

Algebraic Families of Harish-Chandra Pairs

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Abstract

Mathematical physicists have studied degenerations of Lie groups and their representations, which they call contractions. In this paper we study these contractions, and also other families, within the framework of algebraic families of Harish-Chandra modules. We construct a family that incorporates both a real reductive group and its compact form, separate parts of which have been studied individually as contractions. We give a complete classification of generically irreducible families of Harish-Chandra modules in the case of the family associated to $SL(2, \mathbb{R})$.

1 Introduction

The purpose of this paper is to examine from an algebraic perspective families of Lie groups and families of representations, including examples that have long been studied in the mathematical physics literature under the name of *contractions* of groups and representations. We shall set up a general framework, but mainly we shall study the simplest examples and identify phenomena that might deserve further study.

Contractions were first studied by Segal in [Seg51], and by Inönü and Wigner in [IW53]. Their main goal was to study how the symmetries of a physical system can change in various limiting circumstances (for example in the limit as the speed of light becomes infinite). This led them towards

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a deformation theory for Lie groups and their representations, which they developed in various examples of physical interest.

A simple example is the contraction of $SL(2, \mathbb{R})$ to the semidirect product group $SO(2) \ltimes \mathbb{R}^2$. It consists of a smooth family of groups $\{G_t\}$ with $G_t = SL(2, \mathbb{R})$ for all $t \neq 0$ and $G_0 = SO(2) \ltimes \mathbb{R}^2$, the Cartan motion group of $SL(2, \mathbb{R})$. It is known that any infinite-dimensional unitary irreducible representation of G_0 can be obtained as a limit of a suitable smooth family of representations of the groups G_t ; see [DR85, SBBM12].

There is also a contraction of $SU(2)$ to the semidirect product group $SO(2) \ltimes \mathbb{R}^2$. Once again, every infinite-dimensional unitary irreducible representation of the semidirect product can be obtained as a suitable limit of representations of $SU(2)$ [DR85, DR83, SBBM12]. But in this case the approximating representations are only defined for a *discrete* sequence of parameter values converging to zero, and the representations that approximate the given infinite-dimensional representation of G_0 are themselves *finite-dimensional*. This makes it a challenge to accurately express the approximation in mathematical terms.

We shall study these phenomena by changing the context in various ways.

First we shall study families of groups that *change type* in the sense that their isomorphism classes will vary from fiber to fiber. One such family will be

$$G_t \cong \begin{cases} SL(2, \mathbb{R}) & t > 0 \\ SO(2) \ltimes \mathbb{R}^2 & t = 0 \\ SU(2) & t < 0, \end{cases}$$

which combines the two contraction families given above.

Secondly, we shall consistently study families that are parametrized *continuously*, rather than by a discrete space. We shall mostly deal with families over a line, or over the completion to a projective line.

Thirdly, we shall study families of representations *algebraically*, as families of Harish-Chandra modules. In this way we obtain sufficiently many families for our purposes, whereas there are too few continuous families of global representations in the examples we study. Our focus will therefore be on algebraic families of Harish-Chandra pairs, rather than smooth families of groups. We shall use the language of algebraic geometry and work over the complex field; as usual, real families will be recovered using involutions.

In summary our starting point will be an algebraic family of Harish-Chandra pairs parametrized by a line or more generally by a complex algebraic variety. Precise definitions will be given in Section 2, where we aim to present a formalism suitable for the study of a wide range of contractions and other families of groups.

We shall give two general constructions of algebraic families of Harish-Chandra pairs. The first constructs from an arbitrary Harish-Chandra pair a canonical *deformation family* of pairs over the projective line using the deformation to the normal cone construction in geometry. See Paragraph 2.1.2. The second construction starts from a Harish-Chandra pair (\mathfrak{g}, K) that is *symmetric* in the sense that it is equipped with a K -equivariant involution of \mathfrak{g} whose fixed subalgebra is the Lie algebra of K . We obtain from this an algebraic *contraction family* of Harish-Chandra pairs over the projective line, which is a version of the smooth family of groups displayed above. See Paragraph 2.1.3. Other interesting constructions are also possible.

In both of the algebraic families that we shall study the generic member of the family will be (isomorphic to) a fixed Harish-Chandra pair (\mathfrak{g}, K) . So when we study families of Harish-Chandra modules we shall be in particular studying families of Harish-Chandra modules for the fixed pair (\mathfrak{g}, K) . It is of course common in representation theory to encounter representations in families rather than individually. Usually this happens within the context of parabolic induction, and the parameter space is an affine variety. One aspect of our study is an investigation of how these parabolically induced affine families may be compactified over a projective variety (in the cases studied here, this is just the projective line).

We shall introduce real structures on the complex families that we consider (associated to a real structure on the Harish-Chandra pair (\mathfrak{g}, K) that we start from) in the usual way. In a companion paper we shall use Jantzen filtration techniques to recover from our complex algebraic families of Harish-Chandra modules the discrete families of finite-dimensional representations of compact groups originally considered in the mathematical physics literature. In this way the formalism of algebraic families and Harish-Chandra pairs places the discrete approximation phenomenon within an algebraic context, and construction of Jantzen recovers the phenomenon from the algebra.

The same techniques have other uses. For instance we can recover the “Mackey bijection” [Mac75, Hig08, Hig11] between the (tempered or ad-

missible) dual of $SL(2, \mathbb{R})$ and that of its Cartan motion group $SO(2) \times \mathbb{R}^2$. Our approach is potentially quite general.

Our focus is on families of Harish-Chandra pairs, but it is interesting to construct families of *groups* that give rise to families of Harish-Chandra pairs. The deformation to the normal cone construction does this for the first of our families. The situation for the contraction family associated to a symmetric pair is more complicated, but for many classical groups we shall give in Section 3 the construction of an underlying family of groups. In the case of groups defined by bilinear forms, it is natural to use the (projectivization of the) space of all forms as a parameter space, and then perhaps restrict to interesting curves within this space, and this is what we do. We do not know how to handle the general case.

In Section 4, which is independent of Section 3, we shall give a classification theorem for generically irreducible families of Harish-Chandra modules for the contraction family over \mathbb{CP}^1 that is associated to the pair

$$(\mathfrak{g}, K) = (\mathfrak{sl}(2, \mathbb{C}), SO(2, \mathbb{C})).$$

Apart from the expected parameters involving the Casimir element and K-types, we shall explain how simple topological invariants fit into the classification of families over \mathbb{CP}^1 . (We shall also see that there are too many generically irreducible families to admit a reasonable classification, and we shall impose some restrictions on the families considered, again of a topological nature.)

The topics covered here are obviously just a sampling from among the phenomena involving families of representations that one might study, and even for families related to $SL(2, \mathbb{R})$ there is still plenty to be done. We hope that this paper will serve as an invitation to the further study of families from an algebraic point of view.

The first and third authors were partially supported by ERC grant 291612. Part of the work on this project was done at Max-Planck Institute for Mathematics, Bonn. We would like to thank MPIM for the very stimulating atmosphere. The second author was partially supported by NSF grant DMS-1101382.

2 Algebraic Families

In this section we shall give the family versions of a number of standard constructions in representation theory. Namely, we shall define families of Lie algebras, families of algebraic groups, families of Harish-Chandra pairs, and families of Harish-Chandra modules. At the end of the section we shall consider involutions and real structures.

All of the objects and morphisms appearing in this section and the rest of the paper will be algebraic and defined over the field of complex numbers. For example, by a vector bundle we shall mean a complex algebraic vector bundle, by a section we shall mean an algebraic section and so on.

By a *variety* we shall always mean an irreducible, nonsingular, quasi-projective, complex algebraic variety, and for any variety, X , we shall denote by \mathcal{O}_X the structure sheaf of regular functions on X .

2.1 Families of Lie algebras

Definition. Let X be a variety. An *algebraic family of Lie algebras over X* is a locally free sheaf of \mathcal{O}_X -modules, that is equipped with \mathcal{O}_X -linear Lie brackets which make it a sheaf of Lie algebras. A *morphism of algebraic families of Lie algebras over X* is a morphism of \mathcal{O}_X -modules that commutes with the Lie brackets.

All of the families of Lie algebras that we shall consider will have finite-dimensional fibers. In the finite-dimensional case, an algebraic family of complex Lie algebras is the same thing as the sheaf of sections of an algebraic vector bundle whose fibers are equipped with Lie algebra structures that vary algebraically.

2.1.1 Constant and nonconstant families

Let X be a variety and let \mathfrak{g} be a complex Lie algebra. The *constant family over X with fiber \mathfrak{g}* is $\mathcal{O}_X \otimes \mathfrak{g}$ (tensor product over \mathbb{C}). A *locally constant algebraic family of complex Lie algebras* is a family that is locally isomorphic to a constant family.

Here is a simple example of an algebraic family that is *not* locally constant. Let $X = \mathbb{C}$ and let \mathfrak{g} be any non-abelian complex Lie algebra. Make

the sheaf $\mathcal{O}_X \otimes \mathfrak{g}$ into an algebraic family of Lie algebras over the variety \mathbb{C} by means of the formula

$$[s_1, s_2](z) = z[s_1(z), s_2(z)]$$

for the Lie bracket of sections. The fiber over 0 is \mathfrak{g} with the trivial Lie bracket.

2.1.2 The deformation family associated to a Lie subalgebra

Let \mathfrak{g} be a complex Lie algebra and let \mathfrak{k} be a Lie subalgebra of \mathfrak{g} . We shall construct a family over the variety

$$X = \mathbb{C}$$

with fibers

$$\begin{cases} \mathfrak{g} & z \neq 0 \\ \mathfrak{k} \ltimes \mathfrak{g}/\mathfrak{k} & z = 0 \end{cases}$$

(it is essentially the deformation to the normal construction in geometry, as described in [Ful84b, Ful84a], for example). Form the sheaf of regular functions on X with values in \mathfrak{g} , and then form the subsheaf that consists of functions whose value at $0 \in X$ belongs to \mathfrak{k} . One can show that this subsheaf is, in its own right, locally free. It is an algebraic family of Lie algebras in our sense, which we shall call the *deformation family* associated to the inclusion $\mathfrak{k} \subseteq \mathfrak{g}$.

The restriction of the deformation family to the complement of $0 \in X$ is the constant family with fiber \mathfrak{g} . The fiber at $0 \in X$ is the semidirect product Lie algebra $\mathfrak{k} \ltimes \mathfrak{g}/\mathfrak{k}$, as indicated above.

2.1.3 The contraction family associated to a symmetric Lie subalgebra

The following construction of a second family over $X = \mathbb{C}$ is based upon the Inönü-Wigner contractions of Lie algebras studied in [IW53]. Let \mathfrak{g} be a complex Lie algebra, let

$$\theta: \mathfrak{g} \longrightarrow \mathfrak{g}$$

be an involution, and let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be the associated “Cartan” decomposition into +1 and −1 eigenspaces of θ , respectively. Let $X = \mathbb{C}$. Give the sheaf $\mathcal{O}_X \otimes \mathfrak{g}$ the structure of a non-constant algebraic family of Lie algebras by using the decomposition of sheaves

$$\mathcal{O}_X \otimes \mathfrak{g} = \mathcal{O}_X \otimes \mathfrak{k} \oplus \mathcal{O}_X \otimes \mathfrak{p}$$

and by defining the Lie bracket operation on sections belonging to the individual summands by

$$[\eta, \zeta](z) = \begin{cases} z[\eta(z), \zeta(z)]_{\mathfrak{g}} & \text{if } \zeta \text{ and } \eta \text{ are sections of } \mathcal{O}_X \otimes \mathfrak{p} \\ [\eta(z), \zeta(z)]_{\mathfrak{g}} & \text{otherwise.} \end{cases}$$

2.1.4 Base change

Given an algebraic family \mathfrak{h} of complex Lie algebras over Y and a morphism of varieties

$$\psi : X \longrightarrow Y,$$

we can pull back \mathfrak{h} to the sheaf of \mathcal{O}_X -modules

$$\psi^* \mathfrak{h} = \mathcal{O}_X \otimes_{\psi^{-1} \mathcal{O}_Y} \psi^{-1} \mathfrak{h}.$$

It has the structure of an algebraic family of Lie algebras over X .

Example. Let \mathfrak{g} be a Lie algebra equipped with an involution and a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as in Paragraph 2.1.3. Since $\mathfrak{k} \ltimes \mathfrak{p} \cong \mathfrak{k} \ltimes \mathfrak{g} / \mathfrak{k}$, the contraction family of Lie algebras strongly resembles the deformation family from Paragraph 2.1.2. But it is not isomorphic to it since the deformation family is isomorphic to the sheaf $\mathcal{O}_{\mathbb{C}} \otimes \mathfrak{g}$ over \mathbb{C} with Lie bracket

$$[\eta, \zeta](z) = \begin{cases} z^2[\eta(z), \zeta(z)] & \text{if } \zeta \text{ and } \eta \text{ are sections of } \mathcal{O}_X \otimes \mathfrak{p} \\ [\eta(z), \zeta(z)] & \text{otherwise} \end{cases}$$

on homogeneous sections (note the appearance of z^2 in place of the monomial z that is used in 2.1.3). Denote this latter family over \mathbb{C} by \mathfrak{g} , and let \mathfrak{h} be the contraction family over \mathbb{C} from Paragraph 2.1.3. As sheaves of \mathcal{O}_X -modules, \mathfrak{g} and \mathfrak{h} are the same. However the identification is not a morphism of algebraic families of Lie algebras. Instead, \mathfrak{g} is isomorphic to $\psi^* \mathfrak{h}$, where

$$\psi : \mathbb{C} \longrightarrow \mathbb{C}, \quad \psi(z) = z^2.$$

2.1.5 Families of modules

Let \mathfrak{g} be an algebraic family of complex Lie algebras over X . We want to distinguish between the concepts *\mathfrak{g} -module* and *family of representations* of \mathfrak{g} . A module, loosely speaking, is just a collection of modules over the individual fibers of \mathfrak{g} :

Definition. Let \mathfrak{g} be an algebraic family of complex Lie algebras over X . A *\mathfrak{g} -module* is a quasicoherent \mathcal{O}_X -module, \mathcal{F} , together with a morphism of \mathcal{O}_X -modules $\rho : \mathfrak{g} \otimes_{\mathcal{O}_X} \mathcal{F} \rightarrow \mathcal{F}$ that respects the Lie brackets.

In contrast, for families of representations we require a degree of continuity from fiber to fiber:

Definition. Let \mathfrak{g} be an algebraic family of complex Lie algebras over X . A *family of representations* over \mathfrak{g} is a \mathfrak{g} -module that is flat as an \mathcal{O}_X -module.

Flatness is the correct technical notion, but it is not easy to understand from a geometric perspective. However in the examples that arise most immediately in representation theory one is presented with \mathfrak{g} -modules that decompose into countable direct sums of coherent \mathcal{O}_X -modules, and it is worth recalling that a coherent and flat \mathcal{O}_X -module is locally free; see for example [Har77, Proposition 9.2 (e)].

Example. If \mathfrak{g} is a complex semisimple Lie algebra, and if $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is a Borel subalgebra, then as λ ranges over \mathfrak{h}^* the Verma modules

$$V_\lambda = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$$

combine to form a module for the constant family of Lie algebras over \mathfrak{h}^* with fiber \mathfrak{g} . Other examples, including examples of modules over non-constant families, will be given later.

2.2 Families of algebraic groups

Definition. An *algebraic family of groups* over a variety X is a smooth morphism of varieties

$$\mathbf{G} \rightarrow X$$

that carries the structure of a group scheme over X (for details about group schemes see [Sta16, Tag 022R]). A morphism of algebraic families of groups over X is a morphism of varieties over X that is compatible with group structures.

Remark. In our context, thanks to our simplifying assumptions on the varieties we are studying, the smoothness condition on the morphism π from \mathbf{G} to X can be checked pointwise: it is equivalent to the surjectivity of the differential of π on every tangent space. That is, a morphism π is smooth if and only if it is a submersion in the C^∞ -sense. See for example [Har77, Proposition 10.4].

Example. If G is a complex algebraic group and X is a variety, then of course we can form the constant family $\mathbf{G} = G \times X$.

Example. Let G be a complex algebraic group, and let K be an algebraic subgroup. Associated to the inclusion of K into G there is the *deformation to the normal cone*, as in [Ful84a, Chapter 5] for example, which is an algebraic family of groups over $\mathbb{C}\mathbb{P}^1$. This is the group-theoretic counterpart of the Lie algebraic deformation family in Paragraph 2.1.2 (we defined the deformation family of complex Lie algebras over \mathbb{C} , but exactly the same definition may be given over $\mathbb{C}\mathbb{P}^1$).

Remark. The construction of a group-theoretic counterpart of the contraction family in Paragraph 2.1.3 is a more delicate matter, which will be considered in Section 3.

Definition. Let \mathbf{G} be an algebraic family of algebraic groups over Y , and let $\psi: X \rightarrow Y$ be a morphism of varieties. The base change of \mathbf{G} with respect to ψ is the group scheme

$$\psi^* \mathbf{G} = X \times_Y \mathbf{G}$$

over X .

2.2.1 The associated family of Lie algebras

Given an algebraic family of groups \mathbf{G} over X as in Section 2.2, the associated algebraic family of Lie algebras, denoted by $\text{Lie}(\mathbf{G})$, is the sheaf of vertical, right-invariant vector fields on \mathbf{G} , or in other words the sheaf of \mathcal{O}_X -linear derivations of $\mathcal{O}_{\mathbf{G}}$ for which the diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathbf{G}} & \longrightarrow & \mathcal{O}_{\mathbf{G}} \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathbf{G}} \\ \xi \downarrow & & \downarrow \xi \otimes 1 \\ \mathcal{O}_{\mathbf{G}} & \longrightarrow & \mathcal{O}_{\mathbf{G}} \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathbf{G}} \end{array}$$

is commutative (the horizontal maps are multiplication in \mathbf{G}).

According to our assumptions the identity section $X \rightarrow \mathbf{G}$ embeds X as a smooth subvariety of the smooth variety \mathbf{G} . The sheaf $\text{Lie}(\mathbf{G})$ is isomorphic to the sheaf of sections of the normal bundle of X in \mathbf{G} by restriction of vector fields to X .

2.2.2 Action of a family of groups on an \mathcal{O}_X -module

If \mathbf{G} is an algebraic family of groups over X , and if \mathcal{F} is a sheaf of \mathcal{O}_X -modules, then an *action* of \mathbf{G} on \mathcal{F} is a morphism of sheaves of \mathcal{O}_X -modules

$$\mathcal{F} \longrightarrow \mathcal{O}_{\mathbf{G}} \otimes_{\mathcal{O}_X} \mathcal{F}$$

that is compatible with the multiplication and unit operations on \mathbf{G} in the usual way. This is a special case of the concept of equivariant sheaf; for further information see [BL94, Vis05].

Definition. A *family of representations* of \mathbf{G} is a flat, quasicohherent sheaf of \mathcal{O}_X -modules that is equipped with an action of \mathbf{G} .

2.2.3 Action of the associated family of Lie algebras

If \mathbf{G} acts on \mathcal{F} , then there is an induced action of $\text{Lie}(\mathbf{G})$ defined by

$$\text{Lie}(\mathbf{G}) \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \text{Lie}(\mathbf{G}) \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathbf{G}} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{O}_{\mathbf{G}} \otimes_{\mathbf{G}} \mathcal{F} \longrightarrow \mathcal{F}$$

where the first arrow comes from the action of \mathbf{G} , the second from the action of $\text{Lie}(\mathbf{G})$ by derivations, and the third from restriction along the identity section $X \rightarrow \mathbf{G}$.

2.2.4 The adjoint action

The identity section

$$X \longrightarrow \mathbf{G}$$

is invariant under the adjoint action

$$\mathbf{G} \times_X \mathbf{G} \longrightarrow \mathbf{G},$$

and so there is an induced adjoint action of \mathbf{G} on the normal bundle for the image of the identity section, or in other words on the associated family of Lie algebras:

$$\mathrm{Lie}(\mathbf{G}) \longrightarrow \mathcal{O}_{\mathbf{G}} \otimes_{\mathcal{O}_X} \mathrm{Lie}(\mathbf{G}).$$

The action of $\mathrm{Lie}(\mathbf{G})$ associated, in turn, to this action of \mathbf{G} is the Lie bracket operation on $\mathrm{Lie}(\mathbf{G})$.

2.2.5 Reductive families and admissible modules

Definition. Let \mathbf{K} be an algebraic family of groups over X . We shall say that \mathbf{K} is *reductive* if for every $x \in X$ the fiber $\mathbf{K}|_x$ is a complex reductive algebraic group.

So a reductive family over X is the same thing as a smooth, reductive group scheme over X .

Definition. Let \mathbf{K} be a reductive family of groups over X and let \mathcal{F} be a family of representations of \mathbf{K} . We shall say that \mathcal{F} is *admissible* if the sheaf of \mathcal{O}_X -modules $[\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F}]^{\mathbf{K}}$ is locally free and of finite rank, for any family \mathcal{L} of representations of \mathbf{K} that is locally free and of finite rank as an \mathcal{O}_X -module.

Example. If $\mathbf{K} = \mathbf{K} \times X$ is a constant reductive family, then \mathcal{F} is admissible if and only if the sheaf

$$\mathcal{H}om_{\mathbf{K}}(\mathcal{O}_X \otimes_{\mathbb{C}} V, \mathcal{F}) \cong [V^* \otimes_{\mathbb{C}} \mathcal{F}]^{\mathbf{K}}$$

is coherent for every finite-dimensional representation V of \mathbf{K} . In this case there is a canonical isotypical decomposition

$$\bigoplus_{\tau \in \widehat{\mathbf{K}}} \mathcal{F}_{\tau} \xrightarrow{\cong} \mathcal{F},$$

indexed by the equivalence classes of irreducible (algebraic) representations of \mathbf{K} , where

$$\mathcal{F}_{\tau} = V_{\tau} \otimes_{\mathbb{C}} \mathcal{H}om_{\mathbf{K}}(\mathcal{O}_X \otimes_{\mathbb{C}} V_{\tau}, \mathcal{F}).$$

2.3 Families of Harish-Chandra pairs

Finally we come to the main definition in this section: that of an algebraic family of Harish-Chandra pairs. There are small variations in the literature on the familiar concept of an individual Harish-Chandra pair. The following definition will make clear which one we wish to adopt here (take X to be a single point).

Definition. Let X be a variety. An *algebraic family of Harish-Chandra pairs* over X consists of the following data:

- (a) an algebraic family \mathfrak{g} of Lie algebras over X , and
- (b) an algebraic family \mathbf{K} of groups over X ,

together with an action of \mathbf{K} on \mathfrak{g} by automorphisms, and a \mathbf{K} -equivariant embedding of algebraic families of Lie algebras over X ,

$$j : \text{Lie}(\mathbf{K}) \longrightarrow \mathfrak{g},$$

such that the action of $\text{Lie}(\mathbf{K})$ on \mathfrak{g} via $\text{ad}_{\mathfrak{g}} \circ j$ coincides with the differential of the action of \mathbf{K} on \mathfrak{g} .

There are the obvious notions of morphism of families Harish-Chandra pairs over X , base change and morphism of families of Harish-Chandra pairs over different bases. We omit the details.

2.3.1 Deformation and contraction families of Harish-Chandra pairs

In Paragraph 2.1.2 we associated to any pair of Lie algebras $\mathfrak{k} \subseteq \mathfrak{g}$ a deformation family \mathfrak{g} of Lie algebras over \mathbb{C} . If \mathfrak{k} is the Lie algebra of a complex algebraic group K , and if (\mathfrak{g}, K) is a Harish-Chandra pair, then we obtain from the deformation family an algebraic family of Harish-Chandra pairs $(\mathfrak{g}, \mathbf{K})$ in which \mathbf{K} is the constant algebraic family of groups with fiber K .

If in addition $\mathfrak{k} \subseteq \mathfrak{g}$ is the fixed-point subalgebra of a K -equivariant involution on \mathfrak{g} , and if instead of the deformation family we consider the contraction family of Lie algebras from Paragraph 2.1.3, then we obtain a second family of Harish-Chandra pairs $(\mathfrak{g}, \mathbf{K})$ over \mathbb{C} , in which \mathbf{K} is once again the constant family of groups with fiber K .

2.4 Algebraic families of Harish-Chandra modules

Definition. Let $(\mathfrak{g}, \mathbf{K})$ be an algebraic family of Harish-Chandra pairs over X . An *algebraic family of Harish-Chandra modules* for $(\mathfrak{g}, \mathbf{K})$ consists of a flat, quasicoherent \mathcal{O}_X -module, \mathcal{F} , that is equipped with

- (a) an action of \mathbf{K} on \mathcal{F} , and
- (b) an action of \mathfrak{g} on \mathcal{F} ,

such that the action morphism

$$\mathfrak{g} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{F}$$

is \mathbf{K} -equivariant, and such that the differential of the \mathbf{K} -action in (a) is equal to the composition of the inclusion of $\mathrm{Lie}(\mathbf{K})$ into \mathfrak{g} with the action of \mathfrak{g} on \mathcal{F} .

2.4.1 Admissible families of Harish-Chandra modules

Let $(\mathfrak{g}, \mathbf{K})$ be an algebraic family of Harish-Chandra pairs over X , and assume that \mathbf{K} is a reductive algebraic family of groups over X .

Definition. An algebraic family \mathcal{F} of Harish-Chandra modules is *admissible* if the \mathbf{K} -action on \mathcal{F} is admissible, and if \mathcal{F} is finitely generated as a \mathfrak{g} -module.

2.5 Real structures on algebraic families

In this section we shall recall some definitions related to real structures on varieties, and make the natural definitions of real structures on algebraic families of Lie algebras and groups.

2.5.1 Real structures on varieties

Recall that if X is a variety, then its *complex conjugate* \bar{X} is the variety whose underlying topological space is the same as that of X , and whose structure sheaf $\mathcal{O}_{\bar{X}}$ is the complex conjugate of the sheaf \mathcal{O}_X ; this is the same sheaf of rings as \mathcal{O}_X , but equipped with the complex conjugate scalar multiplication.

The operation of complex conjugation is a functor from the category of varieties to itself, and the composition of complex conjugation with itself is the identity functor.

Definition. Let X and Y be varieties. An *antiholomorphic morphism* from X to Y is a morphism of algebraic varieties $X \rightarrow \bar{Y}$.

Definition. A *real structure*, or *antiholomorphic involution*, on X is a morphism $\sigma_X : X \rightarrow \bar{X}$ such that the composition

$$X \xrightarrow{\sigma_X} \bar{X} \xrightarrow{\overline{\sigma_X}} X$$

is the identity.

Compare for example [Bor91, Chapter 1] or [Spr98, Chapter 11] for all this.

2.5.2 Real structures on O_X -modules

Let X be a variety and let \mathcal{F} be a sheaf of O_X -modules. The complex conjugate sheaf $\bar{\mathcal{F}}$ is a sheaf of $O_{\bar{X}}$ -modules, and complex conjugation is a functor from O_X -modules to $O_{\bar{X}}$ -modules.

Definition. Given a real structure σ_X on X , a *real structure*, or *antiholomorphic involution*, on \mathcal{F} is a morphism of O_X -modules

$$\sigma_{\mathcal{F}} : \mathcal{F} \rightarrow \sigma_X^* \bar{\mathcal{F}}$$

such that the composition

$$\mathcal{F} \xrightarrow{\sigma_{\mathcal{F}}} \sigma_X^* \bar{\mathcal{F}} \xrightarrow{\sigma_X^* [\overline{\sigma_{\mathcal{F}}}] } \sigma_X^* \left[\overline{\sigma_X^* \bar{\mathcal{F}}} \right] \xrightarrow{\cong} \mathcal{F}$$

is the identity morphism.

Example. If \mathcal{F} is the sheaf of sections of a vector bundle V , then a real structure on \mathcal{F} is the same thing as a real structure on the variety underlying V that yields a morphism of vector bundles

$$\begin{array}{ccc} V & \xrightarrow{\sigma_V} & \bar{V} \\ \pi \downarrow & & \downarrow \bar{\pi} \\ X & \xrightarrow{\sigma_X} & \bar{X}. \end{array}$$

2.5.3 Real structures on algebraic families of Lie algebras and groups

Let X be a variety and let us specialize the above discussion to our algebraic families.

If \mathbf{G} is an algebraic family of groups over X , then $\overline{\mathbf{G}}$ is a family of complex algebraic groups over the conjugate variety \overline{X} .

Similarly, if \mathfrak{g} is an algebraic family of complex Lie algebras over X then the complex conjugate sheaf $\overline{\mathfrak{g}}$ is an algebraic family of complex Lie algebras over \overline{X} .

Finally, if $(\mathfrak{g}, \mathbf{K})$ is an algebraic family of Harish-Chandra pairs over X then $(\overline{\mathfrak{g}}, \overline{\mathbf{K}})$ is an algebraic family of Harish-Chandra pairs over \overline{X} .

Definition. We shall refer to the families above as the *conjugate families* associated to \mathbf{G} , \mathfrak{g} and $(\mathfrak{g}, \mathbf{K})$. A *real structure* in each case is a morphism σ for which the composition of σ with $\overline{\sigma}$ is the identity.

2.5.4 Real families associated to real structures

The *set of real points* of a real structure on X is the set of fixed points for σ_X in the underlying topological space of X (it may of course be empty).

Given a real structure on an algebraic family of groups or Lie algebras over X , we obtain by restriction a family of complex groups or Lie algebras over the real points in X . Each member of the family carries its own real structure, and by further passing to fixed sets we obtain families of Lie groups or real Lie algebras over the real points in X . It is in this way that we shall recover families originally studied in the mathematical physics literature.

3 Algebraic families of classical groups

In this section we shall construct algebraic families of symmetric pairs of groups whose underlying families of Harish-Chandra pairs are the contraction families considered in Paragraphs 2.1.3 and 2.3.1. The construction will not apply to every case, but for instance it will apply to the examples

- (a) $(GL(q+p, \mathbb{C}), GL(q, \mathbb{C}) \times GL(p, \mathbb{C}))$
- (b) $(O(q+p, \mathbb{C}), O(q, \mathbb{C}) \times O(p, \mathbb{C}))$

(c) $(\mathrm{Sp}(2n, \mathbb{C}), \mathrm{GL}(n, \mathbb{C}))$

as well as the determinant-one versions of (a) and (b) (we do not know how to handle the most general case).

Thus we shall begin with a complex affine algebraic group G , together with an involution θ with fixed subgroup K , and construct (in the cases above) an algebraic family \mathbf{G} of groups over $\mathbb{C}\mathbb{P}^1$. The fiber over all but two points in $\mathbb{C}\mathbb{P}^1$ will be isomorphic to G , and the remaining two fibers will be isomorphic to the semidirect product $K \ltimes \mathfrak{p}$, where \mathfrak{p} is the minus one eigenspace of θ as it acts on the Lie algebra of G (we treat \mathfrak{p} as an additive group, using its vector space structure). The family \mathbf{G} will carry an involution, and the fixed-point subfamily will be the constant family of groups with fiber K .

Our families will carry natural real structures that are compatible with the standard real structure on $\mathbb{C}\mathbb{P}^1$. We obtain a family of real groups over $\mathbb{R}\mathbb{P}^1 \subseteq \mathbb{C}\mathbb{P}^1$. For example in case (i) the family has the form

$$\mathbf{G}^\sigma|_x \cong \begin{cases} \mathrm{U}(p, q) & x > 0 \\ \mathrm{U}(p) \times \mathrm{U}(q) \ltimes \mathfrak{p}^\sigma & x = 0, \infty \\ \mathrm{U}(p+q) & x < 0, \end{cases}$$

where \mathfrak{p}^σ is the minus one eigenspace of the Cartan involution of the real Lie algebra $\mathfrak{u}(p, q)$ with fixed subalgebra $\mathfrak{u}(p) \times \mathfrak{u}(q)$, and x is the usual coordinate on $\mathbb{R}\mathbb{P}^1$.

When $p = q = 1$ the determinant-one version of this family will give

$$\mathbf{G}^\sigma|_x \simeq \begin{cases} \mathrm{SU}(1, 1) & x > 0 \\ \mathrm{U}(1) \ltimes \mathfrak{p}^\sigma & x = 0, \infty \\ \mathrm{SU}(2) & x < 0, \end{cases}$$

which is the family that we mentioned in the introduction.

3.1 The general construction

We shall start with an algebraic group G and an involution θ with fixed subgroup K . We shall assume that G is embedded in a larger complex affine algebraic group H . We shall obtain an algebraic family of groups by first conjugating G inside of H by inner automorphisms of H , so as to

obtain a family of subgroups of H that are all isomorphic to G , and then forming a closure so as to obtain the family that we want.

We are interested in families of groups that contain the *constant* family K as a subfamily, and so we shall consider only inner automorphisms that fix K (in fact it will be sufficient for our purposes to consider only automorphisms that fix K pointwise, and indeed only some of these automorphisms).

To form the closure, we shall work in the Grassmannian $\text{Gr}_d(\mathfrak{h})$ of linear subspaces of \mathfrak{h} of dimension $d = \dim(\mathfrak{g})$ where \mathfrak{h} is the Lie algebra of H .

Let $Z \subseteq H$ be a subvariety consisting of elements in H that centralize K (we emphasize that Z need not be the full centralizer of K in H , and it need not even be a subgroup). Denote by

$$X_0 \subseteq \text{Gr}_d(\mathfrak{h})$$

the image inside the Grassmannian of the morphism

$$\alpha: Z \longrightarrow \text{Gr}_d(\mathfrak{h})$$

defined by $\alpha(z) = \text{Ad}_z[\mathfrak{g}]$, and denote by

$$X \subseteq \text{Gr}_d(\mathfrak{h})$$

the closure of X_0 in $\text{Gr}_d(\mathfrak{h})$. Consider now the restriction to X of the tautological bundle E over $\text{Gr}_d(\mathfrak{h})$, as in the diagram:

$$\begin{array}{ccccc} \mathfrak{g} & \hookrightarrow & E & \hookrightarrow & \text{Gr}_d(\mathfrak{h}) \times \mathfrak{h} \\ \downarrow & & \downarrow & & \downarrow \\ X & \hookrightarrow & \text{Gr}_d(\mathfrak{h}) & \xlongequal{\quad} & \text{Gr}_d(\mathfrak{h}). \end{array}$$

The fibers of \mathfrak{g} are Lie subalgebras of \mathfrak{h} , and we obtain in this way an algebraic family \mathfrak{g} of Lie algebras over the projective variety X .

As for algebraic families of groups, if G is connected (or if K meets every component of G), then the subgroup $\text{Ad}_z[G] \subseteq H$ depends only on the image of z under the morphism α , and so we obtain a collection of algebraic subgroups of H parametrized by X_0 . It may be checked on a case by case basis for our examples that this is an algebraic family of groups,

namely a subfamily of the constant family with fiber H , and that the closure of this family in $\text{Gr}_d(\mathfrak{h}) \times H$ is an algebraic family of groups G over X , whose family of Lie algebras is the family \mathfrak{g} above. We shall summarize the calculations in one case in the next section.

Our construction of real structures will be as follows. We shall begin with a real structure σ on G that commutes with the involution θ of G .¹ We shall assume that σ extends to H and also determines a real structure on $Z \subseteq H$. Then all of the families described above acquire real structures from σ .

3.2 Special cases

Let us examine the case of the pair

$$G = \text{GL}(q+p, \mathbb{C}) \quad \text{and} \quad K = \text{GL}(q, \mathbb{C}) \times \text{GL}(p, \mathbb{C}).$$

We shall embed G diagonally in the product $H = G \times G$, and we shall take $Z \subseteq H$ to be the set of all pairs $z = (g, g^{-1})$, where

$$g = \begin{pmatrix} \mu I_q & 0 \\ 0 & \nu I_p \end{pmatrix},$$

and where $\mu, \nu \in \mathbb{C}^\times$. The Lie subalgebra $\text{Ad}_z[\mathfrak{g}] \subseteq \mathfrak{h}$ decomposes as

$$\text{Ad}_z[\mathfrak{g}] = \mathfrak{k} \oplus \mathfrak{p}_z,$$

where \mathfrak{p}_z consists of all pairs of matrices of the form

$$\left(\begin{bmatrix} 0 & \mu^2 B \\ \nu^2 C & 0 \end{bmatrix}, \begin{bmatrix} 0 & \nu^2 B \\ \mu^2 C & 0 \end{bmatrix} \right).$$

This only depends on $\mu^2 \nu^{-2} \in \mathbb{C}^\times$, and the variety $X_0 \subseteq \text{Gr}_d(\mathfrak{h})$ from the previous section is isomorphic to \mathbb{C}^\times in this way.

The closure of X_0 is isomorphic to \mathbb{CP}^1 ; the points in \mathbb{CP}^1 with homogeneous coordinates $[0, 1]$ and $[1, 0]$ correspond to the spaces \mathfrak{p}_0 and \mathfrak{p}_∞ of matrix pairs of the types

$$\left(\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}, \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \right) \quad \text{and} \quad \left(\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} \right),$$

¹It is worth recalling that real structures on a reductive G , which are the main groups of interest to us, are in correspondence, up to conjugation, with involutions on G , in such a way that the involution and real structure commute, and the composition determines a compact form of G ; see [OV90, Theorems 3&4, pp.230–231].

respectively. We obtain an algebraic family of Lie algebras over $X = \mathbb{CP}^1$ that is isomorphic over $\mathbb{C} \subseteq \mathbb{CP}^1$ to the contraction family from Paragraph 2.1.3 associated to the involution

$$\theta = \text{Ad}_{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} : \mathfrak{g} \longrightarrow \mathfrak{g}.$$

Remark. In Paragraph 2.1.3 we defined the contraction family over the affine line rather than over the projective line. The computation above shows that the family can be extended to the projective line in at least the special case considered there, but it is a simple matter to do so in general. First, the formula

$$[\eta, \zeta](w) = \begin{cases} \frac{1}{w}[\eta(w), \zeta(w)] & \text{if } \zeta \text{ and } \eta \text{ are sections of } \mathcal{O}_X \otimes \mathfrak{p} \\ [\eta(w), \zeta(w)] & \text{otherwise.} \end{cases}$$

defines an algebraic family over the complement of 0 in \mathbb{CP}^1 . Second, the formula

$$\eta \longmapsto \begin{cases} z \cdot \eta & \text{if } \eta \text{ is a section of } \mathcal{O}_X \otimes \mathfrak{p} \\ \eta & \text{if } \eta \text{ is a section of } \mathcal{O}_X \otimes \mathfrak{k} \end{cases}$$

defines an isomorphism from the family over $\mathbb{CP}^1 \setminus \{\infty\}$ to the family over $\mathbb{CP}^1 \setminus \{0\}$, when both are restricted to $\mathbb{CP}^1 \setminus \{0, \infty\}$. We can use this isomorphism to glue the two families (sheaves) together.

Continuing with the special case, the closure of the family of groups G_z over X_0 is the algebraic family of groups over $X = \mathbb{CP}^1$, a subfamily of the constant family with fiber H , whose fibers over 0 and ∞ are the subgroups

$$G_{0,\infty} = \{k + X : k \in K \text{ and } X \in \mathfrak{p}_{0,\infty}\}$$

(recall that K is diagonally embedded in $H = G \times G$). As algebraic groups, these are isomorphic to the semidirect products $K \ltimes \mathfrak{p}_{0,\infty}$.

The antiholomorphic involution of G defined by

$$\sigma(g^*) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} g^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(where g^* is the conjugate-transpose) commutes with θ and the corresponding group of real points is $U(\mathfrak{p}, \mathfrak{q})$. If we extend σ to $H = G \times G$ by the formula

$$\sigma : (g_1, g_2) \longmapsto (\sigma(g_2), \sigma(g_1))$$

then σ maps $Z \subseteq H$ to its conjugate, determines the standard real structure on $X = \mathbb{CP}^1$, for which the antiholomorphic involution is complex conjugation on homogeneous coordinates, and gives real structures on the groups $\mathbf{G}|_x$, with $x \in \mathbb{RP}^1$, exactly as described at the beginning of this section.

Remark. All of the constructions pass to the determinant-one subgroups of G and H , and in the particular case $p = q = 1$ we obtain the family of real groups

$$\mathbf{G}^\sigma|_x \cong \begin{cases} \mathrm{SU}(1, 1) & x > 0 \\ \mathrm{U}(1) \times \mathfrak{p}^\sigma & x = 0, \infty \\ \mathrm{SU}(2) & x < 0, \end{cases}$$

over \mathbb{RP}^1 we were seeking.

4 A classification problem

In this final section of the paper we shall study a classification problem in order to explore a little further the concept of algebraic family of Harish-Chandra modules in a simple, concrete case.

The setup is as follows. We shall study one particular algebraic family of Harish-Chandra pairs, namely the family $(\mathfrak{g}, \mathbf{K})$ from Section 3.2, with $p = q = 1$. In this case the fibers of \mathfrak{g} are generically isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ and \mathbf{K} is the constant family of groups with fiber \mathbb{C}^* (embedded as the diagonal subgroup in $\mathrm{SL}(2, \mathbb{C})$). The sheaf of Lie algebras \mathfrak{g} decomposes as a direct sum of invertible sheaves

$$\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-2}$$

according to the action of \mathbf{K} , and we shall analyze algebraic families of Harish-Chandra modules by breaking down the \mathfrak{g} -action into three parts according to this decomposition. The family of Lie algebras associated to \mathbf{K} coincides with \mathfrak{g}_0 and it is isomorphic to the constant family $\mathcal{O}_X \otimes_{\mathbb{C}} \mathrm{Lie}(\mathbb{C}^*)$.

An algebraic family of Harish-Chandra modules \mathcal{F} has a \mathbf{K} -isotypical decomposition

$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n.$$

The action of \mathbf{K} on \mathcal{F} is of course given by the isotypical decomposition, and since $\mathfrak{g}_0 = \text{Lie}(\mathbf{K})$, the action of \mathfrak{g}_0 is determined too. The full action of \mathfrak{g} determines, and is determined by, morphisms of sheaves

$$\mathfrak{g}_{\pm 2} \otimes_{\mathcal{O}_X} \mathcal{F}_n \longrightarrow \mathcal{F}_{n\pm 2}.$$

They satisfy compatibility conditions that are related in part to the fact that \mathfrak{g}_0 is the Lie algebra of \mathbf{K} . For the generically irreducible families \mathcal{F} that we shall study, each nonzero \mathcal{F}_n is in fact an invertible sheaf, and the degrees of these sheaves further constrain the morphisms above.

The main lessons learned from this classification problem will be as follows:

- (a) The interesting classification problem appears to involve generically irreducible modules, defined below.
- (b) There are invariants for the classification problem, namely K-types, and infinitesimal characters (in our case the infinitesimal characters will be associated to certain elements of the ring $\mathbb{C}[z, z^{-1}]$), that are similar to those in the standard classification problem for individual Harish-Chandra modules.
- (c) There are also additional geometric invariants, namely the degrees of the invertible sheaves \mathcal{F}_n .
- (d) These invariants together are enough to solve some cases of the classification problem, but they are not enough in general. In fact the general problem is unwieldy without the imposition of further hypotheses beyond generic irreducibility.

The classification results we obtain will be used in the sequel to this paper when we consider representations of real groups and the contraction families of representations from mathematical physics.

4.1 Generically irreducible and Quasi-simple families

4.1.1 Generically irreducible modules

Let $(\mathfrak{g}, \mathbf{K})$ be an algebraic family of Harish-Chandra pairs over a variety X . Informally speaking, an algebraic family of Harish-Chandra modules

\mathcal{F} for $(\mathfrak{g}, \mathbf{K})$ is *generically irreducible* if for almost any $x \in X$ the fiber $\mathcal{F}|_x$ is an irreducible $(\mathfrak{g}|_x, \mathbf{K}|_x)$ -module.² For the family of Harish-Chandra pairs that we will work with in this section, generic irreducibility means that all except at most countably many fibers are irreducible.

4.1.2 Quasi-simple families

Let \mathfrak{g} be an algebraic family of Lie algebras over X . The *sheaf of universal enveloping algebras* $\mathcal{U}(\mathfrak{g})$ is the sheaf of \mathcal{O}_X -algebras characterized by the usual universal property: it is equipped with a morphism

$$\mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})$$

that is compatible with Lie and commutator brackets, and is initial among such morphisms.

In the context of algebraic families, the sheaf of universal enveloping algebras is locally free as a sheaf of \mathcal{O}_X -modules (in fact there is a Poincaré-Birkhoff-Witt isomorphism from the sheaf of symmetric algebras $S(\mathfrak{g})$ to the sheaf of enveloping algebras).

Definition. Let $\mathcal{Z}(\mathfrak{g})$ be the center of $\mathcal{U}(\mathfrak{g})$. A $(\mathfrak{g}, \mathbf{K})$ -module \mathcal{F} is *quasisimple* if the morphism

$$\mathcal{U}(\mathfrak{g}) \longrightarrow \text{End}(\mathcal{F})$$

maps $\mathcal{Z}(\mathfrak{g})$ into the subsheaf $\mathcal{O}_X \cdot I_{\mathcal{F}} \subseteq \text{End}(\mathcal{F})$.

This means that a family is quasisimple if every fiber has an infinitesimal character. The usual Schur's Lemma argument implies:

Lemma. *A generically irreducible admissible $(\mathfrak{g}, \mathbf{K})$ -module is quasisimple. \square*

4.2 The contraction family for $SL(2)$

Throughout the rest of Section 4 we shall focus our attention on a specific family $(\mathfrak{g}, \mathbf{K})$ of Harish-Chandra pairs over $X = \mathbb{C}\mathbb{P}^1$, namely the family associated to the determinant one case of the family of groups constructed in Section 3.2, with $p = q = 1$.

²Formally, an algebraic family of Harish-Chandra modules \mathcal{F} over $(\mathfrak{g}, \mathbf{K})$ is *generically irreducible* if $\mathbb{k} \otimes_{\mathcal{O}_X} \mathcal{F}$ is irreducible as a module for the Lie algebra $\mathbb{k} \otimes_{\mathcal{O}_X} \mathfrak{g}$, where \mathbb{k} is the algebraic closure of the field of rational functions on X .

To be explicit, \mathfrak{g} is the following subfamily of the constant family of Lie algebras over \mathbb{CP}^1 with fiber $\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$:

$$\left\{ \left(\begin{bmatrix} a & \alpha b \\ \beta c & -a \end{bmatrix}, \begin{bmatrix} a & \beta b \\ \alpha c & -a \end{bmatrix}, [\alpha : \beta] \right) \in \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) \times \mathbb{CP}^1 \right\}.$$

It can be checked that

$$\mathfrak{g}|_z \cong \begin{cases} \mathfrak{sl}(2, \mathbb{C}) & z \neq 0, \infty \\ \mathbb{C}^* \ltimes \mathbb{C}^2 & z = 0, \infty, \end{cases}$$

where $z = \alpha/\beta$. The family \mathbf{K} is the constant family of groups with fiber $\mathbf{K} = \Delta H = \{ (h, h) | h \in H \}$ where

$$H = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \in \mathrm{SL}(2, \mathbb{C}) \right\}.$$

Our aim is to classify the equivalence classes of generically irreducible and admissible $(\mathfrak{g}, \mathbf{K})$ -modules in this case.

Restricted to the affine line $\beta \neq 0$, this is the contraction family discussed in Paragraph 2.1.3. We could have chosen to work with the deformation family from Paragraph 2.1.2, extended to a family over \mathbb{CP}^1 . The outcome would be essentially the same.

4.3 Isotypical decomposition of \mathfrak{g}

In the following lemma we shall write k_a for the diagonal element (h_a, h_a) , where

$$h_a = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}.$$

Lemma. *The sheaf \mathfrak{g} decomposes into a direct sum*

$$\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-2}$$

under the action of \mathbf{K} , in such a way that

$$\mathrm{Ad}_{k_a} \cdot \sigma_n = a^n \cdot \sigma_n$$

for any section σ_n of \mathfrak{g}_n . Each of the sheaves \mathfrak{g}_n is invertible, with

$$\mathrm{deg}(\mathfrak{g}_2) = -1, \quad \mathrm{deg}(\mathfrak{g}_0) = 0 \quad \text{and} \quad \mathrm{deg}(\mathfrak{g}_{-2}) = -1.$$

Proof. For the existence of the weight decomposition see Paragraph 2.2.5. The formulas

$$\begin{aligned} X: [\alpha : \beta] &\longmapsto \left(\begin{bmatrix} 0 & \alpha/\beta \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \\ Y: [\alpha : \beta] &\longmapsto \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \alpha/\beta & 0 \end{bmatrix} \right) \end{aligned}$$

define nowhere vanishing rational sections of \mathfrak{g}_2 and \mathfrak{g}_{-2} respectively, each with simple poles at $\beta = 0$, while the formula

$$H: [\alpha : \beta] \longmapsto \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

defines a nowhere vanishing regular section of \mathfrak{g}_0 . These sections determine the degrees. \square

4.4 Invariants

We shall start by attaching the usual invariants, more or less, to generically irreducible algebraic families of Harish-Chandra modules, namely an invariant associated to the *Casimir subsheaf* of $\mathcal{Z}(\mathfrak{g})$, which we shall define below, and an invariant consisting of the set of the K-types appearing in the isotypical decomposition of a module.

4.4.1 The Casimir sheaf

The sheaf $\mathcal{Z}(\mathfrak{g})$ is filtered by the usual notion of order in enveloping algebras.

Definition. The *Casimir sheaf* $\mathcal{C} \subseteq \mathcal{Z}(\mathfrak{g})$ is the subsheaf of the order 2 part of $\mathcal{Z}(\mathfrak{g})$ consisting of sections that act trivially on the family of trivial modules.

The Casimir sheaf is an invertible sheaf, isomorphic to $\mathcal{O}(-2)$ (this will be clear from the explicit construction of rational sections in the next paragraph). By Schur's lemma its action on a generically irreducible algebraic family of Harish-Chandra modules is determined by a morphism of \mathcal{O}_X -modules

$$\mathcal{C} \longrightarrow \mathcal{O}_X.$$

This is our first invariant.

4.4.2 Sections of the Casimir sheaf

The sections X , H and Y in the proof of the lemma in Section 4.3 satisfy the relations

$$[H, X] = 2X, \quad [H, Y] = -2Y \quad \text{and} \quad [X, Y] = zH.$$

where $z = \alpha/\beta$ as before. The formula

$$C = zH^2 + 2XY + 2YX = zH^2 + 2zH + 4YX$$

defines a rational section of the Casimir sheaf, which we shall call the *Casimir section*. From the formula it is evident that $\text{ord}_\infty(C) = -2$ and $\text{ord}_z(C) = 0$ elsewhere. Hence:

Lemma. *Let \mathcal{F} be a generically irreducible admissible $(\mathfrak{g}, \mathbf{K})$ -module. The Casimir section, C , acts on \mathcal{F} by multiplication by a polynomial function on X of the form*

$$c_2z^2 + c_1z + c_0. \quad \square$$

This particular rational section of the Casimir sheaf will be the most convenient one for our computations. But since the restriction of \mathfrak{g} to the complement of $z=0$ and $z=\infty$ is semisimple, there is also a *canonical Casimir section* associated to the Killing forms away from these two points (actually we shall divide the Killing form by the constant 4 to simplify formulas). In terms of our chosen Casimir section above, the canonical Casimir section is

$$C_{\text{canonical}} = z^{-1}C.$$

4.4.3 Weights of generically irreducible modules

In this section we shall study the restriction to \mathbf{K} of a generically irreducible and admissible algebraic family of Harish-Chandra modules for $(\mathfrak{g}, \mathbf{K})$.

Recall that the algebraic family of groups \mathbf{K} in our algebraic family of Harish-Chandra pairs $(\mathfrak{g}, \mathbf{K})$ is the constant family with fiber the diagonal subgroup $\mathbf{K} = \Delta H$ in $H \times H$, where H is the group of diagonal matrices in $SL(2, \mathbb{C})$. The irreducible representations of \mathbf{K} are the one-dimensional weights

$$k_a \longmapsto \alpha^n$$

for $n \in \mathbb{Z}$. We shall use the word “weight” rather than “K-type” in what follows, and if \mathcal{F} is an algebraic family of $(\mathfrak{g}, \mathbf{K})$ -modules, then we shall refer to the K-isotypical decomposition

$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n$$

of \mathcal{F} , as in Paragraph 2.2.5, as its *weight decomposition*.

Since an admissible family \mathcal{F} is generically irreducible if and only if the fiber $\mathcal{F}|_x$ is an irreducible $(\mathfrak{g}|_x, \mathbf{K}|_x)$ -module for all but countably many points x , there are (infinitely many) fibers $\mathcal{F}|_x$ which are irreducible as $(\mathfrak{g}|_x, \mathbf{K}|_x)$ -modules. These have the same weights as \mathcal{F} itself, and hence:

Lemma. *The list of weights with multiplicities, of a generically irreducible admissible $(\mathfrak{g}, \mathbf{K})$ -module coincides with the list of weights, with multiplicities of some irreducible admissible $(\mathfrak{sl}(2, \mathbb{C}), \mathbf{H})$ -module. \square*

We list all possibilities for weights of such modules in Table 1.³ Each weight appears with multiplicity at most one and all the weights of a given irreducible admissible module share the same parity. This list of weights is our second invariant of a generically irreducible admissible $(\mathfrak{g}, \mathbf{K})$ -module.

Module	Weights
even principal series	$2\mathbb{Z}$
odd principal series	$2\mathbb{Z} + 1$
positive discrete series, $\ell \geq 1$	$\ell + 2\mathbb{N}$
negative discrete series, $\ell \leq -1$	$\ell - 2\mathbb{N}$
finite-dimensional, $k \geq 0$	$k, k-2, \dots, -k$

Table 1: The weights of admissible irreducible $(\mathfrak{sl}(2, \mathbb{C}), \mathbf{H})$ -modules.

4.5 Further invariants

The action of the Casimir sheaf and the weight decomposition are not enough to completely determine a generically irreducible algebraic family of Harish-Chandra modules \mathcal{F} up to isomorphism. That is, they are not

³We have included limit discrete series as discrete series, since for the purposes of the current work there is really no difference between the two.

enough to determine the morphisms of sheaves

$$(4.5.1) \quad \mathfrak{g}_2 \otimes \mathcal{F}_n \longrightarrow \mathcal{F}_{n+2}$$

and

$$(4.5.2) \quad \mathfrak{g}_{-2} \otimes \mathcal{F}_{n+2} \longrightarrow \mathcal{F}_n$$

for all n . The action of the Casimir sheaf *is* sufficient to determine the composition

$$\mathfrak{g}_{-2} \otimes \mathfrak{g}_2 \otimes \mathcal{F}_n \longrightarrow \mathcal{F}_n,$$

as we shall see below, and so each of the morphisms (4.5.1) and (4.5.2) determines the other. We need further invariants, and, as we shall see, in at least some cases these are conveniently provided by the degrees of the sheaves \mathcal{F}_n .

4.5.1 Degrees of the weight components

If $\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n$ is the weight decomposition of a generically irreducible admissible $(\mathfrak{g}, \mathbf{K})$ -module, then we have seen that each nonzero \mathcal{F}_n is a locally free sheaf of \mathcal{O}_X -modules of rank one. Since $X = \mathbb{C}\mathbb{P}^1$, the sheaf of \mathcal{O}_X -modules \mathcal{F}_n is therefore determined up to isomorphism by its degree, $\deg(\mathcal{F}_n)$.

The function $n \mapsto \deg(\mathcal{F}_n)$, which is defined on the set of weights of \mathcal{F} , will be our third invariant.

Lemma. *If $\mathcal{F} = \bigoplus \mathcal{F}_n$ is the weight decomposition of a generically irreducible admissible $(\mathfrak{g}, \mathbf{K})$ -module, and if \mathcal{F}_n and \mathcal{F}_{n+2} are nonzero, then:*

(a) $\deg(\mathcal{F}_n) - 1 \leq \deg(\mathcal{F}_{n+2}) \leq \deg(\mathcal{F}_n) + 1$.

(b) *If $\deg(\mathcal{F}_n) - 1 = \deg(\mathcal{F}_{n+2})$, then the morphism*

$$\mathfrak{g}_2 \otimes \mathcal{F}_n \longrightarrow \mathcal{F}_{n+2}$$

is an isomorphism.

(c) *If $\deg(\mathcal{F}_{n+2}) = \deg(\mathcal{F}_n) + 1$, then the morphism*

$$\mathfrak{g}_{-2} \otimes \mathcal{F}_{n+2} \longrightarrow \mathcal{F}_n$$

is an isomorphism.

Proof. In the generically irreducible case the action determines non-zero morphisms of sheaves

$$\mathfrak{g}_2 \otimes \mathcal{F}_n \longrightarrow \mathcal{F}_{n+2}$$

and

$$\mathfrak{g}_{-2} \otimes \mathcal{F}_{n+2} \longrightarrow \mathcal{F}_n$$

Such nonzero morphisms exist only when

$$\deg(\mathfrak{g}_2) + \deg(\mathcal{F}_n) \leq \deg(\mathcal{F}_{n+2})$$

and

$$\deg(\mathfrak{g}_{-2}) + \deg(\mathcal{F}_{n+2}) \leq \deg(\mathcal{F}_n),$$

respectively. So the first part of the lemma follows from the fact that the sheaves $\mathfrak{g}_{\pm 2}$ have degree minus one. The rest of the lemma follows from the fact that for $X = \mathbb{C}\mathbb{P}^1$, any nonzero morphism between two invertible sheaves of O_X -modules with equal degrees is an isomorphism. \square

4.6 Basis for a generically irreducible module

Let \mathcal{F} be a generically irreducible admissible $(\mathfrak{g}, \mathbf{K})$ -module, with weight decomposition

$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n$$

We have seen that each nonzero weight space \mathcal{F}_n is a locally free sheaf of O_X -modules of rank one. Keeping in mind that $X = \mathbb{C}\mathbb{P}^1$, we find that each non-zero weight space is isomorphic to some $O(k)$, and we can choose a section of the following sort that will be convenient for computations.

Lemma. *Let \mathcal{F} be a generically irreducible $(\mathfrak{g}, \mathbf{K})$ -module.*

(i) *Each nonzero weight space \mathcal{F}_n has a rational section f_n that is regular except perhaps at $z = \infty$, and also nowhere vanishing except perhaps at $z = \infty$. Moreover $\text{ord}_\infty(f_n) = \deg(\mathcal{F}_n)$.*

(ii) *The section f_n is unique, up to multiplication by a nonzero complex scalar.* \square

Now consider the rational sections X , H and Y of the weight spaces \mathfrak{g}_2 , \mathfrak{g}_0 and \mathfrak{g}_{-2} introduced in Paragraph 4.4.1. If we fix sections f_n as in the lemma above, then

$$\begin{aligned} Hf_n &= \mathfrak{n}f_n \\ Xf_n &= A_n f_{n+2} \\ Yf_{n+2} &= B_n f_n \end{aligned}$$

for some polynomial functions A_n and B_n on the affine z -line.

The degrees of $A_n(z)$ and $B_n(z)$ are constrained by the degrees of the sheaves \mathcal{F}_n and \mathcal{F}_{n+2} (whenever these sheaves are nonzero). Namely the formulas above lead to the inequalities

$$1 - \deg(\mathcal{F}_n) \geq \deg(A_n) - \deg(\mathcal{F}_{n+2})$$

and

$$1 - \deg(\mathcal{F}_{n+2}) \geq \deg(B_n) - \deg(\mathcal{F}_n).$$

Now from its formula we find that the Casimir section C (see Paragraph 4.4.1 again) acts on f_n as multiplication by the polynomial function

$$z \longmapsto (\mathfrak{n}^2 + 2\mathfrak{n})z + 4A_n(z)B_n(z).$$

This must of course be independent of n , which is a constraint on the collection of all A_n and B_n .

4.7 The case of varying degrees

Let \mathcal{F} be a generically irreducible $(\mathfrak{g}, \mathbf{K})$ -module. We have seen that

$$\deg(\mathcal{F}_n) - 1 \leq \deg(\mathcal{F}_{n+2}) \leq \deg(\mathcal{F}_n) + 1$$

whenever \mathcal{F}_n and \mathcal{F}_{n+2} are nonzero. The following computation shows that if $\deg(\mathcal{F}_n)$ always differs from $\deg(\mathcal{F}_{n+2})$, then the invariants that we have identified serve to parametrize generically irreducible algebraic families of Harish-Chandra modules.

Theorem. *The isomorphism class of a generically irreducible $(\mathfrak{g}, \mathbf{K})$ -module \mathcal{F} for which $\deg(\mathcal{F}_n)$ always differs from $\deg(\mathcal{F}_{n+2})$ is determined by:*

- (i) *the list of weights of \mathcal{F} ,*

(ii) the degrees $\deg(\mathcal{F}_n)$, and

(iii) the action of the Casimir section.

Proof. We know the weights of \mathcal{F} constitute a finite, semi-infinite or infinite arithmetic progression in \mathbb{Z} with interval 2. Consider any two consecutive weight spaces \mathcal{F}_n and \mathcal{F}_{n+2} . If

$$\deg(\mathcal{F}_{n+2}) = \deg(\mathcal{F}_n) - 1$$

then, referring to the notation and the inequalities in the previous paragraph, we find that A_n is a constant polynomial. It must be nonzero since \mathcal{F} is generically irreducible. As for B_n , the formula at the end of previous section tells us that it is determined by the action of the Casimir section and the value of the constant A_n .

The other possibility is that

$$\deg(\mathcal{F}_n) = \deg(\mathcal{F}_{n+2}) - 1,$$

in which case B_n is a nonzero constant, while A_n is determined by this constant and the value of the Casimir section.

We can adjust the sections f_n by multiplication with nonzero scalars so that in either case the constants above are always $A_n = 1$ in the first case and $B_n = 1$ in the second case. Given any two such generically irreducible modules with the same weights, degrees of weight spaces, and actions of the Casimir section, there is a unique isomorphism of sheaves of \mathcal{O}_X -modules that corresponds these adjusted sections f_n , and it is an isomorphism of $(\mathfrak{g}, \mathbf{K})$ -modules. \square

4.8 The case of equal degrees

At the other extreme are the generically irreducible algebraic families of Harish-Chandra modules for which all the degrees $\deg(\mathcal{F}_n)$ are equal. We shall briefly study them in this paragraph.

In this case an analysis like the one in the previous paragraph shows only that

$$\deg(A_n) \leq 1 \quad \text{and} \quad \deg(B_n) \leq 1.$$

And indeed, given a generically irreducible admissible module with equal degrees, we can obtain a new one by simply exchanging any number of the

A_n with B_n , so that the new action of \mathfrak{g} is defined by

$$Xf_n = B_n f_{n+2} \quad \text{and} \quad Yf_{n+2} = A_n f_n$$

for these chosen values of n . The new module will be isomorphic to the old one if and only if A_n is a multiple of B_n for all these chosen n , which is impossible for more than two choices of n .

So by starting with one generically irreducible module with equal degrees, we construct in this way an uncountable family of modules all with the same Casimir, weight and degree invariants.

4.9 Some classes of generically irreducible modules

The observations made in the previous two paragraphs show that there are too many generically irreducible modules to admit a reasonable classification (although it is indeed possible to carry out that classification). For this reason it seems worthwhile to consider some subclasses, and this is what we shall do in this section.

All of the subclasses will consist of modules for which $\deg(\mathcal{F}_n)$ always differs from $\deg(\mathcal{F}_{n+2})$. The first (which is actually an infinite family, parametrized by weights k) is as follows:

I_k . Modules with \mathcal{F}_k nonzero and

- (i) $\deg(\mathcal{F}_{n+2}) = \deg(\mathcal{F}_n) - 1$ whenever $n \geq k$,
- (ii) $\deg(\mathcal{F}_{n-2}) = \deg(\mathcal{F}_n) - 1$ whenever $n \leq k$.

To say that $\deg(\mathcal{F}_{n\pm 2}) = \deg(\mathcal{F}_n) - 1$ is to say that the morphism

$$\mathfrak{g}_{\pm 2} \otimes_{\mathcal{O}_X} \mathcal{F}_n \longrightarrow \mathcal{F}_{n\pm 2}$$

is an isomorphism. As a result, the modules in class I_k can alternately be described as those modules that are generated, as algebraic families of $(\mathfrak{g}, \mathbf{K})$ -modules or equivalently as ordinary Harish-Chandra modules in every fiber, by their k -th isotypical parts.

Similarly we can consider the “dual” class

II_k . Modules with \mathcal{F}_k nonzero and

- (i) $\deg(\mathcal{F}_{n+2}) = \deg(\mathcal{F}_n) + 1$ whenever $n \geq k$,

(ii) $\deg(\mathcal{F}_{n-2}) = \deg(\mathcal{F}_n) + 1$ whenever $n \leq k$.

These are the modules \mathcal{F} for which every $(\mathfrak{g}, \mathbf{K})$ -submodule contains the k -th isotypical part of \mathcal{F} .

The classes I_k and II_k exhibit phenomena seen in individual (reducible) principal series representations, which is why they might deserve further study. On the other hand the following two classes seem very natural in their own rights:

III. Modules with $\deg(\mathcal{F}_{n+2}) = \deg(\mathcal{F}_n) + 1$ for all n .

IV. Modules with $\deg(\mathcal{F}_{n+2}) = \deg(\mathcal{F}_n) - 1$ for all n .

One might say that these are the modules generated by high enough, or low enough, isotypical parts, respectively.

Remark. The Picard group of isomorphism classes of invertible sheaves on X acts by tensor product on algebraic families of Harish-Chandra modules over $(\mathfrak{g}, \mathbf{K})$. For this reason it would be harmless to assume in addition that for example the k -th weight space has degree zero in cases I_k and II_k , or that the degree of \mathcal{F}_n is precisely $\lfloor \frac{n}{2} \rfloor$ or $-\lfloor \frac{n}{2} \rfloor$ in cases III or IV, respectively.

4.9.1 The principal series case

In this paragraph we shall show how to construct a generically irreducible $(\mathfrak{g}, \mathbf{K})$ -module \mathcal{F} with weights $2\mathbb{Z}$ or $2\mathbb{Z}+1$ that lies in any of the four classes above and has any value of the canonical Casimir section

$$c_1 z + c_0 + c_{-1} z^{-1},$$

except for certain constant integral values, which can never occur.

The excluded values are the constants $m(m+2)$, where $m \in \mathbb{Z}$ has the same parity as the weights of \mathcal{F} . To see this, we refer to the notation from Section 4.6. When the Casimir section assumes the constant value $m(m+2)$, the *Casimir equation* relating the Casimir section to the polynomials A_n and B_n is

$$m(m+2)z = n(n+2)z + A_n(z)B_n(z).$$

So one of A_m or B_m is zero, which cannot happen in a generically irreducible module.

If the excluded values are avoided, then a module in each of the classes I-IV can be constructed using the methods of Section 4.6, as follows. The Casimir equation never forces the product $A_n B_n$ to be the zero polynomial, and we can always choose a solution with one of A_n or B_n to be 1, and the other to be a quadratic polynomial, in such a way that the degrees of A_n and B_n satisfy the inequalities in Section 4.6.

When the inequalities are satisfied, the polynomials A_n and B_n define morphisms of sheaves

$$\mathfrak{g}_2 \otimes \mathcal{F}_n \longrightarrow \mathcal{F}_{n+2}$$

and

$$\mathfrak{g}_{-2} \otimes \mathcal{F}_{n+2} \longrightarrow \mathcal{F}_n.$$

The fact that A_n and B_n solve the Casimir equation implies that these morphisms define a generically irreducible, admissible family of $(\mathfrak{g}, \mathbf{K})$ -modules.

4.9.2 The non-principal series cases

When the weights of an admissible and generically irreducible family of $(\mathfrak{g}, \mathbf{K})$ -modules fall into one of the non-principal series cases listed in Table 1, the situation is completely different from the one described above.

Lemma. *If the weights of a generically irreducible $(\mathfrak{g}, \mathbf{K})$ -module, \mathcal{F} , coincide with those of a discrete series, limit discrete series, or irreducible finite dimensional $(\mathfrak{sl}_2(\mathbb{C}), \mathbf{H})$ -module, then the canonical Casimir section C acts by multiplication by a constant function. The constant is*

$$(4.9.1) \quad \begin{cases} \ell^2 - 2|\ell| & \text{for discrete or limit discrete series, and} \\ k^2 + 2k & \text{for finite-dimensional representations,} \end{cases}$$

where we refer to Table 1 for the indexing.

Remark. Note that $\ell^2 - 2|\ell| = k(k+2)$ when $k > 0$ and $|\ell| = k+2$.

Proof. In all of these cases there is an extreme (either highest or lowest) weight space, and it is a simple matter to directly compute the action of the Casimir for $SL(2)$ on this weight space, with the results indicated. \square

As for the existence of modules as above, it is a simple matter to construct a generically irreducible $(\mathfrak{g}, \mathbf{K})$ -module with the indicated Casimir section in each of the classes I-IV above (and indeed in any case where the degrees of successive weight modules differ by no more than 1).

4.9.3 Classification of generically irreducible modules

Let us summarize.

Theorem. *In each of the classes I-IV, the classification of generically irreducible admissible $(\mathfrak{g}, \mathbf{K})$ -modules up to isomorphism and an action of the Picard group of X is as follows:*

- (i) *The set of weights of any generically irreducible admissible module is one of the possibilities listed in Table 1.*
- (ii) *When there is an extreme weight, the Casimir section is determined by the extreme weight, as in Paragraph 4.9.2, and there is a unique module, up to isomorphism, with the given weights in the given class.*
- (iii) *When there is no extreme weight, the set of weights is either $2\mathbb{Z}$ or $2\mathbb{Z}+1$. The Casimir section cannot be one of the excluded possibilities listed in Paragraph 4.9.1, but there is a unique module, up to isomorphism, in each of the classes, with the given weights and with any other value of the Casimir section. \square*

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