§1. Introduction

Let \( \mathfrak{g} \) be a complex semisimple Lie algebra. Despite the fact that finite representations of the algebra \( \mathfrak{g} \) were one of the very first objects of study, there exist even simpler representations. Here a host of properties of finite representations are essentially consequences of analogous properties of these simpler modules. Categories of these modules, such as the so-called \( \mathcal{O} \)-modules, were introduced by the authors, and their definition was given in [1]. In this work there will be studied elementary objects of that character, the modules \( M_\chi \) (see [1]). This work can be read independently of [1]. Study of the \( M_\chi \) module was begun in Verma’s work* which obtained a series of deep results on \( M_\chi \) modules.

The \( M_\chi \) modules (for a precise definition, see below) are of interest since they are the simplest modules generated by a single vector of highest weight \( \chi - \rho \), where \( \rho \) as usual denotes a half-sum of positive roots. All other modules generated by a single vector of highest weight, including also irreducible finite representations, are factor modules of the modules \( M_\chi \). In the present work there is a complete description of the categories of the modules \( M_\chi \). This is to say, on the basis of the Verma Theorem, \( \text{Hom}(M_{\chi_1}, M_{\chi_2}) \) is either 0 or \( \mathbb{C} \). The question arises, for such \( \chi_1, \chi_2 \) pairs, whether there exists a nontrivial mapping of \( M_{\chi_1} \) to \( M_{\chi_2} \). The main result of our article is the establishment of necessary conditions for the existence of such a mapping (Theorem 2). The proof of Theorem 2 is fairly involved; the authors have been unable to come upon an easier proof. The form of the hypothesis in [2], for Theorem 2, was retained.†

The results obtained on \( M_\chi \) modules permit one to understand from a single point of view the greater part of the classical results in complex semisimple Lie algebras, in particular the Kostant theorem or the equivalence to it of the Weyl formula for characters, the Borel-Weil Theorem, etc.

§2. Notation and Background †

\( \mathfrak{g} \) is a complex semisimple Lie algebra of rank \( r \), \( \mathfrak{h} \) is a Cartan subalgebra of the algebra \( \mathfrak{g} \):

\( \Delta \) is a system of roots of \( \mathfrak{g} \) relative to \( \mathfrak{h} \) with a fixed ordering; \( \Sigma \) is the sum of the simple roots; \( \Delta_+ \)

is the set of positive roots, \( \rho = \frac{1}{2} \sum_{\gamma \in \Delta_+} \gamma \);

\( E_\gamma \) is the root vector corresponding to the root \( \gamma \in \Delta \), where \( \gamma ([E_\gamma, E_{-\gamma}]) = 2 \). It is known that

\[ [E_\alpha, E_{-\beta}] = 0 \text{ for } \alpha, \beta \in \Sigma, \alpha \neq \beta; \]

*The authors wish to express their gratitude to Prof. J. Dixmier for bringing the exceptionally interesting work [2] to their notice.
†We note that as formulated in [2] the theorem asserting that every submodule \( M \) in \( M_\chi \) has the form \( M = \bigcup M_\chi \) is false. Counterexamples can be constructed, in particular for Lie algebra of the group \( SL(4, \mathbb{C}) \); see our last page.
†For precise definitions and proofs, see also [4] and [5].

\( n_+ \) is the subalgebra of \( g \), natural in \( E_\gamma, \gamma \in \Delta^+ \);
U(\( g \)), U(\( n_+ \)) are enveloping algebras of \( g \) and \( n_+ \), respectively; \( Z(\( g \)) \) is the center of U(\( g \));
U(\( g \)) (\( x_1, \ldots, x_k \)) for elements \( x_1, \ldots, x_k \) from the U(\( g \))-module M denotes the U(\( g \))-submodule in M generated by \( x_1, \ldots, x_k \). The meaning of U(\( n_+ \)) (\( x_1, \ldots, x_k \)) is analogous;
\( \mathfrak{h}^* \) is the space dual to \( \mathfrak{h} \); \( \mathfrak{h}^*_R \) is the real linear subspace in \( \mathfrak{h}^* \), natural for all roots \( \gamma \in \Delta \); \( \mathfrak{h}^* = \mathfrak{h}^*_R + i\mathfrak{h}^*_R \);
\( \langle \cdot, \cdot \rangle \) is the scalar product in \( \mathfrak{h}^* \) constructed according to the Killing form of the algebra \( g \), \(|\cdot|\) is the corresponding norm in \( \mathfrak{h}^*_R \). We note that \( \chi (\langle E_\gamma, E_{-\gamma} \rangle) = 2\langle \chi, \gamma \rangle / \langle \gamma, \gamma \rangle \), \( \chi \in \mathfrak{h}^* \), \( \gamma \in \Delta^+ \);
\( \mathfrak{h}^*_R \) is the lattice in \( \mathfrak{h}^* \) constructed from those \( \chi \) for which \( 2\langle \chi, \gamma \rangle / \langle \gamma, \gamma \rangle \in \mathbb{Z} \) for all \( \gamma \in \Delta \);
\( f(\alpha) \) is the fundamental weight corresponding to the root \( \alpha \in \Sigma \), i.e., \( 2\langle f(\alpha), \alpha \rangle / \langle \alpha, \alpha \rangle = 1 \), \( \langle f(\alpha), \beta \rangle = 0 \) for \( \beta \in \Sigma \backslash \alpha \). The weights \( f(\alpha), \alpha \in \Sigma \), generate \( \mathfrak{h}^*_Z \);
\( x_1 \gg x_2 \) for \( x_1, x_2 \in \mathfrak{h}^* \) signifies that \( x_1 - x_2 = \sum_{\alpha \in \Sigma} n_\alpha \), \( n_\alpha \in \mathbb{Z}, n_\alpha > 0 \);
\( W \) is the Weyl group of the algebra \( g \);
\( \sigma_\gamma \in W \) is the reflection corresponding to the root \( \gamma \in \Delta \), i.e., \( \sigma_\gamma x = x - 2\langle x, \gamma \rangle \gamma \), \( \gamma \in \mathfrak{h}^* \). We recall that if \( \alpha \in \Sigma \), \( \gamma \in \Delta^+ \), \( \gamma \neq \alpha \), then \( \sigma_\alpha \gamma \in \Delta^+ \);
\( l(w) \) is the length of the element \( w \in W \), i.e., the least number of factors in a representation \( w = \sigma_{\alpha_1}, \ldots, \sigma_{\alpha_k}a_1 \in \Sigma \);
\( x_1 \sim x_2 \) for \( x_1, x_2 \in \mathfrak{h}^* \) signifies the existence of an element \( w \in W \) for which \( x_1 = wx_2 \);
\( \Sigma \subseteq \mathfrak{h}^* \) is the union of hyperplanes \( \langle \text{Re } \chi, \gamma \rangle = 0 \) for all \( \gamma \in \Delta \); Weyl chambers are connected components \( \mathfrak{h}^* \backslash \Sigma; \overline{C} \) is the closure of the Weyl chamber \( C \); \( C^+ \) is the Weyl chamber containing \( \rho \). The group \( W \) acts simply transitively on the set of Weyl chambers. It can be shown that if \( \chi \in \overline{C} \) and \( w_\chi \in \mathbb{C} \), then \( \text{Re } \chi = \text{Re } w_\chi \). Two Weyl chambers \( C_1 \) and \( C_2 \) are called adjacent if \( \dim_R(\overline{C_1} \cap \overline{C_2}) = 2r-1 \). There then exists an element \( \gamma \in \Delta^+ \), such that \( \sigma_\gamma C_1 = C_2 \);
\( P(\( F \)) \subseteq \mathfrak{h}^*_Z \) is the set of weights of the finite-dimensional module \( F \).

\section*{The Modules M_\chi}

**Definition 1.** Suppose \( \chi \in \mathfrak{h}^* \). We denote by \( J_\chi \) the left ideal (in U(\( g \))), generated by the elements \( E_\gamma, \gamma \in \Delta^+, \) and \( H - \chi(H) + \rho(H), H \in \mathfrak{h} \). We further put \( M_\chi = U(\mathfrak{g})/J_\chi \).

The following readily verified lemma describes the simplest properties of the modules \( M_\chi \).

**Lemma 1.** 1) A U(\( g \))-module \( M_\chi \) is generated by one generator \( f_\chi \) of weight \( -\rho \) such that \( E_\alpha f_\chi = 0 \) for \( \alpha \in \Sigma \); 2) \( U(\mathfrak{n}_+) \) acts on \( M_\chi \) without zero divisors; 3) \( U(\mathfrak{n}_+) (f_\chi) = M_\chi \).

Besides, Properties 1) and 2) uniquely characterize the module \( M_\chi \).

**Theorem 1.** (Verma, [2]). Let \( \chi, \psi \in \mathfrak{h}^* \). Two cases are possible: 1) \( \text{Hom}_U(\mathfrak{g}) (M_\chi, M_\psi) = 0 \), 2) \( \text{Hom}_U(\mathfrak{g}) (M_\chi, M_\psi) = \mathbb{C} \), and every nontrivial homomorphism \( M_\chi \to M_\psi \) is an imbedding.

In the second case we can consider \( M_\chi \) as a submodule of \( M_\psi \).

We shall study in the present note for what \( \chi, \psi \in \mathfrak{h}^* \) the module \( M_\chi \) can be imbedded in \( M_\psi \). This is obviously possible only in case \( \chi \ll \psi \).

**Lemma 2.** Let \( \alpha \in \Sigma, \psi = \alpha \chi \) and \( \psi - \chi = n \alpha \), where \( n \in \mathbb{Z}, n \geq 0 \). Then \( M_\chi \subseteq M_\psi \).

**Proof.** It is easy to show that \( E_\alpha (E_k f_\psi) = k(n-k) E_k^{-1} f_\psi \). Put \( f = E_{-\alpha} f_\psi \). Then \( f \) has weight \( \chi - \rho \), \( E_{-\alpha} f = 0 \) and, for any \( \beta \in \Sigma, \beta \neq \alpha \), \( E_\beta f = E_\beta E_{-\alpha} f_\psi = E_{-\alpha} E_\beta f_\psi = 0 \). By Lemma 1, the submodule in \( M_\psi \), natural on \( f \) is isomorphic to \( M_\chi \), which demonstrates the lemma.

Suppose \( z \in Z(\mathfrak{g}) \). Then \( z \cdot f_\chi \) is proportional to \( f_\chi \), i.e., \( z \cdot f_\chi = \theta_\chi(z) f_\chi, \theta_\chi(z) \in \mathbb{C} \). Since \( f_\chi \) is a generator in \( M_\chi \), \( z \cdot x = \theta_\chi(z) x \) for all \( x \in M_\chi \). In this way we get the homomorphism \( \theta_\chi: Z(\mathfrak{g}) \to \mathbb{C} \).
LEMMA 3. $\theta x_1 = \theta x_2$ if and only if $x_1 \sim x_2$.

The proof of Lemma 3 follows readily from a theorem of Harish-Chandra on eigenvalues of Laplace operators (see [3] and [8]).

COROLLARY. If $M_X \subset M_\Psi$ then $\chi \sim \Psi$.

Central to this work is Theorem 2 giving, along with Theorem 3, necessary and sufficient conditions that $M_X$ is imbedded in $M_\Psi$.

§ 4. Necessity of Condition (A)

Suppose $\chi, \Psi \in \mathfrak{g}^*$, $\gamma_1, \gamma_2, \ldots, \gamma_k \in \Delta_+$. We shall say that the sequence $\gamma_1, \ldots, \gamma_k$ satisfies Condition (A) for the pair $(\chi, \Psi)$, if

1) $\chi = \sigma_{\gamma_k} \ldots \sigma_{\gamma_1} \Psi$.

2) Putting $x_0 = \Psi$, $x_i = \sigma_{\gamma_i} \ldots \sigma_{\gamma_1} \Psi$, it is then true that $2 < \frac{x_{i-1}}{x_i}, \frac{\gamma_i}{x_i} > \in \mathbb{Z}$.

3) $< x_{i-1}, \gamma_i > \geq 0$.

THEOREM 2. Let $\chi, \Psi \in \mathfrak{g}^*$ be such that $M_X \subset M_\Psi$. Then there exists a sequence $\gamma_1, \ldots, \gamma_k \in \Delta_+$ satisfying Condition (A) for the pair $(\chi, \Psi)$.

We preface the proof of the theorem by a series of lemmas.

LEMMA 4. Let $M$ be some $U(\mathfrak{g})$-module, and $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ such that its filtration by submodules $M_i$ obeys $M_k/M_{k-1} \approx M_X$. Let $x_0$ be maximal in the set of those $x_k$, for which $x_0 \sim x_k$.

Then there is, in $M$, a submodule isomorphic to $M_X$.

Proof. Let $J_1 \subset Z(\mathfrak{g})$ be the kernel of the homomorphism $\theta x_i$ and $J = J_1 \ldots J_n$. Then $J$ annihilates $M$, which is to say that there operates in $M$ a commutative finite-dimensional algebra $Z(\mathfrak{g})/J$. (Finiteness of $Z(\mathfrak{g})/J$ follows from the Noether character of $Z(\mathfrak{g})$.) Since $Z(\mathfrak{g})/J$ decomposes into a direct sum of local rings (see [12]), $M = \oplus N(\mathfrak{f})$ where the submodules $N(\mathfrak{f})$ correspond to the distinct ones among the maximal ideals $J_i \subset Z(\mathfrak{g})$. Suppose submodule $N(\mathfrak{f})$ corresponds to $J_0$; we put $L_k = N(\mathfrak{f}) \cap M_k$.

If $J_k = J_0$ (i.e., $x_k \sim x_0$), and $L_k/L_{k-1} = 0$, if $J_k \neq J_0$. Hence there is in $N(\mathfrak{f})$ an element $f$ of weight $x_0 - \rho$ and no elements of greater weight (since there are no such from the module $L_k/L_{k-1}$). By Lemma 1, the submodule in $M$ generated by $f$ is isomorphic to $M_X$.

LEMMA 5. Let $F$ be a finite $U(\mathfrak{g})$-module and let $\chi \in \mathfrak{g}^*$. Then in $M_X \otimes F$ there exists a filtration by submodules $0 = L_0 \subset L_1 \subset L_n = M_X \otimes F$ such that $L_k/L_{k-1} \approx M_X + \lambda_k$, where $\lambda_k \in P(F)$.

Proof. Let $e_1, \ldots, e_n$ be a basis of $F$ consisting of weight elements with weights $\lambda_1, \ldots, \lambda_n$, and suppose the indexing chosen so that $i \leq j$, if $\lambda_i \gg \lambda_j$. Let $x_i = f_X \otimes e_i \in M_X \otimes F$ and $L_k = U(\mathfrak{g}) (x_1, \ldots, x_k)$.

We verify that the modules $L_k$ satisfy the hypotheses of the lemma. In fact the image $\bar{F}_k$ of the element $x_k$ in $L_k/L_{k-1}$ is a generator in $L_k/L_{k-1}$ of weight $x + \lambda_k - \rho$ and $E_{\alpha} \bar{F}_k = 0$ for $\alpha \in \Sigma$. Hence $L_k = U(\mathfrak{g}) (x_1, \ldots, x_k)$. We verify that $L_k$ is a free $U(\mathfrak{g})$-module. Suppose $x_i \in U(\mathfrak{g})$, $1 \leq i \leq k$, and $l$ is the largest of the degrees of elements $X_i$ (relative to the natural filtration in $U(\mathfrak{g})$). Then $\sum_{i=1}^n X_i \otimes e_i = \sum_{i=1}^n X_i \otimes e_i = \sum_{i=1}^n Y_i \otimes e_i$ for $e_i = 0$, since the degrees of the elements $Y_j \in U(\mathfrak{g})$ are less than $l$. Applying Lemma 1, we conclude that $L_k/L_{k-1} \approx M_X + \lambda_k$.

The following lemma provides the key to the proof of Theorem 2.

LEMMA 6. Suppose $\chi, \Psi \in \mathfrak{g}^*$ are such that $M_X \subset M_\Psi$, that $F$ is a finite-dimensional $U(\mathfrak{g})$-module, and that $\lambda \in P(F)$ is a weight such that the weight $x + \lambda$ is maximal in the set $W(x + \lambda) \cap (x + P(F))$. Then there exists a weight $\mu \in P(F)$ such that $M_X + \lambda \subset M_\Psi + \mu$.

Proof. It follows from Lemmas 4 and 5 that there is a submodule $M$ in $M_X \otimes F$ which is isomorphic to $M_X + \lambda$. Clearly, $M_X \otimes F$ is imbedded in $M_\Psi \otimes F$. Suppose that $0 = L_0 \subset L_1 \subset \ldots \subset L_n = M_\Psi \otimes F$ is the filtration entering in Lemma 5, and that $L_k$ is the least submodule containing the image of $M$. Then the image of $M$, isomorphic to $M_X + \lambda$, maps nontrivially to $L_k/L_{k-1} \approx M_\Psi + \mu$.

According to Theorem 1, this mapping is an imbedding.

*We say that weight $\chi$ is maximal in the set $\mathfrak{g} \subset \mathfrak{g}^*$ if, from the condition $\varphi \in \mathfrak{g}$, $\varphi \gg \chi$ there follows $\varphi = \chi$. 
LEMMA 7. Let $C_I$ and $C_z$ be two Weil chambers, $X \in C_I$, $X' \in C_I$, $\psi' \in C_z$, $\chi' - \chi \in \mathfrak{h}^*_R$, $\chi \sim \psi$, $\chi' \sim \psi'$, and let the roots $\gamma_1, \ldots, \gamma_k \in \Delta_+$ satisfy Condition (A) for the pair $(X, \psi)$. Then they satisfy Condition (A) for the pair $(X', \psi')$ too.

Proof. Point 2) of Condition (A) is obvious, Point 3) follows from the fact that for any $\gamma$ the sign of $<\text{Re } X, \gamma>$ is constant in a Weil chamber, and Point 1) follows from the fact that $\chi' - \chi_k = \sigma_{\gamma_k} \ldots \sigma_{\gamma_1}$, $\chi' \in C_I$, $\chi_k' \in C_I$, whence $\text{Re } X' = \text{Re } \chi'$. Since $\chi' - \chi_k \in \mathfrak{b}^*_R$, $\text{Im } X' = \text{Im } \chi' \delta$ which is to say $X' = \chi'_{\delta}$.

Lemma 7 shows us how to extend Condition (A) for one pair $(X, \psi)$ to its confirmation for another pair $(X', \psi')$.

LEMMA 8. Let $C_I$ and $C_z$ be adjacent Weil chambers, and let $\varphi_1 \in C_I$, $\varphi_2 \in C_z$ and $\gamma \in A_+$ be such that $<\text{Re } \varphi_1, \gamma> < 0$, $<\text{Re } \varphi_2, \gamma> > 0$. Then $\sigma \gamma \varphi_1 = \varphi_2$.

Proof. Suppose the assertion of the lemma does not hold. Then we can choose $\varphi_1 \in C_I \cap \mathfrak{b}^*_R$ and $\varphi_2' \in C_z \cap \mathfrak{b}^*_R$ such that $|<\varphi_1', \gamma> - <\varphi_2', \gamma>|$ is much smaller than $<\varphi_2', \gamma>$. Since the sign of $<\text{Re } \varphi, \gamma>$ is constant in a Weil chamber, the inequalities $<\varphi_1', \gamma> < 0$, $<\varphi_2', \gamma> > 0$, hold, whence $<\varphi_2' - \varphi_1', \gamma> > <\varphi_2', \gamma>$, which leads to a contradiction.

Proof of the Theorem. Suppose $\chi \in \mathfrak{h}^*$. Denote by $Y(\chi)$ the following assertion:

$Y(\chi)$: for every $\psi \in \mathfrak{h}^*$ for which $M_\chi \subset M_\psi$, there exists a sequence $\gamma_1, \ldots, \gamma_k \in \Delta_+$, satisfying Condition (A) for the pair $(\chi, \psi)$.

We proceed to prove $Y(\chi)$ for all $\chi$. We first note that $Y(\chi)$ is true for all $\chi \in \overline{C}^+$, since in this case there follows from $\chi \gg \chi$ and $\chi \sim \chi$ the fact that $\psi = \chi$. The arbitrary case is handled by the following steps.

Step 1. Let $C$ be a Weil chamber, let $\chi \in C$ and let $\gamma \in \Delta_+$ be a root such that $<\gamma, \text{Re } \chi> < 0$; let $\sigma \gamma \varphi_1$ be a Weil chamber adjoint to $C$. Let $F$ be a finite dimensional $U(\chi)$-module for which $\chi + P(F) \subset C \cup \sigma \gamma \varphi_1$ and $\gamma \in P(F)$. Then, if $\chi + \lambda \in \sigma \gamma \varphi_1$, there follows from $Y(\chi + \lambda)$ the truth of $Y(\chi)$.

Proof. Suppose $M_\chi \subset M_\psi$ and $w \in W$ are such that $\psi = w \chi$. It follows from the hypothesis that there exist no more than two weights $\mu$ of the module $F$ such that $\chi + \lambda \sim \psi + \mu$. Suppose $\mu_1$ is one of these weights for which $\psi + \mu_1 \in w \sigma \gamma \varphi_1$ (it always exists) and $\mu_2$ is the one for which $\psi + \mu_2 \in \mathfrak{c} \psi$ (it may not exist). By Lemma 6, either $M_\chi + \lambda \subset M_\psi + \mu_1$, or $M_\chi + \lambda \subset M_\psi + \mu_2$. We give detailed consideration to both cases.

1. Suppose $M_\chi + \lambda \subset M_\psi + \mu_1$. It follows from $Y(\chi + \lambda)$ that there is a sequence $\gamma_1, \ldots, \gamma_k$ of elements of $\Delta_+$, satisfying Condition (A) for the pair $(\chi + \lambda, \psi + \mu_1) = (\chi, \psi)$. We shall construct a sequence satisfying Condition (A) for the pair $(\chi, \psi)$. Put $\chi_0 = \gamma_1, \ldots, \gamma_k \psi, \chi_0 = \gamma_1 \psi, \ldots, \gamma_k \psi$. We remark that $\chi_0$ and $\chi_0$ are always different on the element $\mathfrak{b}^*_R$. Hence for any $i$, either $\chi_0 < \chi_0$, or $\chi_0 > \chi_0$. If $\chi_0 < \chi_0$, then the sequence $\gamma_1, \ldots, \gamma_k$ satisfies Condition (A) for the pair $(\chi, \psi)$. In the contrary case we denote by $i_0$ the first number for which $\chi_0 > \chi_0$. We show that the sequence $\gamma_1, \ldots, \gamma_{i_0}$, $\chi_0 < \chi_0$, $\chi_0 > \chi_0$, $\gamma_{i_0+1}, \ldots, \gamma_k$ satisfies Condition (A) for the pair $(\chi, \psi)$. It follows from Lemma 8 that the elements $\chi_0$ and $\chi_0$ lie in the same Weil chamber. This means that $\sigma \gamma \sigma \gamma_1 \ldots \sigma \gamma_{i_0+1} \ldots \sigma \gamma_{i_0} \lambda$, and Point 1) of Condition (A) is fulfilled. Moreover, $\sigma \gamma = \sigma \gamma_1 \ldots \sigma \gamma_{i_0+1} \ldots \sigma \gamma_{i_0} \lambda = \sigma \gamma_{i_0} \lambda \ldots \sigma \gamma_{i_0} \lambda \lambda$. Hence $2 <\gamma, \gamma>/<\gamma, \gamma> = 0 <\gamma_{i_0}, \gamma_{i_0}> <\gamma_{i_0}, \gamma_{i_0}> \in \mathfrak{z}$. Fulfillment of Point 2) of Condition (A) for the remaining elements of the sequence is obvious. Fulfillment of Point 3) of Condition (A) follows at once from Lemma 6 and Lemma 7.

2. Now suppose that $M_\chi + \lambda \subset M_\psi + \mu_2$ and that $\gamma_1, \ldots, \gamma_k$ is a sequence of roots satisfying Condition (A) for the pair $(\chi + \lambda, \psi + \mu_2)$. From Lemma 7 and from the condition $<\text{Re } X, \gamma> < 0$ it at once follows that the sequence $\gamma_1, \ldots, \gamma_k$ satisfies this condition for the pair $(\chi, \psi)$.

Step 2. Let $C$ be a Weil chamber, let $\chi \in C$, and let $F$ be a finite-dimensional module such that $\chi + P(F) \subset C$, and $\lambda \in P(F)$. Then $Y(\chi)$ follows from $Y(\chi + \lambda)$.

The proof of this assertion is analogous to the one conducted above, but simpler.

We note that applying Steps 1 and 2 in the required order, we can show $Y(\chi)$ for every $\chi \in \mathfrak{h}^*$, "sufficiently far" from the set $\mathfrak{z}$. More precisely, $d$ being distance in $\mathfrak{b}^*_R$, suppose that $\chi \in \mathfrak{h}^*$ is a weight such that $d(\text{Re } \chi, \mathfrak{z}) > 3 |\beta|$. Then we can construct a sequence $\chi = \chi_0, \chi_1, \ldots, \chi_k$ of elements of $\mathfrak{h}^*$ such that $d(\text{Re } \chi_i, \mathfrak{z}) > 2 |\beta|, \chi_i - \chi_{i+1} \in \mathfrak{b}^*_R, \chi_k \in \mathfrak{c}^+$ and such that for every $i$ one of the following two conditions is fulfilled:
1) \( x_i \) and \( x_{i+1} \) lie in two adjacent Weil chambers \( C \) and \( a_3/C \) and moreover \( \Re x_i, \Re x_{i+1} \leq 0 \) and \( |x_{i+1} - x_i| \) is a much smaller distance from \( x_i \) than all other Weil chambers;

2) \( x_i \) and \( x_{i+1} \) lie in the same Weil chamber and \( |x_{i+1} - x_i| < 2|\rho| \).

Considering the finite-dimensional modules \( F_\lambda \) of least weights \( x_{i+1} - x_i \) and applying respectively Steps 1 and 2, we extend \( Y(x) \) to truth of the assertion \( Y(x_1) \). Here the fact is used that the length of all weights \( F_\lambda \) does not surpass \( |x_{i+1} - x_i| \) (see [5]).

The case of arbitrary \( \chi \) is analyzed with the help of the following considerations. Step 3. Let \( \chi \in \mathfrak{b}^* \), \( \Re \chi \neq 0 \). For each \( \alpha > 0 \) we denote by \( D_\alpha \) the cone in \( \mathfrak{b}_R^+ \) consisting of those non-zero \( \varphi \) such that the angle between \( \varphi \) and \( \Re \chi \) is less than \( \alpha \). We can choose \( \alpha \) small enough that the Weil chamber \( C \) intersects \( D_\alpha \) if and only if \( \chi \in \mathbb{C} \). We choose \( \lambda \in \mathfrak{b}^*_Z \) such that (1) \( \delta (\Re(\chi + \lambda), \Xi) > 3|\rho| \), (2) \( \Re(\chi + \lambda) \) is maximal in the set \( W(\Re(\chi + \lambda)) \cap D_\alpha \).

We now show that \( Y(\chi) \) follows from \( Y(\chi + \lambda) \).

Suppose \( F \) is a finite-dimensional module with least weight \( \lambda \). We show that for all weights \( \chi + \lambda \) the premises of Lemma 6 are fulfilled. Suppose the weight \( \varphi \in P(F) \) is such that \( \Re (\chi + \lambda) \sim \Re (\chi + \mu) \). Since \( |\Re (\chi + \lambda)| = |\Re (\chi + \mu)| \) and \( |\lambda| \leq |\mu| \), it follows that \( \Re (\chi + \mu) \in D_\alpha \), which means the inequality \( \Re (\chi + \lambda) \ll \Re (\chi + \mu) \) cannot be fulfilled. Hence we can take Lemma 6 and the assertion that if \( M_\lambda \subseteq M_\mu \), then \( M_{\chi + \lambda} \subseteq M_{\chi + \mu} \) for some \( \mu \in P(F) \). Reasoning analogous to that adduced above shows that the angle between \( \Re (\psi + \mu) \) and \( \Re \psi \) is not greater than \( \alpha \) and hence \( \psi + \mu \) and \( \psi \) lie in the same Weil chamber. We may now apply Lemma 7 which completes the proof of Step 3 and of the whole of Theorem 2.

§5. Sufficiency of Condition (A)

THEOREM 3 (see [11]). Let \( \gamma, \psi \in \mathfrak{b}^* \). If there exists a sequence of roots \( \gamma_1, \ldots, \gamma_k \in \Delta_+ \), satisfying Condition (A) for the pair \( (\gamma, \psi) \), then \( M_\gamma \subseteq M_\psi \).

Proof. It suffices to consider the case where the sequence consists of a single root \( \gamma \in \Delta_+ \).

LEMMA 9. If \( \gamma, \psi \in \mathfrak{b}^* \) and \( \alpha \in \Sigma \) are such that \( M_\gamma \subseteq M_\psi \) and \( M_\alpha \chi \subseteq M_\chi \), then \( \alpha \in \mathfrak{m}_\alpha \psi \).

Proof. It follows immediately from the conditions of the lemma that \( \psi - \sigma_\alpha \psi = n\alpha \), where \( n \in \mathbb{Z} \). By Lemma 2 it is enough to consider the case \( n > 0 \), where \( M_\alpha \psi \subseteq M_\psi \). Suppose \( \tilde{f}_\psi \) is the image of \( f_\psi \) in \( M_\psi/M_\alpha \psi \). Then \( E^{n-\alpha}_\alpha \tilde{f}_\psi = 0 \). The image of the element \( f_\chi \) in \( M_\psi/M_\alpha \psi \) has the form \( f_\chi = xf_\psi \), where \( X \in U(\mathfrak{g}) \). The element \( E^n_{-\alpha} X \) can be written in the form \( E^n_{-\alpha} X = X E^n_\alpha X \), where \( k \) increases unboundedly with \( k \). Hence \( E^n_{-\alpha} f_\chi = 0 \) for large \( k \). Since \( E^n_{-\alpha} f_\chi = k(n-k)E^n_{-\alpha} f_\chi \), it follows that \( E^n_{-\alpha} f_\chi = 0 \), i.e., that \( M_\alpha \psi \subseteq M_\psi \).

LEMMA 10. The assertion of the theorem is true for \( \chi \in \mathfrak{b}_Z^\ast \), \( \psi = \sigma_\chi \psi \).

Proof. We can suppose that \( \chi \neq \psi \). We choose elements \( \alpha_1, \ldots, \alpha_k \in \Sigma \), such that in the sequence of weights \( \chi_1 = \sigma_{\alpha_1} \ldots \sigma_{\alpha_k} \chi \) the relations \( \chi_{i+1} \gg \chi_i, \chi_k \in \mathbb{C} \) are fulfilled. Suppose \( \psi_1 = \sigma_{\alpha_1} \ldots \sigma_{\alpha_k} \psi \). Each \( \psi_1 \) is obtained from \( \chi_1 \) by some mapping which we denote by \( \sigma_{\gamma_1} \), \( \gamma_1 \in \Delta_+ \). By Lemma 9 it is enough to show that \( M_\chi \subseteq M_\psi \) for some \( i \).

We take as \( \psi \) the last index for which \( \psi \gg x_i \). Then \( \psi_1 - x_i = \sigma_{\alpha_1} \ldots \sigma_{\alpha_k} \psi \), where \( n > 0 \), and \( \psi_1 - x_i \ll \sigma_{\alpha_1} \ldots \sigma_{\alpha_k} \psi \). Then \( \psi_1 - x_i = n\gamma_1 \), where \( n > 0 \), and \( \psi_1 - x_i \ll \sigma_{\alpha_1} \ldots \sigma_{\alpha_k} \psi \). Then \( \psi_1 - x_i = n\gamma_1 \), which is only possible if \( \alpha_1 \neq \gamma_1 \), but then \( M_\chi \subseteq M_\psi \) by Lemma 2.

We now prove Theorem 3. Let \( \gamma \in \Delta_+ \), \( \chi, \psi \in \mathfrak{b}_Z^\ast \) be such that \( \chi = \sigma_\gamma \psi, \psi \ll \gamma = n\gamma \), where \( n \in \mathbb{Z} \). Denote by \( n_{\gamma, \gamma} \) the finite dimensional subspace in \( U(\mathfrak{n}_\gamma) \), consisting of elements of weight \(-n_{\gamma, \gamma} \).

The module \( M_\psi \), considered as a \( U(\mathfrak{n}_\gamma) \)-module, is a free module with image \( f_\psi \). Hence for the proof of the theorem we need to find a nonnull element \( \chi \in n_{\gamma, \gamma} \) such that \( \chi x = \sigma_\gamma f_\psi \). The equations written give a system of linear homogeneous equations in the space \( n_{\gamma, \gamma} \), whose coefficients depend linearly on \( \gamma \). Consider the hyperplane \( S \) in \( \mathfrak{b}_Z^\ast \) consisting of those weights \( \varphi \) such that \( 2 < \varphi, \gamma >, \gamma = n \). By Lemma 10 the given system has a nontrivial solution for all \( \varphi \in S \cap \mathfrak{b}_Z^\ast \). If \( \gamma = \omega \), \( w \in W, \alpha \in \Delta \), then

*This proof differs from that in [11]. In particular it does not use enumerations of simple Lie algebras.
\[ S \cap b_2^S = \left\{ \left( n_i^{(a)} + \sum_{\beta \in \Phi^{(a)}} n_\beta^{(b)} \right), n_\beta \in \mathbb{Z} \right\}, \]

i.e., it constitutes an \((n-1)\)-dimensional lattice in \(S\). Hence our system has a nontrivial solution for all \(\varphi \in S\) and in particular for \(\varphi = \psi\). This completely proves Theorem 3.

Theorems 2 and 3 permit introduction of the following partial ordering into the group \(W\).

**Definition 2.** Let \(x_0 \in C^+ \cap b_2^S\). We put \(w_1 < w_2\) for elements \(w_1, w_2 \in W\), if \(M_{w_1}x_0 \subset M_{w_2}x_0\).

**Theorem 4.** 1) The ordering introduced in Theorem 4 does not depend on the choice of \(x_0 \in C^+ \cap b_2^S\).

2) Suppose \(w_1, w_2 \in W\). The inequality \(w_1 < w_2\) is fulfilled if and only if there exists a sequence of images \(\sigma_{\gamma_1}, \ldots, \sigma_{\gamma_k} \in W\), such that \(w_1 = \sigma_{\gamma_k} \cdots \sigma_{\gamma_1} w_2\) and \(l(\sigma_{\gamma_1} \cdots \sigma_{\gamma_k}) > l(\sigma_{\gamma_1} \cdots \sigma_{\gamma_1} w_2)\) for every \(i\). Here \(l(w)\) is the length of the element \(w \in W\).

The proof of the first part of the theorem follows immediately from Theorems 2 and 3 and Lemma 7. For proving the second part we need the following result on constructing the system of roots (see [9]). Suppose \(w \in W\). Then \(l(w)\) coincides with the number of those roots \(\gamma \in \Delta^+\), for which \(w_\gamma \not\in -\Delta^+\).

As remarked in [2], the adjoined ordering can be given the following geometrical meaning. Let \(G\) be a simply-connected Lie group with Lie algebra \(g\), let \(B \subset G\) be the Borel subgroup corresponding to the subalgebra generated by \(\mathfrak{b}\) and \(E_\gamma, \gamma \in \Delta^+\). Suppose moreover that \(P = G/B\) is the fundamental projective space of the group \(G\). Each element \(w \in W\) we put in correspondence with the subspace \(P_w = BwB \subset P\). These subspace are the key to the splitting of \(P\) (see [10]). Here \(w_1 < w_2\) if and only if \(P_{w_1} \supset P_{w_2}\). We also note the connection of this ordering with the Paley-Wiener Theorem for complex semisimple Lie groups \(G\) ([6]).

§6. Appendix. Multiplicity of the Weight of a Finite-Dimensional Representation

We introduce here a simple algebraic proof of the Kostant formula for the multiplicity of a weight of a finite-dimensional representation (see [7]). To do this we shall use only the definition of the module \(M_\chi\) (Def. 1) and Lemmas 1 and 3. To make it possible to use Lemma 3 we note that in [8] there was introduced a proof of the Harish-Chandra Theorem on the eigenvalues of a Laplace operator without using the formula of \(H\). Weil for characters.

Let \(\mathfrak{g}\) be the space of all functions in \(\mathfrak{g}^*\). For \(u \in \mathfrak{g}^*\) we place \(\text{supp } u = \{X \in \mathfrak{g}^* | u(x) = 0\}\). The group \(W\) acts in \(\mathfrak{g}^*\) by the formula \((w u)(x) = u(w^{-1}x)\). Let \(\mathfrak{g} \subset \mathfrak{g}^*\) be a subspace consisting of all functions \(u\) for which \(\text{supp } u\) is contained in the union of a finite number of sets of the form \(\{X - \sum_{\gamma \in \Delta^+} n_\gamma \gamma, n_\gamma \in \mathbb{Z}, n_\gamma > 0\}\).

The space \(\mathfrak{g}\) is a commutative algebra with respect to the convolution operation \(u_1 * u_2(x) = \sum_{\gamma \in \Delta^+} u_1(x - \gamma)u_2(x)\), \(u_2(\Phi)\) (in this sum only a finite number of terms are different from zero). Let \(\delta_\chi \in \mathfrak{g}\) be a function such that \(\delta_\chi(X) = 1, \delta_\chi(\Phi) = 0\) for \(\Phi \not\in X\). Then \(\delta_0\) is the unit of the ring \(\mathfrak{g}\).

**Definition 3.** 1) \(Q(\chi)\) is the number of families \(\{n_\gamma\}_{\gamma \in \Delta^+}, \gamma \in \mathbb{Z}, n_\gamma > 0\), such that \(\sum_{\gamma \in \Delta^+} n_\gamma = -\chi\).

2) \(L = \prod_{\gamma \in \Delta^+} \left( \frac{\delta_\chi + \delta_{-\gamma}}{\delta_\gamma} \right)\).

We shall call the function \(Q(\chi)\) the Kostant function.*

**Lemma 11.** \(L * Q * \delta_{-\rho} = \delta_0\).

**Proof.** Let \(a_\gamma = \delta_0 + \delta_{-\gamma} + \delta_{-2\gamma} + \ldots \in \mathfrak{g}\). Then clearly \(Q = \prod_{\gamma \in \Delta^+} a_\gamma \text{ if } (\delta_0 - \delta_{-\gamma}) \cdot a_\gamma = \delta_0\). The lemma at once follows from the fact that \(L = \prod_{\gamma \in \Delta^+} (\delta_0 - \delta_{-\gamma}) \cdot \delta_0\).

**Lemma 12.** \(w L = (-1)^l(w)L\) for \(w \in W\).

*Our definition of \(Q(\chi)\) differs from that provided in [7] in replacement of \(\chi\) by \(-\chi\).
Proof. It suffices to verify that \( \sigma_\alpha L = -L \) for \( \alpha \in \Sigma \). Since \( \sigma_\alpha \) transposes elements of the set \( \Delta^+ \setminus \alpha \) and carries \( \alpha \) to \( -\alpha \), \( \sigma_\alpha L = \left( \delta_{\frac{\alpha}{2}} - \delta_{\frac{-\alpha}{2}} \right) \cdot \prod_{\gamma \leq \alpha, \alpha} \left( \frac{\delta_{\gamma} - \delta_{-\gamma}}{\delta_{\gamma}} \right) = -L \).

Definition 4. Suppose the \( \mathfrak{u}(g) \)-module \( M \) is a direct sum of weights of spaces \( V_\chi \) of weight \( \chi \) and \( \dim V_\chi \leq \infty \) for all \( \chi \in \mathfrak{h}^* \). We call the function \( \pi_M \in \mathfrak{h}^* \), given by the formula \( \pi_M(\chi) = \dim V_\chi \) by the name character of \( M \).

**LEMMA 13.** \( \pi_{M_\chi}(\gamma) = Q(\gamma - \chi + \rho) \).

**Proof.** Suppose \( \gamma_1, \ldots, \gamma_k \) is an arbitrary indexing of the roots \( \gamma \in \Delta^+ \). From the theorem of Poincaré-Birkhoff-Witt (see [4], Ch. I) it follows from Lemma 1 that the elements \( E_{\gamma_1} \ldots E_{\gamma_k} \mathfrak{h}_x (n_i \in \mathbb{Z}, \sum n_i \geq 0) \) form a basis in \( M_\chi \), and Lemma 13 follows from the definition of \( Q \).

**COROLLARY.** \( L * \pi_{M_\chi} = Qh \).

**LEMMA 14.** Let \( \mathfrak{u}(g) \)-module \( M \) and element \( x_0 \in \mathfrak{c} \) be such that 1) there is a basis in \( M \) of weight vectors; 2) \( z \cdot x = \theta_x (z)x \) for every \( x \in M, z \in \mathfrak{z}(g); \) 3) \( \pi_M \) exists and \( \pi_M \in \mathfrak{h}^* \).

Let \( D_M = \{ x \in \mathfrak{h}^* | \chi \sim x_0, \chi \leq \rho \} \) for some \( \gamma \in \text{supp} \pi_M \).

Then \( \pi_M = \sum_{x \in D_M} c_x \pi_{M_\chi}, c_x \in \mathbb{Z} \).

**Proof.** In view of Condition 3) there exists in \( \text{supp} \pi_M \) at least one maximal element \( \chi \). Then \( E_{\chi} x = 0 \) for any \( x \in V_\chi, \chi \in \mathfrak{c} \). Hence for any \( z \in \mathfrak{z}(g), v \in \mathfrak{v} \) we have \( z \cdot x = \theta_x (z)x \), and hence, \( x + \rho \sim \chi \), i.e., \( x + \rho \in \text{DM} \). Let \( k = \dim V_\chi \). By definition of \( \pi_M \) it follows that we can construct a mapping \( \chi: (M_\chi + \rho) \to M \), translating the generators \( f_{\chi + \rho} \) to \( k \) linearly independent elements \( V_\chi \). Suppose \( L \) and \( N \) are the kernel and co-kernel of \( \chi \). It follows from the exact sequence \( 0 \to L \to (M_\chi + \rho) \to M \to N \to 0 \) that \( \pi_M = \pi_L - \pi_N - k \pi_{M_\chi + \rho} \).

The moduli \( L \) and \( M \) satisfy the hypotheses of Lemma 14 and in addition \( D_L \) and \( D_N \) lie in \( D_M \). Induction on the number of elements of \( D_M \) concludes the proof of Lemma 14.

**Remark.** The foregoing reasoning establishes that the moduli \( M_\chi \) generate the group of Grothendieck category of the moduli \( M \) satisfying the conditions of Lemma 14.

**COROLLARY.** Under the hypotheses of Lemma 14, \( L * \pi_M \subset D_M \).

**COROLLARY.** Suppose now that \( F \) is a continuous finite-dimensional \( \mathfrak{u}(g) \)-module with highest weight \( \lambda \). Then \( \pi_F \) is invariant with respect to \( W \) (see [5]). By Lemma 12, \( w(L * \pi_F) = (-1)^{l(w)} (L * \pi_F) \).

Since \( \dim V_\lambda = 1 \) in the module \( F \), it follows that \( c_0 = 1 \) (see the proof of Lemma 14), hence \( L * \pi_F = \sum_{v \in W} (-1)^{l(w)} \delta_w (\lambda + \rho) \). Applying Lemma 11, we get \( \pi_F = Q * \delta_{\rho} \sum_{v \in W} (-1)^{l(w)} \delta_w (\lambda + \rho) \). In conclusion, we get the following theorem.

**THEOREM 5 (Kostant formula).** Let \( F \) be a finite-dimensional continuous representation of \( g \) with highest weight \( \lambda \). Then

\[
\pi_F (\mu) = \sum_{w \in W} (-1)^{l(w)} Q (\mu + \rho - w (\lambda + \rho)).
\]

**Note.** In conclusion we append an example of a submodule \( M \) in \( M_\chi \) which does not have the form \( U M_\chi \). Let \( g \) be the Lie algebra of the group \( SL(4, \mathbb{C}), E_{1k}(i \neq k) \), and let \( E_{i} - E_\bar{i} \) be a basis of \( g \) with \( E_{i} \) (\( i < k \)) corresponding to roots from \( \Delta^+ \). We consider the module \( M_\chi \), wherein all weights \( \chi \) are given by the equations \( \chi (E_{11} - E_{12}) = \chi (E_{22} - E_{23}) = 0, \chi (E_{33} - E_{34}) = 1 \). Suppose \( f_1 = E_{23}n_1 \) and \( M = U(\mathfrak{g}) (f_1) \). It follows from Theorem 2 that if \( E_{i} \) \( a = 0 \) for \( i < k \), then \( a = c_1 + c_2 \), where \( a_1 \in \mathbb{M}, c_2 \in \mathbb{C} \). Put \( x = E_{24}E_{32}n_1 + E_{34}E_{23}n_1 \). Then \( x \notin \mathbb{M} \) and \( E_{ik} x \in \mathbb{M} \) for \( i < k \). Hence the submodule generated by \( x \) and \( \mathbb{M} \) is a proper submodule of \( M_\chi \) not having the form \( U M_\chi \).

**LITERATURE CITED**

9. N. Bourbaki, Groupes et Algebres de Lie, Hermann (1968), Chaps. 4-6.