Stuff I skipped in the lecture of January 12.

In the proof of the Kerckhoff-Masur-Smillie theorem, I reduced the proof to the verification of the following condition:

$(*)$ *for any translation surface q, interval $I \subset \mathbb{R}$ and $\rho_0 > 0$ there exists $\rho \in (0, \rho_0]$ and t_0 such that for all $t \geq t_0$, for any saddle connection σ on q,*

$$\sup_{s' \in e^{2t} I} \ell_{u_{s'} a_t q}(\sigma) = \sup_{s \in I} \ell_{a_t u_s q}(\sigma) \geq \rho.$$

Here $a_t = \mathrm{diag}(e^t, e^{-t})$, u_s is the horocycle flow to time s, $e^{2t} I$ is the dilation of I by a factor of e^{2t}, and $\ell_q(\sigma) = \|\mathrm{hol}_q(\sigma)\|$. With no loss of generality we can assume that the norm used here is the sup norm on $\mathbb{R}^2$.

Verifying $(*)$ justifies the application of quantitative nondivergence for horocycles that was used in the proof (Theorem B in the lecture of December 29, in that lecture the condition was labelled $(**)$).

To see that $(*)$ holds, let $I = [s_1, s_2]$. By replacing q with $u_{s_1} q$ we can assume $I = [0, b]$ where $b = s_2 - s_1$. Let ρ_1 denote the minimal length of a saddle connection on q. That is, for any saddle connection σ on q, if we write $\mathrm{hol}_q(\sigma) = (x_\sigma, y_\sigma)$, then $\max\{|x_\sigma|, |y_\sigma|\} \geq \rho_1$. Now let $\rho = \min\{\rho_0, \rho_1\}$ and choose $t_0 > 0$ large enough so that for any $t \geq t_0$ we have $e^t b - e^{-t} \geq 1$. We have

$$\mathrm{hol}_{a_t u_s q}(\sigma) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_\sigma \\ y_\sigma \end{pmatrix} = \begin{pmatrix} e^t (x_\sigma + s y_\sigma) \\ e^{-t} y_\sigma \end{pmatrix}.$$

So the supremum in $(*)$ is bounded below by

$$\max\{\ell_{a_t q}(\sigma), \ell_{a_t u_b q}(\sigma)\} \geq \max\{e^t |x_\sigma|, e^t |x_\sigma + b y_\sigma|\}.$$

For a given σ, we divide into two cases. If $|x_\sigma| \geq e^{-t}\rho$ then $e^t |x_\sigma| \geq \rho$, giving the required estimate. If $|x_\sigma| < e^{-t}\rho < \rho$ then by the choice of ρ_1, $|y_\sigma| \geq \rho$, and then

$$e^t |x_\sigma + b y_\sigma| \geq e^t (b |y_\sigma| - e^{-t}\rho) \geq \rho(e^t b - e^{-t}).$$

This also gives the desired estimate by the choice of t_0.