COXETER FUNCTORS AND GABRIEL'S THEOREM

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It has recently become clear that a whole range of problems of linear algebra can be formulated in a uniform way, and in this common formulation there arise general effective methods of investigating such problems. It is interesting that these methods turn out to be connected with such ideas as the Coxeter–Weyl group and the Dynkin diagrams.

We explain these connections by means of a very simple problem. We assume no preliminary knowledge. We do not touch on the connections between these questions and the theory of group representations or the theory of infinite-dimensional Lie algebras. For this see [3]–[5].

Let Γ be a finite connected graph; we denote the set of its vertices by Γ_0 and the set of its edges by Γ_1 (we do not exclude the cases where two vertices are joined by several edges or there are loops joining a vertex to itself). We fix a certain orientation Λ of the graph Γ; this means that for each edge $l \in \Gamma_1$ we distinguish a starting-point $\alpha(l) \in \Gamma_0$ and an end-point $\beta(l) \in \Gamma_0$.

With each vertex $\alpha \in \Gamma_0$ we associate a finite-dimensional linear space V_α over a fixed field K. Furthermore, with each edge $l \in \Gamma_1$ we associate a linear mapping $f_l\colon V_{\alpha(l)} \to V_{\beta(l)}$ ($\alpha(l)$ and $\beta(l)$ are the *starting-point and* end-point of the edge l). We impose no relations on the linear mappings f_l. We denote the collection of spaces V_α and mappings f_l by (V, f).

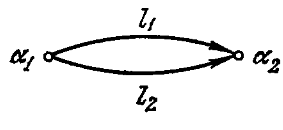
DEFINITION 1. Let (Γ, Λ) be an oriented graph. We define a *category* $\mathscr{L}(\Gamma, \Lambda)$ in the following way. An *object* of $\mathscr{L}(\Gamma, \Lambda)$ is any collection (V, f) of spaces V_α $(\alpha \in \Gamma_0)$ and mappings f_l $(l \in \Gamma_1)$. A morphism φ: $(V, f) \to (W, g)$ is a collection of linear mappings $\varphi_\alpha\colon V_\alpha \to W_\alpha$ $(\alpha \in \Gamma_0)$ such that for any edge $l \in \Gamma_1$ the following diagram

$$\begin{tikzcd} V_{\alpha(l)} \arrow[r, "f_l"] \arrow[d, "\varphi_{\alpha(l)}"] & V_{\beta(l)} \arrow[d, "\varphi_{\beta(l)}"] \\ W_{\alpha(l)} \arrow[r, "g_l"'] & W_{\beta(l)} \end{tikzcd}$$

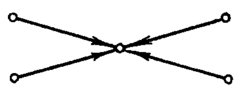
is commutative, that is, $\varphi_{\beta(l)} f_l = g_l \varphi_{\alpha(l)}$.

Many problems of linear algebra can be formulated in these terms. For example, the question of the canonical form of a linear transformation $f\colon V \to V$ is connected with the diagram

The classification of a pair of linear mappings $f_1\colon V_1 \to V_2$ and $f_2\colon V_1 \to V_2$ leads to the graph



A very interesting problem is that of the classification of quadruples of subspaces in a linear space, which corresponds to the graph



This last problem contains several problems of linear algebra.[1]

Let (Γ, Λ) be an oriented graph. The direct sum of the objects (V, f) and (U, g) in $\mathscr{L}(\Gamma, \Lambda)$ is the object (W, h), where $W_\alpha = V_\alpha \oplus U_\alpha$, $h_l = f_l \oplus g_l$ $(\alpha \in \Gamma_0,\ l \in \Gamma_1)$.

We call a non-zero object $(V, f) \in \mathscr{L}(\Gamma, \Lambda)$ indecomposable if it cannot be represented as the direct sum of two non-zero objects. The simplest indecomposable objects are the irreducible objects L_α $(\alpha \in \Gamma_0)$, whose structure is as follows: $(L_\alpha)_\gamma = 0$ for $\gamma \neq \alpha$, $(L_\alpha)_\alpha = K$, $f_l = 0$ for all $l \in \Gamma_1$.

It is clear that each object (V, f) of $\mathscr{L}(\Gamma, \Lambda)$ is isomorphic to the direct sum of finitely many indecomposable objects.[2]

In many cases indecomposable objects can be classified.[3]

In his article [1] Gabriel raised and solved the following problem: to find all graphs (Γ, Λ) for which there exist only finitely many non-isomorphic indecomposable objects $(V, f) \in \mathscr{L}(\Gamma, \Lambda)$. He made the following

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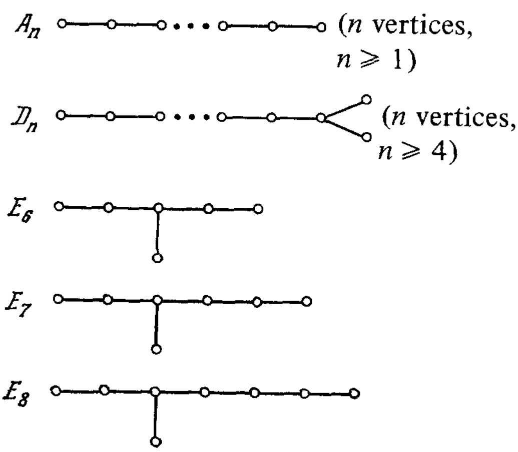
1 Let us explain how the problem of the canonical form of a linear operator $f\colon V \to V$ reduces to that of a quadruple of subspaces. Consider the space $W = V \oplus V$ and in it the graph of f, that is, the subspace E_4 of pairs $(\xi, f\xi)$, where $\xi \in V$. The mapping f is described by a quadruple of subspaces in W, namely $E_1 = V \oplus 0$, $E_2 = 0 \oplus V$, $E_3 = \{(\xi, \xi) \mid \xi \in V\}$ (E_3 is the diagonal) and $E_4 = \{(\xi, f\xi) \mid \xi \in V\}$ – the graph of f. Two mappings f and f' are equivalent if and only if the quadruples corresponding to them are isomorphic. In fact, E_1 and E_2 define "coordinate planes" in W, E_3 establishes an identification between them, and then E_4 gives the mapping.

2 It can be shown that such a decomposition is unique to within isomorphism (see [6], Chap. II, 14, the Krull–Schmidt theorem).

3 We believe that a study of cases in which an explicit classification is impossible is by no means without interest. However, we should find it difficult to formulate precisely what is meant in this case by a "study" of objects to within isomorphism. Suggestions that are natural at first sight (to consider the subdivision of the space of objects into trajectories, to investigate versal families, to distinguish "stable" objects, and so on) are not, in our view, at all definitive.

surprising observation. For the existence of finitely many indecomposable objects in $\mathscr{L}$ (Γ, Λ) it is necessary and sufficient that Γ should be one of the following graphs:



(this fact does not depend on the orientation Λ).The surprising fact here is that these graphs coincide exactly with the Dynkin diagrams for the simple Lie groups.[1]

However, this is not all. As Gabriel established, the indecomposable objects of $\mathscr{L}$ (Γ, Λ) correspond naturally to the positive roots, constructed according to the Dynkin diagram of Γ.

In this paper we try to remove to some extent the "mystique" of this correspondence. Whereas in Gabriel's article the connection with the Dynkin diagrams and the roots is established a posteriori, we give a proof of Gabriel's theorem based on exploiting the technique of roots and the Weyl group. We do not assume the reader to be familiar with these ideas, and we give a complete account of the necessary facts.

An essential role is played in our proof by the functors defined below, which we call Coxeter functors (the name arises from the connection of these functors with the Coxeter transformations in the Weyl group). For the particular case of a quadruple of subspaces these functors were introduced in [2] (where they were denoted by Φ^+ and Φ^-). Essentially, our paper is a synthesis of Gabriel's idea on the connection between the categories of diagrams $\mathscr{L}(\Gamma, \Lambda)$ with the Dynkin diagrams and the ideas of the first part of [2], where with the help of the functors Φ^+ and Φ^- the "simple" indecomposable objects are separated from the more "complicated" ones.

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[1] More precisely, Dynkin diagrams with single arrows.

We hope that this technique is useful not only for the solution of Gabriel's problem and the classification of quadruples of subspaces, but also for the solution of many other problems (possibly, not only problems of linear algebra).

Some arguments on Gabriel's problem, similar to those used in this article, have recently been expressed by Roiter. We should also like to draw the reader's attention to the articles of Roiter, Nazarova, Kleiner, Drozd and others (see [3] and the literature cited there), in which very effective algorithms are developed for the solution of problems in linear algebra. In [3], Roiter and Nazarova consider the problem of classifying representations of ordered sets; their results are similar to those of Gabriel on the representations of graphs.

§ 1. Image functors and Coxeter functors

To study indecomposable objects in the category $\mathscr{L}(\Gamma, \Lambda)$ we consider "image functors", which construct for each object $V \in \mathscr{L}(\Gamma, \Lambda)$ some new object (in another category); here an indecomposable object goes either into an indecomposable object or into the zero object. We construct such a functor for each vertex α at which all the edges have the same direction (that is, they all go in or all go out). Furthermore, we construct the "Coxeter functors" Φ^+ and Φ^-, which take the category $\mathscr{L}(\Gamma, \Lambda)$ into itself.

For each vertex $\alpha \in \Gamma_0$ we denote by Γ^α the set of edges containing α. If Λ is some orientation of the graph Γ, we denote by $\sigma_\alpha\Lambda$ the orientation obtained from Λ by changing the directions of all edges $l \in \Gamma^\alpha$.

We say that a vertex α is (–)-accessible (with respect to the orientation Λ) if $\beta(l) \neq \alpha$ for all $l \in \Gamma_1$ (this means that all the edges containing α start there and that there are no loops in Γ with vertex at α). Similarly we say that the vertex β is (+)-accessible if $\alpha(l) \neq \beta$, for all $l \in \Gamma_1$.

DEFINITION 1.1 1) Suppose that the vertex β of the graph Γ is (+)-accessible with respect to the orientation Λ. From an object (V, f) in $\mathscr{L}(\Gamma, \Lambda)$ we construct a new object (W, g) in $\mathscr{L}(\Gamma, \sigma_\beta\Lambda)$.

Namely, we put $W_\gamma = V_\gamma$ for $\gamma \neq \beta$.

Next we consider all the edges $l_1, l_2, \ldots, l_k$ that end at β (that is, all edges of Γ^β). We denote by W_β the subspace in the direct sum $\bigoplus_{i=1}^{k} V_{\alpha(l_i)}$ consisting of the vectors $v = (v_1, \ldots, v_k)$ (here $v_i \in V_{\alpha(l_i)}$) for which $f_{l_i}(v_1) + \ldots + f_{l_k}(v_k) = 0$. In other words, if we denote by h the mapping $h\colon \bigoplus_{i=1}^{k} V_{\alpha(l_i)} \to V_\beta$ defined by the formula

$h(v_1, v_2, \ldots, v_k) = f_{l_1}(v_1) + \ldots + f_{l_k}(v_k)$, then $W_\beta = \operatorname{Ker} h$.

We now define the mappings g_l. For $l \notin \Gamma^\beta$ we put $g_l = f_l$. If $l = l_j \in \Gamma^\beta$, then g_l is defined as the composition of the natural embedding of W_β in $\oplus\, V_{\alpha(l_i)}$ and the projection of this sum onto the term $V_{\alpha(l_j)} = W_{\alpha(l_j)}$. We note that on all edges $l \in \Gamma^\beta$ the orientation has been changed, that is, the resulting object (W, g) belongs to $\mathscr{L}(\Gamma, \sigma_\beta\Lambda)$. We denote the object (W, g) so constructed by $F_\beta^+(V, f)$.

2) Suppose that the vertex $\alpha \in \Gamma_0$ is $(-)$-accessible with respect to the orientation Λ. From the object $(V, f) \in \mathscr{L}(\Gamma, \Lambda)$ we construct a new object $F_\alpha^-(V, f) = (W, g) \in \mathscr{L}(\Gamma, \sigma_\alpha\Lambda)$. Namely, we put

$$W_\gamma = V_\gamma \text{ for } \gamma \neq \alpha$$
$$g_l = f_l \text{ for } l \notin \Gamma^\alpha$$

$W_\alpha = \bigoplus_{i=1}^{k} V_{\beta(l_i)}/\operatorname{Im} \widetilde{h}$, where $\{l_1, \ldots, l_k\} = \Gamma^\alpha$, and the mapping $\widetilde{h}: V_\alpha \to \bigoplus_{i=1}^{k} V_{\beta(l_i)}$ is defined by the formula $\widetilde{h}(v) = (f_{l_1}(v), \ldots, f_{l_k}(v))$.

If $l \in \Gamma^\alpha$, then the mapping $g_l\colon W_{\beta(l)} \to W_\alpha$ is defined as the composition of the natural embedding of $W_{\beta(l)} = V_{\beta(l)}$ in $\bigoplus_{i=1}^{k} V_{\beta(l_i)}$ and the projection of this direct sum onto W_α.

It is easy to verify that F_β^+ (and similarly F_α^-) is a functor from $\mathscr{L}(\Gamma, \Lambda)$ into $\mathscr{L}(\Gamma, \sigma_\beta\Lambda)$ (or $\mathscr{L}(\Gamma, \sigma_\alpha\Lambda)$, respectively). The following property of these functors is basic for us.

THEOREM 1.1 1) *Let (Γ, Λ) be an oriented graph and let $\beta \in \Gamma_0$ be a vertex that is $(+)$-accessible with respect to Λ. Let $V \in \mathscr{L}(\Gamma, \Lambda)$ be an indecomposable object. Then two cases are possible:*

a) *$V \approx L_\beta$ and $F_\beta^+ V = 0$ (we recall that L_β is an irreducible object, defined by the condition $(L_\beta)_\gamma = 0$ for $\gamma \neq \beta$, $(L_\beta)_\beta = K$, $f_l = 0$ for all $l \in \Gamma_1$).*

b) *$F_\beta^+(V)$ is an indecomposable object, $F_\beta^- F_\beta^+(V) = V$, and the dimensions of the spaces $F_\beta^+(V)_\gamma$ can be calculated by the formula*

$$\text{(1.1.1)} \qquad \begin{aligned} \dim F_\beta^+(V)_\gamma &= \dim V_\gamma \quad \text{for } \gamma \neq \beta, \\ \dim F_\beta^+(V)_\beta &= -\dim V_\beta + \sum_{l \in \Gamma^\beta} \dim V_{\alpha(l)}. \end{aligned}$$

2) *If the vertex α is $(-)$-accessible with respect to Λ and if $V \in \mathscr{L}(\Gamma, \Lambda)$ is an indecomposable object, then two cases are possible:*

a) *$V \approx L_\alpha$, $F_\alpha^-(V) = 0$.*

b) $F_\alpha^-(V)$ *is an indecomposable object,* $F_\alpha^+F_\alpha^-(V) = V$,

(1.1.2) $$\dim F_\alpha^-(V)_\gamma = \dim V_\gamma \text{ for } \gamma \neq \alpha,$$
$$\dim F_\alpha^-(V)_\alpha = -\dim V_\alpha + \sum_{l\in\Gamma^\alpha} \dim V_{\beta(l)}.$$

PROOF. If the vertex β is (+)-accessible with respect to Λ, then it is (−)-accessible with respect to $\sigma_\beta\Lambda$, and so the functor $F_\beta^-F_\beta^+$: $\mathscr{L}(\Gamma, \Lambda) \to \mathscr{L}(\Gamma, \Lambda)$ is defined. For each object $V \in \mathscr{L}(\Gamma, \Lambda)$ we construct a morphism i_V^β: $F_\beta^-F_\beta^+(V) \to V$ in the following way.

If $\gamma \neq \beta$, then $F_\beta^-F_\beta^+(V)_\gamma = V_\gamma$, and we put $(i_V^\beta)_\gamma = \mathrm{Id}$, the identity mapping.

For the definition of $(i_V^\beta)_\beta$ we note that in the sequence of mappings

$F_\beta^+(V)_\beta \xrightarrow{\widetilde{h}} \bigoplus_{l\in\Gamma^\beta} V_{\alpha(l)} \xrightarrow{h} V_\beta$ (see definition 1.1) $\operatorname{Ker} h = \operatorname{Im} \widetilde{h}$; we take for $(i_V^\beta)_\beta$ the natural mapping
$$F_\beta^-F_\beta^+(V)_\beta = \bigoplus_{l\in\Gamma^\beta} V_{\alpha(l)}/\operatorname{Im}\widetilde{h} = \bigoplus_{l\in\Gamma^\beta} V_{\alpha(l)}/\operatorname{Ker} h \to V_\beta.$$
It is easy to verify that i_V^β is a morphism. Similarly, for each (−)-accessible vertex α we construct a morphism p_V^α: $V \to F_\alpha^+F_\alpha^-(V)$. Now we state the basic properties of the functors F_α^-, F_β^+ and the morphisms p_V^α, i_V^β.

LEMMA 1.1. 1) $F_\alpha^\pm(V_1 \oplus V_2) = F_\alpha^\pm(V_1) \oplus F_\alpha^\pm(V_2)$. 2) p_V^α *is an epimorphism and* i_V^β *is a monomorphism.* 3) *If* i_V^β *is an isomorphism, then the dimensions of the spaces* $F_\beta^+(V)_\gamma$ *can be calculated from* (1.1.1). *If* p_V^α *is an isomorphism, then the dimensions of the spaces* $F_\alpha^-(V)_\gamma$ *can be calculated from* (1.1.2). 4) *The object* $\operatorname{Ker} p_V^\alpha$ *is concentrated at* α *(that is,* $(\operatorname{Ker} p_V^\alpha)_\gamma = 0$ *for* $\gamma \neq \alpha$*). The object* $V/\operatorname{Im} i_V^\beta$ *is concentrated at* β. 5) *If the object* V *has the form* F_α^+W *(*F_β^-W*, respectively), then* p_V^α *(*i_V^β*) is an isomorphism.* 6) *The object* V *is isomorphic to the direct sum of the objects* $F_\beta^-F_\beta^+(V)$ *and* $V/\operatorname{Im} i_V^\beta$ *(similarly,* $V \approx F_\alpha^+F_\alpha^-(V) \oplus \ker p_V^\alpha$*).*

PROOF. 1), 2), 3), 4) and 5) can be verified immediately. Let us prove 6).

We have to show that $V \approx F_\beta^-F_\beta^+(V) \oplus \widetilde{V}$, where $\widetilde{V} = V/\operatorname{Im} i_V^\beta$. The natural projection φ_β': $V_\beta \to \widetilde{V}_\beta$ has a section φ_β: $\widetilde{V}_\beta \to V_\beta$ $(\varphi_\beta' \cdot \varphi_\beta = \mathrm{Id})$. If we put $\varphi_\gamma = 0$ for $\gamma \neq \beta$, we obtain a morphism φ: $\widetilde{V} \to V$. It is clear that the morphisms φ: $\widetilde{V} \to V$ and i_V^β: $F_\beta^-F_\beta^+(V) \to V$ give a decomposition of V into a direct sum. We can prove similarly that $V \approx F_\alpha^+F_\alpha^-(V) \oplus \operatorname{Ker} p_V^\alpha$.

We now prove Theorem 1.1. Let V be an indecomposable object of the category $\mathscr{L}(\Gamma, \Lambda)$, and β a (+)-accessible vertex with respect to Λ. Since

$V \approx F_\beta^- F_\beta^+(V) \oplus V/\mathrm{Im}\, i_V^\beta$ and V is indecomposable, V coincides with one of the terms.

CASE I). $V = V/\mathrm{Im}\, i_V^\beta$. Then $V_\gamma = 0$ for $\gamma \neq \beta$ and, because V is indecomposable, $V \approx L_\beta$.

CASE II). $V = F_\beta^- F_\beta^+(V)$, that is, i_V^β is an isomorphism. Then (1.1.1) is satisfied by Lemma 1.1. We show that the object $W = F_\beta^+(V)$ is indecomposable. For suppose that $W = W_1 \oplus W_2$. Then $V = F_\beta^-(W_1) \oplus F_\beta^-(W_2)$ and so one of the terms (for example, $F_\beta^-(W_2)$) is 0. By 5) of Lemma 1.1, the morphism $p_V^\beta : W \to F_\beta^+ F_\beta^-(W)$ is an isomorphism, but $p_V^\beta(W_2) \subset F_\beta^+ F_\beta^-(W_2) = 0$, that is, $W_2 = 0$.

So we have shown that the object $F_\beta^+(V)$ is indecomposable. We can similarly prove 2) of Theorem 1.1.

We say that a sequence of vertices $\alpha_1, \alpha_2, \ldots, \alpha_k$ is (+)-accessible with respect to Λ if α_1 is (+)-accessible with respect to Λ, α_2 is (+)-accessible with respect to $\sigma_{\alpha_1}\Lambda$, α_3 is (+)-accessible with respect to $\sigma_{\alpha_2}\sigma_{\alpha_1}\Lambda$, and so on. We define a (−)-accessible sequence similarly.

COROLLARY 1.1. *Let* (Γ, Λ) *be an oriented graph and* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *a (+)-accessible sequence.*

1) *For any* i $(1 \leqslant i \leqslant k)$, $F_{\alpha_1}^- \cdot \ldots \cdot F_{\alpha_{i-1}}^- (L_{\alpha_i})$ *is either* 0 *or an indecomposable object in* $\mathscr{L}(\Gamma, \Lambda)$ *(here* $L_{\alpha_i} \in \mathscr{L}(\Gamma, \sigma_{\alpha_{i-1}}\sigma_{\alpha_{i-2}} \ldots \sigma_{\alpha_1}\Lambda)$*).*[1]

2) *Let* $V \in \mathscr{L}(\Gamma, \Lambda)$ *be an indecomposable object, and*

$$F_{\alpha_k}^+ F_{\alpha_{k-1}}^+ \cdot \ldots \cdot F_{\alpha_1}^+(V) = 0.$$

Then for some i

$$V \approx F_{\alpha_1}^- F_{\alpha_2}^- \cdot \ldots \cdot F_{\alpha_{i-1}}^- (L_{\alpha_i}).$$

We illustrate the application of the functors F_β^+ and F_α^- by the following theorem.

THEOREM 1.2. *Let* Γ *be a graph without cycles (in particular, without loops), and* Λ, Λ' *two orientations of it.*

1) *There exists a sequence of vertices* $\alpha_1, \ldots, \alpha_k$, *(+)-accessible with respect to* Λ, *such that* $\sigma_{\alpha_k}\sigma_{\alpha_{k-1}} \cdot \ldots \cdot \sigma_{\alpha_1}\Lambda = \Lambda'$.

2) *Let* $\mathscr{M}$, $\mathscr{M}'$ *be the sets of classes (to within isomorphism) of indecomposable objects in* $\mathscr{L}(\Gamma, \Lambda)$ *and* $\mathscr{L}(\Gamma, \Lambda')$, $\widetilde{\mathscr{M}} \subset \mathscr{M}$ — *the set of classes of objects* $F_{\alpha_1}^- F_{\alpha_2}^- \cdot \ldots \cdot F_{\alpha_{i-1}}^- (L_{\alpha_i})$ $(1 \leqslant i \leqslant k)$, *and* $\widetilde{\mathscr{M}}' \subset \mathscr{M}'$ *the set of classes of objects* $F_{\alpha_k}^+ \cdot \ldots \cdot F_{\alpha_{i+1}}^+ (L_{\alpha_i})$ $(1 \leqslant i \leqslant k)$. *Then the functor* $F_{\alpha_k}^+ \cdot \ldots \cdot F_{\alpha_1}^+$ *sets up a one-to-one correspondence between* $\mathscr{M} \setminus \widetilde{\mathscr{M}}$ *and* $\mathscr{M}' \setminus \widetilde{\mathscr{M}}'$.

[1] Where it cannot lead to misunderstanding, we denote by the same symbol L_α irreducible objects in all categories $\mathscr{L}(\Gamma, \Lambda)$, omitting the indication of the orientation Λ.

This theorem shows that, knowing the classification of indecomposable objects for Λ, we can easily carry it over to Λ'; in other words, problems that can be obtained from one another by reversing some of the arrows are equivalent in a certain sense.

Examples show that the same is true for graphs with cycles, but we are unable to prove it.

PROOF OF THEOREM 1.2. It is clear that 2) follows at once from 1) and Corollary 1.1. Let us prove 1).

It is sufficient to consider the case when the orientations Λ and Λ' differ in only one edge l. The graph $\Gamma \setminus l$ splits into two connected components. Let Γ' be the one that contains the vertex $\beta(l)$ ($\beta(l)$ is taken with the orientation of Λ). Let $\alpha_1, \ldots, \alpha_k$ be a numbering of the vertices of Γ' such that for any edge $l' \in \Gamma'_1$ the index of the vertex $\alpha(l')$ is greater than that of $\beta(l')$. (Such a numbering exists because Γ' is a graph without cycles.) It is easy to see that the sequence of vertices $\alpha_1, \ldots, \alpha_k$ is the one required (that is, it is (+)-accessible and $\sigma_{\alpha_k} \cdot \ldots \cdot \sigma_{\alpha_1}\Lambda = \Lambda'$). This proves Theorem 1.2.

It is often convenient to use a certain combination of functors $F_\alpha^\pm$ that takes the category $\mathcal{L}(\Gamma, \Lambda)$ into itself.

DEFINITION 1.2. Let (Γ, Λ) be an oriented graph without oriented cycles. We choose a numbering $\alpha_1, \ldots, \alpha_n$ of the vertices of Γ such that for any edge $l \in \Gamma_1$ the index of the vertex $\alpha(l)$ is greater than that of $\beta(l)$. We put $\Phi^+ = F^+_{\alpha_n} \cdot \ldots \cdot F^+_{\alpha_2} F^+_{\alpha_1}$, $\Phi^- = F^-_{\alpha_1} \cdot F^-_{\alpha_2} \cdot \ldots \cdot F^-_{\alpha_n}$. We call Φ^+ and Φ^- *Coxeter functors.*

LEMMA 1.2. 1) *The sequence* $\alpha_1, \ldots, \alpha_n$ *is (+)-accessible and* $\alpha_n, \ldots, \alpha_1$ *is (−)-accessible.* 2) *The functors* Φ^+ *and* Φ^- *take the category* $\mathcal{L}(\Gamma, \Lambda)$ *into itself.* 3) Φ^+ *and* Φ^- *do not depend on the freedom of choice in numbering the vertices.*

The proof of 1) and 2) is obvious. We prove 3) for Φ^+. We note firstly that if two different vertices $\gamma_1, \gamma_2 \in \Gamma_0$ are not joined by an edge and are (+)-accessible with respect to some orientation, then the functors $F^+_{\gamma_1}$ and $F^+_{\gamma_2}$ commute (that is, $F^+_{\gamma_2} F^+_{\gamma_1} = F^+_{\gamma_1} F^+_{\gamma_2}$).

Let $\alpha_1, \ldots, \alpha_n$ and $\alpha'_1, \ldots \alpha'_n$ be two suitable numberings and let $\alpha_1 = \alpha'_m$. Then the vertices $\alpha'_1, \alpha'_2, \ldots, \alpha'_{m-1}$ are not joined to α_1 by an edge (if α_1 and α'_i $(i < m)$ are joined by an edge l, then $\alpha(l) = \alpha'_m = \alpha_1$ by virtue of the choice of the numbering of $\alpha'_1, \ldots, \alpha'_n$, but this contradicts the choice of the numbering of $\alpha_1, \ldots, \alpha_n$). Therefore $F^+_{\alpha'_m} \cdot \ldots \cdot F^+_{\alpha'_1} = F^+_{\alpha'_{m-1}} \cdot \ldots \cdot F^+_{\alpha'_1} F^+_{\alpha_1}$. Carrying out a similar argument with α_2, then with α_3, and so on, we prove that $F^+_{\alpha'_n} \cdot \ldots \cdot F^+_{\alpha'_1}$ $F^+_{\alpha'_n} \cdot \ldots \cdot F^+_{\alpha'_1} = F^+_{\alpha_n} \cdot \ldots \cdot F^+_{\alpha_1}$.

The proof is similar for the functor Φ^-.

Following [2] we can introduce the following definition.

DEFINITION 1.3. Let (Γ, Λ) be an oriented graph without oriented cycles. We say that an object $V \in \mathcal{L}(\Gamma, \Lambda)$ is (+)-(respectively, (−)-) irregular if $(\Phi^+)^k V = 0$ $((\Phi^-)^k V = 0)$ for some k. We say that an object V is regular if $V \approx (\Phi^-)^k (\Phi^+)^k V \approx (\Phi^+)^k (\Phi^-)^k V$ for all k.

NOTE 1. Using the morphisms p_V^α and i_V^β introduced in the proof of Theorem 1.1, we can construct a canonical epimorphism $p_V^k\colon V \to (\Phi^+)^k (\Phi^-)^k V$ and monomorphism $i_V^k\colon (\Phi^-)^k (\Phi^+)^k V \to V$. The object V is regular if and only if for all k these morphisms are isomorphisms.

NOTE 2. If an object V is annihilated by the functor $F_{\alpha_s}^+ \cdot \ldots \cdot F_{\alpha_1}^+$ ($\alpha_1, \ldots, \alpha_s$ is some (+)-accessible sequence), then this object is (+)-irregular. Moreover, the sequence $\alpha_1, \ldots, \alpha_s$ can be extended to $\alpha_1, \ldots, \alpha_s, \alpha_{s+1}, \ldots, \alpha_m$ so that $F_{\alpha_m}^+ \cdot \ldots \cdot F_{\alpha_{s+1}}^+ \cdot F_{\alpha_s}^+ \cdot \ldots \cdot F_{\alpha_1}^+ = (\Phi^+)^s$.

THEOREM 1.3. *Let (Γ, Λ) be an oriented graph without oriented cycles.* 1) *Each indecomposable object $V \in \mathcal{L}(\Gamma, \Lambda)$ is either regular or irregular.* 2) *Let $\alpha_1, \ldots, \alpha_n$ be a numbering of the vertices of Γ such that for any $l \in \Gamma_1$ the index of $\alpha(l)$ is greater than that of $\beta(l)$. Put $V_i = F_{\alpha_1}^- F_{\alpha_2}^- \cdot \ldots \cdot F_{\alpha_{i-1}}^- (L_{\alpha_i}) \in \mathcal{L}(\Gamma, \Lambda)$, $\hat{V}_i = F_{\alpha_n}^+ \cdot \ldots \cdot F_{\alpha_{i+1}}^+ (L_{\alpha_i}) \in \mathcal{L}(\Gamma, \Lambda)$ (here $1 \leqslant i \leqslant n$). Then $\Phi^+(V_i) = 0$ and any indecomposable object $V \in \mathcal{L}(\Gamma, \Lambda)$ for which $\Phi^+(V) = 0$ is isomorphic to one of the objects V_i. Similarly, $\Phi^-(\hat{V}_i) = 0$, and if V is indecomposable and $\Phi^-(V) = 0$, then $V \approx \hat{V}_i$ for some i.* 3) *Each (+)-(respectively, (−)-) irregular indecomposable object V has the form $(\Phi^-)^k V_i$ (respectively, $(\Phi^+)^k \hat{V}_i$) for some i, k.*

Theorem 1.3 follows immediately from Corollary 1.1.

With the help of this theorem it is possible, as was done in [2] for the classification of quadruples of subspaces, to distinguish "simple" (irregular) objects from more "complicated" (regular) objects; other methods are necessary for the investigation of regular objects.

§ 2. Graphs, Weyl groups and Coxeter transformations

In this section we define Weyl groups, roots, and Coxeter transformations, and we prove results that are needed subsequently. We mention two differences between our account and the conventional one.

a) We have only Dynkin diagrams with single arrows.

b) In the case of graphs with multiple edges we obtain a wider class of groups than, for example, in [7].

DEFINITION 2.1. Let Γ be a graph without loops.

1) We denote by $\mathscr{E}_\Gamma$ the linear space over $\mathbf{Q}$ consisting of sets $x = (x_\alpha)$ of rational numbers x_α $(\alpha \in \Gamma_0)$.

For each $\beta \in \Gamma_0$ we denote by $\bar{\beta}$ the vector in $\mathscr{E}_\Gamma$ such that $(\bar{\beta})_\alpha = 0$ for $\alpha \neq \beta$ and $(\bar{\beta})_\beta = 1$.

We call a vector $x = (x_\alpha)$ *integral* if $x_\alpha \in \mathbf{Z}$ for all $\alpha \in \Gamma_0$.

We call a vector $x = (x_\alpha)$ *positive* (written $x > 0$) if $x \neq 0$ and

$x_\alpha \geqslant 0$ for all $\alpha \in \Gamma_0$.

2) We denote by B the quadratic form on the space $\mathscr{E}_\Gamma$ defined by the formula $B(x) = \sum_{\alpha \in \Gamma_0} x_\alpha^2 - \sum_{l \in \Gamma_1} x_{\gamma_1(l)} \cdot x_{\gamma_2(l)}$, where $x = (x_\alpha)$, and $\gamma_1\ (l)$ and $\gamma_2\ (l)$ are the ends of the edge l. We denote by $\langle\, ,\, \rangle$ the corresponding symmetric bilinear form.

3) For each $\beta \in \Gamma_0$ we denote by σ_β the linear transformation in $\mathscr{E}_\Gamma$ defined by the formula $(\sigma_\beta x)_\gamma = x_\gamma$ for $\gamma \neq \beta$, $(\sigma_\beta x)_\beta = - x_\beta + \sum_{l \in \Gamma^\beta} x_{\gamma(l)}$, where $\gamma(l)$ is the end-point of the edge l other than β.

We denote by W the semigroup of transformations of $\mathscr{E}_\Gamma$ generated by the σ_β $(\beta \in \Gamma_0)$.

LEMMA 2.1. 1) *If $\alpha, \beta \in \Gamma_0$, $\alpha \neq \beta$, then $\langle \bar{\alpha}\ \bar{\alpha} \rangle = 1$ and $2 \langle \bar{\alpha}\ \bar{\beta} \rangle$ is the negative of the number of edges joining α and β.* 2) *Let $\beta \in \Gamma_0$. Then $\sigma_\beta(x) = x - 2 \langle \bar{\beta}, x \rangle \bar{\beta}$, $\sigma_\beta^2 = 1$. In particular, W is a group.* 3) *The group W preserves the integral lattice in $\mathscr{E}_\Gamma$ and preserves the quadratic form B.* 4) *If the form B is positive definite (that is, $B(x) > 0$ for $x \neq 0$), then the group W is finite.*

PROOF. 1), 2) and 3) are verified immediately; 4) follows from 3).

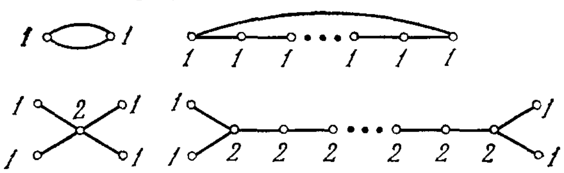
For the proof of Gabriel's theorem the case where B is positive definite is interesting.

PROPOSITION 2.1. *The form B is positive definite for the graphs A_n, D_n, E_6, E_7, E_8 and only for them* (see [7], Chap. VI).

We give an outline of the proof of this proposition.

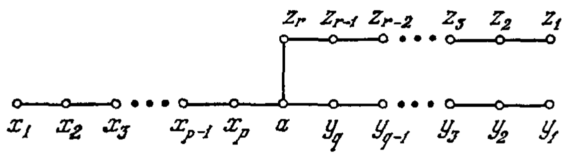
1. If Γ contains a subgraph of the form

(*)



then the form B is not positive definite, because when we complete the numbers at the vertices in Fig. (*) by zeros, we obtain a vector $x \in \mathscr{E}_\Gamma$ for which $B(x) \leqslant 0$. Hence, if B is positive definite, then Γ has the form

(**)



where p, q, r are non-negative integers.

2 For each non-negative integer p we consider the quadratic form in $(p+1)$ variables $x_1, \ldots, x_{p+1}$

$$C_p(x_1, \ldots, x_{p+1}) = -x_1x_2 - x_2x_3 - \ldots - x_px_{p+1} + x_1^2 + \ldots + x_p^2 + \frac{p}{2(p+1)} x_{p+1}^2.$$

This form is non-negative definite, and the dimension of its null space is 1. Moreover, any vector $x \neq 0$ for which $C_p(x) = 0$ has all its coordinates non-zero.

To prove these facts it is sufficient to rewrite $C_p(x)$ in the form

$$C_p(x) = \sum_{i=1}^{p} \frac{i}{2(i+1)} \left(x_{i+1} - \frac{i+1}{i} x_i\right)^2.$$

3. We place the numbers $x_1, \ldots, x_p, y_1, \ldots, y_q, z_1, \ldots, z_r$ a at the vertices of Γ in accordance with Fig. (**). Then

$$B(x_i, y_i, z_i, a) = C_p(x_1, \ldots, x_p, a) + C_q(y_1, \ldots, y_q, a) + \\ + C_r(z_1, \ldots, z_r, a) + \left(1 - \frac{p}{2(p+1)} - \frac{q}{2(q+1)} - \frac{r}{2(r+1)}\right) a^2.$$

Hence it is clear that B is positive definite if and only if

$\frac{p}{2(p+1)} + \frac{q}{2(q+1)} + \frac{r}{2(r+1)} < 1$, that is, $\frac{1}{p+1} + \frac{1}{q+1} + \frac{1}{r+1} > 1$.

4. We may suppose that $p \leqslant q \leqslant r$. We examine possible cases.

a) $p = 0$, q and r arbitrary. $A = \frac{1}{p+1} + \frac{1}{q+1} + \frac{1}{r+1} > 1$, that is, B is positive definite (series A_n).

b) $p = 1$, $q = 1$, r arbitrary. $A > 1$ (series D_n),

c) $p = 1$, $q = 2$, $r = 2, 3, 4$. $A > 1$ (E_6, E_7, E_8),

d) $p = 1$, $q = 2$, $r \geqslant 5$. $A \leqslant 1$,

$p = 1$, $q = 3$, $r \geqslant 3$. $A \leqslant 1$,

$p \geqslant 2, q \geqslant 2, r \geqslant 2$. $A \leqslant 1$.

Thus B is positive definite for the graphs A_n, D_n, E_6, E_7, E_8 and only for them.

DEFINITION 2.2 A vector $x \in \mathscr{E}_\Gamma$ is called a *root* if for some $\beta \in \Gamma_0$, $w \in W$ we have $x = w\bar{\beta}$. The vectors $\bar{\beta}$ ($\beta \in \Gamma_0$) are called *simple roots*. A root x is called positive if $x > 0$ (see Definition 2.1).

LEMMA 2.2 1) *If x is a root, then x is an integral vector and* $B(x) = 1$. 2) *If x is a root, then $(-x)$ is a root.* 3) *If x is a root, then either $x > 0$ or $(-x) > 0$.*

PROOF. 1) follows from Lemma 2.1; 2) follows from the fact that $\sigma_\alpha(\bar{\alpha}) = -\bar{\alpha}$ for all $\alpha \in \Gamma_0$.

3) is needed only when B is positive definite and we prove it only in this case.

We can write the root x in the form $\sigma_{\alpha_1}\sigma_{\alpha_2} \cdot \ldots \cdot \sigma_{\alpha_k}\bar{\beta}$, where $\alpha_1, \ldots, \alpha_k, \beta \in \Gamma_0$. It is therefore sufficient to show that if $y > 0$ and $\alpha \in \Gamma_0$, then either $\sigma_\alpha y > 0$ or $y = \bar{\alpha}$ (and $-\sigma_\alpha y = +\bar{\alpha} > 0$).

Since $\|y\| = \|\bar{\alpha}\| = 1$, we have $|\langle\bar{\alpha}, y\rangle| \leqslant 1$. Moreover, $2\langle\bar{\alpha}, y\rangle \in \mathbf{Z}$. Hence $2\langle\bar{\alpha}, y\rangle$ takes one of the five values $2, 1, 0, -1, -2$.

a) $2\langle\bar{\alpha}, y\rangle = 2$. Then $\langle\bar{\alpha}, y\rangle = 1$, that is, $y = \bar{\alpha}$.

b) $2\langle\bar{\alpha}, y\rangle \leqslant 0$. Then $\sigma_\alpha(y) = y - 2\langle\bar{\alpha}, y\rangle\,\bar{\alpha} > 0$.

c) $2\langle\bar{\alpha}, y\rangle = 1$. Since $2\langle\bar{\alpha}, y\rangle = 2y_\alpha - \sum_{l\in\Gamma^\alpha} y_{\gamma(l)}$ ($\gamma(l)$ is the other end-point of the edge l), we have $y_\alpha > 0$, that is, $y_\alpha \geqslant 1$. Hence $\sigma_\alpha y = y - \bar{\alpha} > 0$.

This proves Lemma 2.2.

DEFINITION 2.3. Let Γ be a graph without loops, and let $\alpha_1, \ldots, \alpha_n$ be a numbering of its vertices. An element $c = \sigma_{\alpha_n} \cdot \ldots \cdot \sigma_{\alpha_1}$ (c depends on the choice of numbering) of the group W is called a *Coxeter transformation.*

LEMMA 2.3. *Suppose that the form B for the graph Γ is positive definite:*

1) *the transformation c in $\mathscr{E}_\Gamma$ has non non-zero invariant vectors;*

2) *if $x \in \mathscr{E}_\Gamma$, $x \neq 0$, then for some i the vector $c^i x$ is not positive.*

PROOF. 1) Suppose that $y \in \mathscr{E}_\Gamma$, $y \neq 0$ and $cy = y$. Since the transformations $\sigma_{\alpha_n}, \sigma_{\alpha_{n-1}}, \ldots, \sigma_{\alpha_2}$ do not change the coordinate corresponding to α_1 (that is, for any $z \in \mathscr{E}_\Gamma$ $(\sigma_{\alpha_i} z)_{\alpha_1} = z_{\alpha_1}$ for $i \neq 1$), we have $(\sigma_{\alpha_1} y)_{\alpha_1} = (cy)_{\alpha_1} = y_{\alpha_1}$. Hence $\sigma_{\alpha_1} y = y$ Similarly we can prove that $\sigma_{\alpha_2} y = y$, then $\sigma_{\alpha_3} y = y$, and so on.

For all $\alpha \in \Gamma_0$, $\sigma_\alpha y = y - 2\langle\bar{\alpha}, y\rangle\bar{\alpha} = y$, that is $\langle\bar{\alpha}, y\rangle = 0$. Since the vectors $\bar{\alpha}$ ($\alpha \in \Gamma_0$) form a basis of $\mathscr{E}_\Gamma$ and B is non-degenerate, $y = 0$.

2) Since W is a finite group, for some h we have $c^h = 1$. If all the vectors $x, cx, \ldots, c^{h-1}x$ are positive, then $y = x + cx + \ldots + c^{h-1}x$ is non-zero. Hence $cy = y$, which contradicts 1).

§ 3. Gabriel's theorem

Let (Γ, Λ) be an oriented graph. For each object $V \in \mathscr{L}(\Gamma, \Lambda)$ we regard the set of dimensions $\dim V_\alpha$ as a vector in $\mathscr{E}_\Gamma$ and denote it by $\dim V$.

THEOREM 3.1 (Gabriel [1]). 1) *If in $\mathscr{L}(\Gamma, \Lambda)$ there are only finitely many non-isomorphic indecomposable objects, then Γ coincides with one of the graphs A_n, D_n, E_6, E_7, E_8.*

2) *Let Γ be a graph of one of the types A_n, D_n, E_6, E_7, E_8, and Λ some orientation of it. Then in $\mathscr{L}(\Gamma, \Lambda)$ there are only finitely many non-isomprphic indecomposable objects. In addition, the mapping $V \mapsto \dim V$ sets up a one-to-one correspondence between classes of isomorphic indecomposable objects and positive roots in $\mathscr{E}_\Gamma$.*

We start with a proof due to Tits of the first part of the theorem.

TITS'S PROOF. Consider the objects $(V, f) \in \mathscr{L}(\Gamma, \Lambda)$ with a fixed dimension $\dim V = m = (m_\alpha)$.

If we fix a basis in each of the spaces V_α, then the object (V, f) is completely defined by the set of matrices A_l ($l \in \Gamma_1$), where A_l is the matrix of the mapping f_l: $V_{\alpha(l)} \to V_{\beta(l)}$. In each space V_α we change the basis by means of a non-singular $(m_\alpha \times m_\alpha)$ matrix g_α. Then the matrices A_l are replaced by the matrices

$$(*) \qquad A_l' = g_{\beta(l)}^{-1} A_l \, g_{\alpha(l)}.$$

Let A be the manifold of all sets of matrices A_l ($l \in \Gamma_1$) and G the group of all sets of non-singular matrices g_α ($\alpha \in \Gamma_0$). Then G acts on A according to (*); clearly, two objects of $\mathcal{L}(\Gamma, \Lambda)$ with given dimension m are isomorphic if and only if the sets of matrices $\{A_l\}$ corresponding to them lie in one orbit of G.

If in $\mathcal{L}(\Gamma, \Lambda)$ there are only finitely many indecomposable objects, then there are only finitely many non-isomorphic objects of dimension m. Therefore the manifold A splits into a finite number of orbits of G. It follows[1] that $\dim A \leqslant \dim G - 1$ (the -1 is explained by the fact that G has a 1-dimensional subgroup $G_0 = \{g(\lambda) \mid \lambda \in K^*\}$, $g(\lambda)_\alpha = \lambda \cdot 1_{V_\alpha}$, which acts on A identically). Clearly, $\dim G = \sum_{\alpha \in \Gamma_0} m_\alpha^2$, $\dim A = \sum_{l \in \Gamma_1} m_{\alpha(l)} m_{\beta(l)}$.

Therefore the condition $\dim A \leqslant \dim G - 1$ can be rewritten in the form[2] $B(m) > 0$ (if $m \neq 0$). In addition, it is easy to verify that $B((x_\alpha)) \geqslant B((|x_\alpha|))$ for all $x = (x_\alpha) \in \mathscr{E}_\Gamma$.

So we have shown that if in $\mathcal{L}(\Gamma, \Lambda)$ there are finitely many indecomposable objects, then the form B in $\mathscr{E}_\Gamma$ is positive definite.

As we have shown in Proposition 2.1, this holds only for the graphs A_n, D_n, E_6, E_7, E_8.

We now prove the second part of Gabriel's theorem.

LEMMA 3.1. *Suppose that* (Γ, Λ) *is an oriented graph,* $\beta \in \Gamma_0$ *a* (+)-*accessible vertex with respect to* Λ, *and* $V \in \mathcal{L}(\Gamma, \Lambda)$ *an indecomposable object. Then either* $F_\beta^+(V)$ *is an indecomposable object and* $\dim F_\beta^+(V) = \sigma_\beta(\dim V)$, *or* $V = L_\beta$, $F_\beta^+(V) = 0$, $\dim F_\beta^+(V) \neq \sigma_\beta(\dim V) < 0$. *A similar statement holds for a* (–)-*accessible vertex* α *and the functor* F_α^-.

This lemma is a reformulation of Theorem 1.1.

COROLLARY 3.1. *Suppose that the sequence of vertices* $\alpha_1, \ldots, \alpha_k$ *is* (+)-*accessible with respect to* Λ *and that* $V \in \mathcal{L}(\Gamma, \Lambda)$ *is an indecomposable object. Put* $V_j = F_{\alpha_j}^+ F_{\alpha_{j-1}}^+ \cdot \ldots \cdot F_{\alpha_1}^+ V$, $m_j = \sigma_{\alpha_j}\sigma_{\alpha_{j-1}} \cdot \ldots \cdot \sigma_{\alpha_1}(\dim V)$ $(0 \leqslant j \leqslant k)$. *Let i be the last index such that* $m_j > 0$ *for* $j \leqslant i$. *Then the* V_j *are indecomposable objects for* $j \leqslant i$, *and* $V = F_{\alpha_1}^- \cdot \ldots \cdot F_{\alpha_j}^- V_j$. *If* $i < k$, *then* $V_{i+1} = V_{i+2} = \ldots = V_k = 0$, $V_i = L_{\alpha_{i+1}}$, $V = F_{\alpha_1}^- \cdot \ldots \cdot F_{\alpha_i}^- (L_{\alpha_{i+1}})$. *Similar statements are true when* (+) *is replaced by* (–).

We now show that in the case of a graph Γ of type A_n, D_n, E_6, E_7 or E_8 (that is, B is positive definite), indecomposable objects correspond to positive roots.

a) Let $V \in \mathcal{L}(\Gamma, \Lambda)$ be an indecomposable object.

[1] This argument is suitable only for an infinite field K. If $K = \mathbf{F}_q$ is a finite field, we must use the fact that the number of non-isomorphic objects of dimension m increases no faster than a polynomial in m, and the number of orbits of G on the manifold A is not less than $C \cdot q^{\dim A - (\dim G - 1)}$.

[2] We can clearly restrict ourselves to graphs without loops.

We choose a numbering $\alpha_1, \alpha_2, \ldots, \alpha_n$ of the vertices of Γ such that for any edge $l \in \Gamma_1$ the vertex $\alpha(l)$ has an index greater than that of $\beta(l)$. Let $c = \sigma_{\alpha_n} \cdot \ldots \cdot \sigma_{\alpha_1}$ be the corresponding Coxeter transformation.

By Lemma 2.3, for some k the vector $c^k(\dim V) \in \mathscr{E}_\Gamma$ is not positive.

If we consider the (+)-accessible sequence $\beta_1, \beta_2, \ldots, \beta_{nk} = (\alpha_1, \ldots, \alpha_n, \alpha_1, \ldots, \alpha_n, \ldots, \alpha_1, \ldots, \alpha_n)$ (k times), then we have $\sigma_{\beta_{nk}} \cdot \ldots \cdot \sigma_{\beta_1}(\dim V) = c^k(\dim V) \ngtr 0$. From Corollary 3.1 it follows that there is an index $i < kn$ (depending only on $\dim V$) such that $V = F^-_{\beta_1} \cdot F^-_{\beta_2} \cdot \ldots \cdot F^-_{\beta_i}(L_{\beta_{i+1}})$, $\dim V = \sigma_{\beta_1} \cdot \ldots \cdot \sigma_{\beta_i}(\bar{\beta}_{i+1})$. It follows that $\dim V$ is a positive root and V is determined by the vector $\dim V$.

b) Let x be a positive root.

By Lemma 2.3, $c^k x \ngtr 0$ for some k. Consider the (+)-accessible sequence $\beta_1, \beta_2, \ldots, \beta_{nk} = (\alpha_1, \ldots, \alpha_n, \ldots, \alpha_1, \ldots, \alpha_n)$ (k times). Then $\sigma_{\beta_{nk}} \cdot \ldots \cdot \sigma_{\beta_1}(x) = c^k(x) \ngtr 0$. Let i be the last index for which $\sigma_{\beta_i}\sigma_{\beta_{i-1}} \cdot \ldots \cdot \sigma_{\beta_1}(x) > 0$. It is obvious from the proof of 3) in Lemma 2.2 that $\sigma_{\beta_i} \cdot \ldots \cdot \sigma_{\beta_1}(x) = \bar{\beta}_{i+1}$.

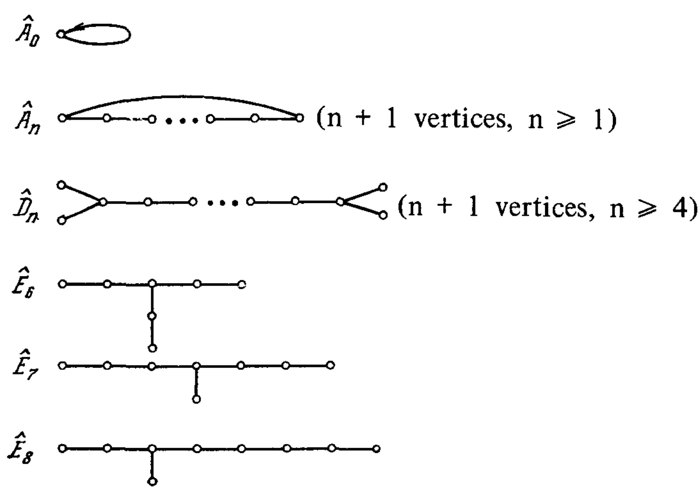
It follows that Corollary 3.1 that $V = F^-_{\beta_1} F^-_{\beta_2} \cdot \ldots \cdot F^-_{\beta_i}(L_{\beta_{i+1}}) \in \mathscr{L}(\Gamma, \Lambda)$ is an an indecomposable object and $\dim V = \sigma_{\beta_1} \cdot \ldots \cdot \sigma_{\beta_i}(\bar{\beta}_{i+1}) = x$.

This concludes the proof of Gabriel's theorem.

NOTE 1. When B is positive definite, the set of roots coincides with the set of integral vectors $x \in \mathscr{E}_\Gamma$ for which $B(x) = 1$ (this is easy to see from Lemma 2.3 and the proof of Lemma 2.2).

NOTE 2. It is interesting to consider categories $\mathscr{L}(\Gamma, \Lambda)$, for which the canonical form of an object of dimension m depends on fewer than $C \cdot |m|^2$ parameters (here $|m| = \Sigma\, |m_\alpha|$, $\alpha \in \Gamma_0$). From the proof it is obvious that for this it is necessary that B should be non-negative definite.

As in Proposition 2.1 we can show that B is non-negative definite for the graphs A_n, D_n, E_6, E_7, E_8 and $\hat{A}_0$, $\hat{A}_n$, $\hat{D}_n$, $\hat{E}_6$, $\hat{E}_7$, $\hat{E}_8$, where



(the graphs $\hat{A}_n$, $\hat{D}_n$, $\hat{E}_6$, $\hat{E}_7$, $\hat{E}_8$ are extensions of the Dynkin diagrams (see [7])).

In a recent article Nazarova has given a classification of indecomposable objects for these graphs. In addition, she has shown there that such a classification for the remaining graphs would contain a classification of pairs of non-commuting operators (that is, in a certain sense it is impossible to give such a classification).

§ 4. Some open questions

Let Γ be a finite connected graph without loops and Λ an orientation of it.

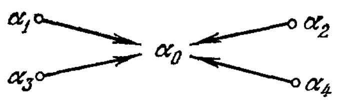
CONJECTURES. 1) Suppose that $x \in \mathscr{E}_\Gamma$ is an integral vector, $x > 0$, $B(x) > 0$ and x is not a root. Then any object $V \in \mathscr{L}(\Gamma, \Lambda)$ for which $\dim V = x$ is decomposable.

2) If x is a positive root, then there is exactly one (to within isomorphism) indecomposable object $V \in \mathscr{L}(\Gamma, \Lambda)$, for which $\dim V = x$.

3) If V is an indecomposable object in $\mathscr{L}(\Gamma, \Lambda)$ and $B(\dim V) \leqslant 0$, then there are infinitely many non-isomorphic indecomposable objects $V' \in \mathscr{L}(\Gamma, \Lambda)$ with $\dim V' = \dim V$ (we suppose that K is an infinite field).

4) If Λ and Λ' are two orientations of Γ and $V \in \mathscr{L}(\Gamma, \Lambda')$ is an indecomposable object, then there is an indecomposable object $V' \in \mathscr{L}(\Gamma, \Lambda')$ such that $\dim V' = \dim V$.

We illustrate this conjecture by the example of the graph (Γ, Λ)



(quadruple of subspaces).

For each $x \in \mathscr{E}_\Gamma$ we put $\rho(x) = -2\langle\bar{\alpha}_0, x\rangle$ (if $x = (x_0, x_1, x_2, x_3, x_4)$, then $\rho(x) = x_1 + x_2 + x_3 + x_4 - 2x_0$).

In [2] all the indecomposable objects in the category $\mathscr{L}(\Gamma, \Lambda)$ are described. They are of the following types.

1. Irregular indecomposable objects (see the end of § 1). Such objects are in one-to-one correspondence with positive roots x for which $\rho(x) \neq 0$.

2. Regular indecomposable objects V for which $B(\dim V) \neq 0$. These objects are in one-to-one correspondence with positive roots x for which $\rho(x) = 0$.

3) Regular objects V for which $B(\dim V) = 0$. In this case $\dim V$ has the form $\dim V = (2n, n, n, n, n)$, $\rho(\dim V) = 0$. Indecomposable objects with fixed dimension $m = (2n, n, n, n, n)$ depend on one parameter. If $m \in \mathscr{E}_\Gamma$ is an integral vector such that $m > 0$ and $B(m) = 0$, then it has the form $m = (2n, n, n, n, n)$ $(n > 0)$ and there are indecomposable objects V for which $\dim V = m$.

If f is a linear transformation in n-dimensional space consisting of one

Jordan block then the quadruple of subspaces corresponding to it (see the Introduction) is a quadruple of the third type.

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