STRUCTURE OF REPRESENTATIONS GENERATED BY VECTORS OF HIGHEST WEIGHT

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§1. Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Despite the fact that finite representations of the algebra $\mathfrak{g}$ were one of the very first objects of study, there exist even simpler representations. Here a host of properties of finite representations are essentially consequences of analogous properties of these simpler modules. Categories of these modules, such as the so-called O-modules, were introduced by the authors, and their definition was given in [1]. In this work there will be studied elementary objects of that character, the modules M_χ (see [1]). This work can be read independently of [1]. Study of the M_χ module was begun in Verma's work* which obtained a series of deep results on M_χ modules.

The M_χ modules (for a precise definition, see below) are of interest since they are the simplest moduli generated by a single vector of highest weight $\chi-\rho$, where ρ as usual denotes a half-sum of positive roots. All other modules generated by a single vector of highest weight, including also irreducible finite representations, are factor modules of the modules M_χ. In the present work there is a complete description of the categories of the modules M_χ. This is to say, on the basis of the Verma Theorem, Hom(M_{χ_1}, M_{χ_2}) is either 0 or **C**. The question arises, for such χ_1, χ_2 pairs, whether there exists a nontrivial mapping of M_{χ_1} to M_{χ_2}. The main result of our article is the establishment of necessary conditions for the existence of such a mapping (Theorem 2). The proof of Theorem 2 is fairly involved; the authors have been unable to come upon an easier proof. The form of the hypothesis in [2], for Theorem 2, was retained.†

The results obtained on M_χ modules permit one to understand from a single point of view the greater part of the classical results in complex semisimple Lie algebras, in particular the Kostant theorem or the equivalence to it of the Weil formula for characters, the Borel-Weil Theorem, etc.

§2. Notation and Background‡

$\mathfrak{g}$ is a complex semisimple Lie algebra of rank r; $\mathfrak{h}$ is a Cartan subalgebra of the algebra $\mathfrak{g}$;

Δ is a system of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$ with a fixed ordering; Σ is the sum of the simple roots; Δ_+ is the set of positive roots, $\rho=\frac{1}{2}\sum_{\gamma\in\Delta_+}\gamma$;

E_γ is the root vector corresponding to the root $\gamma\in\Delta$, where $\gamma([E_\gamma, E_{-\gamma}])=2$. It is known that $[E_\alpha, E_{-\beta}]=0$ for $\alpha, \beta\in\Sigma$, $\alpha\neq\beta$;

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*The authors wish to express their gratitude to Prof. J. Dixmier for bringing the exceptionally interesting work [2] to their notice.

†We note that as formulated in [2] the theorem asserting that every submodule M in M_χ has the form $M=\cup M_{\chi_i}$ is false. Counterexamples can be constructed, in particular for Lie algebra of the group SL (4, **C**); see our last page.

‡For precise definitions and proofs, see also [4] and [5].

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$\mathfrak{n}_-$ is the subalgebra of $\mathfrak{g}$, natural in $E_{-\gamma}$, $\gamma \in \Delta_+$;

$U(\mathfrak{g})$, $U(\mathfrak{n}_-)$ are enveloping algebras of $\mathfrak{g}$ and $\mathfrak{n}_-$, respectively; $Z(\mathfrak{g})$ is the center of $U(\mathfrak{g})$;

$U(\mathfrak{g})\,(x_1, \ldots, x_k)$ for elements $x_1, \ldots, x_k$ from the $U(\mathfrak{g})$-module M denotes the $U(\mathfrak{g})$-submodule in M generated by $x_1, \ldots, x_k$. The meaning of $U(\mathfrak{n}_-)\,(x_1, \ldots, x_k)$ is analogous;

$\mathfrak{h}^*$ is the space dual to $\mathfrak{h}$; $\mathfrak{h}^*_R$ is the real linear subspace in $\mathfrak{h}^*$, natural for all roots $\gamma \in \Delta$; $\mathfrak{h}^* = \mathfrak{h}^*_R + i\mathfrak{h}^*_R$;

$\langle \cdot, \cdot \rangle$ is the scalar product in $\mathfrak{h}^*$ constructed according to the Killing form of the algebra $\mathfrak{g}$, $|\cdot|$ is the corresponding norm in $\mathfrak{h}^*_R$. We note that $\chi([E_\gamma, E_{-\gamma}]) = 2\langle \chi, \gamma\rangle / \langle \gamma, \gamma\rangle$, $\chi \in \mathfrak{h}^*$, $\gamma \in \Delta_+$;

$\mathfrak{h}^*_Z$ is the lattice in $\mathfrak{h}^*$ constructed from those χ for which $2\langle \chi, \gamma\rangle / \langle \gamma, \gamma\rangle \in Z$ for all $\gamma \in \Delta$;

$f^{(\alpha)}$ is the fundamental weight corresponding to the root $\alpha \in \Sigma$, i.e., $2\langle f^{(\alpha)}, \alpha\rangle / \langle \alpha, \alpha\rangle = 1$, $\langle f^{(\alpha)}, \beta\rangle = 0$ for $\beta \in \Sigma \backslash \alpha$. The weights $f^{(\alpha)}$, $\alpha \in \Sigma$, generate $\mathfrak{h}^*_Z$;

$\chi_1 \gg \chi_2$ for $\chi_1, \chi_2 \in \mathfrak{h}^*$ signifies that $\chi_1 - \chi_2 = \sum_{\alpha \in \Sigma} n_\alpha \alpha$, $n_\alpha \in Z$, $n_\alpha \geqslant 0$;

W is the Weil group of the algebra $\mathfrak{g}$;

$\sigma_\gamma \in W$ is the reflection corresponding to the root $\gamma \in \Delta$, i.e., $\sigma_\gamma \chi = \chi - 2\langle \chi, \gamma\rangle \langle \gamma, \gamma\rangle^{-1} \gamma$, $\chi \in \mathfrak{h}^*$. We recall that if $\alpha \in \Sigma$, $\gamma \in \Delta_+$, $\gamma \neq \alpha$, then $\sigma_\alpha \gamma \in \Delta_+$;

$l(w)$ is the length of the element $w \in W$, i.e., the least number of factors in a representation $w = \sigma_{\alpha_1}, \ldots, \sigma_{\alpha_k}$, $\alpha_i \in \Sigma$;

$\chi_1 \sim \chi_2$ for $\chi_1, \chi_2 \in \mathfrak{h}^*$ signifies the existence of an element $w \in W$ for which $\chi_1 = w\chi_2$;

$\Xi \subset \mathfrak{h}^*$ is the union of hyperplanes $\langle \mathrm{Re}\, \chi, \gamma\rangle = 0$ for all $\gamma \in \Delta$; Weil chambers are connected components $\mathfrak{h}^* \backslash \Xi$; $\bar{C}$ is the closure of the Weil chamber C; C^+ is the Weil chamber containing ρ. The group W acts simply transitively on the set of Weil chambers. It can be shown that if $\chi \in \bar{C}$ and $w\chi \in \bar{C}$, then $\mathrm{Re}\, \chi = \mathrm{Re}\, w\chi$. Two Weil chambers C_1 and C_2 are called adjacent if $\dim_R(\bar{C}_1 \cap \bar{C}_2) = 2r-1$. There then exists an element $\gamma \in \Delta_+$, such that $\sigma_\gamma C_1 = C_2$;

$P(F) \in \mathfrak{h}^*_Z$ is the set of weights of the finite-dimensional module F.

§3. The Modules M_χ

Definition 1. Suppose $\chi \in \mathfrak{h}^*$. We denote by J_χ the left ideal (in $U(\mathfrak{g})$), generated by the elements E_γ, $\gamma \in \Delta_+$, and $H - \chi(H) + \rho(H)$, $H \in \mathfrak{h}$. We further put $M_\chi = U(\mathfrak{g})/J_\chi$.

The following readily verified lemma describes the simplest properties of the modules M_χ.

LEMMA 1. 1) A $U(\mathfrak{g})$-module M_χ is generated by one generator f_χ of weight $\chi - \rho$ such that $E_\alpha f_\chi = 0$ for $\alpha \in \Sigma$; 2) $U(\mathfrak{n}_-)$ acts on M_χ without zero divisors; 3) $U(\mathfrak{n}_-)\,(f_\chi) = M_\chi$.

Besides, Properties 1) and 2) uniquely characterize the module M_χ.

THEOREM 1. (Verma, [2]). Let $\chi, \psi \in \mathfrak{h}^*$. Two cases are possible: 1) $\mathrm{Hom}_{U(\mathfrak{g})}(M_\chi, M_\psi) = 0$, 2) $\mathrm{Hom}_{U(\mathfrak{g})}(M_\chi, M_\psi) = \mathbf{C}$, and every nontrivial homomorphism $M_\chi \to M_\psi$ is an imbedding.

In the second case we can consider M_χ as a submodule of M_ψ.

We shall study in the present note for what $\chi, \psi \in \mathfrak{h}^*$ the module M_χ can be imbedded in M_ψ. This is obviously possible only in case $\chi \ll \psi$.

LEMMA 2. Let $\alpha \in \Sigma$, $\psi = \sigma_\alpha \chi$ and $\psi - \chi = n\alpha$, where $n \in Z$, $n \geq 0$. Then $M_\chi \subset M_\psi$.

Proof. It is easy to show that $E_\alpha(E^k_{-\alpha} f_\psi) = k(n-k)E^{k-1}_{-\alpha} f_\psi$. Put $f = E^n_{-\alpha} f_\psi$. Then f has weight $\chi - \rho$, $E_\alpha f = 0$ and, for any $\beta \in \Sigma$, $\beta \neq \alpha$, $E_\beta f = E_\beta E^n_{-\alpha} f_\psi = E^n_{-\alpha} E_\beta f_\psi = 0$. By Lemma 1, the submodule in M_ψ, natural on f is isomorphic to M_χ, which demonstrates the lemma.

Suppose $z \in Z(\mathfrak{g})$. Then $z \cdot f_\chi$ is proportional to f_χ, i.e., $z \cdot f_\chi = \theta_\chi(z) f_\chi$, $\theta_\chi(z) \in \mathbf{C}$. Since f_χ is a generator in M_χ, $z \cdot x = \theta_\chi(z)x$ for all $x \in M_\chi$. In this way we get the homomorphism θ_χ: $Z(\mathfrak{g}) \to \mathbf{C}$.

LEMMA 3. $\theta\chi_1 = \theta\chi_2$ if and only if $\chi_1 \sim \chi_2$.

The proof of Lemma 3 follows readily from a theorem of Harish-Chandra on eigenvalues of Laplace operators (see [3] and [8]).

COROLLARY. If $M\chi \subset M\psi$ then $\chi \sim \psi$.

Central to this work is Theorem 2 giving, along with Theorem 3, necessary and sufficient conditions that $M\chi$ is imbedded in $M\psi$.

§ 4. Necessity of Condition (A)

Suppose $\chi, \psi \in \mathfrak{h}^*$, $\gamma_1, \gamma_2, \ldots, \gamma_k \in \Delta_+$. We shall say that the sequence $\gamma_1, \ldots, \gamma_k$ satisfies Condition (A) for the pair (χ, ψ), if

1) $\chi = \sigma_{\gamma_k} \ldots \sigma_{\gamma_1}\psi$.

2) Putting $\chi_0 = \psi$, $\chi_i = \sigma_{\gamma_i} \ldots \sigma_{\gamma_1}\psi$, it is then true that $2\langle \chi_{i-1}, \gamma_i\rangle / \langle \gamma_i, \gamma_i\rangle \in \mathbf{Z}$.

3) $\langle \chi_{i-1}, \gamma_i\rangle \geq 0$.

THEOREM 2. Let $\chi, \psi \in \mathfrak{h}^*$ be such that $M\chi \subset M\psi$. Then there exists a sequence $\gamma_1, \ldots, \gamma_k \in \Delta_+$, satisfying Condition (A) for the pair (χ, ψ).

We preface the proof of the theorem by a series of lemmas.

LEMMA 4. Let M be some $U(\mathfrak{g})$-module, and $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ such that its filtration by submodules M_i obeys $M_k/M_{k-1} \approx M\chi_k$. Let χ_s be maximal* in the set of those χ_k, for which $\chi_s \sim \chi_k$. Then there is, in M, a submodule isomorphic to $M\chi_s$.

Proof. Let $J_i \subset Z(\mathfrak{g})$ be the kernel of the homomorphism $\theta\chi_i$ and $J = J_1 \ldots J_n$. Then J annihilates M, which is to say that there operates in M a commutative finite-dimensional algebra $Z(\mathfrak{g})/J$. (Finiteness of $Z(\mathfrak{g})/J$ follows from the Noether character of $Z(\mathfrak{g})$.) Since $Z(\mathfrak{g})/J$ decomposes into a direct sum of local rings (see [12]), $M = \oplus N^{(j)}$ where the submodules $N^{(j)}$ correspond to the distinct ones among the maximal ideals $J_i \in Z(\mathfrak{g})$. Suppose submodule $N^{(1)}$ corresponds to J_s; we put $L_k = N^{(1)} \cap M_k$. Then $L_k/L_{k-1} = M\chi_k$, if $J_k = J_s$ (i.e., $\chi_k \sim \chi_s$), and $L_k/L_{k-1} = 0$, if $J_k \neq J_s$. Hence there is in $N^{(1)}$ an element f of weight $\chi_s - \rho$ and no elements of greater weight (since there are no such from the module L_k/L_{k-1}). By Lemma 1, the submodule in M generated by f is isomorphic to $M\chi_s$.

LEMMA 5. Let F be a finite $U(\mathfrak{g})$-module and let $\chi \in \mathfrak{h}^*$. Then in $M\chi \otimes F$ there exists a filtration by submodules $0 = L_0 \subset L_1 \subset L_n = M\chi \otimes F$ such that $L_k/L_{k-1} \approx M\chi + \lambda_k$, where $\lambda_k \in P(F)$.

Proof. Let $e_1, \ldots, e_n$ be a basis of F consisting of weight elements with weights $\lambda_1, \ldots, \lambda_n$, and suppose the indexing chosen so that $i \leq j$, if $\lambda_i \gg \lambda_j$. Let $a_i = f\chi \otimes e_i \in M\chi \otimes F$ and $L_k = U(\mathfrak{g})(a_1, \ldots, a_k)$. We verify that the modules L_k satisfy the hypotheses of the lemma. In fact the image $\bar{a}_k$ of the element a_k in L_k/L_{k-1} is a generator in L_k/L_{k-1} of weight $\chi + \lambda_k - \rho$ and $E_\alpha \bar{a}_k = 0$ for $\alpha \in \Sigma$. Hence $L_k = U(\mathfrak{n}_-)(a_1, \ldots, a_k)$. We verify that L_k is a free $U(\mathfrak{n}_-)$-module. Suppose $X_i \in U(\mathfrak{n}_-)$, $1 \leq i \leq k$, and l is the largest of the degrees of elements X_i (relative to the natural filtration in $U(\mathfrak{n}_-)$). Then $\sum_{i=0}^{k} X_i a_i = \sum_{i=0}^{k} X_i f_\chi \otimes e_i + \sum_{j=1}^{n} Y_j f_\chi \otimes e_j \neq 0$, since the degrees of the elements $Y_j \in U(\mathfrak{n}_-)$ are less than l. Applying Lemma 1, we conclude that $L_k/L_{k-1} \approx M\chi + \lambda_k$.

The following lemma provides the key to the proof of Theorem 2.

LEMMA 6. Suppose $\chi, \psi \in \mathfrak{h}^*$ are such that $M\chi \subset M\psi$, that F is a finite-dimensional $U(\mathfrak{g})$-module, and that $\lambda \in P(F)$ is a weight such that the weight $\chi + \lambda$ is maximal in the set $W(\chi + \lambda) \cap (\chi + P(F))$. Then there exists a weight $\mu \in P(F)$ such that $M\chi+\lambda \subset M\psi+\mu$.

Proof. It follows from Lemmas 4 and 5 that there is a submodule M in $M\chi \otimes F$ which is isomorphic to $M\chi+\lambda$. Clearly, $M\chi \otimes F$ is imbedded in $M\psi \otimes F$. Suppose that $0 = L_0 \subset L_1 \subset \ldots \subset L_n = M\psi \otimes F$ is the filtration entering in Lemma 5, and that L_k is the least submodule containing the image of M. Then the image of M, isomorphic to $M\chi+\lambda$, maps nontrivially to $L_k/L_{k-1} \approx M\psi+\mu$, $\mu \in P(F)$, and, according to Theorem 1, this mapping is an imbedding.

*We say that weight χ is maximal in the set $D \subset \mathfrak{h}$ if, from the condition $\varphi \in D$, $\varphi \gg \chi$ there follows $\varphi = \chi$.

LEMMA 7. Let C_1 and C_2 be two Weil chambers, $\chi \in C_1$, $\chi' \in \overline{C}_1$, $\psi \in C_2$, $\psi' \in \overline{C}_2$, $\chi'-\chi \in \mathfrak{h}^*_Z$, $\psi'-\psi \in \mathfrak{h}^*_Z$, $\chi \sim \psi$, $\chi' \sim \psi'$ and let the roots $\gamma_1, \ldots, \gamma_k \in \Delta_+$ satisfy Condition (A) for the pair (χ, ψ). Then they satisfy Condition (A) for the pair (χ', ψ') too.

Proof. Point 2) of Condition (A) is obvious, Point 3) follows from the fact that for any γ the sign of $\langle \mathrm{Re}\ \chi, \gamma\rangle$ is constant in a Weil chamber, and Point 1) follows from the fact that $\chi' \sim \chi'_k = \sigma_{\gamma_k} \ldots \sigma_{\gamma_1}\psi'$, $\chi' \in \overline{C}_1$, $\chi'_k \in \overline{C}_1$, whence $\mathrm{Re}\ \chi' = \mathrm{Re}\ \chi'_k$. Since $\chi'-\chi'_k \in \mathfrak{h}^*_Z$, $\mathrm{Im}\ \chi' = \mathrm{Im}\ \chi'_k$, which is to say $\chi' = \chi'_k$.

Lemma 7 shows us how to extend Condition (A) for one pair (χ, ψ) to its confirmation for another pair (χ', ψ').

LEMMA 8. Let C_1 and C_2 be adjacent Weil chambers, and let $\varphi_1 \in C_1$, $\varphi_2 \in C_2$ and $\gamma \in \Delta_+$ be such that $\langle \mathrm{Re}\ \varphi_1, \gamma\rangle < 0$, $\langle \mathrm{Re}\ \varphi_2, \gamma\rangle > 0$. Then $\sigma_\gamma C_1 = C_2$.

Proof. Suppose the assertion of the lemma does not hold. Then we can choose $\varphi_1 \in C_1 \cap \mathfrak{h}^*_R$ and $\varphi'_2 \in C_2 \cap \mathfrak{h}^*_R$ such that $|\varphi'_2 - \varphi'_1|$ is much smaller than $\langle\varphi'_2, \gamma\rangle$. Since the sign of $\langle \mathrm{Re}\ \varphi, \gamma\rangle$ is constant in a Weil chamber, the inequalities $\langle\varphi'_1, \gamma\rangle < 0$, $\langle\varphi'_2, \gamma\rangle > 0$, hold, whence $\langle\varphi'_2 - \varphi'_1, \gamma\rangle > \langle\varphi'_2, \gamma\rangle$, which leads to a contradiction.

Proof of the Theorem. Suppose $\chi \in \mathfrak{h}^*$. Denote by $Y(\chi)$ the following assertion:

$Y(\chi)$: for every $\psi \in \mathfrak{h}^*$ for which $M\chi \subset M\psi$, there exists a sequence $\gamma_1, \ldots, \gamma_k \in \Delta_+$, satisfying Condition (A) for the pair (χ, ψ).

We proceed to prove $Y(\chi)$ for all χ. We first note that $Y(\chi)$ is true for all $\chi \in \overline{C}^+$, since in this case there follows from $\psi \gg \chi$ and $\psi \sim \chi$ the fact that $\psi = \chi$. The arbitrary case is handled by the following steps.

Step 1. Let C be a Weil chamber, let $\chi \in C$ and let $\gamma \in \Delta_+$ be a root such that $\langle\gamma, \mathrm{Re}\ \chi\rangle < 0$; let $\sigma_\gamma C$ be a Weil chamber adjoint to C. Let F be a finite dimensional $U(\mathfrak{g})$-module for which $\chi + P(F) \subset \overline{C} \cup \sigma_\gamma\overline{C}$ and $\lambda \in P(F)$. Then, if $\chi + \lambda \in \sigma_\gamma C$, there follows from $Y(\chi + \lambda)$ the truth of $Y(\chi)$.

Proof. Suppose $M\chi \subset M\psi$ and $w \in W$ are such that $\psi = w\chi$. It follows from the hypothesis that there exist no more than two weights μ of the module F such that $\chi + \lambda \sim \psi + \mu$. Suppose μ_1 is that one of these weights for which $\psi + \mu_1 \in w\sigma_\gamma C$ (it always exists) and μ_2 is the one for which $\psi + \mu_2 \in wC$ (it may not exist). By Lemma 6, either $M\chi+\lambda \subset M\psi+\mu_1$, or $M\chi+\lambda \subset M\psi+\mu_2$. We give detailed consideration to both cases.

1. Suppose $M\chi+\lambda \subset M\psi+\mu_1$. It follows from $Y(\chi + \lambda)$ that there is a sequence $\gamma_1, \ldots, \gamma_k$ of elements of Δ_+, satisfying Condition (A) for the pair $(\chi + \lambda, \psi + \mu_1) = (\overline{\chi}, \overline{\psi})$. We shall construct a sequence satisfying Condition (A) for the pair (χ, ψ). Put $\chi_i = \sigma_{\gamma_i}, \ldots, \sigma_{\gamma_1}\psi$, $\overline{\chi}_i = \sigma_{\gamma_i}, \ldots, \sigma_{\gamma_1}\overline{\psi}$. We remark that χ_i and $\overline{\chi}_i$ are always different on the element $\mathfrak{h}^*_Z$. Hence for any i, either $\chi_i \ll \chi_{i-1}$, or $\chi_i \gg \chi_{i-1}$. If $\chi_i \ll \chi_{i-1}$ for every i, then the sequence $\gamma_1, \ldots, \gamma_k$ satisfies Condition (A) for the pair (χ, ψ). In the contrary case we denote by i_0 the first number for which $\chi_{i_0} \gg \chi_{i_0-1}$. We show that the sequence $\gamma_1, \ldots, \gamma_{i_0-1}, \gamma_{i_0+1}, \ldots, \gamma_k, \gamma$ satisfies Condition (A) for the pair (χ, ψ). It follows from Lemma 8 that the elements χ_{i_0-1} and $\overline{\chi}_{i_0}$ lie in the same Weil chamber. This means that $\sigma_\gamma\sigma_{\gamma_k} \ldots \sigma_{\gamma_{i_0+1}} \cdot \sigma_{\gamma_{i_0-1}} \ldots \sigma_{\gamma_1} = w^{-1}$, and Point 1) of Condition (A) is fulfilled. Moreover, $\sigma_\gamma = \sigma_{\gamma_k} \ldots \sigma_{\gamma_{i_0+1}}\sigma_{\gamma_{i_0}}\sigma_{\gamma_{i_0+1}} \ldots \sigma_{\gamma_k}$, i.e., $\gamma = \pm\sigma_{\gamma_k} \ldots \sigma_{\gamma_{i_0+1}}\gamma_{i_0}$. Hence $2\langle\chi, \gamma\rangle/\langle\gamma, \gamma\rangle = \pm 2\langle\chi_{i_0-1}, \gamma_{i_0}\rangle/\langle\gamma_{i_0}, \gamma_{i_0}\rangle \in Z$. Fulfillment of Point 2) of Condition (A) for the remaining elements of the sequence is obvious. Fulfillment of Point 3) of Condition (A) follows at once from Lemma 8 and Lemma 7.

2. Now suppose that $M\chi+\lambda \subset M\psi+\mu_2$ and that $\gamma_1, \ldots, \gamma_k$ is a sequence of roots satisfying Condition (A) for the pair $(\chi + \lambda, \psi + \mu_2)$. From Lemma 7 and from the condition $\langle \mathrm{Re}\ \chi, \gamma\rangle < 0$ it at once follows that the sequence $\gamma_1, \ldots, \gamma_k, \gamma$ satisfies this condition for the pair (χ, ψ).

Step 2. Let C be a Weil chamber, let $\chi \in C$, and let F be a finite-dimensional module such that $\chi + P(F) \subset C$, and $\lambda \in P(F)$. Then $Y(\chi)$ follows from $Y(\chi + \lambda)$.

The proof of this assertion is analogous to the one conducted above, but simpler.

We note that applying Steps 1 and 2 in the required order, we can show $Y(\chi)$ for every $\chi \in \mathfrak{h}^*$, "sufficiently far" from the set Ξ. More precisely, d being distance in $\mathfrak{h}^*_R$, suppose that $\chi \in \mathfrak{h}^*$ is a weight such that $d(\mathrm{Re}\ \chi, \Xi) > 3|\rho|$. Then we can construct a sequence $\chi = \chi_0, \chi_1, \ldots, \chi_k$ of elements of $\mathfrak{h}^*$ such that $d(\mathrm{Re}\ \chi_i, \Xi) > 2|\rho|$, $\chi_i - \chi_{i+1} \in \mathfrak{h}^*_Z$, $\chi_k \in C^+$ and such that for every i one of the following two conditions is fulfilled:

1) χ_i and χ_{i+1} lie in two adjacent Weil chambers C and $\sigma_\gamma C$, $\gamma \in \Delta_+$, and moreover $<\mathrm{Re}\,\chi_i, \gamma> < 0$ and $|\chi_{i+1} - \chi_i|$ is a much smaller distance from χ_i than all other Weil chambers;

2) χ_i and χ_{i+1} lie in the same Weil chamber and $|\chi_{i+1} - \chi_i| < 2|\rho|$.

Considering the finite-dimensional modules F_i of least weights $\chi_{i+1} - \chi_i$ and applying respectively Steps 1 and 2, we extend $Y(\chi)$ to truth of the assertion $Y(\chi_k)$. Here the fact is used that the length of all weights F_i does not surpass $|\chi_{i+1} - \chi_i|$ (see [5]).

The case of arbitrary χ is analyzed with the help of the following considerations.

Step 3. Let $\chi \in \mathfrak{h}^*$, $\mathrm{Re}\,\chi \neq 0$. For each $a > 0$ we denote by D_a the cone in $\mathfrak{h}^*_{\mathbb{R}}$ consisting of those nonzero φ such that the angle between φ and $\mathrm{Re}\,\chi$ is less than a. We can choose a small enough that the Weil chamber C intersects D_a if and only if $\chi \in \overline{C}$. We choose $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, such that 1) $d\,(\mathrm{Re}(\chi + \lambda), \Xi) > 3\,|\rho|$, 2) $\mathrm{Re}(\chi + \lambda)$ is maximal in the set $W(\mathrm{Re}(\chi + \lambda)) \cap D_a$.

We now show that $Y(\chi)$ follows from $Y(\chi + \lambda)$.

Suppose F is a finite-dimensional module with least weight λ. We show that for all weights $\chi + \lambda$ the premises of Lemma 6 are fulfilled. Suppose the weight $\mu \in P(F)$ is such that $\mathrm{Re}\,(\chi + \lambda) \sim \mathrm{Re}\,(\chi + \mu)$. Since $|\mathrm{Re}\,(\chi + \lambda)| = |\mathrm{Re}\,(\chi + \mu)|$ and $|\lambda| \geq |\mu|$, it follows that $\mathrm{Re}(\chi + \mu) \in D_a$, which means the inequality $\mathrm{Re}(\chi + \lambda) \ll \mathrm{Re}(\chi + \mu)$ cannot be fulfilled. Hence we can take Lemma 6 and the assertion that if $M_\chi \subset M_\psi$, then $M_{\chi+\lambda} \subset M_{\psi+\mu}$ for some $\mu \in P(F)$. Reasoning analogous to that adduced above shows that the angle between $\mathrm{Re}\,(\psi + \mu)$ and $\mathrm{Re}\,\psi$ is not greater than a and hence $\psi + \mu$ and ψ lie in the same Weil chamber. We may now apply Lemma 7 which completes the proof of Step 3 and of the whole of Theorem 2.

§5. Sufficiency of Condition (A)

THEOREM 3 (see [11]). Let $\chi, \psi \in \mathfrak{h}^*$. If there exists a sequence of roots $\gamma_1, \ldots, \gamma_k \in \Delta_+$, satisfying Condition (A) for the pair (χ, ψ), then $M_\chi \subset M_\psi$.

Proof.* It suffices to consider the case where the sequence consists of a single root $\gamma \in \Delta_+$.

LEMMA 9. If $\chi, \psi \in \mathfrak{h}^*$ and $\alpha \in \Sigma$ are such that $M_\chi \subset M_\psi$ and $M_{\sigma_\alpha \chi} \subset M_\chi$, then $M_{\sigma_\alpha \chi} \subset M_{\sigma_\alpha \psi}$.

Proof. It follows immediately from the conditions of the lemma that $\psi - \sigma_\alpha \psi = n\alpha$, where $n \in \mathbb{Z}$. By Lemma 2 it is enough to consider the case $n > 0$, where $M_{\sigma_\alpha \psi} \subset M_\psi$. Suppose $\bar{f}_\psi$ is the image of f_ψ in $M_\psi / M_{\sigma_\alpha \psi}$. Then $E^n_{-\alpha} \bar{f}_\psi = 0$. The image of the element f_χ in $M_\psi / M_{\sigma_\alpha \psi}$ has the form $\bar{f}_\chi = X \bar{f}_\psi$, where $X \in U(\mathfrak{g})$. The element $E^k_{-\alpha} X$ can be written in the form $E^k_{-\alpha} X = X_1 E^{k_1}_{-\alpha}$, where k_1 increases unboundedly with k. Hence $E^k_{-\alpha} \bar{f}_\chi = 0$ for large k. Since $E_\alpha E^k_{-\alpha} f_\chi = k(n' - k) E^{k-1}_{-\alpha} f_\chi$ (where $n' = 2<\chi, \alpha>/<\alpha, \alpha> \in \mathbb{Z}$), it follows that $E^{n'}_{-\alpha} \bar{f}_\chi = 0$, i.e., that $M_{\sigma_\alpha \chi} \subset M_{\sigma_\alpha \psi}$.

LEMMA 10. The assertion of the theorem is true for $\chi \in \mathfrak{h}^*_{\mathbb{Z}}$, $\psi = \sigma_\gamma \chi$.

Proof. We can suppose that $\chi \neq \psi$. We choose elements $\alpha_1, \ldots, \alpha_k \in \Sigma$, such that in the sequence of weights $\chi_i = \sigma_{\alpha_i} \ldots \sigma_{\alpha_1} \chi$ the relations $\chi_{i+1} \gg \chi_i$, $\chi_k \in \overline{C}^+$ are fulfilled. Suppose $\psi_i = \sigma_{\alpha_i} \ldots \sigma_{\alpha_1} \psi$. Each ψ_i is obtained from χ_i by some mapping which we denote by σ_{γ_i}, $\gamma_i \in \Delta_+$. By Lemma 9 it is enough to show that $M_{\chi_i} \subset M_{\psi_i}$ for some i.

We take as i the last index for which $\psi_i \gg \chi_i$ ($i < k$, since $\chi_k \gg \psi_k$ and $\chi_k \neq \psi_k$). Then $\psi_i - \chi_i = n\gamma_i$, where $n > 0$, and $\psi_{i+1} - \chi_{i+1} = \sigma_{\alpha_{i+1}}(\psi_i - \chi_i) = n\sigma_{\alpha_{i+1}}\gamma_i \ll 0$, which is only possible if $\alpha_{i+1} = \gamma_i$, but then $M_{\chi_i} \subset M_{\psi_i}$ by Lemma 2.

We now prove Theorem 3. Let $\gamma \in \Delta_+$, $\chi, \psi \in \mathfrak{h}^*$ be such that $\chi = \sigma_\gamma \psi$, $\psi - \chi = n\gamma$, where $n \in \mathbb{Z}$. Denote by $\mathfrak{n}_{n\gamma}$ the finite dimensional subspace in $U(\mathfrak{n}_-)$, consisting of elements of weight $-n\gamma$.

The module M_ψ, considered as a $U(\mathfrak{n}_-)$-module, is a free module with image f_ψ. Hence for the proof of the theorem we need to find a nonnull element $X \in \mathfrak{n}_{n\gamma}$ such that $E_\alpha X f_\psi = 0$ for all $\alpha \in \Sigma$. The equations written give a system of linear homogeneous equations in the space $\mathfrak{n}_{n\gamma}$, whose coefficients depend linearly on ψ. Consider the hyperplane S in $\mathfrak{h}^*$ consisting of those weights φ such that $2<\varphi, \gamma>/<\gamma, \gamma> = n$. By Lemma 10 the given system has a nontrivial solution for all $\varphi \in S \cap \mathfrak{h}^*_{\mathbb{Z}}$. If $\gamma = w\alpha$, $w \in W$, $\alpha \in \Sigma$, then

*This proof differs from that in [11]. In particular it does not use enumerations of simple Lie algebras.

$S \cap \mathfrak{h}^*_Z = \left\{ w\left(nf^{(\alpha)} + \sum_{\beta \in \Sigma \setminus \alpha} n_\beta f^{(\beta)} \right),\ n_\beta \in Z \right\}$, i.e., it constitutes an (n−1)-dimensional lattice in S. Hence our system has a nontrivial solution for all $\varphi \in S$ and in particular for $\varphi = \psi$. This completely proves Theorem 3.

Theorems 2 and 3 permit introduction of the following partial ordering into the group W.

Definition 2. Let $\chi_0 \in C^+ \cap \mathfrak{h}^*_Z$. We put $w_1 < w_2$ for elements $w_1, w_2 \in W$, if $M_{w_1\chi_0} \subset M_{w_2\chi_0}$.

THEOREM 4. 1) The ordering introduced in Theorem 4 does not depend on the choice of $\chi_0 \in C^+ \cap \mathfrak{h}^*_Z$.

2) Suppose $w_1, w_2 \in W$. The inequality $w_1 < w_2$ is fulfilled if and only if there exists a sequence of images $\sigma_{\gamma_1}, \ldots, \sigma_{\gamma_k} \in W$, such that $w_1 = \sigma_{\gamma_k} \ldots \sigma_{\gamma_1} w_2$ and $l(\sigma_{\gamma_{i+1}} \ldots \sigma_{\gamma_1} w_2) > l(\sigma_{\gamma_i} \ldots \sigma_{\gamma_1} w_2)$ for every i. Here $l(w)$ is the length of the element $w \in W$.

The proof of the first part of the theorem follows immediately from Theorems 2 and 3 and Lemma 7. For proving the second part we need the following result on constructing the system of roots (see [9]). Suppose $w \in W$. Then $l(w)$ coincides with the number of those roots $\gamma \in \Delta_+$, for which $w_\gamma \in -\Delta_+$.

As remarked in [2], the adduced ordering can be given the following geometrical meaning. Let G be a simply-connected Lie group with Lie algebra $\mathfrak{g}$, let $B \subset G$ be the Borel subgroup corresponding to the subalgebra generated by $\mathfrak{h}$ and E_γ, $\gamma \in \Delta_+$. Suppose moreover that $P = G/B$ is the fundamental projective space of the group G. Each element $w \in W$ we put in correspondence with the subspace $P_w = BwB \subset P$. These subspaces are the key to the splitting of P (see [10]). Here $w_1 < w_2$ if and only if $P_{w_1} \supset P_{w_2}$. We also note the connection of this ordering with the Paley-Wiener Theorem for complex semisimple Lie groups G ([6]).

§6. Appendix. Multiplicity of the Weight of a Finite-Dimensional Representation

We introduce here a simple algebraic proof of the Kostant formula for the multiplicity of a weight of a finite-dimensional representation (see [7]). To do this we shall use only the definition of the module M_χ (Def. 1) and Lemmas 1 and 3. To make it possible to use Lemma 3 we note that in [8] there was introduced a proof of the Harish-Chandra Theorem on the eigenvalues of a Laplace operator without using the formula of H. Weil for characters.

Let $\tilde{\mathscr{E}}$ be the space of all functions in $\mathfrak{h}^*$. For $u \in \tilde{\mathscr{E}}$ we place $\operatorname{supp} u = \{\chi \in \mathfrak{h}^* | u(\chi) = 0\}$. The group W acts in $\tilde{\mathscr{E}}$ by the formula $(wu)(\chi) = u(w^{-1}\chi)$. Let $\mathscr{E} \subset \tilde{\mathscr{E}}$ be a subspace consisting of all functions u for which supp u is contained in the union of a finite number of sets of the form $\left\{\chi - \sum_{\gamma \in \Delta_+} n_\gamma \gamma \,|\, n_\gamma \in Z,\ n_\gamma \geqslant 0\right\}$.

The space $\mathscr{E}$ is a commutative algebra with respect to the convolution operation $u_1 * u_2(\chi) = \sum_{\psi \in \mathfrak{h}^*} u_1(\chi - \psi)\, u_2(\psi)$ (in this sum only a finite number of terms are different from zero). Let $\delta_\chi \in \mathscr{E}$ be a function such that $\delta_\chi(\chi) = 1$, $\delta_\chi(\psi) = 0$ for $\psi \neq \chi$. Then δ_0 is the unit of the ring $\mathscr{E}$.

Definition 3. 1) $Q(\chi)$ is the number of families $\{n_\gamma\}_{\gamma \in \Delta_+}$, $n_\gamma \in Z$, $n_\gamma \geq 0$, such that $\sum_{\gamma \in \Delta_+} n_\gamma \gamma = -\chi$.

2) $L = \prod_{\gamma \in \Delta_+} \left(\delta_{\frac{\gamma}{2}} - \delta_{-\frac{\gamma}{2}} \right)$.

We shall call the function $Q(\chi)$ the Kostant function.*

LEMMA 11. $L * Q * \delta_{-\rho} = \delta_0$.

Proof. Let $a_\gamma = \delta_0 + \delta_{-\gamma} + \delta_{-2\gamma} + \ldots \in \mathscr{E}$. Then clearly $Q = \prod_{\gamma \in \Delta_+} a_\gamma$ и $(\delta_0 - \delta_\gamma) * a_\gamma = \delta_0$. The lemma at once follows from the fact that $L = \prod_{\gamma \in \Delta_+} (\delta_0 - \delta_{-\gamma}) * \delta_\rho$.

LEMMA 12. $wL = (-1)^{l(w)} L$ for $w \in W$.

*Our definition of $Q(\chi)$ differs from that provided in [7] in replacement of χ by $-\chi$.

Proof. It suffices to verify that $\sigma_\alpha L = -L$ for $\alpha \in \Sigma$. Since σ_α transposes elements of the set $\Delta_+\backslash\alpha$ and carries α to $-\alpha$, $\sigma_\alpha L = \left(\delta_{-\frac{\alpha}{2}} - \delta_{\frac{\alpha}{2}}\right) * \prod\limits_{\gamma\in\Delta_+\backslash\alpha} \left(\delta_{\frac{\gamma}{2}} - \delta_{-\frac{\gamma}{2}}\right) = -L.$

Definition 4. Suppose the $U(\mathfrak{g})$-module M is a direct sum of weights of spaces V_χ of weight χ and $\dim V_\chi < \infty$ for all $\chi \in \mathfrak{h}^*$. We call the function $\pi_M \in \mathscr{E}$, given by the formula $\pi_M(\chi) = \dim V_\chi$ by the name character of M.

LEMMA 13. $\pi_{M_\chi}(\psi) = Q(\psi - \chi + \rho)$.

Proof. Suppose $\gamma_1, \ldots, \gamma_s$ is an arbitrary indexing of the roots $\gamma \in \Delta_+$. From the theorem of Poincaré-Birkhoff-Witt (see [4], Ch. I) it follows from Lemma 1 that the elements $E^{n_1}_{-\gamma_1} \ldots E^{n_s}_{-\gamma_s} f_\chi$ ($n_i \in Z$, $n_i \geq 0$) form a basis in M_χ, and Lemma 13 follows from the definition of Q.

COROLLARY. $L * \pi_{M_\chi} = \delta_\chi$.

LEMMA 14. Let $U(\mathfrak{g})$-module M and element $\chi_0 \in \bar{C}^+$ be such that 1) there is a basis in M of weight vectors; 2) $z \cdot x = \theta_{\chi_0}(z)x$ for every $x \in M$, $z \in Z(\mathfrak{g})$; 3) π_M exists and $\pi_M \in \mathscr{E}$.

Let $D_M = \{\chi \in \mathfrak{h}^* \mid \chi \sim \chi_0,\ \chi \ll \psi + \rho$ for some $\psi \in \operatorname{supp} \pi_M\}$.

Then $\pi_M = \sum\limits_{\chi\in D_M} c_\chi \pi_{M_\chi},\ c_\chi \in Z.$

Proof. In view of Condition 3) there exists in supp π_M at least one maximal element χ. Then $E_\alpha \chi = 0$ for any $x \in V_\chi$, $\alpha \in \Sigma$. Hence for any $z \in Z(\mathfrak{g})$, $v \in V_\chi$ we have $z \cdot x = \theta_{\chi+\rho}(z)x$, and hence, $\chi + \rho \sim \chi_0$, i.e., $\chi + \rho \in D_M$. Let $k = \dim V_\chi$. By definition of M_χ it follows that we can construct a mapping $\varkappa$: $(M_{\chi+\rho})^k \to M$, translating the generators $f_{\chi+\rho}$ to k linearly independent elements V_χ. Suppose L and N are the kernel and co-kernel of $\varkappa$. It follows from the exact sequence $0 \to L \to (M_{\chi+\rho})^k \xrightarrow{\varkappa} M \to N \to 0$ that $\pi_M = \pi_L - \pi_N - k\pi_{M_{\chi+\rho}}$. The moduli L and M satisfy the hypotheses of Lemma 14 and in addition D_L and D_N lie in $D_M \backslash (\chi + \rho)$. Induction on the number of elements of D_M concludes the proof of Lemma 14.

Remark. The foregoing reasoning establishes that the moduli M_χ generate the group of Grothendieck category of the moduli M satisfying the conditions of Lemma 14.

COROLLARY. Under the hypotheses of Lemma 14, $L * \pi_M \subset D_M$.

Suppose now that F is a continuous finite-dimensional $U(\mathfrak{g})$-module with highest weight λ. Then π_F is invariant with respect to W (see [5]). By Lemma 12, $w(L * \pi_F) = (-1)^{l(w)} (L * \pi_F)$. By the Corollary to Lemma 14, $L * \pi_F = \sum c_w \delta_{w(\lambda+\rho)}$. Since $\dim V_\lambda = 1$ in the module F, it follows that $c_e = 1$ (see the proof of Lemma 14), hence $L * \pi_F = \sum\limits_{w\in W} (-1)^{l(w)} \delta_{w(\lambda+\rho)}$. Applying Lemma 11, we get $\pi_F = Q * \delta_{-\rho} \sum\limits_{w\in W} (-1)^{l(w)} \delta_{w(\lambda+\rho)}$. In conclusion, we get the following theorem.

THEOREM 5 (Kostant formula). Let F be a finite-dimensional continuous representation of $\mathfrak{g}$ with highest weight λ. Then

$$\pi_F(\mu) = \sum_{w\in W} (-1)^{l(w)} Q(\mu + \rho - w(\lambda + \rho)).$$

Note. In conclusion we append an example of a submodule M in M_χ which does not have the form $\cup M_{\chi_i}$. Let $\mathfrak{g}$ be the Lie algebra of the group $SL(4, C)$, E_{ik} ($i \neq k$), and let $E_{ii} - E_{jj}$ be a basis of $\mathfrak{g}$ with E_{ik} ($i < k$) corresponding to roots from Δ_+. We consider the module M_χ, wherein all weights χ are given by the equations $\chi(E_{11} - E_{12}) = \chi(E_{33} - E_{44}) = 0$, $\chi(E_{22} - E_{33}) = 1$. Suppose $f_1 = E_{32} f_\chi$ and $\tilde{M} = U(\mathfrak{g})(f_1)$. It follows from Theorem 2 that if $E_{ik}a = 0$ for $i < k$, then $a = a_1 + c f_\chi$, where $a_1 \in \tilde{M}$, $c \in C$. Put $x = E_{42}E_{21} f_\chi + E_{43}E_{31} f_\chi$. Then $x \notin \tilde{M}$ and $E_{ik}x \in \tilde{M}$ for $i < k$. Hence the submodule generated by x and $\tilde{M}$ is a proper submodule of M_χ not having the form $\cup M_\chi$.

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