p,q,r) \;:=\; \langle R_1, R_2, R_3 \mid R_1^2 = R_2^2 = R_3^2 = \mathbb{I}, \\ (R_1R_2)^p = (R_2R_3)^q = (R_3R_1)^r = \mathbb{I}\rangle. \end{aligned}$$

Let $\triangle \subset \mathbb{R}\mathsf{P}^2$ with sides s_1, s_2, s_3 respectively, and consider representations

$$\rho \in \mathsf{Hom}\big(\Gamma(p,q,r), \mathsf{SL}(3,\mathbb{R})\big)$$

such that $\rho(R_i)$ is reflection fixing the line containing s_i. Furthermore assume that the differential

$$\mathsf{D}\rho(R_1, R_2)_{s_1 \cap s_2} \;\in\; \mathsf{GL}\big(\mathsf{T}_{s_1\cap s_2}\mathbb{R}\mathsf{P}^2\big)$$

is conjugate to a rotation of angle $2\pi/p$, with similar statements for $\rho(R_2R_3)$ and $\rho(R_3R_3)$, the respective vertices $s_2 \cap s_3$ and $s_3 \cap s_1$, and exponents p, q, r. (Compare Exercise 2.5.9.) Denote by $\mathcal{R}(p,q,r)$ the set of equivalence classes of such representations. Suppose that $1/p + 1/q + 1/r \leq 1$.

- Show that such a ρ determines a proper free action of $\Gamma(p,q,r)$ on an open domain $\Omega \subset \mathbb{R}\mathsf{P}^2$ with fundamental domain $\triangle$.
- Show that $\Gamma(p,q,r)$ admits torsionfree finite index subgroups Γ_f such that Ω/Γ_f is a convex $\mathbb{R}\mathsf{P}^2$-manifold.
- Compute the dimension of $\mathcal{R}(p,q,r)$.

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- If $1/p + 1/q + 1/r \leq 1$, then show $\mathcal{R}(p, q, r)$ is a cell of dimension 0 or 1 depending on the number of p, q, r equal to 2. In particular if $p, q, r \geq 3$, then $\mathcal{R}(p, q, r)$ is homeomorphic to $\mathbb{R}$.
- When $p = q = r = 3$, relate this to the tessellations described in §2.5.2.

This is one of the first examples of a *thin subgroup* of a simple Lie group. Compare also Long–Reid–Thistlethwaite [**247**], where this example is embedded in a (discrete) one-parameter family of subgroups of $\mathsf{SL}(3, \mathbb{Z})$. For the complete classification of convex $\mathbb{R}\mathsf{P}^2$-orbifolds, see Choi–Goldman [**92**]

For the deformation theory of convex structures on the three-holed sphere P see [**153**], which is the first step in the construction of coordinates on $\mathbb{R}\mathsf{P}^2_{\mathsf{convex}}(\Sigma)$.

One notable new feature in the projective theory of pants not seen for the classical case of hyperbolic structures is that the geometric structure on P is not determined by the structure ∂P. The structure at ∂P is determined by the conjugacy classes of the respective holonomies around the boundary components. The conjugacy classes range over a 2-dimensional space, giving 6 dimensions to the structures on ∂P. However the full deformation space has dimension 8, so there are two more *internal parameters* involved in the deformations of a pants. Finding geometric meaning to these internal parameters has been an intriguing and tantalizing problem. Once the boundary parameters (2 dimensions for each of the three boundary components) are prescribed, there are two internal parameters, and the *relative* deformation space is a 2-cell. See also Zhang [**355**], Wienhard–Zhang [**347**], Bonahon–Dreyer [**57**] and Bonahon–Kim [**58**] for a discussion of the internal parameters.

In terms of the parameters for triangle groups, Guichard [**179**] estimates the Hölder exponent of the limit set in terms of ratios of eigenvalues of elements, and bounds it from below. Lukyanenko [**249**] conjectures that this bound is obtained for the Coxeter element $\rho(R_1 R_2 R_3)$ in the family described in Exercise 13.1.1.

Exercise 13.1.2. Formulate and prove Lukyanenko's conjecture.

13.1.2. Generalized Fenchel–Nielsen earthquakes. One would like to develop a theory of twist flows for $\mathbb{R}\mathsf{P}^2$-manifolds analogous to the earthquake flows described in §7.3.2.

As described in §7.3.2, deformations supported in a tubular neighborhood of a curve $\mathcal{C}$ correspond to paths $\mathfrak{z}_t$ in the centralizer of the holonomy $\rho(c)$, where c is a based loop freely homotopic to $\mathcal{C}$. For example, suppose

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that $\rho(\gamma) \in \mathsf{SL}(3,\mathbb{R})$ is the diagonal matrix

$$\gamma := \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \in \mathsf{SL}(3,\mathbb{R}) \tag{13.1}$$

with $\lambda_1 > \lambda_2 > \lambda_3$. The identity component of its centralizer consists of diagonal matrices

$$A(s,t) := \begin{bmatrix} e^s & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-s-t} \end{bmatrix}$$

where $s, t \in \mathbb{R}$.

Analogous to the Fenchel–Nielsen earthquake flows discussed in §7.3.2 and the formula for the geodesic length function on $\mathfrak{F}(\Sigma)$ in terms of the invariant function ℓ discussed in Exercise 7.4.5 are functions and flows on $\mathbb{R}\mathsf{P}^2_{\mathsf{convex}}(\Sigma)$. If $\rho \in \mathsf{Hom}(\pi, \mathsf{SL}(3,\mathbb{R}))$ is the holonomy representation of a marked convex $\mathbb{R}\mathsf{P}^2$-structure on Σ, then for every $c \in \pi \setminus \{1\}$, the holonomy $\rho(c)$ conjugate to the diagonal matrix γ in (13.1). Denote the invariant open set comprising such matrices by $\mathfrak{D}^+$, and define an invariant function

$$\mathfrak{D}^+ \xrightarrow{f} \mathbb{R}_+$$
$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \longmapsto \log(\lambda_1) - \log(\lambda_3).$$

Exercise 13.1.3. Let $\rho \in \mathsf{Hom}(\pi, \mathsf{SL}(3,\mathbb{R}))$ be a holonomy representation of a marked convex $\mathbb{R}\mathsf{P}^2$-manifold M in $\mathbb{R}\mathsf{P}^2_{\mathsf{convex}}(\Sigma)$.

- Show that the function $f_c(\rho)$ is the length of the unique closed geodesic homotopic to $\mathcal{C}$ computed with respect to the Hilbert metric on M (the *Hilbert geodesic length function*).
- Compute the variation function F of f and identify the corresponding flow on $\mathbb{R}\mathsf{P}^2_{\mathsf{convex}}(\Sigma)$ in terms of the Hilbert metric on the corresponding $\mathbb{R}\mathsf{P}^2$-manifold, analogous to Exercise 7.4.6.

13.1.3. Bulging deformations. We describe here a general construction of such convex domains as limits of *piecewise conic* curves.

If Ω/Γ is a convex $\mathbb{R}\mathsf{P}^2$-manifold homeomorphic to a closed surface S with $\chi(S) < 0$, then every element $\gamma \in \Gamma$ is *positive hyperbolic,* that is,

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conjugate in $\mathsf{SL}(3,\mathbb{R})$ to a diagonal matrix of the form

$$A(s,t) := \begin{bmatrix} e^s & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-s-t} \end{bmatrix}.$$

where $s > t > -s-t$. Its centralizer is the *maximal* $\mathbb{R}$*-split torus* A consisting of all diagonal matrices in $\mathsf{SL}(3,\mathbb{R})$. It is isomorphic to the Cartesian product $\mathbb{R}^\times \times \mathbb{R}^\times$ and has four connected components. Its identity component A^+ consists of diagonal matrices with positive entries.

The orbits of H_t are arcs of conics depicted in Figure 13.2.

Associated to any measured geodesic lamination λ on a hyperbolic surface S is *bulging deformation* as an $\mathbb{R}\mathsf{P}^2$-surface. Namely, apply a one-parameter group of collineations

$$e^{-t/3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

to the coordinates on either side of a leaf. This extends Thurston's *earthquake* deformations (the analog of *Fenchel–Nielsen twist deformations* along possibly infinite geodesic laminations), and the *bending deformations* in $\mathsf{PSL}(2,\mathbb{C})$. See Bonahon–Dreyer [**57**]. This real 2-parameter family is analogous to McMullen's *quakebend* deformations [**264**].

In general, if S is a convex $\mathbb{R}\mathsf{P}^2$-manifold, then deformations are determined by a geodesic lamination with a transverse measure taking values in the Weyl chamber of $\mathfrak{sl}(3,\mathbb{R})$. When S is itself a hyperbolic surface, all the deformations in the singular directions become earthquakes and deform $\partial\tilde{S}$ trivially (just as in $\mathsf{PSL}(2,\mathbb{C})$).

Exercise 13.1.4. Let M be a marked convex $\mathbb{R}\mathsf{P}^2$-manifold with holonomy representation $\rho \in \mathsf{Hom}(\pi, \mathsf{SL}(3,\mathbb{R}))$.

- If $\mathcal{C}$ is a simple closed curve, show that the generalized Fenchel–Nielsen flow commutes with the bulging flow, generating an $\mathbb{R}^2$-action associated to $\mathcal{C}$.
- If $\mathcal{C}_1, \mathcal{C}_2$ are disjoint simple closed curves, show that the corresponding $\mathbb{R}^2$-actions commute. Therefore a pants decomposition $\mathcal{P}$ of Σ determines an $(\mathbb{R}^2)^{3g-3}$-action on $\mathbb{R}\mathsf{P}^2_{\mathsf{convex}}(\Sigma)$.
- Define a mapping

$$\mathbb{R}\mathsf{P}^2_{\mathsf{convex}}(\Sigma) \xrightarrow{(\ell,\beta)_{\mathcal{P}}} (\mathbb{R}^+ \times \mathbb{R})^{3g-3}$$

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which has the structure of a principal $(\mathbb{R}^2)^{3g-3}$-bundle with this action.

- Identify the fibers with Cartesian products of the deformation spaces of marked convex $\mathbb{R}\mathsf{P}^2$-structures on the components P_i of the complement $\Sigma \setminus \mathcal{P}$ with prescribed boundary structures (where $i = 1, \ldots, 2g-2$).

This is the analog of Fenchel–Nielsen coordinates on $\mathbb{R}\mathsf{P}^2$, once the internal parameters of the P_i are identified.

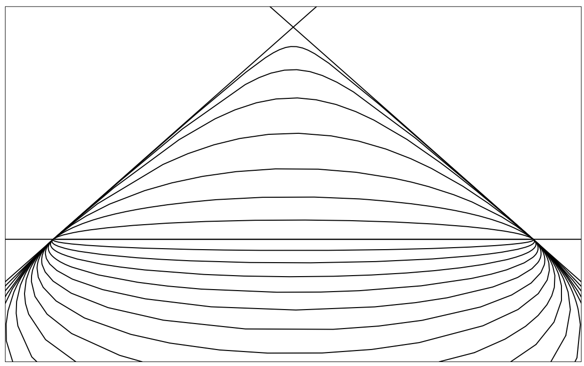

Figure 13.2. Conics tangent to a triangle

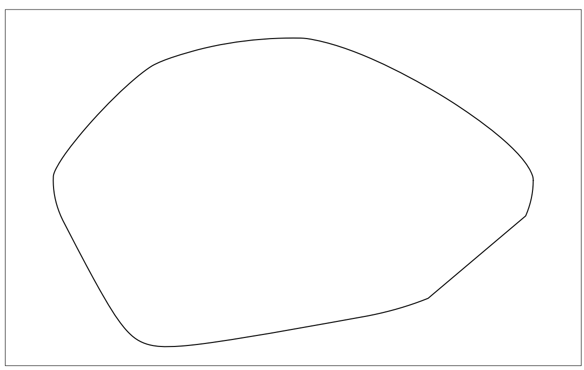

Figure 13.3. Deforming a conic

13.2. Coordinates for convex structures

To describe generalized Fenchel–Nielsen coordinates on $\mathbb{R}\mathsf{P}^2_{\mathsf{convex}}(\Sigma)$, one needs coordinates in the case Σ is a 3-holed sphere (a *pair of pants*).

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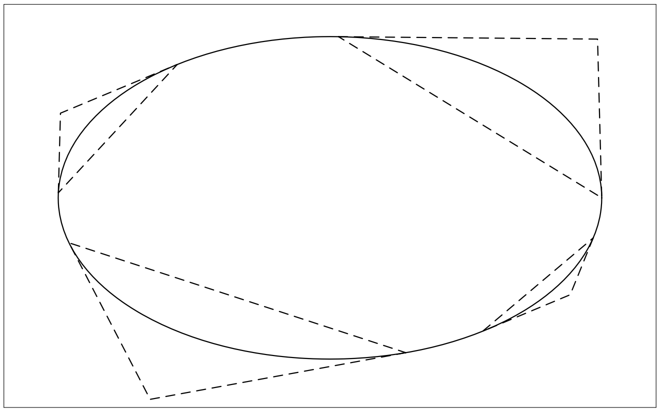

Figure 13.4. Bulging data

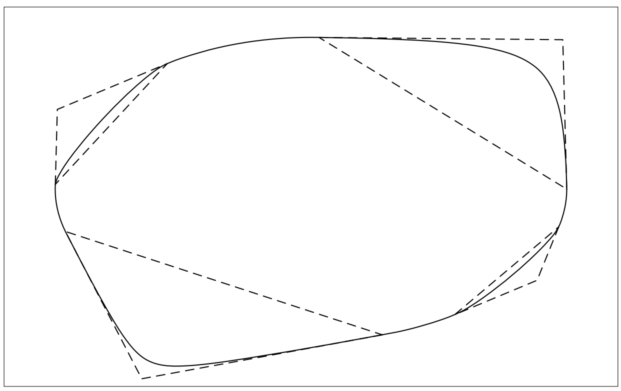

Figure 13.5. The deformed conic

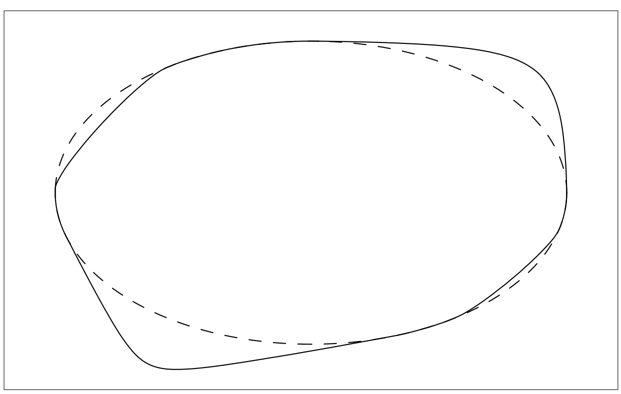

Figure 13.6. The conic with its deformation

13.2.1. Fock–Goncharov coordinates. Fock and Goncharov [**129**] develop an ambitious program for studying surface group representations into

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split $\mathbb{R}$-forms, and develop natural coordinates on certain components discovered by Hitchin [**194**] and studied by Labourie [**242**].

A version when $G = \mathsf{SL}(3, \mathbb{R})$ is developed in Fock–Goncharov [**130**], giving coordinates on the deformation space of convex $\mathbb{R}\mathsf{P}^2$-structures. Compare also Ovsienko–Tabachnikov [**284**].

A new object in their theory is the *triple ratio,* a projective invariant of three flags in $\mathbb{R}\mathsf{P}^2$ in general position. A *flag* in $\mathbb{R}\mathsf{P}^2$ is an *incident pair* (p, ℓ), where $p \in \mathbb{R}\mathsf{P}^2$ and $\ell \in (\mathbb{R}\mathsf{P}^2)^*$. Incidence here simply means that $p \in \ell$. Two flags (p_1, ℓ_1) and (p_2, ℓ_2) are in *general position* if and only if $p_i \notin \ell_j$ for $i \neq j$.

Exercise 13.2.1. Show that the projective group acts transitively on the set of pairs of flags in general position.

Now suppose that Ψ is a polarity. A Ψ-flag is a flag of the form (p, ℓ) where $\ell = \Psi(p)$. Convex $\mathbb{R}\mathsf{P}^2$-structures arising from hyperbolic structures occur when the extra flag data consist of Ψ-flags, where Ψ is the polarity corresponding to the hyperbolic structure.

Exercise 13.2.2. Show that the stabilizer of Ψ in the projective group does *not* act transitively on the set of general position Ψ-flags, and show that the quotient space is 1-dimensional, with a coordinate defined by a cross ratio

The *triple ratio* of *three* flags

$$(p_1, \ell_1),\ (p_2, \ell_2),\ (p_3, \ell_3)$$

in general position is defined as follows. Find vectors $\mathbf{v}_i$ representing p_i for $i = 1, 2, 3$ and covectors ψ_j representing ℓ_j for $j = 1, 2, 3$.

Exercise 13.2.3. Show that the scalar quantity

$$\frac{\psi_1(\mathbf{v}_2)\ \psi_2(\mathbf{v}_3)\ \psi_3(\mathbf{v}_1)}{\psi_1(\mathbf{v}_3)\ \psi_2(\mathbf{v}_1)\ \psi_3(\mathbf{v}_2)}$$

is independent of the choices of ψ_i and $\mathbf{v}_j$, and describes a complete projective invariant of triples of flags in general position.

Starting with an ideal triangulation τ of surface S, Fock and Goncharov attach parameters to the ideal simplices in τ: two for each side of a simplex (corresponding the hyperbolic conjugacy classes in $\mathsf{SL}(3, \mathbb{R})$) and a triple ratio invariant for each simplex. This gives the correct dimension and indeed a set of global coordinates for $\mathbb{R}\mathsf{P}^2_{\mathsf{convex}}(\Sigma)$.

13.3. Affine spheres and Labourie–Loftin parametrization

Another more analytic approach is due independently to Labourie [**241**] and Loftin [**245**] which we only briefly mention.

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We suppose here that Σ is a closed oriented surface of genus $g > 1$ with a marking $\Sigma \to M$, where M is a convex $\mathbb{R}\mathsf{P}^2$-manifold. Fix a holonomy homomorphism

$$\pi_1(M) \xrightarrow{\approx} \Gamma < \mathsf{SL}(3, \mathbb{R})$$

and developing map

$$\widetilde{M} \xrightarrow{\mathsf{dev}} \Omega \subset \mathbb{R}\mathsf{P}^2.$$

Associated to a convex $\mathbb{R}\mathsf{P}^2$-manifold is an Γ-equivariant lift of dev to a convex surface in the convex cone $\Omega' \subset \mathsf{A}^3$ covering Ω. The corresponding embedding

$$\widetilde{M} \xrightarrow{f} \Omega' \subset \mathbb{R}^3$$

is *convex* and transverse to the radiant vector field.

In affine differential geometry (see, e.g. Nomizu–Sasaki [**282**]), an *affine normal* at a point p in a convex surface $S \subset \mathsf{A}^3$ is the line tangent to the curve γ through p formed by the centroids of sections $S \cap P_t$ where P_t are planes parallel to the affine tangent plane $\mathsf{T}_p S$.

Exercise 13.3.1. Show that γ has a natural parametrization which can be characterized in terms of the connection on A^3 and the geometry of S.

The convex surface S is an *affine sphere,* appropriately normalized, if the affine normals to S all coincide with the position vector with respect to a point in A^3, which we identify with the origin in $\mathbb{R}^3$. For *hyperbolic affine spheres,* the affine normal vector field coincides with the radiant vector field. (There are elliptic and parabolic affine spheres as well, where the affine normals all point towards the origin or are parallel, respectively. This is the analog of *umbilic points* in Euclidean differential geometry.

By using the solution of a projectively invariant differential equation, Loftin and Labourie show that a Γ-equivariant affine sphere in A^3 exists. The equation used by Labourie and Cheng–Yau [**85**] is of Monge–Ampére type and its solutions enjoy many strong properties.

Exercise 13.3.1 implies the affine normal line field over $\widetilde{M}$ is generated by an *affine normal vector field* $\xi \in \Gamma(f^*\mathsf{T}\mathsf{A}^3)$. This defines a splitting of the trivial $\mathbb{R}^3$-bundle $f^*\mathsf{T}\mathsf{A}^3$ over M as a direct sum

$$f^*\mathsf{T}\mathbb{R}^3 \;=\; \mathsf{T}M \oplus \langle \xi \rangle. \tag{13.2}$$

Analogous to the Gauss equation in classical surface theory, decompose the ambient connection D on $f^*\mathsf{T}\mathsf{A}^3$ into an affine connection ∇ and a bilinear form h on $\mathsf{T}M$:

$$D_X\big(f_*Y\big) \;=\; f_*\nabla_X Y + h(X,Y)\xi \tag{13.3}$$

where the *affine fundamental form* h is analogous to the second fundamental form of a regular surface and ∇ is analogous to the Levi–Civita connection of

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the induced Riemannian structure (the first fundamental form). Vanishing of Tor_D implies that h is symmetric and convexity implies that h is positive definite, obtaining a Riemannian structure on M. It underlying conformal structure makes M into a Riemann surface X. From the affine sphere equation the pair (∇, h) determines a *holomorphic cubic differential* c on X. For more details see Nomizu–Sasaki [**282**] and Loftin [**245**].

Furthermore such an affine sphere is determined by the pair (X, c). In this way $\mathbb{R}\mathsf{P}^2_{\mathsf{convex}}(\Sigma)$ identifies with the holomorphic vector bundle over $\mathfrak{T}_g$ whose fibers comprise holomorphic cubic differentials (Labourie [**241**], Loftin [**245**]). In particular this fibration over $\mathfrak{T}_g$ is Mod_g-invariant.

The Vinberg metric constructed in §4 also determines a Riemann surface (by taking the conformal structure underlying the Riemannian metric) and also defines a Mod_g-equivariant mapping $\mathbb{R}\mathsf{P}^2_{\mathsf{convex}}(\Sigma) \longrightarrow \mathfrak{T}_g$. The relation between the Labourie–Loftin metric and the Vinberg metric seems intriguing.

13.4. Pathological developing maps and grafting

The nonconvex (grafted) structures have *pathological developing maps,* even on $\mathbb{T}^2$. That is, the developing maps are generally not covering spaces onto their images.

In genus one, these Smillie–Sullivan–Thurston structures have pathological developing maps onto the complement of three points in $\mathbb{R}\mathsf{P}^2$. Their radiant suspensions are affine 3-manifolds whose developing maps surject to the complement of three lines in $\mathbb{R}^3$.

The higher genus examples are constructed by grafting convex structures on higher genus surfaces with annuli. The developing maps for these $\mathbb{R}\mathsf{P}^2$-structures surject onto $\mathbb{R}\mathsf{P}^2$ and the developing maps for their radiant suspensions surject onto $\mathbb{R}^3 \setminus \{\mathbf{0}\}$.

Exercise 13.4.1. Prove these surjective developing maps are pathological. (Hint: Show that the 2-manifolds and 3-manifolds are aspherical.)

We first describe $\mathbb{R}\mathsf{P}^2$-manifolds with cyclic holonomy generated by a *positive diagonal matrix*

$$\gamma := \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \in \mathsf{SL}(3, \mathbb{R}) \tag{13.4}$$

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with $a > b > c$. The corresponding collineation fixes three points

$$p_3 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad p_2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad p_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and preserves the corresponding lines

$$l_3 := \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad l_2 := \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad l_1 := \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

The dynamics vary in the three corresponding affine patches.

$$(x, y) \xmapsto{\mathcal{A}_3} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix},$$

γ acts as the affine expansion fixing the origin $(x, y) = (0, 0) \longleftrightarrow p_3$:

$$(x, y) \xmapsto{\gamma} \begin{bmatrix} a/c\ x \\ b/c\ y \\ 1 \end{bmatrix}$$

with eigenvalues $a/c > b/c > 1$. Similarly, in the affine chart $\mathcal{A}_1$ with ideal line l_1,

$$(u, v) \xmapsto{\mathcal{A}_1} \begin{bmatrix} 1 \\ u \\ v \end{bmatrix},$$

γ acts as the affine *contraction* fixing the origin $(u, v) = (0, 0) \longleftrightarrow p_1$: having eigenvalues $1 > c/a > b/a$. Denote the corresponding affine patches $\mathsf{A}_3 := \mathbb{R}\mathsf{P}^2 \setminus l_3$ and $\mathsf{A}_1 := \mathbb{R}\mathsf{P}^2 \setminus l_1$ respectively.

The first example of a closed $\mathbb{R}\mathsf{P}^2$-manifold with pathological developing map was discovered in 1976, independently by Sullivan–Thurston [**320**] and Smillie [**310**].[2] This example lives on $\mathbb{T}^2$ and arises as follows. The collineation γ generates a discrete cyclic subgroup $\Gamma := \langle\gamma\rangle < \mathsf{PGL}(3, \mathbb{R})$ which acts properly on the complements $\mathsf{A}_1 \setminus \{p_1\}$ and $\mathsf{A}_3 \setminus \{p_3\}$ respectively. The quotients are Hopf tori modeled on the affine spaces A_1 and A_3 respectively, which we denote:

$$T_i := \big(\mathsf{A}_i \setminus \{p_i\}\big)/\Gamma \text{ for } i = 1, 3$$

[2] Actually [**310**] describes its radiant suspension.

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and regard them as $\mathbb{R}\mathsf{P}^2$-manifolds. We may choose developing maps

$$\widetilde{T}_i \xrightarrow{\mathsf{dev}_i} \mathbb{R}\mathsf{P}^2.$$

The line l_2 is invariant under Γ, and the fixed points p_1, p_3 of Γ separate l_2 into two open intervals. Choose one such interval $\mathcal{I}$; then the image

$$c_i = \Pi_i \mathsf{dev}_i^{-1}(\mathcal{I})$$

is a closed geodesic on T_i. There exist tubular neighborhoods $\mathcal{N}_i$ of $c_i \subset T_i$ for $i = 1, 3$ such that a projective isomorphism $\mathcal{N}_1 \xrightarrow{j} \mathcal{N}_3$ of $\mathbb{R}\mathsf{P}^2$-manifolds exists.

Let M be the $\mathbb{R}\mathsf{P}^2$-manifold obtained by *grafting* T_1 to T_3 along j:

$$M := (T_1|c_1) \bigcup_j (T_3|c_3).$$

Then M is an $\mathbb{R}\mathsf{P}^2$ manifold homeomorphic to a 2-torus with holonomy group Γ and whose developing map dev_M surjects onto the complement $\mathbb{R}\mathsf{P}^2 \setminus \{p_1, p_2, p_3\}$.

Exercise 13.4.2. Prove that dev_M is *not* a covering space onto its image.

Figures 13.7 and 13.8 depict developing maps for these structures. Denote by b_i a simple closed curve on T_i with trivial holonomy which intersects c_i exactly once, and transversely. Corresponding to $b_1 \cup b_2$ is a curve $b \subset M$ which is the image of

$$(b_1 \setminus c_1) \coprod (b_2 \setminus c_2) \subset (T_1|c_1) \coprod (T_3|c_3)$$

under the quotient map

$$(T_1|c_1) \coprod (T_3|c_3) \longrightarrow M.$$

Since the holonomy around b is trivial, it lifts to a simple closed curve $\widetilde{b} \subset \widetilde{M}$ and its developing image $\mathsf{dev}(\widetilde{b})$ is the figure-eight curve (a leminscate) depicted in Figure 13.7. Its images under γ and γ^{-1} are also drawn. Figure 13.7 depicts more images of this curve, suggesting how the developing map passes down to the infinite cyclic covering space of M corresponding to the kernel of the holonomy representation.

Exercise 13.4.3. Prove that dev_M is *not* a covering space onto its image.

This example gives a counterexample to the main technical lemma of [**151**], (Theorem 2.2). This lemma asserts that if M is a closed (G, X)-manifold with holonomy $\Gamma < G$ and $\Omega \subset X$ is a Γ-invariant open subset of X with a Γ-invariant complete Riemannian metric $\mathfrak{g}_\Omega$, then the region $M_\Omega \subset M$ corresponding to Ω inherits a complete Riemannian metric from $\mathfrak{g}_\Omega$. Compare Exercise5.2.6.

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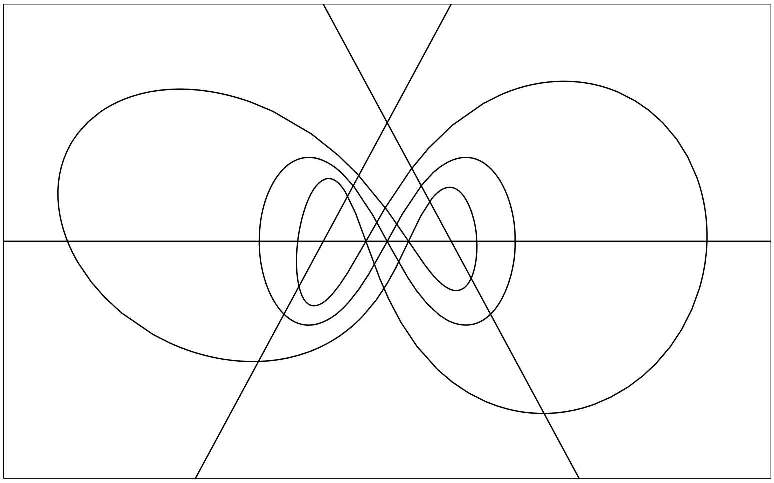

Figure 13.7. The first few iterations of the developing map of the Smillie–Sullivan–Thurston example, which is an $\mathbb{R}\mathsf{P}^2$-manifolds $M \approx \mathbb{T}^2$. The developing map is defined on a cyclic covering space $\widehat{M}$ of M which has a fundamental domain an annulus. This annulus is immersed in $\mathbb{R}\mathsf{P}^2 \setminus \{p_1, p_2, p_3\}$ winding around the fixed points p_1 and p_3 (in opposite senses). This picture shows two adjacent copies of this annulus.

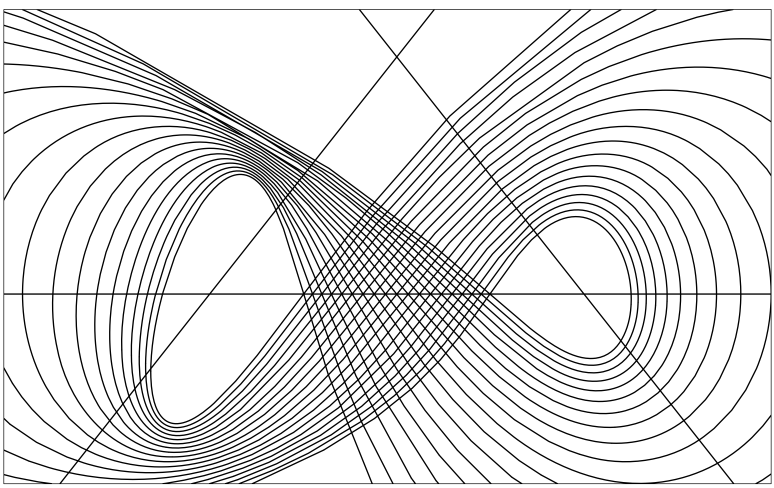

Figure 13.8. Depicted here are more iterations of the developing map, obtained by applied the holonomy transformation A to the immersion of a fundamental annulus depicted in Figure 13.7. As one moves to one end of $\widehat{M}$ the development of one part of the image approaches p_1; as one moves to the other end, the development of the other part of the images approached p_3.

Exercise 13.4.4. Find a counterexample to this assertion inside the Smillie–Sullivan–Thurston example.

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See §14.2 for a correct proof of the main theorem of [**151**].

A characterization of $\mathbb{R}\mathsf{P}^2$-manifolds with Fuchsian holonomy follows from the same arguments as for $\mathbb{C}\mathsf{P}^1$-manifolds; see [**151**] and the discussion in §14.2.

In higher dimensions, Smillie and Benoist discuss compact affine manifolds M whose holonomy lies in the subgroup $\Delta < \mathsf{GL}(n, \mathbb{R})$ of *diagonal matrices.* The Δ-invariant decomposition of $\mathbb{R}^n$ into orthants decomposes M into submanifolds modeled on the facets of a standard orthant $[0, \infty)^n \subset \mathbb{R}^n$. The n-dimensional submanifolds in this stratification are quotients of the positive orthant $(0, \infty)^n$ (which Benoist calls "bricks") and have boundary components quotients of codimension one facets. The holonomy embeds in a subgroup of Δ whose identity component (isomorphic to $(\mathbb{R}^+)^r$) acts properly on the interior and the boundary components of the brick and defines the structure of a $\mathbb{T}^r$-bundle over a Hausdorff manifold.

Exercise 13.4.5. (Smillie [**308**], Benoist [**36, 39**] For every $g > 1$, find an affine structure with diagonal holonomy on $\Sigma_g \times \mathbb{T}^2$, where Σ_g denotes a closed orientable surface of genus g.

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Chapter 14

Complex-projective structures

From the general viewpoint of locally homogeneous geometric structures, $\mathbb{C}\mathsf{P}^1$-manifolds occupy a central role. Historically these objects arose from the applying the theory of second-order holomorphic linear differential equations to conformal mapping of plane domains. Theoretically these objects seem to be fundamental in so many homogeneous spaces extend the geometry of $\mathbb{C}\mathsf{P}^1$. Furthermore $\mathbb{C}\mathsf{P}^1$-manifolds play a fundamental role in the theory of hyperbolic 3-manifolds and classical Kleinian groups.

A $\mathbb{C}\mathsf{P}^1$-manifold has the underlying structure as a *Riemann surface.* Starting from a Riemann surface M, a compatible $\mathbb{C}\mathsf{P}^1$-structure is a (holomorphic) *projective structure* on the Riemann surface M. Remarkably, projective structures on a Riemann surface M admit an extraordinarily clean classification: the deformation space of projective structures on a fixed Riemann surface M is a complex affine space whose underlying vector space is the space $H^0(M; \kappa^2)$ of *holomorphic quadratic differentials* on M. We describe this parametrization, following Gunning [**181**, **182**].

However, the geometry of the developing map can become extremely complicated, despite this clean determination of the deformation space.

An alternate synthetic-geometry parametrization of $\mathbb{C}\mathsf{P}^1(\Sigma)$ is due to Thurston, involves locally convex developments into hyperbolic 3-manifolds. In this case $\mathbb{C}\mathsf{P}^1(\Sigma)$ identifies with the product of the space $\mathfrak{F}(\Sigma)$ of marked hyperbolic structures on Σ and the *Thurston cone* $\mathcal{ML}(\Sigma)$ of *measured*

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geodesic laminations on Σ. Although Thurston's parametrization and Poincaré's parametrization have the same crude topological consequence:

$$\mathbb{C}\mathsf{P}^1(\Sigma) \approx \mathbb{R}^{12g-12},$$

they are extremely different, both in content and context. For an exposition of Thurston's parametrization, see Kamishima–Tan [**208**], and for more recent developments [**65**]. This generalizes more directly to higher dimensions, and more closely relates to topological constructions with the developing map, such as *grafting.*

The grafting construction was first developed by Hejhal [**188**] and, independently, Maskit [**260**] (Theorem 5) and Sullivan–Thurston [**320**]. As for $\mathbb{R}\mathsf{P}^2$-surfaces studied in the previous chapter, this construction yields pathological developing maps, typically local homeomorphisms from the universal covering space *onto* all of $\mathbb{C}\mathsf{P}^1$. Grafting is also responsible for the *non-injectivity* of the holonomy mapping

$$\mathbb{C}\mathsf{P}^1(\Sigma) \xrightarrow{\mathsf{hol}} \mathsf{Hom}\big(\pi_1(\Sigma), \mathsf{PSL}(2,\mathbb{C})\big)/\mathsf{PSL}(2,\mathbb{C}),$$

that is, when the holonomy representation does *not* determine the structure. This consequence of grafting was already noted in §5.5.5 for closed $\mathbb{R}\mathsf{P}^1$-manifolds. (Compare also Baba [**16–19**], Baba–Gupta [**20**], Gallo–Kapovich–Marden [**142**] and Goldman [**151**].)

We will only touch on the subject, which has a vast and ever-expanding literature. We refer to the excellent survey article of Dumas [**117**] for more details. See also Kapovich [**210**], §7, Hubbard [**197**], §6.3 and Marden [**251**], §6–8 for other perspectives on the subject.

These structures extend, in higher dimensions, to *flat conformal structures,* upon which we only discuss briefly in §15.6. We refer to the excellent survey article of Matsumoto [**261**] for more details.

Another direction in which these structures generalize is to *holomorphic projective structures* on complex manifolds. We do not discuss these structures here, referring instead to Klingler [**218**], Dumitrescu [**118**], and McKay [**263**]. Closely related is the theory of *complex Kleinian groups,* for which we refer to Cano–Navarrete–Seade [**76**], Cano–Seade [**77**] and Barrera–Cano–Navarette–Seade [**29**].

14.1. Schwarzian parametrization

For the remainder of this chapter, we denote by X the Riemann surface underlying a $\mathbb{C}\mathsf{P}^1$-manifold. Denote by P^1 the complex projective line, with automorphism group $\mathsf{PGL}(2,\mathbb{C})$. Since P^1 is the Riemann sphere $\mathbb{C}\cup\{\infty\}$, holomorphic mappings $X \to \mathsf{P}^1$ are simply meromorphic functions on X.

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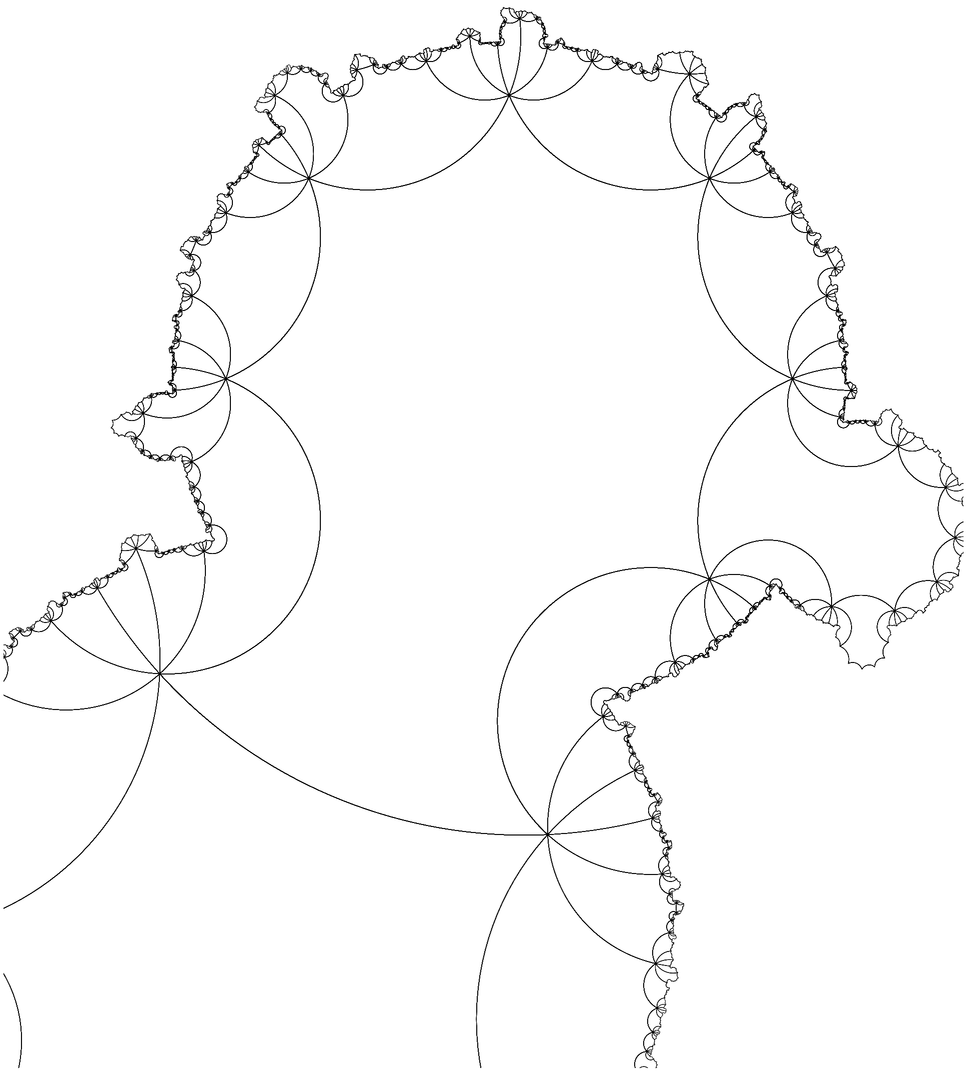

Figure 14.1. A quasi-Fuchsian structure on a genus 2 surface with an octagonal fundamental domain

Indeed, we choose a marked Riemann surface

$$\Sigma \xrightarrow{\approx} X$$

representing a point $\mathfrak{T}(\Sigma)$ (which we abusively call X) and consider the subset $\mathcal{P}(X)$ of $\mathbb{C}\mathsf{P}^1(\Sigma)$ with underlying marked Riemann surface X. This is the fiber of the forgetful map $\mathbb{C}\mathsf{P}^1(\Sigma) \longrightarrow \mathfrak{T}(\Sigma)$ over X.

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We show that $\mathcal{P}(X)$ is a (complex) affine space whose underlying vector space identifies with the vector space $H^0(X, \kappa^2)$ comprising *holomorphic quadratic differentials* on X. The key to this construction is the *Schwarzian differential* $\mathcal{S}$ which assigns to a holomorphic mapping f defined on a subset $\Omega \subset \mathsf{P}^1$ a quadratic differential $\Phi \in H^0(\Omega, \kappa^2)$. In terms of a local coordinate z, the quadratic differential is

$$\Phi = \phi(z) dz^2$$

where $\Omega \xrightarrow{\phi} \mathbb{C}$ is holomorphic. We apply $\mathcal{S}$ to the developing map to obtain the *Schwarzian parameter* Φ. This operator is $\mathsf{PSL}(2,\mathbb{C})$-invariant and satisfies a *transformation law* making it a cocycle from the pseudogroup of local biholomorphisms f defined on Ω, taking values in the presheaf $\Omega \longmapsto H^0(\Omega, \kappa^2)$ of holomorphic quadratic differentials:

$$\mathcal{S}(f \circ g) \;=\; g^*\mathcal{S}(f) \;+\; \mathcal{S}(f) \tag{14.1}$$

It *defines* the *projective sub-pseudogroup* in the sense that $\mathcal{S}(f) = 0$ if and only if f is locally projective.

14.1.1. Affine structures and the complex exponential map. We begin with the easier case of the differential operator, the *pre-Schwarzian* $\mathfrak{A}$, defining the *affine sub-pseudogroup.* Namely, let $\Omega \subset \mathsf{P}^1$ and $\Omega \xrightarrow{f} \mathsf{P}^1$ a holomorphic mapping which is a *local biholomorphism* at every $z \in \Omega$, that is,

$$f'(z) \neq 0,$$

for all $z \in \Omega$. The differential operator $\mathfrak{A}$ associates to f the *Abelian differential,* that is, the holomorphic 1-form

$$\mathfrak{A}(f) \;:=\; d\log f' \;=\; \frac{f''(z)}{f'(z)} dz, \tag{14.2}$$

defined on the affine patch (and analytically extended to the rest of Ω. If g is another local biholomorphism, the Chain Rule implies that wherever the composition $g \circ f$ is defined,

$$(g \circ f)' = (g' \circ f) \cdot f',$$

so

$$\log(g \circ f)' = \log(g' \circ f) + \log f',$$

and differentiating

$$\mathfrak{A}(f \circ g) \;=\; g^*\mathfrak{A}(f) + \mathfrak{A}(f) \tag{14.3}$$

where $g^*\big(\phi(z)dz\big)$ denotes the natural action of g on the Abelian differential $\phi(z)dz$:

$$g^*\big(\phi(z)dz\big) := \phi\big(g(z)\big)g'(z)dz$$

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The transformation law (14.3) asserts that $\mathfrak{A}$ is a *cocycle* from the pseudogroup of biholomorphisms to the presheaf $\Omega \mapsto H^0(\Omega, \kappa)$ of Abelian differentials.

The cocycle condition (14.3) asserts $\mathsf{Ker}(\mathfrak{A})$ is closed under composition (wherever defined) and defines a subpseudogroup. These are the *locally affine* mappings defined by $f''(z) = 0$.

Furthermore, if X is a Riemann surface and $X \xrightarrow{f} \mathbb{C}$ is a local biholomorphism, (14.3) implies that the restrictions $\mathfrak{A}(f|_U)$ to coordinate patches $U \subset \Omega$ extend to a globally defined Abelian differential $\mathfrak{A}(f) \in H^0(X; \kappa)$.

14.1.2. Projective structures and quadratic differentials. Now we deduce the form of the Schwarzian from its projective invariance, using the cocycle property of the differential operator $\mathfrak{A}$ and the Bruhat decomposition of the projective group $G := \mathsf{PSL}(2,\mathbb{C})$. Our strategy is to reduce the calculation of $\mathfrak{A}(g)$ for general $g \in G$ to that of $\mathfrak{A}(\mathfrak{J})$ where $\mathfrak{J}(z) := 1/z$

Exercise 14.1.1. The projective group G is generated by the affine group $\mathfrak{B} := \mathsf{Aff}(1,\mathbb{C})$ (its Borel subgroup) and the inversion

$$z \overset{\mathfrak{J}}{\longmapsto} -1/z,$$

(the generator of the Weyl group).

- Deduce the *Bruhat decomposition*

  $$G \;=\; \mathfrak{B} \coprod \mathfrak{B}\,\mathfrak{J}\,\mathfrak{B}$$

  explicitly from the following identity in $\mathsf{SL}(2,\mathbb{C})$:

  $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1/c & a \\ c & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix}$$

  when $c \neq 0$ and $ad - bc = 1$.
- Show that

  $$\mathfrak{A}(\mathfrak{J}) = \frac{-2}{z} dz.$$
- Suppose

  $$g(z) = \frac{az+b}{cz+d}.$$

  Then

  $$\mathfrak{A}(g) \;=\; \frac{-2}{z + d/c} dz.$$
- Define a differential operator:

  $$H^0(\Omega,\kappa) \xrightarrow{\mathsf{D}} H^0(\Omega,\kappa^2)$$
  $$\phi(z)dz \longmapsto \phi'(z)dz^2.$$

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Show that

$$\mathsf{D}\mathfrak{A}(g) - \frac{1}{2}\mathfrak{A}(g)^2 \;=\; 0. \tag{14.4}$$

The formula (14.4) is reminiscent of the expression of the curvature of a connection in terms of a connection 1-form and the *Fundamental Theorem of Calculus* for Lie group-valued differential forms (Sharpe [**305**]).

Now define the *Schwarzian differential*

$$\mathsf{Bihol}(\Omega, \mathsf{P}^1) \xrightarrow{\mathcal{S}} H^0(\Omega, \kappa^2)$$
$$g \longmapsto \mathsf{D}\mathfrak{A}(g) - \frac{1}{2}\mathfrak{A}(g)^2 \;=\; \left\{ \left(\frac{g''}{g'}\right)' - \frac{1}{2}\left(\frac{g''}{g'}\right)^2 \right\} dz^2$$

where $\mathsf{Bihol}(\Omega, \mathsf{P}^1)$ comprises holomorphic maps $\Omega \longrightarrow \mathsf{P}^1$ which are *local biholomorphisms,* that is meromorphic functions g on Ω for which $g'(z) \neq 0$ everywhere on Ω. (The usual *Schwarzian derivative,* denoted classically by $\{g, z\}$, is the coefficient

$$\left(\frac{g''}{g'}\right)' - \frac{1}{2}\left(\frac{g''}{g'}\right)^2 \;=\; \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2$$

of the Schwarzian derivative $\mathcal{S}(g)$.

Exercise 14.1.2. Prove the cocycle property (14.1).

Although (14.1) follows from a slightly messy but straightforward calculation (see Gunning [**181**] for example), a more conceptual treatment is discussed in Hubbard [**197**]. If $g \in \mathsf{Bihol}(\Omega, \mathsf{P}^1)$ is a local biholomorphism, and $z_0 \in \Omega$, denote the element of G which agrees with $g \in G$ to *second order* at z_0 by $\mathcal{O}(g)_{z_0} \in G$, the *osculating Möbius transformation* for g at z_0.

Explicitly, write

$$g(z) \;=\; a_0 \,+\, a_1(z - z_0) \,+\, \frac{a_2}{2}(z - z_0)^2 \,+\, \frac{a_3}{6}(z - z_0)^3 \\ + \cdots + \frac{a_n}{n!}(z - z_0)^n \,+ \dots$$

with $a_0 = g(z_0)$, $a_1 = g'(z_0) \neq 0$, $a_2 = g''(z_0)$ and $a_3 = g'''(z_0)$. The Möbius transformation

$$z \xmapsto{\mathcal{O}(g)_{z_0}} \frac{a_1 z}{1 + (a_2/a_1) z}$$

osculates g at z_0 and

$$f(z) \;=\; \mathcal{O}(g)_{z_0}(z) + \left(\frac{a_3}{6} - \frac{a_2^2}{4a_1}\right) z^3 + \dots$$

The leading term equals $1/a_1$ times the coefficient of $\mathcal{S}(g)$ at z_0, that is, the Schwarzian derivative of g at z_0. From this conceptual description of $\mathcal{S}(g)$ follows the cocycle property (14.1).

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14.1.3. Solving the Schwarzian equation. To show that $\mathsf{P}(X)$ is an $H^0(X,\kappa^2)$-torsor, one must show that for any holomorphic quadratic differential $\Phi = \phi(z)dz^2$, a meromorphic function $f \in \mathsf{Bihol}(X, \mathsf{P}^1)$ exists with $\mathcal{S}(f) = \Phi$. Furthermore f is unique up to a Möbius transformation. That is, given ϕ, find f solving the nonlinear third order differential equation

$$\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 \;=\; \phi. \tag{14.5}$$

The solution involves relating this nonlinear equation to the second order *linear* differential equation

$$u''(z) + \frac{1}{2}\phi(z)u(z) \;=\; 0 \tag{14.6}$$

often called Hill's equation or a Sturm–Liouville equation.

Exercise 14.1.3. Define the *projective solution* $\widetilde{X} \xrightarrow{f} \mathsf{P}^1$ of (14.6) as follows. Every $x \in X$ has a neighborhood Ω such that the solutions of the linear equation (14.6) on Ω forms a 2-dimensional (complex) vector space. Choose a basis $u_1(z), u_2(z)$ of this vector space and define

$$\begin{aligned} \Omega &\xrightarrow{f} \mathsf{P}^1 \\ z &\longmapsto u_1(z)/u_2(z) \end{aligned}$$

- Define the *monodromy representation*

  $$\pi_1(X) \longrightarrow \mathsf{GL}(2,\mathbb{C})$$

  of (14.6) by analytically continuing the local solutions u_1, u_2 over loops in X.
- f is well-defined up to composition with a projective automorphism $\mathsf{P}^1 \to \mathsf{P}^1$.
- The derivative $f'(z) \neq 0$ for all $z \in \Omega$.
- $\mathcal{S}(f) = \phi(z)dz^2$.
- The projective solution f on Ω analytically continues to a local biholomorphism

  $$\widetilde{X} \xrightarrow{\widetilde{f}} \mathsf{P}^1$$

  equivariant with respect to the projectivization $\pi_1(X) \longrightarrow \mathsf{PGL}(2,\mathbb{C})$ of the monodromy representation of (14.6).

14.2. Fuchsian holonomy

A related idea is the classification of projective structures with Fuchsian holonomy [**151**]. This is the converse to the *grafting construction,* whereby grafting is the only construction yielding geometric structures with the same

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holonomy. Recall that a *Fuchsian representation* of a surface group π is an embedding of π as a discrete subgroup of the group $\mathsf{PGL}(2,\mathbb{R}) \cong \mathsf{Isom}(\mathsf{H}^2)$. Equivalently, ρ is the holonomy representation of a hyperbolic structure on a surface Σ with $\pi_1(\Sigma) \cong \pi$. The main result is:

Theorem 14.2.1 (Grafting Theorem)**.** Let M be a closed $\mathbb{C}\mathsf{P}^1$-manifold whose holonomy representation $\pi_1(M) \xrightarrow{\rho} \mathsf{PSL}(2,\mathbb{C})$ is a composition

$$\pi_1(M) \xrightarrow{\rho_0} \mathsf{PGL}(2,\mathbb{R}) \hookrightarrow \mathsf{PSL}(2,\mathbb{C})$$

where ρ_0 is Fuchsian representation. Let M_0 be a hyperbolic structure with holonomy representation ρ_0, regarded as a $\mathbb{C}\mathsf{P}^1$-manifold. Then there is a unique multicurve $S \subset M_0$ such that M is obtained from M_0 by grafting along S.

Also in [**151**] is a characterization of $\mathbb{C}\mathsf{P}^1$-manifolds whose holonomy is a (not necessarily Fuchsian) representation into $\mathsf{PGL}(2,\mathbb{R})$. Such structures are obtained from hyperbolic structures by *folding* along simple closed geodesics. That is, the local charts/developing maps/developing sections have *fold singularities,* which can be interpreted as *bending deformations* in the sense of Thurston, Kamishima–Tan [**208**], Kulkarni–Pinkall [**237**]. The "normal form" in Gallo–Kapovich–Marden [**142**] implies that every nonelementary $\mathsf{PSL}(2,\mathbb{R})$-representation of even Euler number arises from a hyperbolic structure folded along closed geodesics.

However the proof of Theorem 14.2.1 contained a gap, which I first learned from M. Kapovich, who pointed me to the paper of Kuiper [**232**] for clarification (see also Kuiper [**234**]). This gap[1] was later filled by Choi–Lee [**96**]. Here we give a correct proof, based on Kulkarni–Pinkall [**237**] (Theorem 4.2). See also Dupont [**119**].

A counterexample to the main lemma in the proof was described in Exercise 13.4.4.

The grafting theorem uses the setup of Exercise 5.2.6, which we briefly recall. Let M be a connected smooth manifold with a universal covering space $\widetilde{M} \xrightarrow{\Pi} M$ with covering group $\pi = \pi_1(M)$. Give M a (G,X)-structure and let (dev,ρ) be a developing pair:

- $\widetilde{M} \xrightarrow{\mathsf{dev}} X$ denotes the developing map;
- $\pi \overset{\rho}{\twoheadrightarrow} \Gamma < G$ denotes the holonomy representation.

Suppose that $\Omega \subset X$ is a Γ-invariant open subset and

$$M_\Omega := \Pi(\mathsf{dev}^{-1}\Omega) \subset M$$

the subdomain of M corresponding to Ω as in Exercise 5.2.6.

[1]The same gap can be found in Goldman–Kamishima [**144**] and Faltings [**127**].

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The author is grateful to Daniele Alessandrini for patiently explaining the details of the following basic result. Recall from §2.6.5 that the *normality domain* $\mathsf{Nor}(\Gamma, X)$ consists of points having open neighborhoods U such that

$$\Gamma|_U := \{\gamma|_U \mid g \in \Gamma\} \subset\subset \mathsf{Map}(U, X).$$

Theorem 14.2.2 (Kulkarni–Pinkall [**237**], Theorem 4.2)**.** Let M be a closed (G, X)-manifold with holonomy group $\Gamma < G$. Let $\Omega \subset \mathsf{Nor}(\Gamma, X)$ be a Γ-invariant connected open subset of the normality domain, and $M_\Omega \subset M$ the corresponding r egion of M. Then for each component $W \subset M_\Omega$, and each component $\widetilde{W}$ of $\Pi^{-1}(W)$, the restriction

$$\widetilde{W} \xrightarrow{\mathsf{dev}|_{\widetilde{W}}} \Omega$$

is a covering space. In particular $\mathsf{dev}|_{\widetilde{W}}$ is onto.

The proof of Theorem 14.2.2 breaks into a sequence of lemmas. We show that $\mathsf{dev}|_{\widetilde{W}}$ satisfies the *path-lifting criterion* for covering spaces: every path in Ω lifts to a path in $\widetilde{W}$. Specifically, let

$$[0, 1] \xrightarrow{\gamma} \Omega$$

be a path in Ω and consider a point

$$\widetilde{w}_0 \in \Pi^{-1}\big(\gamma(0)\big) \cap \widetilde{W}$$

We seek a path

$$[0, 1] \xrightarrow{\widetilde{c}} \widetilde{W}$$

satisfying:

- $\widetilde{c}(0) = \widetilde{w}_0$;
- $\mathsf{dev} \circ \widetilde{c} = \gamma$.

Since dev is a local homeomorphism, the set T of $t \in [0, 1]$ such that $[0, t] \xrightarrow{\gamma|_{[0,t]}} \Omega$ lifts to

$$[0, t] \xrightarrow{\widetilde{c}|_{[0,t]}} \widetilde{W}$$

is open. Since dev is a local homeomorphism and $\widetilde{W}$ is Hausdorff, any extension $\widetilde{c}|_{[0,t]}$ of the lift to $[0, t]$ is necessarily unique. Thus T is a *connected* open neighborhood of 0 in $[0, 1]$. Reparametrize $\widetilde{c}$ so that $\widetilde{c}$ is defined on $[0, 1)$. It suffices to show that $\widetilde{c}$ lifts to $[0, 1]$.

Let $c = \Pi \circ \widetilde{c}$ be the corresponding curve in M. Since M is compact and

$$[0, 1) \xrightarrow{c} \Pi(\widetilde{W}) \subset M,$$

the curve c *accumulates* in M. That is, $\exists$ a sequence $t_n \in [0, 1)$ with $t_n \nearrow 1$ and $\exists z \in M$ such that

$$\lim_{n\to\infty} c(t_n) = z.$$

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Employ z as the basepoint in M. Fix the corresponding universal covering space $\widetilde{M} \xrightarrow{\Pi} M$, where $\widetilde{M}$ comprises relative homotopy of paths γ in M starting at z. Recall that the deck transformation of $\widetilde{M}$ corresponding to the relative homotopy class $[\beta] \in \pi_1(M, z)$ of a loop β based at z is:

$$[\gamma] \longmapsto [\gamma \star \beta].$$

Choose a developing map $\widetilde{M} \xrightarrow{\mathsf{dev}} X$.

Let $U \ni z$ be an evenly covered coordinate patch[2] in M such that the restriction of dev to some (and hence every) component of $\Pi^{-1}(U)$ is a homeomorphism. Passing to a subsequence if necessary, we may assume that $c(t_n) \in U$ for all n. Choose paths

$$z \overset{\alpha_n}{\rightsquigarrow} c(t_n)$$

in U. The concatenation

$$\alpha_1 \ \star \ c|_{[t_1,t_n]} \ \star \ \alpha_n^{-1}$$

is a based loop β_n in M having relative homotopy class $[\beta_n] \in \pi_1(M, z)$. Let $\widetilde{U}$ be the component of $\Pi^{-1}(U)$ containing $\widetilde{c}(t_1)$, and $\widetilde{z}$ the unique element of $\widetilde{U} \cap \Pi^{-1}(z)$. To simplify notation, denote the deck transformation $[\beta_n]$ by β_n.

Lemma 14.2.3.

$$\lim_{n \longrightarrow \infty} \rho(\beta_n)^{-1} \gamma(t_n) \ = \ \mathsf{dev}(\widetilde{z}).$$

Proof. For $n \gg 1$, each $c(t_n) \in U$. The definition of the deck transformation β_n implies that $\widetilde{c}(t_n) \in \beta_n \widetilde{U}$. Hence

$$\beta_n^{-1} \widetilde{c}(t_n) \in \widetilde{U}.$$

Since $c(t_n) \longrightarrow z$ and $\Pi|_{\widetilde{U}}$ is bijective,

$$\lim_{n \to \infty} \beta_n^{-1} \widetilde{c}(t_n) = \widetilde{z}.$$

Continuity of dev implies:

$$\begin{aligned} \lim_{n \to \infty} \rho(\beta_n^{-1}) \gamma(t_n) = \lim_{n \to \infty} \rho(\beta_n^{-1}) \mathsf{dev}(\widetilde{c}(t_n)) &= \lim_{n \to \infty} \mathsf{dev}\big(\beta_n^{-1} \widetilde{c}(t_n)\big) \\ &= \mathsf{dev}\Big(\lim_{n \to \infty} \beta_n^{-1} \widetilde{c}(t_n) \Big) = \mathsf{dev}(\widetilde{z}), \end{aligned}$$

as claimed. ☐

Conclusion of the proof of Theorem 14.2.2. Apply the condition of normality to the images

$$\rho(\beta_n)^{-1} \circ \gamma$$

[2] in the sense of Greenberg–Harper [**172**] , p.21.

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of the curve $[0,1] \xrightarrow{\gamma} \Omega$. By the definition of Ω, each $\gamma(s)$ has an open neighborhood U_s for which the set of restrictions $\rho(\beta_n)|_{U_s}$ is precompact in $\mathsf{Map}(U_s, X)$. Compactness of $[0,1]$ guarantees that finitely many s_i exist so that the U_{s_i} cover $[0,1]$. It follows that the images $\rho(\beta_n)^{-1} \circ \gamma$ form a precompact sequence in $\mathsf{Map}\big([0,1], X\big)$. After passing to a subsequence, we may assume that $\rho(\beta_n)^{-1} \circ \gamma$ converges uniformly to a continuous map $[0,1] \xrightarrow{\delta} X$.

Now apply Lemma 14.2.3, using the uniform convergence

$$\rho(\beta_n)^{-1} \circ \gamma \rightrightarrows \delta,$$

obtaining

$$\lim_{t \to 1} \mathsf{dev}\big(\widetilde{c}(t)\big) = \delta(1) \in \mathsf{dev}(\widetilde{U}).$$

Since $\mathsf{dev}|_{\widetilde{U}}$ is injective, defining

$$\widetilde{c}(1) := \big(\mathsf{dev}|_{\beta_N \widetilde{U}}\big)^{-1}\big(\delta(1)\big)$$

is the desired continuous extension of $\widetilde{c}$. The proof of Theorem 14.2.2 is complete. □

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Chapter 15

Geometric structures on 3-manifolds

This final chapter collects results on geometric structures on closed 3-manifolds.

The general theory is very much in its infancy. Many innocent-sounding questions seem to be inaccessible with present tools. To my knowledge, the only known example of a closed 3-manifold known *not* to admit an $\mathbb{R}\mathsf{P}^3$-structure is the connected sum $\mathbb{R}\mathsf{P}^3 \# \mathbb{R}\mathsf{P}^3$ (Cooper–Goldman [**99**]).[1] Flat conformal structures are known not to exist on closed 3-manifolds modeled on Heis and Sol (Goldman [**148**] and Schoen–Yau [**300**]). Spherical CR-structures are known *not* to exist on manifolds 3-manifolds modeled on Euclidean space and Sol ([**148**]).

A bit more is known about affine structures on 3-manifolds. Compact complete affine 3-manifolds were discussed in §10.6, following Fried–Goldman [**139**]. The present chapter ends with a brief survey of the classification of *noncompact* complete affine 3-manifolds — the theory of *Margulis spacetimes.* The final word on this due to Danciger–Guéritaud–Kassel [**108**], who classified Margulis spacetimes by the Cartesian product of the Fricke space of a noncompact hyperbolic surface and its arc complex ([**105**].)

Radiant affine 3-manifolds are radiant suspensions $\mathbb{R}\mathsf{P}^2$-manifolds (Barbot–Choi [**23**, **90**]). Similar techniques to Choi's convex decomposition theorem (§13.1) extend to effectively classify radiant affine 3-manifolds.

[1] This also follows from the classification of $\mathbb{R}\mathsf{P}^3$ manifolds with nilpotent holonomy (Benoist [**39**]) by taking a double covering.

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Projective structures on 3-manifolds with solvable fundamental group are well understood, and this chapter begins with a discussion of these manifolds. The most difficult case occurs for hyperbolic torus bundles (quotients of Sol) and we briefly summarize the Dupont's classification [**120**] of these manifolds.

15.1. Affine 3-manifolds with nilpotent holonomy

The structure theory developed in §11.2 applies, and yields the following topological classification.[2]

Exercise 15.1.1. Let M^3 be a closed affine 3-manifold with virtually nilpotent affine holonomy group. Show that M is finitely covered by one of the following manifolds:

- A torus $\mathbb{T}^3$;
- A nilmanifold $\mathsf{Heis}_{\mathbb{Z}}\backslash\mathsf{Heis}_{\mathbb{R}}$;
- $\mathbb{S}^3$
- $\mathbb{S}^3 \times \mathbb{S}^1$;

In particular, the fundamental group $\pi_1(M)$ is virtually nilpotent.

Geometrically, such an affine 3-manifold can be perturbed to an affine bundle over a complete affine nilmanifold with radiant fibers. Unlike 2-dimensional affine manifolds, the developing maps may fail to be covering, namely the Smillie–Sullivan–Thurston examples discussed in §13.4). However they admit a Smillie–Benoist decomposition into well understood pieces.

In particular the developing image stratifies in flat affine submanifolds, which is no longer true for hyperbolic torus bundles.

15.2. Dupont's classification of hyperbolic torus bundles

A closed 3-manifold M where $\pi_1(M)$ is solvable but not virtually nilpotent is finitely covered by a hyperbolic torus bundle (Evans–Moser [**125**]. The projective structures on hyperbolic torus bundles were classified by Dupont [**120**]):

Theorem 15.2.1 (Dupont[**120**])**.** Let M^3 be a hyperbolic torus bundle with an $\mathbb{R}\mathsf{P}^3$ structure. The projective structure is affine and M^3 is a quotient $\Gamma\backslash G$ where G is an affine Lie group isomorphic to $\mathsf{Sol}_{\mathbb{R}}$ and Γ is a lattice in G. Furthermore the developing image is one of the following:

- All of A^3 (M is complete);
- A halfspace in A^3;

[2] A group is *virtually nilpotent* if and only if it has a nilpotent subgroup of finite index.

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- The product with A^1 with a convex parabolic domain

$$\mathcal{P} \;=\; \big\{(x,y) \in \mathsf{A}^2 \mid y > x^2\big\};$$

- The product with A^1 with a concave parabolic domain

$$\mathsf{A}^2 \setminus \mathcal{P} \;=\; \big\{(x,y) \in \mathsf{A}^2 \mid y < x^2\big\}.$$

In particular, a developing map embeds $\widetilde{M}$ as one of these domains and the holonomy homomorphism is an isomorphism $\pi_1(M) \xrightarrow{\cong} \Gamma < G$.

Although the class of affine structures on closed 3-manifolds with *nilpotent* holonomy is understood, the general case of *solvable* holonomy remains mysterious. Theorem 15.2.1 is a first step in this direction. A key ingredient in its proof is the classification of affine structures on closed manifolds whose holonomy factors through $\mathsf{Aff}_+(1,\mathbb{R})$ in Dupont [**119**]. The holonomy must factor through a 1-dimensional subgroup, and is necessarily cyclic.

Two structures are particularly interesting for the behavior of geodesics in light of the results of Vey [**339**]. Recall that a properly convex domain $\Omega \subset \mathsf{A}^n$ is *divisible* if it admits a discrete group $\Gamma < \mathsf{Aut}(\Omega)$ acting properly such that Ω/Γ is compact. (Equivalently, the quotient space Ω/Γ by a discrete subgroup $\Gamma \subset \mathsf{Aut}(\Omega)$ is compact and Hausdorff.) Vey proved that a divisible domain is a cone. However, dropping the properness of the action of Γ on Ω allows counterexamples: $\mathcal{P}$ is a properly convex domain which is not a cone, but admits a group Γ of automorphisms such that $\mathcal{P}/\Gamma$ is compact but not Hausdorff.[3]

Now take the product $\mathcal{P} \times \mathbb{R} \subset \mathsf{A}^3$. Let $G < \mathsf{Aff}(\mathsf{A}^3)$ be the subgroup acting simply transitively (isomorphic to the 3-dimensional unimodular exponential non-nilpotent solvable Lie group discussed in §10.8.1), and let $\Gamma < G$ be a lattice. Then Γ acts properly on $\mathcal{P} \times \mathbb{R}$ and:

- The quotient $M = (\mathcal{P} \times \mathbb{R})/\Gamma$ is a hyperbolic torus bundle (and in particular compact and Hausdorff);
- $\mathcal{P} \times \mathbb{R}$ is not a cone.

$\mathcal{P} \times \mathbb{R}$ is not properly convex, so Vey's result is sharp [**147**].

The Kobayashi pseudometric degenerates along a 1-dimensional foliation of M, and defines the hyperbolic structure transverse to this foliation discussed by Thurston [**323**], Chapter 4.

[3] Exercise 4.1.1 and §10.5.6 describe geometric properties and Lie algebraic properties of $\mathcal{P}$ respectively.

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15.3. Complete affine 3-manifolds

A more extensive recent survey of this subject is Danciger–Drumm–Goldman–Smilga [**105**].

A more extensive recent survey of this subject is Danciger–Drumm–Goldman–Smilga [**105**]. According to [**139**], a *compact* complete affine 3-manifold is finitely covered by a complete affine solvmanifold (see §8.6.2). In turn, these are classified by left-invariant complete affine structures on 3-dimensional Lie groups (see Chapter 10).

The first step is the Auslander–Milnor question: Namely, if $M^3 = \mathsf{A}^3/\Gamma$, show that Γ is solvable. Let $\mathbb{A}(\Gamma)$ denote the Zariski closure of Γ in $\mathsf{Aff}(\mathsf{A}^3)$; clearly it suffices to show that $\mathbb{A}(\Gamma)$ is solvable. This is equivalent to showing that the Zariski closure $\mathbb{A}\big(\mathsf{L}(\Gamma)\big)$ in $\mathsf{GL}(\mathbb{R}^3)$ of the linear holonomy group $\mathsf{L}(\Gamma) \subset \mathsf{GL}(\mathbb{R}^3)$ is solvable. The proof is a case-by-case analysis of the possible Levi factors of $\mathbb{A}\big(\mathsf{L}(\Gamma)\big)$.

Complete affine structures on Euclidean 3-manifolds are classified using 3-dimensional commutative associative algebras; see Exercise 10.6.3.

In his 1977 paper [**272**], Milnor set the record straight caused by the confusion surrounding Auslander's flawed proof of Conjecture 8.6.6. Influenced by Tits's work [**329**] on free subgroups of linear groups and amenability, Milnor observed, that for an affine space A of given dimension, the following conditions are all equivalent:

- Every discrete subgroup of $\mathsf{Aff}(\mathsf{A})$ which acts properly on A is amenable.
- Every discrete subgroup of $\mathsf{Aff}(\mathsf{A})$ which acts properly on A is virtually solvable.
- Every discrete subgroup of $\mathsf{Aff}(\mathsf{A})$ which acts properly on A is virtually polycyclic.
- A nonabelian free subgroup of $\mathsf{Aff}(\mathsf{A})$ admits no proper action on A.
- The Euler characteristic $\chi(\Gamma\backslash\mathsf{A})$ (when defined) of a complete affine manifold $\Gamma\backslash\mathsf{A}$ must vanish (unless $\Gamma = \{1\}$ of course).
- A complete affine manifold $\Gamma\backslash\mathsf{A}$ has finitely generated fundamental group Γ.

(If these conditions were met, one would have a satisfying structure theory, similar to, but somewhat more involved, than the Bieberbach structure theory for flat Riemannian manifolds.)

In [**272**], Milnor provides abundant "evidence" for this "conjecture". For example, let $G \subset \mathsf{Aff}(\mathsf{A})$ be a connected Lie group which acts properly on

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A. Then G must be an amenable Lie group, which simply means that it is a compact extension of a solvable Lie group. (Equivalently, its Levi subgroup is compact.) Furthermore, he provides a *converse:* Milnor shows that every virtually polycyclic group admits a proper affine action.[4]

However convincing as his "evidence" is, Milnor still proposes a possible way of constructing counterexamples:

> *"Start with a free discrete subgroup of* $\mathsf{O}(2,1)$ *and add translation components to obtain a group of affine transformations which acts freely. However it seems difficult to decide whether the resulting group action is properly discontinuous."*

This is clearly a geometric problem: Schottky showed in 1907 that free groups act properly by isometries on hyperbolic 3-space, and hence by diffeomorphisms of A^3. These actions are *not* affine.

One might try to construct a proper affine action of a free group by a construction like Schottky's. Recall that a *Schottky group of rank g* is defined by a system of g open half-spaces $H_1, \dots, H_g$ and isometries $A_1, \dots, A_g$ such that the $2g$ half-spaces

$$H_1, \dots, H_g, A_1(H_1^c), \dots A_g(H_g^c)$$

are all disjoint (where H^c denotes the *complement* of the closure $\bar{H}$ of H). The *slab*

$$\mathsf{Slab}_i := H_i^c \cap A_i(H_i)$$

is a fundamental domain for the action of the cyclic group $\langle A_i \rangle$. The *ping-pong lemma* then asserts that the intersection of all the slabs

$$\Delta := \mathsf{Slab}_1 \cap \cdots \cap \mathsf{Slab}_g$$

is a fundamental domain for the group $\Gamma := \langle A_1, \dots, A_g \rangle$.

Furthermore $A_1, \dots, A_g$ freely generate Γ. The basic idea is the following. For each $i = 1, \dots, g$ choose a halfspace H_i and define $B_i^+ := A_i(H_i^c)$ (respectively $B_i^- := H_i$). We assume that *all the* $B_i^\pm$ *are disjoint.* Then A_i maps all of H_i^c to B_i^+ and A_i^{-1} maps all of $A_i(H_i)$ to B_i^-. Let $w(a_1, \dots, a_g)$ be a reduced word in abstract generators $a_1, \dots, a_g$, with initial letter $a_i^\pm$. Then

$$w(A_1, \dots, A_g)(\Delta) \subset B_i^\pm.$$

Since all the $B_i^\pm$ are disjoint, $w(A_1, \dots, A_g)$ maps Δ off itself. Therefore $w(A_1, \dots, A_g) \neq 1$.

[4]Milnor's actions do *not* have compact quotient. Benoist [**35, 37**] found finitely generated nilpotent groups which admit no affine crystallographic action. Benoist's examples are 11-dimensional and filiform.

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Freely acting discrete cyclic groups of affine transformations have fundamental domains which are *parallel slabs,* that is, regions bounded by two parallel affine hyperplanes. One might try to combine such slabs to form "affine Schottky groups," but immediately one sees this idea is doomed: parallel slabs have disjoint complements only if they are parallel to each other, in which case the group is necessarily cyclic anyway. From this viewpoint, a discrete group of affine transformations seems very unlikely to act properly.

15.4. Margulis spacetimes

In the early 1980's Margulis, while trying to prove that a nonabelian free group can't act properly by affine transformations, discovered that discrete free groups of affine transformations can indeed act properly!

Around the same time, Fried and I tried to extend our classification [**139**] of complete affine structures on *compact* 3-manifolds to general 3-manifolds. We encountered what seemed at the time to one annoying case which we could not handle. However, we were able to show the following:

Proposition 15.4.1. Let A be a 3-dimensional affine space and $\Gamma < \mathsf{Aff}(\mathsf{A})$ a discrete subgroup. Suppose that Γ acts properly on A. Then either Γ is polycyclic or the restriction of the linear holonomy homomorphism

$$\Gamma \xrightarrow{\mathsf{L}} \mathsf{GL}(\mathsf{A})$$

discretely embeds Γ onto a subgroup of $\mathsf{GL}(\mathsf{A})$ conjugate to the orthogonal group $\mathsf{O}(2,1)$.

In particular the complete affine manifold $M^3 = \Gamma\backslash\mathsf{A}$ is a *complete flat Lorentz* 3*-manifold* after one passes to a finite-index torsion-free subgroup of Γ to ensure that Γ acts freely. In particular the restriction $\mathsf{L}|_\Gamma$ defines a free properly discrete isometric action of Γ on the hyperbolic plane H^2 and the quotient $F^2 := \mathsf{H}^2/\mathsf{L}(\Gamma)$ is a complete hyperbolic surface with a homotopy equivalence

$$M^3 \simeq F^2.$$

Already this excludes the case when M^3 is compact, since Γ is the fundamental group of a closed aspherical 3-manifold (and has cohomological dimension 3) and the fundamental group of a hyperbolic surface (and has cohomological dimension ≤ 2). This is a crucial step in the proof of Conjecture 8.6.6 in dimension 3. It is the only case in dimension 3 where compactness is needed for the Auslander–Milnor question.

That the hyperbolic surface F^2 is *noncompact* is a much deeper result due to Geoffrey Mess [**267**]. Later proofs and a generalization have been found by Goldman–Margulis [**168**] and Labourie [**240**]. (Compare the discussion in §15.4.3.) Since the fundamental group of a noncompact surface

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is free, Γ is a free group. Furthermore $\mathsf{L}|_\Gamma$ embeds Γ as a free discrete group of isometries of hyperbolic space. Thus Milnor's suggestion is the *only* way to construct nonsolvable examples *in dimension three.*

15.4.1. Affine boosts and crooked planes. Since L embeds Γ_0 as the fundamental group of the hyperbolic surface F, $\mathsf{L}(\gamma)$ is elliptic only if $\gamma = 1$. Thus, if $\gamma \neq 1$, then $\mathsf{L}(\gamma)$ is either hyperbolic or parabolic. Furthermore $\mathsf{L}(\gamma)$ is hyperbolic for most $\gamma \in \Gamma_0$.

When $\mathsf{L}(\gamma)$ is hyperbolic, γ is an *affine boost,* that is, it has the form

$$\gamma = \left[\begin{array}{ccc|c} e^{\ell(\gamma)} & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha(\gamma) \\ 0 & 0 & e^{-\ell(\gamma)} & 0 \end{array}\right] \tag{15.1}$$

in a suitable coordinate system. The affine transformation γ leaves invariant a unique (spacelike) line C_γ. For (15.1), this is the y-axis (the second coordinate line). Its image in $\mathsf{E}^{2,1}/\Gamma$ is a *closed geodesic,* and identifies with $C_\gamma/\langle\gamma\rangle$. Just as for hyperbolic surfaces, most loops in M^3 are freely homotopic to such closed geodesics.[5]

Margulis observed that C_γ inherits a natural orientation and metric, arising from an orientation on A, as follows. Choose repelling and attracting eigenvectors $\mathsf{L}(\gamma)^\pm$ for $\mathsf{L}(\gamma)$ respectively; choose them so they lie in the same component of the nullcone. Then the orientation and metric on C_γ is determined by a choice of nonzero vector $\mathsf{L}(\gamma)^0$ spanning $\mathsf{Fix}\big(\mathsf{L}(\gamma)\big)$. this vector is uniquely specified by requiring that:

- $\mathsf{L}(\gamma)^0 \cdot \mathsf{L}(\gamma)^0 \; = \; 1$;
- $\big(\mathsf{L}(\gamma)^0, \mathsf{L}(\gamma)^-, \mathsf{L}(\gamma)^+\big)$ is a positively oriented basis.

The restriction of γ to C_γ is a translation by displacement $\alpha(\gamma)$ with respect to this natural orientation and metric.

Compare this to the more familiar *geodesic length function* $\ell(\gamma)$ associated to a class γ of closed curves on the hyperbolic surface Σ. The linear part $\mathsf{L}(\gamma)$ acts by *transvection* along a geodesic $c_{\mathsf{L}(\gamma)} \subset \mathsf{H}^2$. The quantity $\ell(\gamma) > 0$ measures how far $\mathsf{L}(\gamma)$ moves points of $c_{\mathsf{L}(\gamma)}$.

This pair of quantities

$$\big(\ell(\gamma), \alpha(\gamma)\big) \in \mathbb{R}_+ \times \mathbb{R}$$

is a complete invariant of the isometry type of the *flat Lorentz cylinder* $\mathsf{A}/\langle\gamma\rangle$. The absolute value $|\alpha(\gamma)|$ is the length of the unique primitive closed geodesic in $\mathsf{A}/\langle\gamma\rangle$.

[5] Goldman–Labourie [**166**] and Ghosh [**143**] directly relate the dynamics of the geodesic flows on Σ^2 and M^3.

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A fundamental domain is the parallel slab

$$(\Pi_{C_\gamma})^{-1}\left(p_0 \;+\; [0,\alpha(\gamma)]\;\gamma^0\right)$$

where

$$\mathsf{A} \xrightarrow{\Pi_{C_\gamma}} C_\gamma$$

denotes orthogonal projection onto

$$C_\gamma \;=\; p_0 + \mathbb{R}\gamma^0.$$

As noted above, however, parallel slabs don't combine to form fundamental domains for Schottky groups, since their complementary half-spaces are rarely disjoint.

In retrospect this is believable, since these fundamental domains are fashioned from the dynamics of the translational part (using the projection Π_{C_γ}). While the effect of the translational part is properness, the dynamical behavior affecting most points is influenced by the *linear part:* While points on C_γ are displaced by γ at a linear rate, all other points move at an exponential rate.

Furthermore, parallel slabs are less robust than slabs in H^2: while small perturbations of one boundary component extend to fundamental domains, this is no longer true for parallel slabs. Thus one might look for other types of fundamental domains better adapted to the exponential growth dynamics given by the linear holonomy $\mathsf{L}(\gamma)$.

Todd Drumm, in his 1990 Maryland thesis [**115, 116**], defined more flexible polyhedral surfaces, which can be combined to form fundamental domains for *Schottky groups* of 3-dimensional affine transformations. A *crooked plane* is a piecewise linear surface in A, separating A into two *crooked halfspaces.* The complement of two disjoint crooked halfspace is a *crooked slab,* which forms a fundamental domain for a cyclic group generated by an affine boost. Drumm proved the remarkable theorem that if $S_1, \dots, S_g$ are crooked slabs whose complements have disjoint interiors, then given any collection of affine boosts γ_i with S_i as fundamental domain, then the intersection $S_1 \cap \dots \cap S_g$ is a fundamental domain for $\langle \gamma_1, \dots, \gamma_g \rangle$ acting on *all* of A.

Modeling a crooked fundamental domain for Γ acting on A on a fundamental polygon for Γ_0 acting on H^2, Drumm proved the following sharp result:

Theorem 15.4.2 (Drumm [**116**])**.** Every *noncocompact* torsion-free Fuchsian group Γ_0 admits a proper affine deformation Γ whose quotient is a solid handlebody.

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15.4.2. Marked length spectra. We now combine the geodesic length function $\ell(\gamma)$ describing the geometry of the hyperbolic surface Σ with the Margulis invariant $\alpha(\gamma)$ describing the Lorentzian geometry of the flat affine 3-manifold M.

As noted by Margulis, $\alpha(\gamma) = \alpha(\gamma^{-1})$, and more generally

$$\alpha(\gamma^n) \;=\; |n|\alpha(\gamma).$$

The invariant ℓ satisfies the same homogeneity condition, and therefore

$$\frac{\alpha(\gamma^n)}{\ell(\gamma^n)} \;=\; \frac{\alpha(\gamma)}{\ell(\gamma)}$$

is constant along hyperbolic cyclic subgroups. Hyperbolic cyclic subgroups correspond to periodic orbits of the geodesic flow ϕ on the unit tangent bundle $U\Sigma$. Periodic orbits, in turn, define ϕ-invariant probability measures on $U\Sigma$. Goldman–Labourie–Margulis [**167**] prove that, for any affine deformation, this function extends to a continuous function Υ_Γ on the space $\mathcal{C}(\Sigma)$ of ϕ-invariant probability measures on $U\Sigma$. Furthermore when Γ_0 is convex cocompact (that is, contains no parabolic elements), then the affine deformation Γ acts properly if and only if Υ_Γ never vanishes. Since $\mathcal{C}(\Sigma)$ is connected, nonvanishing implies either all $\Upsilon_\Gamma(\mu) > 0$ or all $\Upsilon_\Gamma(\mu) < 0$. From this follows Margulis's *Opposite Sign Lemma,*[6] first proved for groups with parabolics by Charette and Drumm [**83**]:

Theorem (Margulis [**253**])**.** If Γ acts properly, then all of the numbers $\alpha(\gamma)$ have the same sign.

15.4.3. Deformations of hyperbolic surfaces. The Margulis invariant may be interpreted in terms of deformations of hyperbolic structures as follows:

Suppose Γ_0 is a Fuchsian group with quotient hyperbolic surface $\Sigma_0 = \Gamma_0\backslash\mathsf{H}^2$. Let $\mathfrak{g}_{\mathsf{Ad}}$ be the Γ_0-module defined by the adjoint representation applied to the embedding $\Gamma_0 \hookrightarrow \mathsf{O}(2,1)$. The coefficient module $\mathfrak{g}_{\mathsf{Ad}}$ corresponds to the Lie algebra of *right-invariant* vector fields on $\mathsf{O}(2,1)$ with the action of $\mathsf{O}(2,1)$ by left-multiplication. Geometrically these vector fields correspond to the infinitesimal isometries of H^2.

A family of hyperbolic surfaces F_t smoothly varying with respect to a parameter t determines an *infinitesimal deformation,* which is a cohomology class $[u] \in H^1(\Gamma_0, \mathfrak{g}_{\mathsf{Ad}})$, The cohomology group $H^1(\Gamma_0, \mathfrak{g}_{\mathsf{Ad}})$ corresponds to *infinitesimal deformations* of the hyperbolic surface F. In particular the

[6]The survey article by Abels [**1**] gives a very readable treatment of the Margulis's original proof.

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tangent vector to the path F_t of marked hyperbolic structures smoothly varying with respect to a parameter t defines a cohomology class

$$[u] \in H^1(\Gamma_0, \mathfrak{g}_{\mathsf{Ad}}).$$

The same cohomology group parametrizes affine deformations. The translational part u of a linear representations of Γ_0 is a cocycle of the group Γ_0 taking values in the corresponding Γ_0-module V. Moreover two cocycles define affine deformations which are conjugate by a translation if and only if their translational parts are cohomologous cocycles. Therefore translational conjugacy classes of affine deformations form the cohomology group $H^1(\Gamma_0, \mathsf{V})$. Inside $H^1(\Gamma_0, \mathsf{V})$ is the subset Proper corresponding to *proper affine deformations.*

The adjoint representation Ad of $\mathsf{O}(2,1)$ identifies with the orthogonal representation of $\mathsf{O}(2,1)$ on V. Therefore the cohomology group $H^1(\Gamma_0, \mathsf{V})$ consisting of translational conjugacy classes of affine deformations of Γ_0 can be identified with the cohomology group $H^1(\Gamma_0, \mathfrak{g}_{\mathsf{Ad}})$ corresponding to infinitesimal deformations of Σ_0.

Theorem. Suppose $u \in Z^1(\Gamma_0, \mathfrak{g}_{\mathsf{Ad}})$ defines an *infinitesimal deformation* tangent to a smooth deformation F_t of F.

- The marked length spectrum ℓ_t of F_t varies smoothly with t.
- Margulis's invariant $\alpha_u(\gamma)$ represents the derivative

$$\left.\frac{d}{dt}\right|_{t=0} \ell_t(\gamma).$$

- (Opposite Sign Lemma) If $[u] \in \mathsf{Proper}$, then all closed geodesics lengthen (or shorten) under the deformation F_t.

Since closed hyperbolic surfaces do not support deformations in which *every* closed geodesic shortens, such deformations only exist when F is noncompact. This leads to a new proof [**168**] of Mess's theorem [**267**], that F is not compact. (For another, somewhat similar proof, which generalizes to higher dimensions, see Labourie [**240**].)

The tangent bundle $\mathsf{T}G$ of any Lie group G has a natural structure as a Lie group, where the fibration $\mathsf{T}G \xrightarrow{\Pi} G$ is a homomorphism of Lie groups, and the tangent spaces

$$\mathsf{T}_x G = \Pi^{-1}(x) \subset \mathsf{T}G$$

are vector groups, each identified with the Lie algebra of $\mathsf{O}(2,1)$. Deformations of a representation $\Gamma_0 \xrightarrow{\rho_0} G$ correspond to representations $\Gamma_0 \xrightarrow{\rho} \mathsf{T}G$ such that $\Pi \circ \rho = \rho_0$. In our case, affine deformations of $\Gamma_0 \hookrightarrow \mathsf{O}(2,1)$

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correspond to representations in the tangent bundle $\mathsf{TO}(2,1)$. When G is the group $\mathsf{G}(\mathbb{R})$ of $\mathbb{R}$-points of an algebraic group G defined over $\mathbb{R}$, then

$$\mathsf{T}G \;\cong\; \mathsf{G}(\mathbb{R}[\epsilon])$$

where ϵ is an indeterminate with $\epsilon^2 = 0$. (Compare [**156**].) This is reminiscent of the classical theory of quasi-Fuchsian deformations of Fuchsian groups, where one deforms a Fuchsian subgroup of $\mathsf{SL}(2,\mathbb{R})$ in

$$\mathsf{SL}(2,\mathbb{C}) = \mathsf{SL}(2,\mathbb{R}[i])$$

where $i^2 = -1$.

15.4.4. Classification. Proposition 15.4.1 and Theorem15.4.2 point to a strategy for classifying Margulis spacetimes. Namely, a Margulis spacetime M^3 defines a hyperbolic surface

$$F^2 := \mathsf{H}^2/\Gamma_0,$$

and marked hyperbolic surfaces $\Sigma \to F^2$ are classified by the Fricke space $\mathfrak{F}(\Sigma)$ of marked hyperbolic structures on Σ.

The main result of Goldman–Labourie–Margulis [**167**] is that the positivity (or negativity) of Υ_Γ on on $\mathcal{C}(\Sigma)$ is necessary and sufficient for properness of Γ. (For simplicity we restrict ourselves to the case when $\mathsf{L}(\Gamma)$ contains no parabolics — that is, when Γ_0 is convex cocompact.) Thus the subset Proper of the deformation space consisting of *proper* affine deformations of Γ_0 identifies with the open convex cone in $H^1(\Gamma_0,\mathsf{V})$ defined by the linear functionals Υ_μ, for μ in the compact space $\mathcal{C}(\Sigma)$. These give uncountably many linear conditions on $H^1(\Gamma_0,\mathsf{V})$, one for each $\mu \in \mathcal{C}(\Sigma)$. Since the invariant probability measures arising from periodic orbits are dense in $\mathcal{C}(\Sigma)$, the cone Proper is the interior of half-spaces defined by the countable set of functional Υ_γ, where $\gamma \in \Gamma_0$.

The zero level sets $\Upsilon_\gamma^{-1}(0)$ correspond to affine deformations where γ does not act freely. Proper defines a component of the interior of the subset of $H^1(\Gamma_0,\mathsf{V})$ corresponding to affine deformations which are *free* actions.

Actually, one may go further. Using ideas from Thurston [**325**], one reduces the consideration to only those measures arising from *multicurves,* that is, unions of disjoint *simple* closed curves. These measures (after scaling) are dense in the *Thurston cone* $\mathcal{ML}(\Sigma)$ of *measured geodesic laminations* on the hyperbolic surface F. One sees the combinatorial structure of the Thurston cone replicated on the boundary of $\mathsf{Proper} \subset H^1(\Gamma_0,\mathsf{V})$.

We describe this structure for the surfaces with $\chi(\Sigma) = -1$, or equivalently when $\pi_1(\Sigma)$ is a 2-generator free group. Figure 15.1 illustrate the four case. Figure 15.2 denotes the deformation space when Σ is a three-holed sphere, Figure 15.3 denotes the deformation space when Σ is a one-holed

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Klein bottle, Figure 15.4 denotes the deformation space when Σ is a one-holed torus, and Figure 15.5 denotes the deformation space when Σ is a two-holed cross-surface.[7]

When Σ is a 3-holed sphere or a 2-holed cross-surface the Thurston cone degenerates to a finite-sided polyhedral cone. In particular properness is characterized by 3 Margulis functionals for the 3-holed sphere, and 4 for the 2-holed cross-surface. Thus the deformation space of equivalence classes of proper affine deformations is a cone on either a triangle or a convex quadrilateral, respectively.

When Σ is a 3-holed sphere, these functionals correspond to the three components of $\partial\Sigma$. The halfspaces defined by the corresponding three Margulis functionals cut off the deformation space (which is a polyhedral cone with 3 faces). The Margulis functionals for the other curves define halfspaces which strictly contain this cone.

When Σ is a 2-holed cross-surface these functionals correspond to the two components of $\partial\Sigma$ and the two orientation-reversing simple closed curves in the interior of Σ. The four Margulis functionals describe a polyhedral cone with 4 faces. All other closed curves on Σ define halfspaces strictly containing this cone.

In both cases, an ideal triangulation for Σ models a crooked fundamental domain for M, and Γ is an affine Schottky group, and M^3 is an open solid handlebody of genus 2.

For the other surfaces where $\pi_1(\Sigma)$ is free of rank two (equivalently $\chi(\Sigma) = -1$), infinitely many functionals Υ_μ are needed to define the deformation space. The deformation space has infinitely many sides. In these cases M^3 admits crooked fundamental domains corresponding to ideal triangulations of Σ. Unlike the preceding cases no single ideal triangulation works for *all* proper affine deformations. Once again M^3 is a genus two handlebody.

15.5. Lorentzian 3-manifolds

Many of these results extend to the broader (and quite fascinating) study of constant curvature Lorentzian manifolds. In particular the study of 3-dimensional *anti-de Sitter manifolds,* and its extension to flat conformal Lorentzian 3-manifolds is quite fascinating. Anti-de Sitter 3-space is the Lorentzian analog of hyperbolic 3-space. A suggestive model arises from the

[7]This terminology was proposed by John H. Conway, who wanted to reserve $\mathbb{R}\mathsf{P}^2$ for the geometric object, and call its underlying topology a *cross-surface.* His motivation was the standard terminology of *cross-cap* for the complement of a disc in $\mathbb{R}\mathsf{P}^2$.

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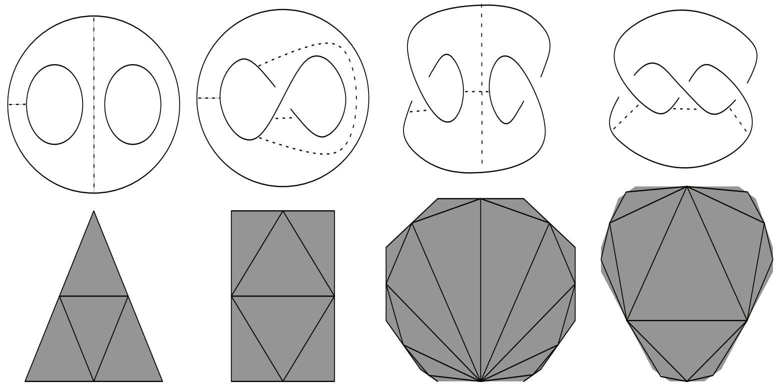

Figure 15.1. Four topological types of surfaces with $\chi = -1$ and their arc complexes: the three-holed sphere, the two-holed cross-surface ($\mathbb{R}\mathsf{P}^2$), the one-holed Klein bottle, and the one-holed torus.

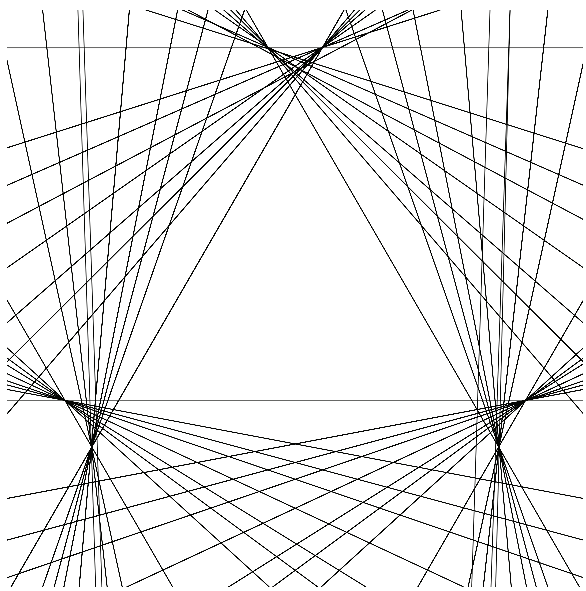

Figure 15.2. Proper affine deformations of the three-holed sphere

universal covering space $X := \widetilde{\mathsf{SL}(2,\mathbb{R})}$ and

$$G := \big(\widetilde{\mathsf{SL}(2,\mathbb{R})} \times \widetilde{\mathsf{SL}(2,\mathbb{R})}\big)/Z,$$

where $Z \hookrightarrow \widetilde{\mathsf{SL}(2,\mathbb{R})} \times \widetilde{\mathsf{SL}(2,\mathbb{R})}$ is the diagonal embedding of the infinite cyclic group

$$\mathsf{center}\big(\widetilde{\mathsf{SL}(2,\mathbb{R})}\big) \cong \pi_1\big(\mathsf{SL}(2,\mathbb{R})\big) \cong \mathbb{Z}.$$

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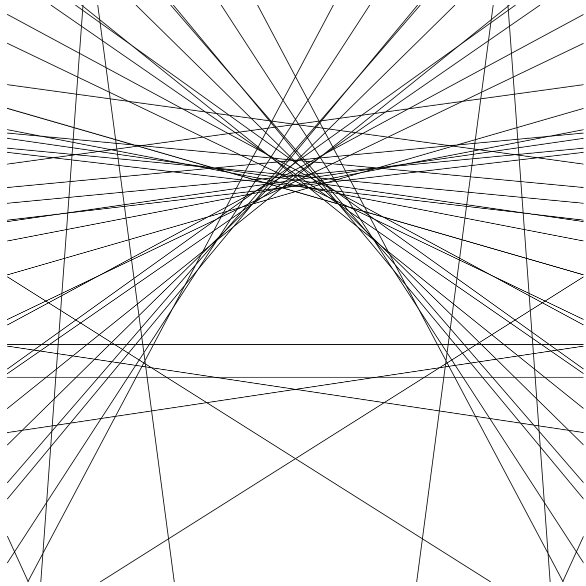

Figure 15.3. Proper affine deformations of the one-holed Klein bottle

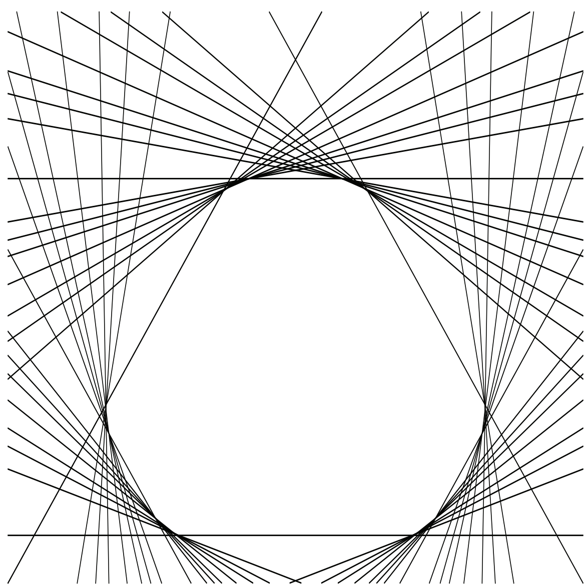

Figure 15.4. Proper affine deformations of the one-holed torus

One of the first papers on this subject is that of Kulkarni–Raymond [**239**], where it is shown that a closed 3-manifold admitting such a structure must be Seifert-fibered, and fits into the Thurston geometry modeled on the unit tangent bundle $\mathsf{T}_1\mathsf{H}^2$. Furthermore every closed 3-manifold

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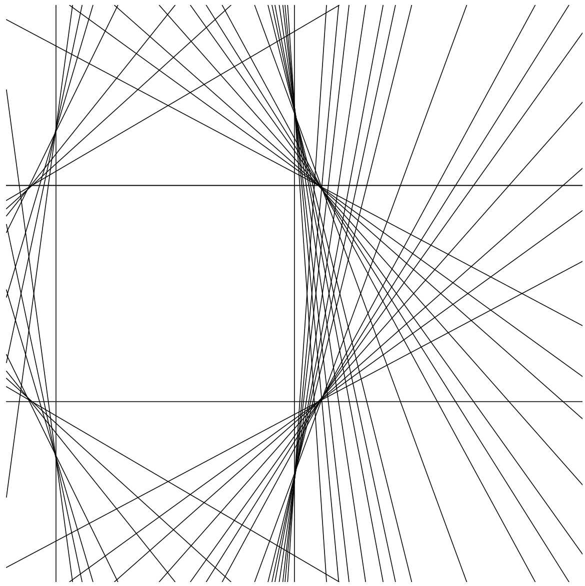

Figure 15.5. Proper affine deformations of the two-holed cross–surface

in this geometry has such a structure. For this geometry, G is the group $\mathsf{SO}(2,2)$ acting on anti-de Sitter space. For reasons of space, we do not discuss these structures, instead referring to recent articles by Barbot [**25**], Barbot–Bonsante–Schlenker [**27**], Bonsante–Danciger–Maloni–Schlenker [**59**], Collier–Tholozan–Toulisse [**98**], Tholozan [**321**], and Danciger–Guéritaud–Kassel [**106, 107**]. This geometry also lies in the flat conformal Lorentzian geometry, where G is the *Einstein Universe,* consisting of null lines in a Lorentzian vector space; compare Barbot–Charette–Drumm–Goldman–Melnick [**28**]. See also Barbot's survey article [**26**] and Frances's basic paper [**132**].

Another interesting geometry is the geometry of the flag manifold for $G = \mathsf{SL}(3,\mathbb{R})$ investigated by Barbot [**24**]. It is very mysterious which 3-manifolds admit such structures, although many Seifert 3-manifolds with hyperbolic base and nonzero Euler number do.

15.6. Higher dimensions: flat conformal and spherical CR-structures

These structures generalize to (G,X)-structures where G is a semisimple Lie group and $X = G/P$, where $P \subset G$ is a parabolic subgroup. The simplest generalization occurs when $G = \mathsf{SO}(n+1,1)$ and $X = \mathbb{S}^n$. The conformal automorphisms of $\mathbb{S}^n$ are just Möbius transformations. In this case X is the model space for *conformal (Euclidean) geometry* and a (G,X)-structure is a

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flat conformal structure, that is, a conformal equivalence class of conformally flat Riemannian metrics.

A key point in this identification is the famous result of Liouville, that, in dimensions > 2, a conformal map from a nonempty connected domain in $\mathbb{S}^n$ is the restriction of a unique Möbius transformation of $\mathbb{S}^n$.

Furthermore this is the boundary structure for hyperbolic structures in dimension $n+1$, since $\mathbb{S}^n = \partial\mathsf{H}^{n+1}_{\mathbb{R}}$ and the group of isometries of H^{n+1} restricts to the group of conformal automorphisms of $\mathbb{S}^n$.

A good general survey of this subject is Matsumoto [**261**]). Kamishima–Tan [**208**] and Dumas [**117**]) describe the unpublished construction of Thurston (using hyperbolic geometry) which identifies a flat conformal structure with a hyperbolic structure with the extra structure of a measured geodesic lamination; roughly speaking the $\mathbb{C}\mathsf{P}^1$-structure is identified with an equivariant map of the universal covering into H^3 which is locally convex and *pleated* (piecewise totally geodesic).

McMullen [**264**] observed that this construction can be obtained as an *intrinsic metric* in the sense of Kobayashi [**223**], for conformal maps of a ball into a flat conformal manifold. This has been extended to higher dimensional flat conformal structures by Kulkarni and Pinkall [**237**, **238**]; compare also Bridegman–Brock–Bromberg [**63**].

Some of the most interesting examples are due to Gromov–Lawson–Thurston [**175**] and Kuiper [**236**]. While products $\Sigma^2 \times \mathbb{S}^1$ (where Σ^2 is a closed hyperbolic surface) admit flat conformal structures, 3-dimensional nilmanifolds and hyperbolic torus bundles do *not* admit such structures (Goldman [**148**]). However, [**175**, **236**] produce examples of flat conformal structures on *twisted* oriented $\mathbb{S}^1$-bundles over Σ^2.

Another interesting example is *spherical CR-geometry,* the boundary structure for complex hyperbolic geometry is a *spherical CR-structure,* where $G = \mathsf{PU}(n,1)$ and $X = \partial\mathsf{H}^n_{\mathbb{C}} \approx \mathbb{S}^{2n-1}$.

One of the first papers on this subject is Burns–Shnider [**72**] which computed the homogenous domains. This geometry is extensively discussed in Goldman [**155**]. For more information on this very active field of research see, for examples, the papers of Deraux–Falbel [**111**], Falbel [**126**], Schwartz [**301**], Parker–Deraux–Paupert [**112**], Will [**348**] and others. A good survey is the recent exposition of Kapovich [**212**]. Geometric constructions of lattices have been given by Mostow [**275**], Deligne and Mostow [**109**] and Thurston [**326**].

While [**148**] shows that $\mathbb{T}^3$ and hyperbolic torus bundles do not admit spherical CR-structures, some twisted $\mathbb{S}^1$-bundes over closed surfaces do.

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Ananin, Grossi and Gusevskii [**8**] produce surprising examples of spherical CR-structures on products $\Sigma \times \mathbb{S}^1$.

Exercise 15.6.1. Find examples of closed 3-manifolds with flat conformal (respectively spherical CR-manifolds) with pathological developing maps.

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Appendices

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Appendix A

Transformation groups

We collect here several general notions about transformation groups. These notions arise in two contexts: First is their central role in defining *geometry* in the sense of Lie–Klein, and the holonomy action of the fundamental group on the model space. These notions make their second appearance in the classification of geometric structures on a fixed topology. Here the action is that of the mapping class group action on the deformation spaces.

A.1. Group actions

We consider *left-actions* of a group on a set, unless otherwise noted. Suppose that G is a group acting on a set X, with the (left-) action denoted by:

$$\begin{aligned} G \times X &\xrightarrow{\alpha} X \\ (g, x) &\longmapsto g \cdot x \end{aligned}$$

We refer to X as a (left-) G-set.

The *kernel* of the action α consists of all g such that $\alpha(g, \cdot) = \mathbb{I}$, that is, $g \cdot x = x, \forall x \in X$. Equivalently, this is the kernel of the homomorphism of G into the group of automorphisms of the set X. The action is *effective* (or *faithful*) if its kernel is trivial.

If $x \in X$, its *stabilizer* is the subgroup:

$$\mathsf{Stab}(x) := \{g \in G \mid g \cdot x = x\}.$$

If $g \in G$, then its *fixed-point set* is the subset:

$$\mathsf{Fix}(g) := \{x \in X \mid g \cdot x = x\}.$$

The action is *free* if and only if $\mathsf{Stab}(x) = \{1\}, \forall x \in X$, or equivalently $\mathsf{Fix}(g) = \emptyset, \forall g \in G, g \neq 1$.

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The *orbit* of a point x is the image

$$G \cdot x := \alpha(G \times \{x\}) = \{g \cdot x \mid g \in G\}.$$

The orbits partition X, so that the group action defines an equivalence relation on X. The equivalence classes are the G-orbits. The action is *transitive* if some (and hence every) orbit equals X.

The action is *simply transitive* if it is transitive and free. In terms of the *orbit map* (or *evaluation map*)

$$\begin{aligned} G &\xrightarrow{\alpha_x} X \\ g &\longmapsto \alpha(g, x) = g \cdot x, \end{aligned}$$

- α is free $x \iff \alpha_x$ is injective $\forall x \in X$;
- α is transitive $\iff \alpha_x$ is injective (for any x);
- α is simply transitive $\iff \alpha_x$ is bijective.

A simply transitive action α of G on X makes X into a *G-torsor.* In particular affine spaces are G-torsors, when G is (the additive group of) a vector space.

Exercise A.1.1. Let X be a left G-set and $x \in X$. Let $H = \mathsf{Stab}(x)$.

- The orbit map α_x defines a G-equivariant isomorphism $G/H \longrightarrow G \cdot x$. where G acts by left-multiplication on the set G/H of left cosets gH for $g \in G$.
- Suppose that $N < H$ is a nontrivial normal subgroup of G. Then G does not act effectively on G/H.

A.2. Proper and syndetic actions

A convenient context in which to work is that of *locally compact Hausdorff topological spaces* and *topological groups.* Recall that a continuous map $X \xrightarrow{f} Y$ is *proper* if for all compact $K \subset Y$, the preimage $f^{-1}(K)$ is a compact subset of X. Recall the following facts from general topology:

- A closed subset of a compact space is compact.
- The continuous image of a compact space is compact.
- Compact subsets of a Hausdorff space are closed.

It follows that a proper map is closed if Y is Hausdorff. Recall that a map $X \xrightarrow{f} Y$ is a *local homeomorphism* if and only if every point $x \in X$ has an open neighborhood U such that the restriction $f|_U$ is a homeomorphism $U \longrightarrow f(U)$.

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Exercise A.2.1. Suppose that X, Y are manifolds and f is a smooth map. Furthermore suppose that f is a local homeomorphism. Then f is a covering map.

Let G be a locally compact Hausdorff topological group and

$$G \times X \xrightarrow{\alpha} X$$
$$(g, x) \longmapsto g \cdot x$$

is a (left) action. We say that the group action α is *proper* if and only if the continuous map

$$G \times X \xrightarrow{f_\alpha} X \times X$$
$$(g, x) \longmapsto (g \cdot x, x)$$

is a proper map.

Intuitively, this means that "going to infinity" in the orbit implies "going to infinity" in the group. That is, the action is *dynamically trivial.* The quotient space becomes tractable and often supports a rich class of functions. When the action is free, an important point is the existence of *local slices* for the action, namely open neighborhoods U in the quotient X/G with local sections $U \to X$. (Compare Koszul [**228**] and Palais [**285**].)

When G has the discrete topology, then properness is just the usual notion of *proper discontinuity.*[1]

Exercise A.2.2. Show that properness is equivalent to the either of the two following conditions: (for the last condition assume that G is second countable)

- For all compact subsets $K_1, K_2 \subset X$, the set
$$G(K_1, K_2) := \{g \in G \mid gK_1 \cap K_2 \neq \emptyset\}$$
is a compact subset of G.
- $G(K, K)$ is a compact subset of G for all compact subsets $K \subset X$.
- For all sequences $g_n \in G$, $x_n \in X$, such that $g_n x_n$ converges, the sequence g_n has a convergent subsequence.

For the last condition, we can say that if x_n stays bounded, but $g_n \to \infty$, then $g_n x_n \to \infty$.

Exercise A.2.3. Suppose that α is a proper action of a locally compact group on a locally compact Hausdorff space X. Then the quotient space $G \backslash X$ is Hausdorff. Is the converse true?

[1]This terminology seems unfortunate since the group actions are all continuous. As Kulkarni has stated, perhaps a better expression is "discretely proper."

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Exercise A.2.4. Suppose that Γ is a discrete group and $\Gamma \times X \xrightarrow{\alpha} X$ is a proper free action. Then the quotient map $X \longrightarrow \Gamma\backslash X$ is a covering space.

Exercise A.2.5. Suppose that $G < \mathsf{Homeo}(X)$.

- Show that if the action of G is proper, then G is a closed subgroup with respect to the compact-open topology on $\mathsf{Homeo}(X)$. In particular, if the action of G is proper *with respect to the discrete topology on G,* then G is a discrete subgroup of $\mathsf{Homeo}(X)$.
- Find an example of a group action where G is discrete as a subgroup of $\mathsf{Homeo}(X)$ but the action is proper. (Hint: the easiest examples are not free actions, and even exist when X is a 1-dimensional manifold. It is much more interesting — and fundamental — to find examples where the actions are free.)

A.3. Topological transformation groupoids

We must relate the actions of $\mathsf{Aff}(\mathsf{A})$ on $\mathfrak{C}(\mathsf{A})$ and G on $\mathfrak{C}(\mathsf{P})$. Recall that a *topological transformation groupoid* consists of a small category $\mathfrak{G}$ whose objects form a topological space X upon which a topological group G acts such that the morphisms $a \to b$ consist of all $g \in G$ such that $g(a) = b$. We write $\mathfrak{G} = (G, X)$. A *homomorphism of topological transformation groupoids* is a functor

$$(X, G) \xrightarrow{(f,F)} (X', G')$$

arising from a continuous map $X \xrightarrow{f} X'$ which is equivariant with respect to a contin uous homomorphism $G \xrightarrow{F} G'$.

The space of isomorphism classes of objects in a category $\mathfrak{G}$ will be denoted $\mathsf{Iso}(\mathfrak{G})$. We shall say that $\mathfrak{G}$ is *proper (respectively syndetic)* if the corresponding action of G on X is proper (respectively syndetic). If $\mathfrak{G}$ and $\mathfrak{G}'$ are topological categories, a functor $\mathfrak{G} \xrightarrow{F} \mathfrak{G}'$ is an *equivalence of topological categories* if the induced map

$$\mathsf{Iso}(\mathfrak{G}) \xrightarrow{\mathsf{Iso}(F)} \mathsf{Iso}(\mathfrak{G}')$$

is a homeomorphism and F is *fully faithful,* that is, for each pair of objects a, b of $\mathfrak{G}$, the induced map

$$\mathsf{Hom}(a, b) \xrightarrow{F_*} \mathsf{Hom}(F(a), F(b))$$

is a homeomorphism. If F is fully faithful it is enough to prove that $\mathsf{Iso}(F)$ is surjective. (Compare Jacobson [**200**].) We have the following general proposition:

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Lemma A.3.1. Suppose that

$$(X, G) \xrightarrow{(f,F)} (X', G')$$

is a homomorphism of topological transformation groupoids which is an equivalence of groupoids and such that f is an open map.

- If (X, G) is proper, so is (X', G').
- If (X, G) is syndetic, so is (X', G').

Proof. An equivalence of topological groupoids induces a homeomorphism of quotient spaces

$$X/G \longrightarrow X'/G'$$

so if X'/G' is compact (respectively Hausdorff) so is X/G. Since (X, G) is syndetic if and only if X/G is compact, this proves the assertion about syndeticity. By Koszul [**228**], p.3, Remark 2, (X, G) is proper if and only if X/G is Hausdorff and the action (X, G) is *wandering (or locally proper):* each point $x \in X$ has a neighborhood U such that

$$G(U, U) = \{g \in G \mid g(U) \cap U \neq \emptyset\}$$

is precompact. Since (f, F) is fully faithful, F maps $G(U, U)$ isomorphically onto $G'(f(U), f(U))$. Suppose that (X, G) is proper. Then X/G is Hausdorff and so is X'/G'. We claim that G' acts locally properly on X'. Let $x' \in X'$. Then there exists $g' \in G'$ and $x \in X$ such that $g'f(x) = x'$. Since G acts locally properly on X, there exists a neighborhood U of $x \in X$ such that $G(U, U)$ is precompact. It follows that $U' = g'f(U)$ is a neighborhood of $x' \in X'$ such that $G'(U', U') \cong G(U, U)$ is precompact, as claimed. Thus G' acts properly on X'. □

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Appendix B

Affine connections

We summarize some of the basic facts about (not necessarily flat) affine connections on a smooth manifold M. We discuss the torsion tensor, interpreting it as the covariant exterior differential of the solder form (the identity endomorphism field), and the Hessian of a smooth function. After discussing the expression of general affine connections in local coordinates is a review of the Levi–Civita (or *Riemannian*) connection. As this general theory only uses the nondegeneracy of the metric connection, it applies equally to *pseudo-Riemannian,* structures defined by a (possibly indefinite) metric tensor. Due to its fundamental importance, this appendix ends with a detailed discussion of the Riemannian connection for the hyperbolic plane H^2.

B.1. The torsion tensor

A connection ∇ on the tangent bundle $\mathsf{T}M$ is called an *affine connection.* It is the right mechanism to define *acceleration* of a smooth curve $\gamma(t)$, since acceleration is the (covariant) derivative of the *velocity* and the velocity $\gamma'(t) \in \mathsf{T}_{\gamma(t)}M$ is a section of the tangent bundle over γ.

The identity isomorphism $\mathsf{T}M \longrightarrow \mathsf{T}M$ is an endomorphism field and defines a $\mathsf{T}M$*-valued* 1*-form,* (sometimes called the *solder form*) which we denote $\mathbb{I}_M \in \mathcal{A}^1(M, \mathsf{W})$. The *torsion* of ∇ is the covariant exterior differential

$$\mathsf{Tor}_\nabla \ := \ D_\nabla \mathbb{I}_M \in \mathcal{A}^2(M; \mathsf{T}M)$$

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Exercise B.1.1. Prove that Tor_∇ is given by the skew-symmetric $\mathsf{C}^\infty(M)$-bilinear mapping:

$$\begin{aligned} \mathsf{Vec}(M) \times \mathsf{Vec}(M) &\xrightarrow{\mathsf{Tor}_\nabla} \mathsf{Vec}(M) \\ (\xi, \eta) &\longmapsto \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta]. \end{aligned}$$

An affine connection ∇ with $\mathsf{Tor}_\nabla = 0$ is *torsionfree.*

In local coordinates $(x^1, \dots, x^n)$, an affine connection is given by *Christoffel symbols.* The coordinates give rise to a *trivial connection* ∇^0 defined by the condition that the coordinate vector fields

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$$

are parallel. Now let ∇ be an arbitrary affine connection. Define the *Christoffel symbols* $\Gamma_{ij}^k(x) \in \mathsf{C}^\infty(M)$ by:

$$\nabla_{\partial_i}(\partial_j) := \Gamma_{ij}^k(x)\partial_k.$$

whereby the general covariant derivative of vector fields is:

$$\nabla_{a_i\partial_i}\big(b_j\partial_j\big) = a_i\bigg(\frac{\partial b_k}{\partial x_i} + b_j\Gamma_{ij}^k(x)\bigg)\partial_k.$$

The $\mathsf{End}(\mathsf{T}M)$-valued 1-form representing the difference $\nabla - \nabla^0$ equals

$$\nabla - \nabla^0 \;=\; \Gamma_{ij}^k(x) dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k}.$$

and the torsion equals

$$\mathsf{Tor}_\nabla \;=\; \big(\Gamma_{ij}^k(x) - \Gamma_{ji}^k(x)\big) dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k},$$

so ∇ is torsionfree if and only if $\Gamma_{ij}^k(x)$ is symmetric in i and j. For this reason torsionfree connections are often called *symmetric* connections.

B.2. The Hessian

Now we relate the covariant differential to the classical notion of the *Hessian,* the quadratic form formed by second partial derivatives.

Exercise B.2.1. Let $\alpha \in \mathcal{A}^1(M)$ be a 1-form. Show that the alternation of the covariant differential $\nabla\alpha$ equals $d\alpha - \iota_{\mathsf{Tor}_\nabla}(\alpha)$, that is, for vector fields $\xi, \eta \in \mathsf{Vec}(M)$,

$$\nabla\alpha(\xi, \eta) - \nabla\alpha(\eta, \xi) \;=\; d\alpha(\xi, \eta) - \mathsf{Tor}_\nabla(\xi, \eta).$$

In particular if α is a closed 1-form and ∇ is torsionfree, then $\nabla\alpha$ is a symmetric 2-tensor.

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This is particularly important when α is exact ($\alpha = df$ for $f \in \mathsf{C}^\infty(M)$. In that case $\nabla df = d^2 f$, the *Hessian* of f. In local coordinates (for the trivial connection),

$$d^2 f \;=\; \frac{\partial^2 f}{\partial x^i \partial x^j}\, dx^i dx^j.$$

In particular f is a convex function if $d^2 f$ is positive definite.

B.3. Geodesics

In local coordinates on $\mathsf{T}M$, where $(x^1, \dots, x^n)$ are local coordinates on M and $(v^1, \dots v^n)$ are local coordinates on $\mathsf{T}_p M$, the geodesic equations are:

$$\begin{aligned} \frac{d}{dt} x^k(t) &= v^k(t) \\ \frac{d}{dt} v^i(t) &= -\Gamma^k_{ij}(x) v^i(t) v^j(t) \end{aligned}$$

(for $k = 1, \dots, n$) and $\Gamma^k_{ij}(x)$ are the *Christoffel symbols*:

$$\nabla_{\partial_i}(\partial_j) = \Gamma^k_{ij}(x)\partial_k.$$

In particular the geodesic flow corresponds to the vector field (called the *geodesic spray*):

$$\phi_\Gamma \;:=\; v^k \frac{\partial}{\partial x^k} \;-\; \Gamma^k_{ij}(x) v^i v^j \frac{\partial}{\partial v^k}$$

on $\mathsf{T}M$. See Kobayashi–Nomizu [**224**], do Carmo [**113**] or O'Neill [**283**] for further details.

That ϕ_Γ is *vertically homogeneous of degree one:* That is, it transforms under the one-parameter group of homotheties

$$(p, \mathbf{v}) \xmapsto{h_t} (p, e^s \mathbf{v})$$

by

$$(h_t)_*(\phi_\Gamma) = e^t \phi_\Gamma. \tag{B.1}$$

Furthermore its trajectories are the velocity vector fields of the *geodesics* on M. Namely, let $p = (x^1, \dots, x^n) \in M$ and $\mathbf{v} = (v^1, \dots v^n) \in \mathsf{T}_p M$ be an initial condition. Let $\gamma_{p,\mathbf{v}}(t) \in \mathsf{T}M$ denote the trajectory of ϕ_M, defined for t in an open interval containing $0 \in \mathbb{R}$, defined by:

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \gamma_{p,\mathbf{v}}(t) = \mathbf{v}, \qquad \gamma_{p,\mathbf{v}}(0) = (p, \mathbf{v}).$$

Then the local flow Φ^t of ϕ_Γ on $\mathsf{T}M$ satisfies

$$\Phi^t(p, \mathbf{v}) = \big(\gamma_{p,\mathbf{v}}(t), \gamma'_{p,\mathbf{v}}(t)\big),$$

where

$$\gamma_{p,\mathbf{v}}(t) := (\Pi \circ \Phi^t)(p, \mathbf{v})$$

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and $\mathsf{T}M \xrightarrow{\Pi} M$ denoting the bundle projection. The homogeneity condition (B.1) implies that

$$\gamma_{p,s\mathbf{v}}(t) = \gamma_{p,\mathbf{v}}(st).$$

We write:

$$\mathsf{Exp}_p(t\mathbf{v}) := \gamma_{p,\mathbf{v}}(t) \tag{B.2}$$

for $t \in \mathbb{R}$ sufficiently near 0. Then, whenever s, t are sufficiently near 0,

$$\mathsf{Exp}_p\big((s+t)\mathbf{v}\big) = \mathsf{Exp}_{\gamma(t)}\big(s\mathbb{P}(t\mathbf{v})\big)$$

where, for clarity, we denote $\gamma(t) := \mathsf{Exp}_p(t\mathbf{v})$ and parallel transport $\Pi_p^{\gamma(t)}$ by:

$$\mathsf{T}_pM \xrightarrow{\Pi} \mathsf{T}_{\gamma(t)}.$$

Observe that $\Pi(\mathbf{v}) = \gamma'(t)$.

Compare §8.3 and standard references.

B.4. Projectively equivalent affine connections

Projective structures can be defined in terms of affine connections. First we remark that the geodesics of an affine connection ∇ are independent of the torsion. Namely a curve $\gamma(t)$ is a geodesic if and only if

$$\frac{d^2}{dt^2}\gamma^j(t) + \Gamma^k_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} \;=\; 0$$

where the second term is symmetric in i, j. By subtracting $\frac{1}{2}\mathsf{Tor}_\nabla$, we may assume that $\mathsf{Tor}_\nabla = 0$, that is, Γ^k_{ij} is symmetric in i, j.

Exercise B.4.1. Show that two torsionfree affine connections have the same set of *parametrized* geodesics if and only if they are equal.

Weyl found an elementary criteria for when two torsionfree affine connections ∇ and $\tilde{\nabla}$ have the same *unparametrized* geodesics. In that case, we say the connections are *projectively equivalent.*

Exercise B.4.2. (Weyl) ∇ and $\tilde{\nabla}$ are projectively equivalent if and only if the difference $\tilde{\nabla} - \nabla$ is the *symmetrization* of a 1-form ω, that is,

$$\tilde{\nabla}_X(Y) - \nabla_X Y \;=\; \omega(X)Y + \omega(Y)X,$$

$\forall X, Y \in \mathsf{Vec}(M)$. In terms of local coordinates $\omega = \omega_l dx^l$, this means

$$\Gamma^k_{ij} - \Gamma^k_{ij} \;=\; \omega_i\delta^k_j + \omega_j\delta^k_i.$$

This defines an action of the vector space $\mathcal{A}^1(M)$ of 1-forms on the space $\mathfrak{A}(M)$ of affine connections on M. For more discussion, see Spivak [**316**], Addendum 1, Proposition 18 and Ovsienko–Tabachnikov [**284**], Proposition A.3.2.

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Exercise B.4.3. In the projective models for affine space and hyperbolic space, the geodesics are segments of projective lines. This gives several examples of projective equivalences. For the invariant affine connection on an affine patches, and the Levi–Civta connection for hyperbolic space in the Beltrami–Klein model, find the 1-forms ω effecting these projective equivalences.

In general for a manifold with an affine connection, the geodesics have a projectively natural notion of *projective parameter.* These correspond to maps $\Omega \to M$ where $\Omega \subset \mathbb{R}\mathsf{P}^1$ is a subdomain (open interval), which — up to composition with transformations in $\mathsf{PGL}(2,\mathbb{R})$ — contain an affinely parametrized geodesic.

A torsion-free affine connection is *projectively flat* if locally it is projectively equivalent to the standard connection on an affine patch. Projective flatness is detected by a contraction of the curvature tensor. The deformation space $\mathbb{R}\mathsf{P}^2(\Sigma)$ can be obtained as a the space of equivalence classes of projectively flat affine connections, where the equivalence relation is generated by isotopy and projective equivalence.

The symplectic structure is a double symplectic quotient of the affine space $\mathfrak{A}(\Sigma)$ of affine connections on Σ as follows: The first symplectic reduction arises from the action of $\mathcal{A}^1(\Sigma)$ generating projective equivalence, and the moment map associates to an affine connection ∇ its torsion Tor_∇:

$$\mathfrak{A}(\Sigma) \xrightarrow{\mathsf{Tor}} \mathcal{A}^2(\Sigma, \mathsf{T}\Sigma) \cong \mathcal{A}^1(\Sigma)^*.$$

The corresponding symplectic quotient is the space $\mathfrak{A}_0(\Sigma)$ of projective equivalence classes of torsionfree affine connections.

The second symplectic reduction arises from the Hamiltonian action of the group $\mathsf{Diff}^0(\Sigma)$ on $\mathfrak{A}_0(\Sigma)$; here the moment map

$$\mathfrak{A}_0(\Sigma) \longrightarrow \mathcal{A}^2(\Sigma, \mathsf{T}^*M) \cong \mathsf{Vec}(\Sigma)^*$$

is defined by projective curvature. The symplectic quotient is the deformation space $\mathbb{R}\mathsf{P}^2(\Sigma)$. See Goldman [**154**] for details.

B.5. The (pseudo-) Riemannian connection

The *fundamental theorem of Riemannian geometry* asserts a pseudo-Riemannian manifold (M, g) admits a unique affine connection ∇ with natural properties:

- (Orthogonality) $\nabla \mathsf{g} = 0$;
- (Symmetry) $\mathsf{Tor}_\nabla = 0$.

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Orthogonality is equivalent to the condition that the parallel transport operator

$$\mathsf{T}_x M \xrightarrow{\mathbb{P}_\gamma} \mathsf{T}_y M$$

along a path $x \overset{\gamma}{y}$ maps $(\mathsf{T}_x, \mathsf{g}_x)$ isometrically to $(\mathsf{T}_y, \mathsf{g}_y)$. When computed with respect to an orthonormal frame, this implies that the Christoffel symbols satisfy:

$$\Gamma_{ij}^k = -\Gamma_{ik}^j,$$

that is, the $n \times n$-matrix of 1-forms $\left[\Gamma_{ij}^k dx^1\right]$ is skew-symmetric.

Exercise B.5.1. The *Koszul formula* is an remarkable explicit formula for the Levi–Civita connection in terms of the metric tensor g and the Lie bracket on $\mathsf{Vec}(M)$. Namely, if $X, Y, Z \in \mathsf{Vec}(M)$, then

$$\begin{aligned} 2\mathsf{g}(\nabla_X Y, Z) = X\mathsf{g}(Y, Z) + Y\mathsf{g}(Z, X) - Z\mathsf{g}(X, Y) \\ - \mathsf{g}(X, [Y, Z]) + \mathsf{g}(Y, [Z, X]) + \mathsf{g}(Z, [X, Y]) \end{aligned}$$

defines the unique affine ∇-orthogonal symmetric connection. In local coordinates $(x^1, \dots, x^n)$,

$$\Gamma_{ij}^k = \frac{1}{2}\mathsf{g}^{km}\left(\frac{\partial \mathsf{g}_{jm}}{\partial x^i} + \frac{\partial \mathsf{g}_{im}}{\partial x^j} - \frac{\partial \mathsf{g}_{ij}}{\partial x^m}\right).$$

B.6. The Levi–Civita connection for the Poincaré metric

We denote complex numbers $z = x + yi \in \mathbb{C}$ where $x, y \in \mathbb{R}$.

Let H^2 denote the upper half-plane

$$\{x + iy \mid x, y \in \mathbb{R},\ y > 0\}$$

with the *Poincaré metric:*

$$g = \frac{|dz|^2}{y^2}$$

We compute the Levi–Civita connection ∇ with respect to several different frames. Although this follows from the Koszul formula, we give a step-by-step derivation which may be suggestive in generalizing this calculation to other contexts.

We follow the discussion of the bi-invariant metric on $\mathsf{Aff}_+(1, \mathbb{R})$ from §10.5 using the model of the upper half-plane $y > 0$, and basepoint

$$p_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longleftrightarrow i.$$

We continue our convention that the basic vector fields X, Y are chosen so that X has value $\partial/\partial x$ at the basepoint p_0 and Y has value $\partial/\partial y$ at p_0. In §10.5, the bi-invariant flat affine connection is computed (Table 10.5)

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Table B.1. Covariant derivatives of left-invariant vector fields with respect to the Levi–Civita connection of the Poincaré metric. Here these vector fields $X = X_{\mathfrak{L}}, Y = Y_{\mathfrak{R}}$ form an orthonormal frame. However, they do *not* forma a coordinate frame since they do not commute: their Lie bracket $[X,Y] = -X$ is nonzero. The nonvanishing curvature is nonzero implies this algebra is not left-symmetric, in contrast to Table 10.5.

	X	Y
X	Y	$-X$
Y	0	0

whereas here we compute the Levi–Civita connection (Table B.1 for the Riemannian structures on H^2, which under the correspondence $\mathsf{Aff}_+(1,\mathbb{R}) \longleftrightarrow \mathsf{H}^2$ is only *left-invariant* on $\mathsf{Aff}_+(1,\mathbb{R})$. Furthermore the curvature of this connection is nonzero and therefore the algebra described in Table B.1 is *not* left-symmetric.

Exercise B.6.1. The action of $G^0 = \mathsf{Aff}_+(1,\mathbb{R})$ on the upper halfplane H^2 is given by the étale representation (10.15) in affine coordinates. The rows of the matrix inverse to the linear part correspond to the dual basis of left-invariant 1-forms:

$$y^{-1}dx, \qquad y^{-1}dy.$$

The sum of their squares is the *left-invariant Poincaré metric* on G^0, regarded as the *upper half-plane* $y > 0$*:*

$$y^{-2}\big(dx^2 + dy^2\big).$$

In terms of the framing (X,Y) by left-invariant vector fields, the Levi–Civita connection is given Table B.1.

Let ∂_x, ∂_y be the coordinate vector fields, so that:

$$\mathsf{g}(\partial_x,\partial_x) = \mathsf{g}(\partial_y,\partial_y) = y^{-2} \tag{B.3}$$

$$\mathsf{g}(\partial_x,\partial_y) = \mathsf{g}(\partial_y,\partial_x) = 0. \tag{B.4}$$

The vector fields

$$X := X_{\mathfrak{L}} := y\partial_x$$
$$Y := Y_{\mathfrak{L}} := y\partial_y$$

define an orthonormal frame field. Therefore, for any vector field ϕ,

$$\phi = \mathsf{g}(\phi, X)X + \mathsf{g}(\phi, Y)Y$$

$$= y^2\Big(\mathsf{g}(\phi,\partial_x)\,\partial_x + \mathsf{g}(\phi,\partial_y)\,\partial_y\Big). \tag{B.5}$$

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Theorem. In terms of the coordinate frame, the Levi–Civita connection is given by:

$$\nabla_x \partial_x = y^{-1} \partial_y \tag{B.6}$$

$$\nabla_x \partial_y = -y^{-1} \partial_x \tag{B.7}$$

$$\nabla_y \partial_x = -y^{-1} \partial_x \tag{B.8}$$

$$\nabla_y \partial_y = -y^{-1} \partial_y \tag{B.9}$$

where ∇_x and ∇_y denote ∇_{∂_x} and ∇_{∂_y} respectively. Table B.1 gives the Levi–Civita connection in orthonormal frame $\{X, Y\}$.

Proof. Symmetry of ∇ and $[\partial_x, \partial_y] = 0$ implies:

$$\nabla_x \partial_y = \nabla_y \partial_x, \tag{B.10}$$

which implies the equivalence (B.7) $\Longleftrightarrow$ (B.8).

Differentiate (B.3) with respect to ∂_x:

$$\mathsf{g}(\nabla_x \partial_x, \partial_x) = 0 \tag{B.11}$$

$$\mathsf{g}(\nabla_x \partial_y, \partial_y) = 0. \tag{B.12}$$

Combine (B.12) with (B.10):

$$\mathsf{g}(\nabla_y \partial_x, \partial_y) = 0 \tag{B.13}$$

Differentiate (B.3) with respect to ∂_y:

$$2\mathsf{g}\big(\nabla_y \partial_x, \partial_x\big) = 2\mathsf{g}\big(\nabla_y \partial_y, \partial_y\big) = -2y^{-3},$$

whence

$$\mathsf{g}\big(\nabla_y \partial_x, \partial_x\big) = -y^{-3} \tag{B.14}$$

and

$$\mathsf{g}\big(\nabla_y \partial_y, \partial_y\big) = -y^{-3.}. \tag{B.15}$$

Now:

$$\begin{aligned} \mathsf{g}\big(\nabla_x \partial_x, \partial_y\big) &= \underbrace{\partial_x \mathsf{g}(\partial_x, \partial_y)}_{0 \text{ by (B.4)}} - \mathsf{g}(\partial_x, \nabla_x \partial_y) \\ &= -\mathsf{g}\big(\partial_x, \nabla_y \partial_x\big) \qquad \text{by (B.10)} \\ &= -\mathsf{g}(\nabla_y \partial_x, \partial_x) = y^{-3} \qquad \text{by (B.13)}, \end{aligned} \tag{B.16}$$

and:

$$\begin{aligned} \mathsf{g}(\nabla_y \partial_y, \partial_x) &= \underbrace{\partial_y \mathsf{g}(\partial_y, \partial_x)}_{0 \text{ by (B.4)}} - \underbrace{\mathsf{g}(\partial_y, \nabla_y \partial_x)}_{0 \text{ by (B.13)}} \\ &= 0. \end{aligned} \tag{B.17}$$

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Now we compute the covariant derivatives $\nabla_x \partial_x, \nabla_x \partial_y, \nabla_y \partial_x, \nabla_y \partial_y$ in terms of their inner products:

(B.8) follows by applying (B.5) to (B.13) and (B.14).

(B.7) follows by applying (B.10) to (B.8), as mentioned above.

(B.6) follows by applying (B.5) to (B.11) and (B.16).

(B.9) follows by applying (B.5) to (B.17) and (B.15).

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Appendix C

Representations of nilpotent groups

We collect here several basic facts about linear representations of nilpotent groups.

C.1. Nilpotent groups

Recall that group G is *nilpotent* if $\exists r \in \mathbb{N}$ such that each iterated commutator

$$[g_1, [g_2, [g_3, \dots g_r]]] = e$$

for $g_1, \dots, g_r \in G$. In other words, the *lower central series*

$$G > [G, G] > [G, [G, G]] > \cdots > [G, \dots [G, [G, G]]] = \{e\}$$

terminates after r steps.[1] The smallest number f is called the *nilpotence* or *class* of G. Nilpotence 0 means the group is trivial and nilpotence 1 is equivalent to commutativity. Sometimes a nilpotent group of nilpotence r is called *r-step nilpotent.*

A nilpotent group is an *iterated central extension* in the following sense.

Exercise C.1.1. A *central extension* of groups is an exact sequence An *extension* of groups is an exact sequence

$$A \rightarrowtail B \twoheadrightarrow C \tag{C.1}$$

[1] $[A, B]$ denotes the subgroup generated by commutators $[a, b] := aba^{-1}b^{-1}$ where $a \in A$ and $b \in B$.

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where $A \lhd C$, the first arrow is inclusion of the normal subgroup $A \lhd C$, and the second arrow defines an isomorphism $B/A \xrightarrow{\approx} C$. The extension (C.1) is *central* if and only if $B/A < \mathsf{center}(C/A)$.

- A *central series* is a series of normal subgroups $G_i \lhd G$:

$$\{e\} = G_0 < G_1 < \cdots < G_r = G$$

  such that each extension

$$G_i \rightarrowtail G_{i+1} \twoheadrightarrow \ G$$

  is central. Equivalently, $[G, G_{i+1}] < G_i$. Show that the lower central series as defined above is a central series.
- Show that the series defined by $G_i/G_{i-1} = \mathsf{center}(G/G_{i-1})$ and $G_1 = \mathsf{center}(G)$ is a central series (called the *upper central series*).
- Show that every central series of a nilpotent group lies between the lower central series and the upper central series. For a nilpotent group, the lower and upper central series have the same length.

A linear transformation is *unipotent* if its only eigenvalue is 1. Unipotent groups of matrices form an important class of nilpotent groups:

Exercise C.1.2. Suppose that $G < \mathsf{GL}(\mathsf{V})$ is a group of matrices which are *unipotent upper-triangular,* that is, the matrices are upper triangular with every diagonal entry equal to 1. Then G is nilpotent. Relate its nilpotence to the $\mathsf{dim}(\mathsf{V})$.

C.2. Simultaneous Jordan canonical form

Theorem C.2.1. Let V be a vector space over $\mathbb{C}$ and $\Gamma < \mathsf{GL}(\mathsf{V})$ a nilpotent group. Then $\exists k \in \mathbb{N}$ and Γ-invariant subspaces $\mathsf{V}_i < \mathsf{V}$ for $i = 1, \dots k$ such that

$$\mathsf{V} \ = \ \bigoplus_{i=1}^{k} \mathsf{V}_i$$

and homomorphisms $\Gamma \xrightarrow{\lambda_i} \mathbb{C}^*$ such that for each $\gamma \in \Gamma$ and $i = 1, \dots, k$, the restriction $\gamma - \lambda_i \mathbf{1}$ to V_i is nilpotent. Furthermore there exists a basis of V_i such that each restriction $g|_{\mathsf{V}_i}$ is upper-triangular with diagonal entry $\lambda_i(g)$.

For a cyclic group $\Gamma = \langle\gamma\rangle$, this is just the *Jordan normal form* of γ. Namely, for each scalar $\lambda \in \mathbb{C}$, the *generalized eigenspace* with eigenvalue λ

$$\mathsf{V}_\lambda \ := \ \mathsf{Ker}\big(\gamma - \lambda\mathbf{1}\big)$$

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is γ-invariant, the restriction $\gamma|_{\mathsf{V}_\lambda}$ equals $\lambda\mathbf{1}$ plus a nilpotent transformation, and V decomposes as a direct sum

$$\mathsf{V} \;=\; \bigoplus_{\lambda\in\mathbb{C}} \mathsf{V}_\lambda.$$

Moreover, $\exists$ an invariant complete flag

$$0 < \mathsf{F}^1_\lambda < \cdots < \mathsf{F}^2_\lambda < \cdots < \mathsf{F}^{\mathsf{dim}(\mathsf{V}_\lambda)}_\lambda \;=\; \mathsf{V}_\lambda$$

such that induced action of $\gamma - \lambda\mathbf{1}$ on each 1-dimensional vector space $\mathsf{F}^i_\lambda/\mathsf{F}^{i-1}\lambda$ is trivial. In other words, a basis exists where $\gamma|_{\mathsf{V}_\lambda}$ is upper triangular with each diagonal entries equal to λ. This generalizes to the *Jordan decomposition,* described in §15.1 Humplreys [**198**] for general algebraic groups. Here the *semisimple part* of γ is the direct sum

$$\gamma_s := . \bigoplus_{\lambda\in\mathbb{C}} \lambda\mathbf{1}_{\mathsf{V}_\lambda}$$

(the *semisimplication* of γ) and the *unipotent part* γ_u is defined by $\gamma = \gamma_s\gamma_u$ and the condition that γ_s and γ_u commute.

When Γ is a more general nilpotent group, all elements have this common normal form. In this case the weights determine a homomorphism

$$\Gamma \xrightarrow{\lambda} \mathbb{C}^\times$$

and the elements of γ have common generalized eigenspaces according to λ. The *upper central series*

$$\{e\} < \zeta_1(\Gamma) < \dots \zeta_k(\Gamma) < \dots$$

is the chain of normal subgroups such that $\zeta_1(\Gamma)$ is the center of Γ and, inductively,

$$\zeta_k(\Gamma) \;=\; \mathsf{center}\big(\Gamma/\zeta_{k-1}\big).$$

Exercise C.2.2. Prove this theorem by induction on the length of the *upper* central series, using the fact that a transformation commuting with γ preserves its Jordan decomposition $\mathsf{V} = \bigoplus_\lambda \mathsf{V}_\lambda$. (Hint: start with the Jordan decomposition of an element of the complement $\Gamma \setminus [\Gamma,\Gamma]$ — the "top" of the lower central series — whose centralizer contains $\zeta_1(\Gamma)$ — and inductively work your way down.)

C.3. Nilpotent Lie groups, algebraic groups, and Lie algebras

This theorem can be deduced from the analogous statement about nilpotent algebraic groups, nilpotent Lie groups and nilpotent Lie algebras. (Compare Sagle–Walde [**298**], Theorem 11.14 for Lie groups and Lie algebras, and a detailed proof for Lie algebras, and Humplreys [**198**],§19.2, for algebraic groups.) In general, it is useful to pass to the *algebraic hull* (Zariski closure)

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$\mathbb{A}(\Gamma)$ of Γ in $\mathsf{GL}(\mathsf{V})$, which remains nilpotent if Γ is. Since an algebraic Lie group has finitely many connected components (in the classical topology), a finite index subgroup of Γ lies in a connected nilpotent Lie group. Then the classic theorems of Engel and Lie–Kolchin on Lie algebras apply: the theorem of Lie–Kolchin implies that in a suitable basis, the matrices are upper-triangular and Engel's theorem implies that if all the elements are unipotent, then the matrices can be put in upper-triangular form with 1's on the diagonal.

Closely related is the useful fact that if G is a nilpotent algebraic group, then the semisimple elements and the unipotent elements form commuting closed normal subgroups G_s and G_u respectively, and $G \cong G_s \times G_u$. Using the Lie–Kolchin theorem that *solvable groups* (which includes nilpotent groups) can be upper triangularized, G_s is represented by a group of diagonal matrices, and G_u will be upper-triangular with each diagonal entry equal to 1. The key fact characterizing nilpotence is that G_s and G_u commute with one another.

C.3.1. Affine representations of nilpotent groups. Now we extend the preceding theory of *linear* representations to affine representations. The linearization of affine representations §1.8 implies that the $n+1$-dimensional linear representation of an n-dimensional affine representation always has 1 as a weight, so there is nontrivial generalized eigenspace upon which the group acts unipotently. Define an affine transformation to be *unipotent* if and only if its linear part is unipotent.

Exercise C.3.1. An affine transformation of A^n is unipotent if and only if its linearization is a unipotent linear transformation of $\mathbb{R}^{n+1}$.

Theorem C.2.1 has the following geometric consequence:

Corollary C.3.2. Let A be an affine space over $\mathbb{C}$ and $\Gamma < \mathsf{Aff}(\mathsf{A})$. Then there exists a unique maximal Γ-invariant affine subspace $\mathsf{A}_1 < \mathsf{A}$ such that the restriction of Γ to A_1 is unipotent.

A_1 is called the *Fitting subspace* in [**140**].

Since Γ preserves the affine subspace A_1, it induces an affine action on the quotient space A/A_1. Denote by V the vector space underlying A. Since the affine action on A/A_1 is radiant (it preserves the coset $\mathsf{A}_1 < \mathsf{A}$), we may describe A as an *affine direct sum:*

$$\mathsf{A} \;=\; \mathsf{A}_1 \oplus \mathsf{V}_1$$

where $\mathsf{V}_1 \subset \mathsf{V}$ is an $\mathsf{L}(\Gamma)$-invariant linear subspace. Compare Fried–Goldman–Hirsch [**140**]] where this is proved cohomologically, using techniques from Hirsch [**192**].

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Appendix D

4-dimensional filiform nilpotent Lie algebras

Auslander [**15**] conjectured that a nilpotent simply transitive affine group contains central translations. Fried [**137**] disproved Auslander's conjecture with a complete left-invariant affine structure on the 4-dimensional filiform nilpotent Lie group.

A nilpotent Lie algebra is said to be *filiform* if it is "maximally non-abelian," in the sense that its degree of nilpotence is one less than its dimension. That is, a k-step nilpotent Lie algebra is fili form if its dimension equals $k+1$. Fried's example lives on the unique 4-dimensional filiform Lie algebra. (Curiously, Benoist's example of an 11-dimensional nilpotent Lie algebra admitting no faithful 12-dimensional linear representation — and hence no simply transitive affine actions — is a filiform Lie algebra.)

Let $t, u, v, w \in \mathbb{R}$ be real parameters. We parametrize the 4-dimensional filiform algebra as a semidirect sum

$$\mathfrak{g} := \mathsf{V}_0 \rtimes_{J_3} \mathbb{R}T,$$

where V_0 is a 3-dimensional abelian ideal with coordinates u, v, w and T acts by the 3-dimensional Jordan block

$$J_3 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

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Table D.1. Complete affine structure on 4-dimensional filiform algebra

	X_0	Y_0	Z_0	W_0
X_0	0	0	Y_0	Z_0
Y_0	0	0	0	0
Z_0	0	0	0	0
W_0	0	0	0	0

That is, $\mathfrak{g}$ admits a basis U, V, W, T subject to nonzero commutation relations

$$[T, U] = V, \qquad [T, V] = W$$

and all other brackets between basic elements are zero.

We start with a simply transitive affine action, where U, V, W act by translations. The Lie algebra representation is:

$$\mathcal{A}_0(t, u, v, w) := \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & t \\ & 0 & t & 0 & u \\ & & 0 & t & v \\ & & & 0 & w \end{array}\right],$$

the group representation is:

$$A_0(t, u, v, w) := \exp\big(\mathcal{A}_0(t, u, v, w)\big) := \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & t \\ & 1 & t & t^2/2 & u + tv/2 + t^2w/6 \\ & & 1 & t & v + tw/2 \\ & & & 1 & w \end{array}\right]$$

and a basis of left-invariant vector fields is:

$$X_0 := \partial_x, \quad Y_0 := \partial_y, \quad Z_0 := \partial_z + x\partial_y, \quad W_0 := \partial_w + x\partial_z + \frac{x^2}{2}\partial_y$$

with covariant derivatives tabulated in Table D.1.

Then X_0, Y_0 are parallel vector fields and $\mathfrak{z}(\mathfrak{g})$ is 1-dimensional, spanned by Y_0.

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Now we deform using a parameter $\lambda \in \mathbb{R}$, in the direction of the parallel vector field $X_0 = \partial_x$. Consider the affine representation $A = A_\lambda$ defined by:

$$A_\lambda(t,u,v,w) := \left[\begin{array}{cccc|c} 0 & -\lambda\, w & \lambda\, v & -\lambda\, u & t \\ & 0 & t & 0 & u \\ & & 0 & t & v \\ & & & 0 & w \end{array}\right].$$

Since

$$A_\lambda(t,u,v,w)^2 = \left[\begin{array}{ccc|c} 0 & -\lambda\, tw & \lambda\, tv & \lambda(-2uw + v^2) \\ & 0 & t^2 & tv \\ & & 0 & tw \\ & & & 0 \end{array}\right],$$

$$A_\lambda(t,u,v,w)^3 = \left[\begin{array}{cc|c} 0 & -\lambda\, t^2 w & 0 \\ & 0 & t^2 w \\ & & 0 \\ & & 0 \end{array}\right],$$

$$A_\lambda(t,u,v,w)^4 = \left[\begin{array}{c|c} 0 & -\lambda t^2 w^2 \\ & 0 \\ & 0 \\ & 0 \end{array}\right]$$

the general group element is:

$$\exp\big(A_\lambda(t,u,v,w)\big) =$$
$$\left[\begin{array}{cccc|c} 1 & -\lambda\, w & \lambda\,(v - tw/2) & \lambda\big(-u + tv/2 - t^2 w/6\big) & t + \lambda\big(-uw + v^2/2 - t^2 w^2/24\big) \\ & 1 & v - tw^w/2 & t^2/2 & u + tv/2 + t^2 w/6 \\ & & 1 & t & v + tw/2 \\ & & & 1 & w \end{array}\right].$$

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The last column of this matrix is the developing map, and we can relate the group coordinates to the affine coordinates, by:

$$\begin{aligned} x &= t + \lambda\big(- uw + v^2/2 - t^2w^2/24\big) \\ y &= u + tv/2 + t^2w/6 \\ z &= v + tw/2 \\ w &= w. \end{aligned}$$

A basis for right-invariant vector fields is:

$$T := \partial_x + z\partial_y + w\partial_z, \quad U := \partial_y - \lambda w \partial_x, , \quad V := \partial_z + \lambda z \partial_x, \quad \widetilde{W} := \partial_w - \lambda y \partial_x$$

with U central. Table D.2 gives the multiplication table for Fried's example.

Table D.2. Fried's counterexample to Auslander's conjecture on central translations

	X	Y	Z	$\widetilde{W}$
X	0	0	Y	Z
Y	0	0	0	λX
Z	0	0	λX	0
$\widetilde{W}$	0	λX	0	0

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Appendix E

Semicontinuous functions

E.1. Definitions and elementary properties

Let X be a topological space and $X \xrightarrow{f} \mathbb{R}$ a function. Then f is *upper semicontinuous* if it satisfies any of the following equivalent conditions:

- For each $x \in X$, f is *upper semicontinuous at x*, that is, for all $\epsilon > 0$,
$$f(y) < f(x) + \epsilon$$
for y in some neighborhood of x.
- f is a continuous mapping from X to $\mathbb{R}$, where $\mathbb{R}$ is given the topology whose nonempty open sets are intervals $(-\infty, a)$ where $a \in \mathbb{R} \cup \{+\infty\}$.

Examples of upper semicontinuous functions include the indicator function of a closed set, or the *greatest integer* (or *floor*) function. A function f is *lower semicontinuous* if and only if $-f$ is upper semicontinuous.

Exercise E.1.1. Show that the following conditions are equivalent:

- f is lower semicontinuous.
- $\forall x \in X$ and $\epsilon > 0$,
$$f(y) > f(x) - \epsilon$$
for y in an open neighborhood of x.
- $X \xrightarrow{f} \mathbb{R}$ is continuous where $\mathbb{R}$ is given the topology generated by infinite open intervals (a, ∞), for $a \in \mathbb{R}$.

Exercise E.1.2. A semicontinuous function on a smooth manifold is Borel. (Hint: see Rudin [**297**], Theorem 1.12(c))

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E.2. Approximation by continuous functions

Let (X, d) be a metric space, $n > 0$ and $X \xrightarrow{f} \mathbb{R}$ any function.

Proposition E.2.1. Let $n \geq 0$. The function

$$X \xrightarrow{h_n} \mathbb{R}$$
$$x \longmapsto \sup\big\{ f(p) - n\,\mathsf{d}(p,x) \mid p \in X \big\}$$

is n-Lipschitz.

Proof. For any $p, y \in X$,

$$f(p) - n\,\mathsf{d}(p,y) \leq \sup\big\{ f(q) - n\,\mathsf{d}(q,y) \mid q \in X \big\} \;=\; h_n(y)$$

so

$$f(p) \;\leq\; h_n(y) \,+\, n\,\mathsf{d}(p,y) \;\leq\; h_n(y) \,+\, n\big(\mathsf{d}(p,x) + \mathsf{d}(x,y)\big)$$

whence

$$f(p) - n\,\mathsf{d}(p,x) \;\leq\; h_n(y) + n\,\mathsf{d}(x,y).$$

Taking the supremum over p,

$$h_n(x) \;\leq h_n(y) + n\,\mathsf{d}(x,y)$$

and

$$h_n(x) - h_n(y) \;\leq\; n\,\mathsf{d}(x,y).$$

Symmetrizing, the result follows. ☐

Lemma E.2.2. If $m < n$, then $h_m(x) \geq h_n(x)$.

Proof. Taking the supremum over p of

$$h_m(x) \geq\; f(p) \;-\; n\,\mathsf{d}(p,x),$$

the result follows. ☐

Taking $p = x$ in the definition of h_n yields $h_n(x) \geq f(x)$. We now prove that a bounded upper semicontinuous function is the pointwise limit of a monotonically nonincreasing sequence of continuous (in fact Lipschitz) functions:

$$h_n(x) \searrow f(x).$$

Exercise E.2.3. Let f be an upper semicontinuous function on a compact space X. Then f is bounded above, that is, $\exists M < \infty$ such that $f(x) \leq M$ for all $x \in X$.

Proposition E.2.4. Suppose that f is an upper semicontinuous function which is bounded above. Let $x \in X$. Then

$$\lim_{n\to\infty} h_n(x) \;=\; f(x).$$

Proof. Suppose that $M < \infty$ and $f(y) < M$ for all $y \in X$. Let $\epsilon > 0$. Then it suffices to prove:

$$h_n(x) \;\leq f(x) + \epsilon \tag{E.1}$$

for sufficiently large n (depending on x, ϵ and M).

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Since f is upper semicontinuous, $\exists\delta > 0$ such that

$$f(y) \ < \ f(x) + \epsilon \tag{E.2}$$

whenever $\mathsf{d}(x, y) < \delta$. We claim:

$$f(y) - n\,\mathsf{d}(y, x) \ < \ f(x) + \epsilon \tag{E.3}$$

whenever

$$n \ > \ \frac{M - f(x)}{\delta}. \tag{E.4}$$

If $\mathsf{d}(x, y) < \delta$, then (E.2) implies:

$$f(y) - n\,\mathsf{d}(y, x) \ < \ f(y) \ < \ f(x) + \epsilon.$$

Otherwise $\mathsf{d}(x, y) \geq \delta$ and (E.4) implies:

$$f(y) - n\,\mathsf{d}(y, x) \ < \ M - n\delta \ \leq \ f(x) \ < \ f(x) + \epsilon,$$

as claimed, proving (E.3). Taking supremum yields (E.1), completing the proof of Proposition E.2.4. □

Exercise E.2.5. Let $X = \mathbb{R}$ and f be the *indicator function* at $0 \in \mathbb{R}$, that is,

$$f(x) \ := \ \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$

Then we may take

$$h_n(x) \ = \ \max(1 - nx, 0)$$

which is supported on the interval $[-1/n, 1/n]$.

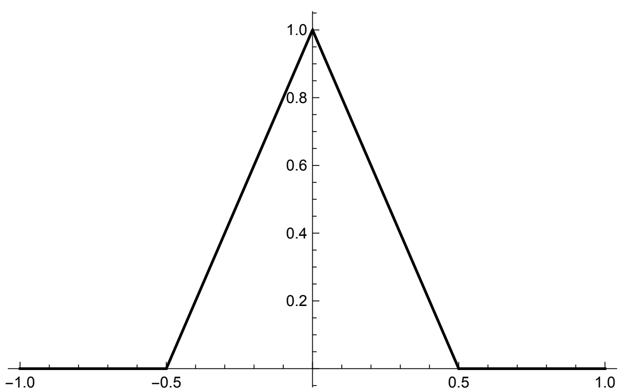

Figure E.1. Continuous approximation of an indicator function

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Appendix F

$\mathsf{SL}(2,\mathbb{C})$ and $\mathsf{O}(3,1)$

We prove the local isomorphism

$$\mathsf{SL}(2,\mathbb{C}) \longrightarrow \mathsf{O}(3,1)$$

mentioned in §3.3.2.

F.1. 2-dimensional complex symplectic vector spaces

Let V be a 2-dimensional vector space over $\mathbb{C}$, and let $\mathsf{V}\times\mathsf{V} \xrightarrow{\Omega} \mathbb{C}$ be a nonzero (and hence nondegenerate) symplectic structure (that is, a skew-symmetric $\mathbb{C}$-bilinear form). We call (V,Ω) a $\mathbb{C}$-*symplectic vector space* of dimension 2.

$\mathsf{SL}(2,\mathbb{C})$ equals the automorphism group $\mathsf{Aut}(\mathsf{V},\Omega)$ of this structure.

Let $\mathsf{V}_{\mathbb{R}}$ be the underlying real vector space. Then V corresponds to the pair $(\mathsf{V}_{\mathbb{R}},\mathbf{J})$ where $\mathsf{V}_{\mathbb{R}} \xrightarrow{\mathbf{J}} \mathsf{V}_{\mathbb{R}}$ is the complex structure, corresponding to scalar multiplication by $i=\sqrt{-1}$. In other words the $\mathbb{C}$-vector space V is equivalent to the $\mathbb{R}$-vector space with complex structure $\mathbf{J}$.

In terms of $(\mathsf{V}_{\mathbb{R}},\mathbf{J})$, the $\mathbb{C}$-symplectic structure Ω on V is equivalent to a pair (ω,ψ) of symplectic structures

$$\mathsf{V}_{\mathbb{R}} \times \mathsf{V}_{\mathbb{R}} \longrightarrow \mathbb{R}$$

on $\mathsf{V}_{\mathbb{R}}$ which are the real and imaginary parts of Ω:

$$\Omega(x,y) \;=\; \omega(x,y) + i\psi(x,y).$$

The three algebraic objects $\mathbf{J},\omega,\psi$ on $\mathsf{V}_{\mathbb{R}}$ are interrelated.

First, both ω and ψ are compatible with $\mathbf{J}$ in the following sense:

$$\omega(\mathbf{J}x,\mathbf{J}y) = -\omega(x,y) \tag{F.1}$$

$$\psi(\mathbf{J}x,\mathbf{J}y) = -\psi(x,y). \tag{F.2}$$

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Furthermore $\mathbf{J}$ and ω determine ψ by:

$$\psi(x, y) = -\omega(\mathbf{J}x, y). \tag{F.3}$$

Thus the pair (V, Ω) is equivalent to the triple $(\mathsf{V}_{\mathbb{R}}, \mathbf{J}, \omega)$ satisfying (F.1) or the quadruple $(\mathsf{V}_{\mathbb{R}}, \mathbf{J}, \omega, \psi)$ satisfying (F.1) and (F.3). Furthermore $\mathsf{SL}(2, \mathbb{C})$ equals the automorphism group $\mathsf{Aut}(\mathsf{V}_{\mathbb{R}}, \mathbf{J}, \omega) = \mathsf{Aut}(\mathsf{V}_{\mathbb{R}}, \mathbf{J}, \omega, \psi)$.

F.2. Split orthogonal 6-dimensional vector spaces

Choose a nonzero element $\mu \in \Lambda^4(\mathsf{V}_{\mathbb{R}})$ of $\mathsf{V}_{\mathbb{R}}$. The second exterior power $\Lambda^2(\mathsf{V}_{\mathbb{R}})$ has dimension 6 and admits a nondegenerate symmetric bilinear form

$$\begin{aligned} \Lambda^2(\mathsf{V}_{\mathbb{R}}) \times \Lambda^2(\mathsf{V}_{\mathbb{R}}) &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto x \cdot y \end{aligned}$$

defined by:

$$x \wedge y = (x \cdot y)\mu.$$

If $g \in \mathsf{GL}(\mathsf{V}_{\mathbb{R}})$, then

$$g(x) \cdot g(y) = \det(g)\, x \cdot y$$

applied to an orientation-reversing element of $\mathsf{GL}(\mathsf{V}_{\mathbb{R}})$ implies this form is equivalent to its negative and therefore has signature $(3, 3)$. This defines a local isomorphism $\mathsf{SL}(4, \mathbb{R}) \longrightarrow \mathsf{O}(3, 3)$.

F.3. Symplectic 4-dimensional real vector spaces

Now introduce the first symplectic structure $\omega \in \Lambda^2(\mathsf{V}_{\mathbb{R}}^*)$. If $g \in \mathsf{Aut}(\mathsf{V}_{\mathbb{R}})$ stabilizes ω, it stabilizes the dual bivector $\omega^* \in \Lambda^2(\mathsf{V}_{\mathbb{R}}) = \mathsf{W}$. Choosing μ so that $(\omega \wedge \omega)(\mu) < 0$ implies that $\omega^* \cdot \omega^* < 0$. It follows that the orthogonal complement

$$\mathsf{W}_1 := (\omega^*)^{\perp} \subset \mathsf{W}$$

is a nondegenerate subspace having signature $(3, 2)$. In particular the stabilizer $\mathsf{Aut}(\mathsf{V}_{\mathbb{R}}, \omega) \cong \mathsf{Sp}(4, \mathbb{R})$ preserves W_1, defining a local isomorphism $\mathsf{Sp}(4, \mathbb{R}) \longrightarrow \mathsf{O}(3, 2)$.

Similarly, the second symplectic structure $\psi \in \Lambda^2(\mathsf{V}_{\mathbb{R}}^*)$ admits a dual bivector $\psi^* \in \Lambda^2(\mathsf{V}_{\mathbb{R}}) = \mathsf{W}$ orthogonal to ω^* also satisfying $\psi^* \cdot \psi^* < 0$. The orthogonal complement $\mathsf{W}_2 := (\omega^*, \psi^*)^{\perp} \subset \mathsf{W}$ has signature $(3, 1)$.

By (F.3), the complex structure $\mathbf{J}$ and the symplectic structure ω determine the symplectic structure ψ, so

$$\mathsf{Aut}(\mathsf{V}_{\mathbb{R}}, \omega, \psi) = \mathsf{Aut}(\mathsf{V}.\Omega) \cong \mathsf{SL}(2, \mathbb{C}).$$

The stabilizer $\mathsf{Aut}(\mathsf{V}_{\mathbb{R}}, \omega, \psi)$ preserves W_2, defining a local isomorphism $\mathsf{SL}(2, \mathbb{C}) \longrightarrow \mathsf{O}(3, 1)$.

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F.4. Lorentzian 4-dimensional vector spaces

It remains to show that the composition

$$\mathsf{SL}(2,\mathbb{C}) \hookrightarrow \mathsf{Sp}(4,\mathbb{R}) \longrightarrow \mathsf{O}(3,2) \tag{F.4}$$

defines a local isomorphism

$$\mathsf{SL}(2,\mathbb{C}) \longrightarrow \mathsf{O}(3,1) < \mathsf{O}(3,2).$$

Clearly the composition of two local isomorphisms is a local isomorphism onto its image.

Exercise F.4.1. Show that the image of (F.4) equals $\mathsf{O}(3,1)$.

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Appendix G

Lagrangian foliations of symplectic manifolds

Affine structures arise naturally in symplectic geometry. Namely, an affine structure is exactly the structure a manifold acquires as a leaf of a Lagrangian foliation of a symplectic manifold. This section discusses this result, due to Alan Weinstein [**344**] (Theorems 7.7 and 7.8). This fundamental fact is a key point both in the representation theory (geometric quantization), and mathematical physics (mirror symmetry), where a pair of transverse Lagrangian foliations is a called *real polarizations.* Compare also Bates–Weinstein [**30**], Corollary 7.8.

G.1. Lagrangian foliations

Recall that a *symplectic structure* on a vector space V is a nondegenerate skew-symmetric bilinear form $\omega \in \Lambda^2(\mathsf{V}^*)$. A pair (V, ω), where ω is a symplectic structure on a vector space V is called a *symplectic vector space.* A linear subspace $L < \mathsf{V}$ is *Lagrangian* in (V, ω) if and only if

- (Isotropy) The restriction of ω to $L \times L$ is zero;
- (Co-isotropy) If $v \in \mathsf{V}$ satisfies $\omega(v, L) = 0$ then $v \in L$.

Suppose now M is a smooth manifold. An exterior 2-form $\omega \in \mathcal{A}^2(M)$ is *nondegenerate* if it defines a nondegenerate pairing on each tangent space T_pM, making each T_pM into a symplectic vector space. If, in addition, ω is *closed*, (that is, $d\omega = 0$), then ω is a *symplectic structure* on M, and (M, ω) is a *symplectic manifold.* An immersed submanifold $L \looparrowright M$ is *Lagrangian* if and only if each tangent space $\mathsf{T}_pL < \mathsf{T}_pM$ is Lagrangian $\forall p \in L$. A foliation $\mathfrak{F}$ of M is *Lagrangian* if each leaf $L \looparrowright M$ is an (immersed) Lagrangian submanifold of (M, ω).

For any manifold N, its cotangent bundle T^*N admits a natural (exact) symplectic structure ω. This structure is *natural* in the sense that it is preserved by the lifts of diffeomorphisms in $\mathsf{Diff}(N)$ to the cotangent bundle. Furthermore the fibers

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and the 0-section of T^*N are *Lagrangian submanifolds* of the symplectic manifold (T^*N, ω).

Now suppose that N is an affine manifold modeled on an affine space A with underlying vector space V. (If $x \in N$ is a basepoint, then V identifies with the tangent space T_xN.) Then then the cotangent bundle T^*N has a flat connection, which is a foliation $\mathcal{F}$ of the total space. Namely, the transpose of the linear holonomy $\pi_1(N) \longrightarrow \mathsf{GL}(\mathsf{V}^*)$ defines a flat bundle, naturally isomorphic to T^*N, whose total space M is the quotient of $\widetilde{L} \times \mathsf{V}^*$ by the above action of $\pi_1(N)$. The leaves of this foliation are the quotients of $\widetilde{N} \times \{\phi\}$ where $\phi \in \mathsf{V}^*$ is a covector. The restriction of the bundle projection to a leaf makes the leaf a covering space of L, with fundamental group the isotropy subgroup of ϕ under the action of $\pi_1(L)$ on V^*. The zero-section $\mathbf{0}_N$ embeds N as a leaf of this foliation. With respect to the canonical symplectic structure ω on T^*N, each leaf L is a Lagrangian submanifold of (T^*L, ω), and $\mathcal{F}$ is a Lagrangian foliation. In particular $\mathbf{0}_N$ is a leaf of $\mathcal{F}$.

Exercise G.1.1. Find an example of a closed affine manifold N for which $\mathbf{0}$ is the only closed leaf of this foliation of T^*N.

G.2. Bott's partial connection on a foliated manifold

The intuition behind Weinstein's result combines ideas from foliation theory and symplectic geometry. A foliation $\mathcal{F}$ of a smooth manifold M is defined by a subbundle, which we call $\mathsf{T}\mathcal{F} < \mathsf{T}M$, such that its space of sections

$$\mathsf{Vec}_{\mathfrak{F}}(M) \;:=\; \Gamma(\mathsf{T}\mathcal{F}) \quad < \quad \Gamma(\mathsf{T}M) \;=\; \mathsf{Vec}(M)$$

is *integrable,* that is, a subalgebra of $\mathsf{Vec}(M)$ under Lie bracket.

A *leaf* is a maximal injectively immersed *integral submanifold,* that is, a submanifold L whose tangent space T_pL equals $(\mathsf{T}\mathcal{F})_p$ for every $p \in L$. The leaves partition M into submanifolds, which locally looks like the partition of $\mathbb{R}^n$ into parallel affine subspaces $\mathbb{R}^{n-q} \times \{y\}$, where $y \in \mathbb{R}^q$. (See, for example, Lee [**244**], §19.)

In proving the vanishing of characteristic classes of vector bundles associated to foliations, Bott [**60, 61**]), introduced a *partial connection* on the normal bundle

$$\nu_{\mathcal{F}} := \mathsf{T}M/\mathsf{T}\mathcal{F}$$

to a foliation $\mathcal{F}$,[1] that is, a mapping

$$\mathsf{Vec}_{\mathcal{F}}(M) \times \Gamma(\nu_{\mathcal{F}}) \longrightarrow \Gamma(\nu_{\mathcal{F}}).$$

The *holonomy pseudogroup* of $\mathcal{F}$ gives rise to a way of parallel transporting "normal vectors" to a leaf L along paths lying on L, in other words lifting a path γ on L to a path $\tilde{\gamma}$ in M. One requires the "endpoint" of the lifted path $\tilde{\gamma}(t)$ remain in

[1] A partial connection is a way of defining parallel transport along *tangential paths,* and, as such, the covariant derivatives are only defined with respect to vector fields in $\mathsf{Vec}_{\mathcal{F}}(M)$. The *Bott connection* is defined by extending the bilinear mapping $\mathsf{Vec}_{\mathcal{F}}(M) \times \Gamma(\nu_{\mathcal{F}}) \to \Gamma(\nu_{\mathcal{F}})$ to $\mathsf{Vec}(M) \times \Gamma(\nu_{\mathcal{F}}) \to \Gamma(\nu_{\mathcal{F}})$ in a relatively arbitrary way. Compare also Haefliger [**184**]. However, we will not need this extension.

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the same leaf as the initial lift $\tilde{\gamma}(0)$, at least for sufficiently small paths and $\tilde{\gamma}(0)$ sufficiently near $\gamma(0)$.

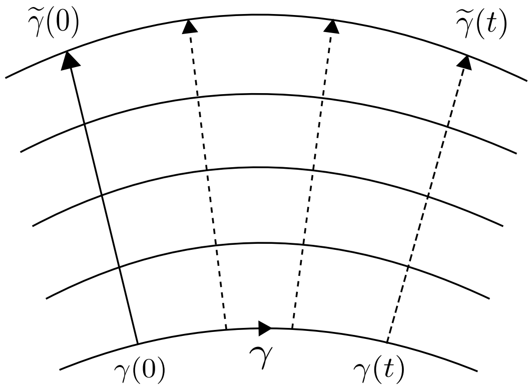

Figure G.1. The holonomy along paths tangential to the leaves parallel transports directions transverse to the leaves. Intuitively, the endpoint of a "normal vector" to the foliation is transported to stay in the same leaf of the foliation.

This is the *infinitesimal Bott partial connection,* which covariantly differentiates "normal vector fields" along tangential vector fields. Integrability of the plane field $\mathsf{T}\mathcal{F} \subset \mathsf{T}M$ implies that its space of sections $\mathsf{Vec}_{\mathcal{F}}(M) < \mathsf{Vec}(M)$ is a (Lie) subalgebra, so

$$\begin{aligned} \mathsf{Vec}_{\mathcal{F}}(M) \times \mathsf{Vec}(M) &\longrightarrow \mathsf{Vec}(M) \\ (\xi, \eta) &\longmapsto [\xi, \eta] \end{aligned} \tag{G.1}$$

maps $\mathsf{Vec}_{\mathcal{F}}(M)$ to itself whenever $\eta \in \mathsf{Vec}_{\mathcal{F}}(M)$ is tangental. Therefore (G.1) defines an $\mathbb{R}$-bilinear pairing

$$\mathsf{Vec}_{\mathcal{F}}(M) \times \Gamma(\nu_{\mathcal{F}}) \longrightarrow \Gamma(\nu_{\mathcal{F}})$$

since

$$\Gamma(\nu_{\mathcal{F}}) = \mathsf{Vec}(M)/\mathsf{Vec}_{\mathcal{F}}(M).$$

This defines, on each leaf L, a connection on the restriction $(\nu_{\mathcal{F}})\big|_L$. The Jacobi identity implies that this connection is flat.

Now we pass from a flat connection on $\nu_{\mathcal{F}}$ to a flat connection on $\mathsf{T}\mathcal{F}$. When ω is a symplectic structure on M and $\mathcal{F}$ is Lagrangian with respect to ω, then the normal bundle $\nu_{\mathcal{F}} = \mathsf{T}M/\mathsf{T}\mathcal{F}$ is isomorphic to the vector bundle $\mathsf{T}^*\mathcal{F}$ dual to $\mathsf{T}\mathcal{F}$. This follows from the following standard fact in symplectic linear algebra:

Exercise G.2.1. If $L < \mathsf{V}$ is a Lagrangian subspace of a symplectic vector space (V, ω), then ω defines an isomorphism $\mathsf{V}/L \longrightarrow L^*$.

When $\mathcal{F}$ is a Lagrangian foliation of a symplectic manifold (M, ω), this construction defines an isomorphism $\mathsf{T}M/\mathsf{T}\mathcal{F} \to \mathsf{T}^*\mathcal{F}$ of vector bundles. The connection on $\mathsf{T}M/\mathsf{T}\mathcal{F} \cong \mathsf{T}^*\mathcal{F}$ defines a *dual connection* on $\mathsf{T}\mathcal{F}$, uniquely specified that the natural pairing $\mathsf{T}^*\mathcal{F} \times \mathsf{T}\mathcal{F} \to \mathbb{R}$ be parallel.

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Exercise G.2.2. A connection is flat if and only if its dual connection is flat.

Thus, if L is a leaf of a Lagrangian foliation $\mathcal{F}$ of a symplectic manifold (M, ω) as above, its tangent bundle $\mathsf{T}L$ identifies with the restriction $\mathsf{T}\mathcal{F}|_L$ and, by duality, the flat connection on $\nu_{\mathcal{F}}$ defines a flat connection on $\mathsf{T}L$.

G.3. Affine connections on the leaves

The next exercises formalize this intuition in terms of vector fields and differential forms. Let (M, ω) be a symplectic manifold and L a leaf of a Lagrangian foliation $\mathcal{F}$. We construct a flat torsion-free affine connection (that is, an *affine structure*) on L.

Exercise G.3.1. Here are the steps to construct an affine connection on L with vanishing curvature and torsion.

- First, suppose M is a smooth manifold and $\omega \in \mathcal{A}^2(M)$ a closed exterior 2-form. Define an $\mathbb{R}$-trilinear mapping

$$\begin{aligned} \mathsf{Vec}(M) \times \mathsf{Vec}(M) \times \mathsf{Vec}(M) &\xrightarrow{\mathcal{W}} \mathsf{C}^\infty(M) \\ (\xi, \eta, \zeta) &\longmapsto \omega([\xi, \eta], \zeta) - \eta\omega(\zeta, \xi) \\ &\qquad + \omega([\eta, \zeta], \xi). \end{aligned}$$

  Show that $\mathcal{W}$ is $\mathsf{C}^\infty(M)$-bilinear in ξ, ζ and

$$\begin{aligned} \mathcal{W}(\xi, f\eta, \zeta) - f\,\mathcal{W}(\xi, \eta, \zeta) \;=\; \\ (\xi f)\,\omega(\eta, \zeta) + \omega(\xi, \eta)(\zeta f). \end{aligned}$$

  for $f \in \mathsf{C}^\infty(M)$. In fact

$$\mathcal{W}(\xi, \eta, \zeta) \;=\; \big(\mathfrak{L}_\eta \omega\big)(\xi, \zeta)$$

  where $\mathfrak{L}_\eta$ denotes Lie derivative with respect to the vector field η.
- Now suppose that ω is nondegenerate. Then $\exists$ an $\mathbb{R}$-bilinear mapping

$$\begin{aligned} \mathsf{Vec}(M) \times \mathsf{Vec}(M) &\longrightarrow \mathsf{Vec}(M) \\ (\xi, \eta) &\longmapsto \widetilde{\nabla}_\xi(\eta) \end{aligned}$$

  such that

$$\mathcal{W}(\xi, \eta, \zeta) \;=\; \omega(\widetilde{\nabla}_\xi \eta, \zeta)$$

  which is $\mathsf{C}^\infty(M)$-linear in ξ and satisfies

$$\widetilde{\nabla}_\xi(f\eta) - f\widetilde{\nabla}_\xi \eta \;=\; (\xi f)\,\eta$$

  whenever $\omega(\xi, \eta) = 0$.

Suppose that $\mathcal{F}$ is a Lagrangian foliation of the symplectic manifold (M, ω), and $L \hookrightarrow M$ is a leaf of $\mathcal{F}$.

Exercise G.3.2. $\widetilde{\nabla}$ defines a affine connection ∇ on L.

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Exercise G.3.3. Show that the torsion of ∇ vanishes. $\big($Hint: Prove that

$$\omega\Big(\widetilde{\nabla}_\xi\eta - \widetilde{\nabla}_\eta\xi - [\xi,\eta],\zeta\Big) = \zeta\omega(\xi,\eta) - d\omega(\xi,\eta,\zeta)$$

whenever $\xi,\eta,\zeta \in \mathsf{Vec}(M)$.$\big)$

Exercise G.3.4. Show that the curvature of ∇ vanishes. $\big($Hint: Rewrite the defining equation for ∇ as:

$$\iota_{\nabla_\xi\eta}\omega \;=\; \iota_\xi d\iota_\eta\omega$$

where ι_V denotes interior multiplication by $V \in \mathsf{Vec}(M)$. Writing

$$\mathsf{Riem}(\xi,\eta) := \nabla_\xi\nabla_\eta - \nabla_\eta\nabla_\xi - \nabla_{[\xi,\eta]}$$

for $\xi,\eta \in \mathsf{Vec}(M)$, show that

$$\iota_{\mathsf{Riem}(\xi,\eta)\zeta}(\omega) \;=\; \big(\iota_\xi d\iota_\eta - \iota_\eta d\iota_\xi - \iota_{[\xi,\eta]}\big)d\iota_\zeta\omega. \tag{G.2}$$

Now (0.3) and (0.2) imply that

$$\big(\iota_\xi d\iota_\eta - \iota_\eta d\iota_\xi - \iota_{[\xi,\eta]}\big)d \;=\; -d\iota_\xi\iota_\eta d$$

and (G.2) implies:

$$\iota_{\mathsf{Riem}(\xi,\eta)\zeta}(\omega) \;=\; -d\iota_\xi\iota_\eta d\iota_\zeta\omega.$$

Since $\mathsf{T}\mathcal{F}$ is integrable, its annihilating ideal

$$\mathsf{Ann}^*(\mathsf{T}\mathcal{F}) \lhd \mathcal{A}^*(\mathsf{T}\mathcal{F})$$

is a *differential ideal,* (that is, closed under d). Furthermore, since $\xi,\eta,\zeta \in \mathsf{Vec}_{\mathcal{F}}(M)$, the corresponding derivations $\iota_\xi,\iota_\eta,\iota_\zeta$ also preserve $\mathsf{Ann}^*(\mathsf{T}\mathcal{F})$. Thus $\mathsf{Ann}^*(\mathsf{T}\mathcal{F})$ contains, successively: the 2-form ω, the 1-form $\iota_\zeta\omega$, the 2-form $d\iota_\zeta\omega$, the 1-form $\iota_\eta d\iota_\zeta\omega$ and, finally the function

$$\iota_\xi\iota_\eta d\iota_\zeta\omega \in \mathsf{Ann}^0(\mathsf{T}\mathcal{F}) = 0.$$

Thus $\iota_{\mathsf{Riem}(\xi,\eta)\zeta}(\omega) = 0$. Nondegeneracy of ω implies

$$\mathsf{Riem}(\xi,\eta)\zeta \;=\; 0$$

as claimed.

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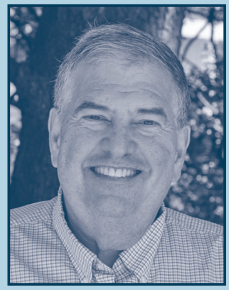

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