We summarize the above results. Every metrized Lie algebra {G, B} can be decomposed into a direct sum of indecomposable nondegenerate mutually orthogonal ideals (1). This decomposition is unique up to isomorphism. The choice of decomposition is completely determined by the center Z(G) of the Lie algebra G. If the hypotheses of Proposition 3 hold then decomposition (1) is unique.

4. We indicate here a method for constructing an infinite series of new indecomposable metrized Lie algebras starting from a fixed indecomposable metrized Lie algebra {G, B}.

If $\mathfrak{G}$ is a connected Lie group, its Lie algebra G is metrizable if and only if there exists on G a biinvariant nondegenerate symmetric bilinear form. It is proved in [2] for such Lie groups $\mathfrak{G}$ that the tangent bundle $T(\mathfrak{G})$ is also a Lie group admitting a form with analogous properties. Thus given any metrized Lie algebra {G, B} we can associate a new metrized Lie algebra $\{T(G), B_T\}$.

THEOREM 5. If dim G > 1, then the metrized Lie algebra $\{T(G), B_T\}$ is decomposable if and only if the original metrized Lie algebra {G, B} is decomposable.

COROLLARY. If dim G > 1 and {G, B} is indecomposable, then for any natural number n the metrizable Lie algebra $T_n(G) = T(T_{n-1}(G))$ is also indecomposable.

If G is simple then $T_n(G)$ is an example of an indecomposable metrizable Lie algebra with a nontrivial Levi–Mal'tsev decomposition.

The author thanks D. V. Alekseevskii for a discussion of the results and for his interest in this work.

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ALGEBRAIC BUNDLES OVER P^n AND PROBLEMS OF LINEAR ALGEBRA

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UDC 513.015.7

1. The description of the algebraic vector bundles over projective space P^n has attracted the attention of many specialists in algebraic geometry (see [1-3]). Recently, interest in this problem has increased even more in connection with the remarkable papers of Atiyah and Ward [4] and Belavin and Zakharov [5], in which the connection of bundles over CP^3 with gauge fields on the four-dimensional sphere is described. In the present note it is shown how the classification of bundles over P^n reduces to a problem of linear algebra, viz., to the classification of finite-dimensional graded representations of the exterior (Grassman) algebra on (n + 1) variables. There are special cases of such a reduction in Barth [2] and Drinfel'd and Manin [3]. Independently obtained, Beilinson [6] is close to our result. We want to express profound gratitude to Yu. I. Manin, whose report on [3] stimulated our interest in these questions.

2. Let Ξ be an (n + 1)-dimensional linear space over an algebraically closed field k, Λ be the exterior algebra on the space Ξ. We introduce a grading on Λ, by setting deg $\xi = -1$ for $\xi \in \Xi$. By a Λ-module we shall mean a finitely generated graded Λ-module; notation $V = \bigoplus_j V_j$. Let $\mathcal{P}$ be the class of free Λ-modules; we shall call Λ-modules V, V' $\mathcal{P}$-equivalent, if $V \oplus P = V' \oplus P'$ for some $P, P' \in \mathcal{P}$.

3. Let **P** be the projective space corresponding to Ξ. We shall construct, for each Λ-module V, a complex L(V) of vector bundles over **P**. Namely, we set $L_j = V_{-j} \otimes \mathcal{O}(j)$, where $\mathcal{O}(j)$ is the j-th power of the Hopf bundle; by definition, a section of the bundle L_j is a homo-

Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 12, No. 3, pp. 66-67, July-September, 1978. Original article submitted January 10, 1978.

 0016-2663/78/1203-0212$07.50

geneous function $f(\xi)$ of degree of homogeneity with values in V_{-j}. We define the differential $d\colon L_j \to L_{j+1}$, by setting $df(\xi) = \xi(f(\xi))$.

If $\xi \in \Xi$, $\xi \neq 0$, then the fiber $L_\xi(V)$ of complex $L(V)$, corresponding to the point $\bar{\xi} \in P$, coincides with the complex vector spaces $L_{\bar{\xi}}(V) = (\ldots \to V_1 \xrightarrow{\xi} V_0 \xrightarrow{\xi} V_{-1} \to \ldots)$. We call the Λ-module V faithful if $H^i(L_\xi(V)) = 0$ for $i \neq 0$ for all $0 \neq \xi \in \Xi$. In this case $H^0(L(V))$ is a vector bundle over P; its fiber at the point $\bar{\xi}$ coincides with $H^0(L_\xi(V))$. We denote this bundle by $\Phi(V)$.

THEOREM 1. Any algebraic vector bundle over P has the form $\Phi(V)$ for some faithful Λ-module V. Here $\Phi(V) \approx \Phi(V')$ if and only if V and V' are $\mathscr{P}$-equivalent.

Remarks. 1) The map $V \mapsto \Phi(V)$ (for exact Λ-modules V) commutes with tensor products, taking symmetric and exterior powers, and passage to the dual module.

2) Let $0 \to V \to P \to V' \to 0$ be an exact sequence of Λ-modules, where $P \in \mathscr{P}$, V is a faithful module. Let W be the Λ-module obtained from V' by the grading shift: $W_j = V'_{j+1}$. Then W is a faithful Λ-module and $\Phi(W) = \Phi(V) \otimes \mathscr{O}(1)$.

3) Let $\xi_0, \ldots, \xi_n$ be a basis in Ξ, $\omega = \xi_0 \ldots \xi_n \in \Lambda$. It is easy to verify that each Λ-module V can be represented in the form $V = V^0 \oplus P$, where $P \in \mathscr{P}$, $\omega V^0 = 0$, to $\mathscr{P}$-equivalent modules V correspond isomorphic modules V^0. Hence vector bundles over P are classified by faithful modules over the algebra $\Lambda/(\omega)$.

3. To formulate a more precise result we need the machinery of derived categories (see [7]). Let Coh be the category of coherent sheaves on P, $C^b(\mathrm{Coh})$ be the category of bounded complexes of objects of Coh and $D^b(\mathrm{Coh})$ be the derived category.

Let $\mathscr{M}(\Lambda)$ be the category of Λ-modules. Considering, for each $V \in \mathscr{M}(\Lambda)$, L(V) as a complex of sheaves on P, we get a functor $L\colon \mathscr{M}(\Lambda) \to C^b(\mathrm{Coh})$. By L_D we denote the composite functor $\mathscr{M}(\Lambda) \to C^b(\mathrm{Coh}) \to D^b(\mathrm{Coh})$. It is easy to verify that for $V \in \mathscr{P}$ the complex L(V) is acyclic, so that $L_D(V) \approx 0$. Hence the functor L_D factors through some functor $L'_D\colon \mathscr{M}(\Lambda)/\mathscr{P} \to D^b(\mathrm{Coh})$, where $\mathscr{M}(\Lambda)/\mathscr{P}$ is the quotient category of $\mathscr{M}(\Lambda)$ by the family of morphisms, factoring through objects $P \in \mathscr{P}$ (see [7, 8]).

THEOREM 2. The functor $L'_D\colon \mathscr{M}(\Lambda)/\mathscr{P} \to D^b(\mathrm{Coh})$ is an equivalence of categories.

Remarks. 1) Let $\mathscr{N}$ be the complete subcategory of $\mathscr{M}(\Lambda)/\mathscr{P}$, consisting of these modules V, such that $H^i(L(V)) = 0$ for $i \neq 0$. Then it follows from Theorem 2 that the functor $V \mapsto H^0(L(V))$ defines an equivalence of the category $\mathscr{N}$ with the category Coh. Whence it is easy to derive Theorem 1.

2) The equivalence L'_D defines on $\mathscr{M}(\Lambda)/\mathscr{P}$ a structure of triangulated **category.** This structure is characterized by the condition that for any exact sequence $0 \to V' \to V \to V'' \to 0$ in $\mathscr{M}(\Lambda)$ the morphisms $V' \to V \to V''$ are included in a triangle in $\mathscr{M}(\Lambda)/\mathscr{P}$, while in this way one gets all pairs of morphisms contained in triangles. In particular, if $V \in \mathscr{P}$, then $V'' = T(V)$, where T is the translation functor.

3) Let k be the trivial Λ-module of degree 0, V be a faithful Λ-module. Then $H^i(P, \Phi(V)) = \mathrm{Hom}_{\mathscr{M}(\Lambda)/\mathscr{P}}(k, T^i V)$. For $i \neq 0$ this group is equal to $\mathrm{Ext}^i_{\mathscr{M}(\Lambda)}(k, V)$.

4. We shall explain the scheme of the proof of Theorem 2. Let $X = \Xi^*$, $S = S(X)$ be the symmetric algebra on the space X with its ordinary grading $S = \bigoplus_{j \geqslant 0} S_j$, $\mathscr{M}(S)$ be the category of graded finitely generated S-modules. We denote by $C^b(S)$ and $C^b(\Lambda)$ the categories of bounded complexes of objects from $\mathscr{M}(S)$ and $\mathscr{M}(\Lambda)$, while in the case $\mathscr{M}(\Lambda)$ it will be assumed that the differential ∂ in the complex satisfies the condition $\partial\xi = -\xi\partial$ for $\xi \in \Xi$.

We construct a function $F\colon C^b(\Lambda) \to C^b(S)$. A complex $(V, \partial) \in C^b(\Lambda)$ will be considered as a bigraded space $V = \oplus V^i_j$, where i is the number of the module in the complex, j is the grading in $\mathscr{M}(\Lambda)$; analogously for complexes $(W, d) \in C^b(S)$. The differentials ∂ and d have bidegree (1, 0). We set $F(V) = W = S \otimes V$ (tensor product over k). We define the differential d in W by the formula $d(s \otimes v) = \Sigma x_i s \otimes \xi_i v + s \otimes \partial v$, where $\{x_i\}$, $\{\xi_i\}$ are dual bases in X and Ξ; we define the bidegree in W as follows: if $s \in S_k$, $v \in V^i_j$, then $s \otimes v \in W^{i-j}_{j+k}$.

Let $D^b(\Lambda)$ and $D^b(S)$ be the derived categories corresponding to $C^b(\Lambda)$ and $C^b(S)$.

THEOREM 3. The functor $F\colon C^b(\Lambda) \to C^b(S)$ extends to a functor $F_D\colon D^b(\Lambda) \to D^b(S)$; the functor F_D is an equivalence of triangulated categories.

To prove Theorem 3 it is necessary to consider the adjoint functor $G: C(S) \to C(\Lambda)$. It is defined as follows: $G(\mathbf{W}) = \mathbf{V} = \mathrm{Hom}_k(\Lambda, \mathbf{W})$; $\partial(v)\lambda = -\Sigma x_i v(\xi_i \lambda) + d(v(\lambda))$; $V_j^i(\Lambda_k) \subset W_{j+k}^{i-j-k}$. Although the image $G(C^b(S))$ does not lie in $C^b(\Lambda)$, G allows one to define a functor $G_D: D^b(S) \to D^b(\Lambda)$. Using the Koszul complex, it is easy to verify that the functor G_D is inverse to the function F_D.

5. Let $\mathscr{F}, \mathscr{I}$ be the full subcategories in $D^b(S)$ and $D^b(\Lambda)$, generated by the complexes, consisting of finite-dimensional (respectively free) modules. It is easy to verify that $F_D^{-1}(\mathscr{F}) = \mathscr{I}$, so that F_D defines an equivalence of categories $D^b(\Lambda)/\mathscr{I} \to D^b(S)/\mathscr{F}$ (the quotient categories in the sense of Verdier [7]).

Using Serre's theorem, describing the category Coh in terms of $\mathscr{M}(S)$ (see [9]), it is easy to get that the category $D^b(\mathrm{Coh})$ is equivalent with $D^b(S)/\mathscr{F}$. Thus, from Theorem 3 follows

THEOREM 4. The categories $D^b(\mathrm{Coh})$ and $D^b(\Lambda)/\mathscr{I}$ are equivalent.

6. Proposition. The natural imbedding $\mathscr{M}(\Lambda) \to D^b(\Lambda)$ defines an equivalence of categories $\mathscr{M}(\Lambda)/\mathscr{P} \to D^b(\Lambda)/\mathscr{I}$.

The proposition follows from the fact that free Λ-modules are projective and injective. Theorem 2 follows from this proposition and Theorem 4.

7. Theorems 1-4 are true for any field k; Theorems 3 and 4 are true if k is replaced by an arbitrary basis Z, Ξ by a locally free sheaf of $\mathscr{O}_Z$-modules, P by a projective spectrum of sheaves of algebras S = S(X), where $X = \Xi^*$.

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COHERENT SHEAVES ON P^n AND PROBLEMS OF LINEAR ALGEBRA

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UDC 513.015.7

The goal of this note is to generalize the results of Horrocks and Barth [1], and Drinfel'd and Manin [2] to the case of projective space of any dimension n. In particular, for any coherent sheaf L on P^n there will be constructed a "two-sided resolution" which is unique up to homotopy (a complex $K^\cdot$ with $H^0(K^\cdot) = L$, $H^i(K^\cdot) = 0$ for $i \neq 0$), the i-th term of which is isomorphic with $\bigoplus_j H^{i+j}(\mathbf{P}^n, L(-j)) \otimes \Omega^j(j)$ (generalized "monads" of Barth). The precise formulation of the result uses the derived categories of Verdier [3].

1. Let C be a triangulated category. We shall say that a family of its objects $\{X_i\}$ generates C, if the smallest full triangulated subcategory containing them is equivalent with C.

LEMMA 1. Let C and D be triangulated categories, F: C → D be an exact functor, $\{X_i\}$ be a family of objects of C. Let us assume that $\{X_i\}$ generates C, $\{F(X_i)\}$ generates D, and for any pair X_i, X_j from the family F: $\mathrm{Hom}^\cdot(X_i, X_j) \to \mathrm{Hom}^\cdot(F(X_i), F(X_j))$ is an isomorphism. Then F is an equivalence of categories.

2. Let $A^\cdot$ be a graded algebra. Notation: $A^\cdot[i]$ is the free one-dimensional graded $A^\cdot$-module with distinguished generator of degree i; $M_{[0,n]}(A^\cdot)$ is the full subcategory of

Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 12, No. 3, pp. 68-69, July-September, 1978. Original article submitted January 10, 1978.

0016-2663/78/1203-0214$07.50