Exercise Sheet, Dynamics on moduli spaces of translation surfaces, Fall 2025

1. A *surface* is a connected second countable Hausdorff topological space S_0 such that every $x \in S_0$ has a neighborhood V homeomorphic to an open set in $\mathbb{R}^2$. We say that a collection of maps $\{(U_\alpha, \varphi_\alpha) : \alpha \in \mathcal{A}\}$ is a *translation atlas on* S_0 if:

- $U_\alpha \subset S_0,\ \varphi_\alpha : U_\alpha \to \mathbb{R}^2,\ \bigcup_{\alpha \in \mathcal{A}} U_\alpha = S_0$;
- for each α, U_α is open in S_0, $\varphi_\alpha(U_\alpha)$ is an open subset of $\mathbb{R}^2$, and φ_α is a homeomorphism between U_α and $\varphi_\alpha(U_\alpha)$;
- For each α, β, the map $\varphi_\beta \circ \varphi_\alpha^{-1}|_{\varphi_\alpha(U_\alpha \cap U_\beta)}$ is a translation.

Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^2$. Given a translation atlas on a surface S_0, for which each $\varphi_\alpha(U_\alpha)$ is bounded and convex, define the *path metric* by letting $d(x,y)$ be

$$\inf \left\{ \sum_{i=1}^{n} \|\varphi_{\alpha_i}(x_i) - \varphi_{\alpha_i}(x_{i-1})\| : x_0 = x, x_n = y, x_{i-1}, x_i \in U_{\alpha_i} \text{ for all } i \right\}.$$

Show that d is a metric on S_0. Let S be the completion of S_0 w.r.t. the path metric, and assume that $x \in S$ has a neighborhood V in S so that $V \smallsetminus \{x\}$ is contained in S_0 and is covered by finitely many of the U_α. Show that V contains a neighborhood of x in S which is translation isomorphic to a plug (i.e. has a map to a plug which is a translation w.r.t. the charts of the translation atlases and the coordinate of the plug). Deduce that S is a surface.

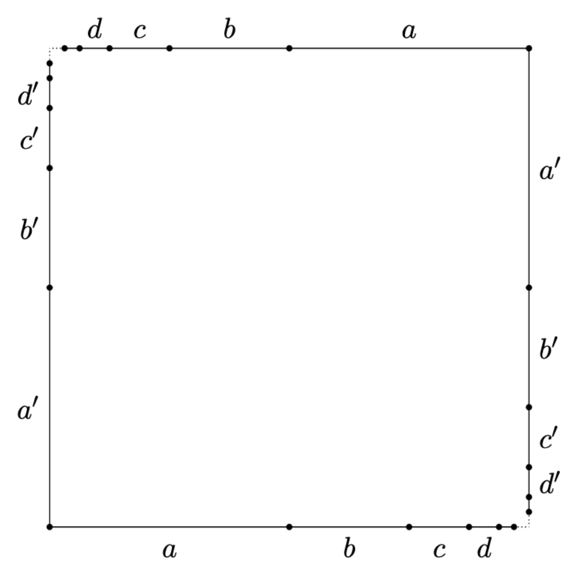
Deduce the following statement given in class. A *simple polygon* is a bounded subset $P \subset \mathbb{R}^2$ homeomorphic to the disk $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$, such that the boundary ∂P consists of finitely many line segments. Let $P_1, \ldots, P_k$ be a finite disjoint collection of polygons and let $e_1, \ldots, e_N$ be the boundary edges. An orientation of a line segment is a choice of initial and terminal endpoints. For each edge, choose the orientation which corresponds to going along ∂P in the counterclockwise direction. Assume that N is even and the boundary edges are partitioned into pairs (e_i, e_j), where $i \neq j$, e_j is the image of e_i under a translation $x \mapsto x + v_{ij}$ for some $v_{ij} \in \mathbb{R}^2$, and this translation is orientation reversing (sends the initial point of e_i to the terminal point of e_j). Let M be the quotient of $\bigcup P_i$ by the equivalence relation $e_i \ni x \sim x + v_{ij} \in e_j$, and let S_0 be the complement in M of points which are endpoints of edges. Define a translation atlas on S_0 which consists of one map to the plane covering the interior of each simple polygon, and one map for each identified pair (e_i, e_j). Show that this is a translation atlas, that M is the metric completion of the translation atlas on S_0, and M is a translation surface.

2. A Möbius strip is the quotient of $[0,1]\times(-1,1)$ (with its topology inherited from $\mathbb{R}^2$) by the relation $(0,y)\sim(1,-y)$ (with the quotient topology). A surface is called *orientable* if it does not contains an open subset homeomorphic to a Möbius strip. Prove that if S_0 is a surface with a translation atlas then S_0 is orientable. Also prove that if S_0 is a surface with a translation atlas and S is the path metric completion of S_0 and satisfies the conditions of Question 1 then S is orientable.

3. Let $Q=[0,1]^2$, let Q_0 be the points of the form $(0,1-2^{-k}),(1-2^{-k},0),(1,2^{-k}),(2^{-k},1)$ for some $k\in\mathbb{N}$, let $Q_1=Q_0\cup\{(0,0),(0,1),(1,0),(1,1)\}$, and let P be the 'polygon with infinitely many edges' given as $[0,1]^2$, where the edges are the connected components of $\partial([0,1]^2)\smallsetminus Q_1$ and each side is glued by translation to the unique side which is parallel and of equal length. That is,

$$\begin{aligned}(x,1)&\sim\Big(x-\textstyle\sum_{j>k}2^{-j}+\sum_{j\le k}2^{-j},0\Big) &&\text{for } x\in\big(2^{-k},2^{-(k-1)}\big)\\ (1,y)&\sim\Big(0,y-\textstyle\sum_{j>k}2^{-j}+\sum_{j\le k}2^{-j}\Big) &&\text{for } y\in\big(2^{-k},2^{-(k-1)}\big).\end{aligned}$$

See picture (edges with the same label are identified by translation).

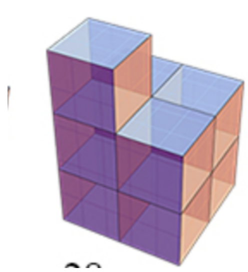


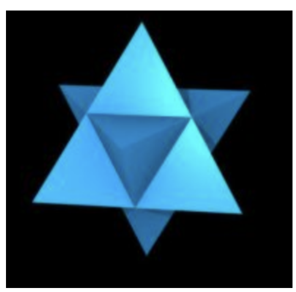
Let S_0 be the quotient topological space. As in Question 1, define a translation atlas on S_0 with one U_α covering $(0,1)^2$ and one U_α for each pair of identified edges. Show that this endows S_0 with the structure of a surface with a translation atlas. How many points are there in the path metric completion? Do they have a neighborhood which is translation isomorphic to a plug? Is the path metric completion a surface?

4. Let r_1, r_2, r_3 be positive rational numbers satisfying $r_1 + r_2 + r_3 = 1$, and let Δ be a triangle whose angles (in radians) are $(r_1\pi, r_2\pi, r_3\pi)$. Let $q = M_\Delta$ be the unfolding of Δ. Compute the number of singularities of q, their orders, and the genus of q.

5. Let Δ be the triangle with angles $(\pi/2n, m\pi/2n, (2n - m - 1)\pi/2n)$, where $n \geq 2$ is an integer and m is an integer $\leq 2n - 2$. How many elements are there in the dihedral group generated by the linear parts of reflections in the sides of Δ? Show that there is no billiard trajectory in Δ that starts at a vertex of angle $\pi/2n$ and ends at the same vertex. *Prove or disprove:* there is a quadrilateral $\mathcal{P}$ with rational angles and a vertex v of $\mathcal{P}$ such that there is no billiard trajectory in $\mathcal{P}$ that starts and ends at v.

6. Let M be the topological space obtained as the boundary of a union of polyhedra glued face to face to form a space homeomorphic to a closed ball. Assume that the faces of the polyhedra have rational angles. See figure for the case of a union of cubes, of tetrahedra, or of one dodecahedron.





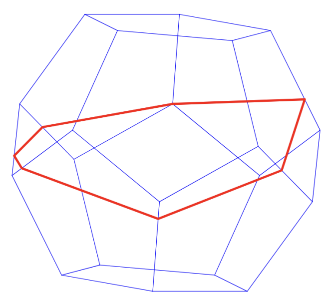


Figure 2. A dodecahedron with a straight-line trajectory.

Let $I \subset \mathbb{R}$ be an open interval, a ray, or all of $\mathbb{R}$, and let $\gamma : I \to M$. We say that γ is a *straightline trajectory* if:

- $\gamma(I)$ does not contain vertices;
- for any boundary face F and any open interval $I_0 \subset I$ such that $\gamma(I_0) \subset F$, $\gamma'(t)$ is constant;
- If e is an edge on the common boundary of two faces F_1, F_2, and $\gamma(t_0) \in e$, then the one-sided derivatives $\lim_{h\to 0\pm} \frac{1}{h}(\gamma(t_0 + h) - \gamma(t_0))$ both exist and make the same angle with e.

Show that there is a translation surface q and a finite-to-one map $\pi : q \to M$, such that the vertices of M are the images of the singular points of q, the restriction of π to the complement of the singular points is a covering map, and the lift of any straightline trajectory on M to q via π is a straightline flow trajectory on q. Two straightline trajectories on M are *parallel* if they have lifts in q which are in the same direction. Show that if M as above is the boundary of a union of cubes (as in the left-hand side of the figure) then, if there is one periodic straightline trajectory on M in direction θ, then all parallel trajectories are either periodic or their lift is a saddle connection. *Prove or disprove:* the same holds if M is obtained from a finite union of identical Platonic solids.

7. A simple polygon P is called *vertex convex* if it is the convex hull of its vertices, and no vertex is a convex combination of other vertices. Suppose M is a translation surface of genus $g \geq 2$ which has a presentation as a polygonal surface composed of one vertex convex polygon with edge identifications. Show that either M has one singular point of order $2g-2$, or two singular points of order $g-1$. Also show that there is a homeomorphism $h : M \to M$ with $h(\Sigma) = \Sigma$ and such that the derivative of h at each nonsingular point is $-\mathrm{Id}$. How many fixed points does h have?

8. Let P be a simple rational polygon, let M_P be the unfolding of P, let V be the set of vertices of P, and let P^+, P^- be identical copies of $P \smallsetminus V$. We consider P^+, P^- as disjoint and identify their points with the corresponding points in P. Let M' be the topological space $P^+ \sqcup P^- / \sim$, where $x \in \partial P^+$ is identified with the same x in ∂P^-. Show that M' is homeomorphic to a sphere with k points removed, where $k = \# V$. Let $\pi : M_P \to P$ be the unfolding map defined in the lecture, and let $\pi' : M_P \to M'$ be the map which sends $x \in P_\delta$ to the copy of $\delta^{-1}(x) \in P^+$ if $\det(\delta) = 1$, and to the copy of $\delta^{-1}(x) \in P^-$ if $\det(\delta) = -1$. Prove that π' is a covering map. Which trajectories in M' are images of straightline trajectories? Extend this discussion to the case that P is a simple irrational polygon.

9. Prove that for $g \geq 3$, there is a translation surface of genus g for which the Veech group is trivial. Prove that for $g = 1$, for surfaces with one marked point, the Veech group is always infinite. In genus 2, and/or in genus 1 with more than one marked point, can the Veech group be trivial?

10. Let M be a translation surface of type $(a_1, \ldots, a_r)$, where $r \geq 1$. Let $\mathrm{Trans}(M)$ and $\mathrm{Trans}_0(M)$ denote respectively the group of translation automorphisms and strict translation automorphisms of M. Give an upper bound on the cardinality of the groups $\mathrm{Trans}(M)$

and $\mathrm{Trans}_0(M)$, in terms of of $(a_1, \dots, a_r)$. Give examples showing that your bound is sharp.

11. Let M be a translation surface of area one, and suppose that $h : M \to M$ is an affine automorphism with derivative $D(h) = \mathrm{diag}(\lambda, \lambda^{-1})$ for $\lambda > 1$. Show that M has a polygonal surface presentation $M = (R_1 \cup \cdots \cup R_k)/\sim$ with the following properties:

- Each of the polygons R_i is a rectangle with horizontal and vertical sides, and with a singularity along each of its sides (possibly at a corner).
- h maps each of the vertical sides of each R_i into one of the vertical sides of one of the R_j, and h^{-1} maps each of the horizontal sides of each R_i into one of the horizontal sides of one of the R_j.
- For any i, j there is ℓ such that $h^\ell(R_i)$ intersects the interior of R_j.

Let Ω be the subset of $\{1, \dots, k\}^{\mathbb{Z}}$ consisting of all sequences (x_ℓ) satisfying that if $h(R_i)$ does not intersect the interior of R_j then there is no ℓ for which $x_\ell = i$ and $x_{\ell+1} = j$. Let $T : \Omega \to \Omega$ be the shift $T((x_\ell)_{\ell\in\mathbb{Z}}) = (x_{\ell+1})_{\ell\in\mathbb{Z}}$. Show that there is a continuous map $\psi : \Omega \to M$ so that $\psi\left((x_\ell)_{\ell\in\mathbb{Z}}\right) = y$ if for every ℓ, $h^\ell(y) \in R_{x_\ell}$. Show that this map is surjective and satisfies $\psi \circ T = h \circ \psi$. Deduce that the set of periodic trajectories for h are dense in M.

12. Let M be a polygonal surface obtained by gluing together d unit squares (i.e., disjoint copies of $[0,1]^2$). Show that the image of M under any element of the group $\mathrm{SL}(2,\mathbb{Z})$ can also be represented by a polygonal surface made of d squares. Deduce that M is a Veech surface.

13. Let $n \geq 6$ be even and let M_n be obtained by gluing together opposite sides of a regular n-gon. How many singular points does M have? What is the genus of M_n? Let θ be the direction of a saddle connection passing once through the regular n-gon representing M_n. Show that in direction θ, the surface M_n is completely periodic, and there is a parabolic affine automorphism of M_n fixing direction θ.

14. Let M be a translation surface, Γ_M the Veech group of M. Prove that Γ_M is not cocompact in $\mathrm{SL}(2,\mathbb{R})$.