In lecture of Dec. 1, I defined:

<u>Def: A translation surface M is completely periodic in direction θ</u> if there are cylinders $C_1, \ldots, C_k$ in direction θ such that $M = \bigcup_{i=1}^{k} \overline{C_i}$.

The following lemma was proved:

<u>Lemma 1.</u> M is completely periodic in direction θ $\Leftrightarrow$ any straightline trajectory in direction θ is either periodic or part of a saddle connection.

The following strengthening was stated but not proved:

<u>Lemma 2</u> If there is no straightline trajectory in direction θ which is an infinite ray then M is completely periodic in direction θ.

By an <u>infinite ray in direction θ</u> we mean a trajectory of the form $\{\phi_t^\theta(z) : t \in I\}$, where $I = (a, \infty)$ or $I = (-\infty, b)$, the trajectory is defined for all $t \in I$, but

$$\lim_{t \to \partial I,\, t \in I} \phi_t^\theta(z) \in \Sigma.$$

<u>Pf of Lemma 2</u> If there is no straightline trajectory which is an infinite ray, then every straightline trajectory which is bounded above is also bounded

below, and is part of a saddle connection. That is, if for some $z \in M \setminus \Sigma$, $\phi_t^\theta(z)$ is define for $t \in [0, b)$ for some $b > 0$, and $\lim_{t \to b-} \phi_t^\theta(z) \in \Sigma$, then there is $a < 0$ s.t. $\lim_{t \to a+} \phi_t^\theta(z) \in \Sigma$; and vice versa.

Let Ξ denote the union of saddle connections on M in direction θ, and let $\varepsilon > 0$ be small enough so that for every ~~$x \in \Xi \setminus \Sigma$, the~~ $x \in \Xi$, the straightline flow trajectory $\{\phi_t^{\theta \pm \pi/2}(x) : |t| \le \varepsilon\}$ is defined. Here, for $x \in \Sigma$, we mean that all trajectories in direction $\theta \pm \pi/2$ starting at x are defined.

Let y be any point which is at distance ε from a point of Ξ in direction $\theta \pm \pi/2$. Then

$\{\phi_t^\theta(y) : t \in \mathbb{R}\}$ is defined for all times

since this trajectory stays at distance ε from

Ξ and Ξ contains Σ. Also, this trajectory is periodic since the total length of Ξ is finite and therefore so is the total length of the set of points obtained as

$\{\phi_z^{\theta \pm \pi/2}(\varepsilon) : z \in \Xi\}$.

We have shown that any connected component of $M \setminus \Xi$ contains a periodic trajectory in direction θ

As in the lecture, the connected components of $M \setminus \Xi$ are all cylinders.