ONE-DIMENSIONAL COHOMOLOGIES OF DISCRETE SUBGROUPS

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Let G be a locally compact unimodular group and Γ a discrete subgroup thereof with the factor group $\Gamma\backslash G$ compact. We shall study the group $H^1(\Gamma, \mathbf{C})$ compact. We shall study the group and its connection with the decomposition into irreducibles of its representation in $L_2(\Gamma\backslash G)$. We shall suppose G to have the following property:

(R) There exists in G a compact subgroup K such that the ring F (with respect to convolution), of continuous finite functions on G and two-way invariant relative to K, is commutative.

We shall prove two theorems with this assumption.

THEOREM 1. There exists a representation H of G such that

$$H^1(\Gamma, \mathbf{C}) = \mathrm{Hom}_G(L_2(\Gamma\backslash G), H).$$

THEOREM 2. If $G = G_1 \times G_2$, with Γ projected everywhere dense on G_1 and G_2, then $H^1(\Gamma, \mathbf{C}) = \mathrm{Hom}\cdot(G, \mathbf{C})$ (i.e., every homomorphism of Γ into $\mathbf{C}$ may be extended to a continuous homomorphism of G into $\mathbf{C}$).

§1. GARDING SPACE AND A DUALITY THEOREM

By a representation T of group G we mean a homomorphism of G into the group of invertible operators of the locally convex complete linear space L (over $\mathbf{C}$) such that the map $G \times L \to L$ given by $(g, l) \to T_g(l)$ is continuous. For each such representation we construct a new representation T^∞ in L^∞ called the representation in Garding space.

It is known from the construction theory of locally compact groups that there is an open subgroup N in G with an admissible subgroup U_i in any unit neighborhood (i.e., a compact subgroup such that N normalizes U_i and N/U_i is a Lie group). We denote by L^{U_i} the subspace of L consisting of vectors x such that

1) $T_{U_i} x = x$;

2) $T_g x$ is an infinitely differentiable vector function on N/U_i.

We specify on L^{U_i} a topology using the system of neighborhoods $V(\rho, \tilde{V})$, where ρ is an element of the enveloping algebra N/U_i and $\tilde{V}$ is a neighborhood in L; viz., we set $V(\rho, \tilde{V}) = \{x \in L^{U_i} \mid \rho(T_g x)(e) \in V\}$. We set $L^\infty = \lim_{U_i \to \{e\}} L^{U_i}$. This is the desired space. The restriction of T to L^∞ is a representation we shall denote by T^∞. It is easy to see that L^∞ is everywhere dense in L. Each continuous representation map $L_1 \to L_2$ induces a continuous map of Garding representations.

DUALITY THEOREM. Let T be a representation of G in L. Then

$$\mathrm{Hom}_G(L^\infty, L_2(\Gamma\backslash G)) = \mathrm{Hom}_G(L^\infty, L_2^\infty(\Gamma\backslash G)) = \mathrm{Hom}_\Gamma(\mathbf{C}, (L^\infty)^*) = H^0(\Gamma, (L^\infty)^*).$$

This theorem is proved in [1]. In particular, if L^∞ is reflexive, then $H^0(\Gamma, (L^\infty)^*) = \mathrm{Hom}_G(L_2(\Gamma\backslash G), (L^\infty)^*)$.

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§ 2. Γ-ACYCLIC MODULES

Let C_0 be the space of continuous finite functions on G with C_0^∞ the corresponding Garding space and $\Omega = (C_0^\infty)^*$ the space of generalized functions on G. We specify on C_0, C_0^∞, and Ω a right representation L of G: $(R_{g_0}f)(g) = f(gg_0)$, $(L_{g_0}f)(g) = f(g_0^{-1}g)$.

LEMMA 1. If Ω is regarded as a module over Γ (using the left representation), then $H^i(\Gamma, \Omega) = 0$ for $i > 0$.

Proof. An admissible subgroup of V may be chosen such that Γ acts on G/V without fixed points. Then $\Gamma\backslash G/V$ is a compact variety. We select a C^∞-triangulation in it. We denote by $\mathcal{R}_i$ the space of functions of Ω with carrier in the preimage of the i-skeleton. $\mathcal{R}_0 \subset \mathcal{R}_1 \subset \ldots \subset \mathcal{R}_n = \Omega$, $C_i = \mathcal{R}_i/\mathcal{R}_{i-1}$. It is then easy to see that C_i is the direct product of the F_{σ_i}, where the σ_i are i-simplexes and F_{σ_i} is the space consisting of Taylor series in variables normal in σ_i with coefficients in functions on the σ_i which are generalized and which do not grow quickly. It is sufficient to show that $H^i(\Gamma, C_K) = 0$, $i > 0$. This follows from the following:

LEMMA 2. If C is the space of functions on Γ taking values in the linear space F, then $H^i(\Gamma, C) = 0$, $i > 0$.

Proof. If $N \subset M$ is a C(Γ)-module , then the sequence $\mathrm{Hom}_\Gamma(M, C) \to \mathrm{Hom}(N, C) \to 0$ is exact, for $\mathrm{Hom}_\Gamma(M, C) = \mathrm{Hom}_C(M, F)$. This means that C is an injective Γ-module (homomorphisms are here considered without the topology).

This proves Lemma 1.

We denote by A the submodule of Ω consisting of functions on G/K. Since A is a direct sum in Ω, it it is injective over Γ and $H^i(\Gamma, A) = 0$, $i > 0$. We examine the exact sequence of modules $0 \to S \to A \xrightarrow{\pi} B \to 0$; the S here consists of constants, and $B = A/S$. Since $H^1(\Gamma, A) = 0$, $H^1(\Gamma, C) = H^0(\Gamma, B)/\pi_*(H^0(\Gamma, A))$. It is easy to see that Ω, A, and B are reflexive, and Ω*, A*, and B* coincide with their Garding spaces. Therefore

$$H^1(\Gamma, C) = \mathrm{Hom}_G(L_2(\Gamma\backslash G), B)/\pi_*(\mathrm{Hom}_G(L_2(\Gamma\backslash G), A).$$

§ 3. HECKE OPERATORS

Let $F \subset C_0$ $(\tilde{F} \subset \Omega_0)$ be finite continuous (generalized) functions two-way invariant relative to K, and F and $\tilde{F}$ convolution rings. We shall consider that (R) holds, i.e., F is commutative. Since F is dense in $\tilde{F}$, the latter is also commutative. It is hence easy to show that if $f \in \Omega$ is a generalized function two-way invariant relative to K, $\varphi \in \tilde{F}$, then $f * \varphi = \varphi * f$. $\tilde{F}$ acts on A according to $R_\varphi(f) = f * \varphi$. This action commutes with L_g. The subspace S is invariant, so therefore $\tilde{F}$ acts on S and B.

We denote by χ the one-dimensional representation (character) of $\tilde{F}$. In particular, we denote by χ_0 the representation in S. For any character χ we examine the subspace $A_\chi \subset A$ ($B_\chi \subset B$, respectively) consisting of elements f such that $\Delta f = \chi(\Delta) f$ for all $\Delta \in \tilde{F}$. This is a closed G-invariant subspace.

Let T be the representation of G in L. For any function $\varphi \in \Omega_0$ we define the operator $T_\varphi = \int_G \varphi(g)$, $T_g dg$. It is defined on the everywhere dense set L^∞. Let φ_K be the Haar measure on K regarded as a generalized function on G. We denote by $P_K = T_{\varphi_K} \cdot P_K$ the projector of the K-invariant vectors in L onto L_K. If $\varphi \in F$, then T_φ carries L_K into itself. If T is a unitary irreducible representation, then L_K is null- or one-dimensional (see [2]). If L_K is one-dimensional, then we denote the corresponding representation of $\tilde{F}$ by χ_T. Let $\mathscr{E}$ be the set of equivalence classes of irreducible unitary representations of G. We decompose $\mathscr{E}$ into three classes:

$$\mathscr{E}_S = \{T \mid \dim L_K = 0\}, \quad \mathscr{E}_0 = \{T \mid \chi_T = \chi_0\},$$
$$\mathscr{E}_1 = \{T \mid \chi_T \text{ definitely, } \chi_T \neq \chi_0\}.$$

Note 1. It is shown in [2] that from (R), $\mathscr{E}_0$ consists only of the unit representation.

We expand $L_2(\Gamma\backslash G)$ into the sum of terms $L_S + L_0 + L_1$ consisting of representations of the corresponding class.

LEMMA 3. Let f be a continuous map of the irreducible unitary representation T into the representation $A: f: L(T) \to A$. Then, if χ_T is defined, then $f(L) \subset A_{\chi_T}$.

It is enough to show that a nontrivial vector from L goes into A_{χ_T}. Let x be a nontrivial vector from L_K. Then $f(x)$ is a two-way function on G invariant relative to K with $\Delta * f(x) = \chi_T(\Delta) f(x) = f(x) * \Delta$ for any $\Delta \in \widetilde{F}$. This means that $f(x) \in A_{\chi_T}$.

LEMMA 4. If V is a nonnull invariant subspace in A, then there is a vector $v \in V$ such that $P_K(v) \neq 0$.

Proof. We may take v to be a smooth function with $v(e) \neq 0$. Then $P_K(v)(e) = v(e) \neq 0$. This means that $Hom_G(T, A) = 0$, if $T \in \mathscr{E}_S$.

LEMMA 5. There is a unique vector in A_{χ_0} invariant relative to K.

Proof. Let $\varphi \in A_{\chi_0}$ and $P_K(\varphi) = \varphi$. We consider the space $V = \{\sum_i a_i L_{g_i}(\varphi) \mid \sum_i a_i = 0\}$; it is an invariant subspace in A. If $v \in V$, then $P_K(v) = (\sum_i a_i P_K L_{g_i})\varphi = (\sum_i a_i P_K L_{g_i} P_K)\varphi = \Delta * \varphi$. Since $\Sigma a_i = 0$, we have that $\Delta * 1 = 0$, i.e., $\chi_0(\Delta) = 0$, and this means that $\Delta * \varphi = \varphi * \Delta = \chi_0(\Delta)\varphi = 0$. It follows from Lemma 4 that V is zero-dimensional, which means that $T_g\varphi = \varphi$ for all $g \in G$, i.e., φ is constant. This proves the lemma.

PROPOSITION. $Hom_G(L_1, A) \xrightarrow{\pi_*} Hom_G(L_1, B)$ is an isomorphism.

Proof. For each $\Delta \in \widetilde{F}$, we denote by $\widetilde{\Delta}$ the operator on L_1 that multiplies the vectors of every irreducible component of T by $\chi_T(\Delta)$. Suppose we have found an element $\Box \in \widetilde{F}$ such that $\widetilde{\Box}$ has a continuous inverse and $\chi_0(\Box) = 0$. Since $\Box(S) = 0$, there exists a unique map δ such that $\delta\pi = \Box$. For any $f \in Hom_G(L_1, B)$, $\bar{f} = \delta \widetilde{\Box}^{-1} f \in Hom_G(L_1, A)$. Then $\pi(\bar{f}) = f$, as required.

How can $\Box$ be found? We construct it such that $\Box$ is strictly positive definite, which it will be if $\widetilde{\Box}$ is such on L_{1K}. There exists an admissible subgroup U_i which acts freely on $\Gamma\backslash G$. Further, $\Gamma\backslash G/U_i$ is a compact variety; L_{1U_i} is a subspace in $L_2(\Gamma\backslash G/U_i)$. We may consider a function $\varphi \in \Omega_0$, such that T_φ takes $L_2(\Gamma\backslash G/U_i)$ into itself, $T_\varphi(1) = 0$, and T_φ is elliptical positive definite. If it equals 0 over a finite number of vectors other than 1, then we supplement it with a nonnegative definite kernel operator of the form T_ψ which is not equal to 0 on these vectors but is on 1. (This can be done easily, since for any $\Delta \in \Omega_0$, $(T_\Delta)^*$ also has the form $T_{\Delta'}$, $\Delta' \in \Omega_0$.) This means that T_φ will be strictly positive definite on $L_2(\Gamma\backslash G/K) - S$ and $P_K \varphi P_K$ will be strictly positive definite on L_{1K}. We have thus proved that

$$H^1(\Gamma, \mathbf{C}) = Hom_G(L_0 + L_S, B)/\pi_*(Hom_G(L_0 + L_S, A)).$$

We have $\pi_*(Hom_G(L_0, A)) = 0$, for $L_0 = \{\lambda\}$ $(\lambda \in \mathbf{C})$; $Hom_G(L_S, A) = 0$ by Lemma 4. This means that $H^1(\Gamma, \mathbf{C}) = Hom_G(L_0 + L_S, B)$.

LEMMA 6. If T is a unitary irreducible representation in the space L, and $f: L \to B$ is a map of representations, then $f(L) \in B_{\chi_0}$ is a map of representations, then $T \in \mathscr{E}_0 \cup \mathscr{E}_S$, and $f(L) \in B_{\chi_T}$ for $T \in \mathscr{E}_1$.

The proof for $\mathscr{E}_0$ and $\mathscr{E}_1$ is the same as in Lemma 3. Let $T \in \mathscr{E}_S$, $\Delta \in F$ for $\chi_0(\Delta) = 0$. Then $\delta f: T \to A$, and by Lemma 4 $\delta f = 0$ (δ is introduced as in the proof of the proposition), $\Delta f = \pi\delta f = 0$, as required.

This means that $H^1(\Gamma, \mathbf{C}) = Hom_G(L_0 + L_S, B_{\chi_0}) = Hom_G(L_2(\Gamma\backslash G), B_{\chi_0})$. This proves Theorem 1.

LEMMA 7. $Hom_G(L_0, B) = Hom(G, \mathbf{C}^+)$ is the set of continuous homomorphisms of G into $\mathbf{C}^+$.

Proof. L_0 is canonically isomorphic to the module $\mathbf{C}$ over G (with trivial action). Let $f: \mathbf{C} \to B$ be continuous, $f(1)$ the image of 1, and v_f any element of A for which $\pi v_f = f(1)$. Then $L_g v_f - v_f \in S \simeq \mathbf{C}$, i.e., we have a map $\varphi_f: G \to \mathbf{C}^+$. It is continuous since the preimage of B_{χ_0} consists of continuous functions. It is easy to see that φ_f is a homomorphism and does not depend on the choice of v_f.

Let a homomorphism $\varphi: G \to \mathbf{C}^+$ be given. We consider the function $v(g) = \varphi(g)$. It is easy to see that $v \in A$ and that $\pi(v)$ is invariant relative to G. Maps are in this way constructed on both sides, and it can be easily verified that they yield an isomorphism.

Note 2. We have thus proved that $H^1(\Gamma, C)$ falls into a sum of two parts: $\mathrm{Hom}(G, C^+)$, which does not depend on Γ, and $\mathrm{Hom}_G(L_S, B_{\chi_0})$, which does.

We now prove Theorem 2. In the case being considered it is enough to prove that $\mathrm{Hom}_G(L_S, B_{\chi_0}) = 0$. Let T be an irreducible unitary representation in L, with $T \subset L_S$. We prove that $\mathrm{Hom}_G(T, B_{\chi_0}) = 0$. It easily follows from the fact that G has property (R) that G_1 and G_2 do also. We may further consider that $K = K_1 \times K_2$. We introduce the subspaces $B_1 = B_{\chi_0 K_1}$ and $B_2 = B_{\chi_0 K_2}$ into B_{χ_0}. It may be proved analogously to Lemma 5 that $B_1 = B_{\chi_0 G_1}$ and $B_2 = B_{\chi_0 G_2}$. In particular, B_1 and B_2 are invariant relative to the action of G.

LEMMA 8. If T is a unitary irreducible representation of G in L and $f: L \to B_{\chi_0}$ is a map of representations, then $f(L) \subset B_1 \cup B_2$.

Proof. We consider an element $v \in L$ which varies according to the representation $\xi_1 \otimes \xi_2$ of group K (ξ_1, ξ_2 irreducible representations of K_1 and K_2, respectively). If $\xi_1 = 1$, then $f(v) \in B_1$, and the result is proved. Suppose $\xi_1 \neq 1$. We denote by B_{ξ_1} (A_{ξ_1} and L_{ξ_1}, respectively) the space of vectors in B (in A and L, respectively) which vary according to the representation ξ_1: $f(L_{\xi_1}) \subset B_{\xi_1} = A_{\xi_1}$. It may be proved analogously to Lemma 4 that since $f(L_{\xi_1}) \neq \{0\}$ in $f(L)$, there is a vector invariant relative to K_2, i.e., one that lies in B_2, as required.

Suppose T occurs in $L_2(\Gamma \backslash G)$ and $f(T) \subset B_1$. Then L of T is realized in the functions φ on G which satisfy the condition

$$\varphi(gg_1) = \varphi(\gamma g) = \varphi(g), \qquad g_1 \in G_1, \quad \gamma \in \Gamma.$$

It follows from this that the φ are constant, since the projection of Γ onto G_2 is everywhere dense. This means that if $T \subset L_S$, then $\mathrm{Hom}G(T, B_{\chi_0}) = 0$, which proves Theorem 2.

LITERATURE CITED

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