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A Proof of Jantzen Conjectures

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To our teacher Israel M. Gelfand

In this paper we will show that the localization functor sends the Jantzen filtration on Verma modules (or, more generally, standard Harish-Chandra modules) to the weight filtration on the corresponding perverse sheaves. This fact immediately implies a lot of remarkable properties of the Jantzen filtration: the hereditary property (conjectured by Jantzen), the socle and cosocle properties, and the Kazhdan-Lusztig algorithm for the computation of multiplicities in consecutive quotients (conjectured in [GJ1] and [GM]).

The paper is divided into two parts. The first part occupies Sections 1 and 2. In Section 1 we consider sheaves of noncommutative algebras on an algebraic variety which are of "local origin" (we call them D-algebras); a typical example is the algebra of differential operators. We list some basic functorial properties of such algebras. Section 2 deals with an important class of D-algebras: rings of twisted differential operators (tdo). We included in Section 2 more material than the minimum needed for the main part of our paper, since the language of tdo's is a convenient gadget in many situations (e.g., in algebraic versions of conformal field theory), and we thought it would be nice to have a review of the subject.

The second part deals with representation theory. It is the heart of this paper. In Section 3 we recall the localization construction [BB1] and write down some properties of K-orbits of flag varieties that will be of use. In Section 4 we define the Jantzen filtration in a geometric setting and describe its intimate relation with the monodromy filtration on vanishing cycles. In Section 5 mixed sheaves appear. We present a proof of Gabber's theorem about the weight filtration on vanishing cycles; since it is transmitted to representation theory, it provides, together with the constructions of Section 4, the Jantzen conjecture.

The main results of this paper were proved in the spring of 1981. The first draft of this paper, which followed notes of a spring 1982 seminar at

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Moscow University, appeared in 1986. The second part of the present paper is an abridged version of this draft (we just added a few recent references); therefore it is a bit archaic in style (we use l-adic mixed sheaves instead of mixed Hodge sheaves of Saito, etc.). We thank Robert Becker for carefully typing this manuscript.

In what follows (except for Section 5) "variety" = "scheme" = "separated scheme of finite type over $\mathbb{C}$", and "algebra" = "associative $\mathbb{C}$-algebra with 1". In fact, one can replace $\mathbb{C}$ by any field of characteristic 0 in any place that has nothing to do with the Riemann-Hilbert correspondence (we deal with arbitrary schemes in 1.1).

If $\pi: X \to Y$ is a morphism of varieties then π_*, π^{-1} denote the sheaf theoretic direct and inverse image functors. If A is an algebra then $\mathscr{M}(A)$ denotes the category of left A-modules, and ${}^r\mathscr{M}(A)$ denotes the category of right A-modules. A subcategory $\mathscr{B}$ of an abelian category $\mathscr{A}$ is a Serre subcategory if $\mathscr{B}$ is a strictly full subcategory closed under extensions and subquotients; in such a case we have the quotient abelian category $\mathscr{A}/\mathscr{B}$.

If M is a sheaf and m is a local section of M, we write $m \in M$.

§1. *D*-calculus

1.1. ***D*-algebras.** Let R be a commutative algebra and M be an R-bimodule. For $r \in R$ we have the endomorphism $\operatorname{ad} r$ of M, $\operatorname{ad} r(m) = rm - mr$. An increasing filtration $M_.$ on M is called a *D-filtration* if $M_{-1} = 0$ and $\operatorname{ad} r(M_i) \subset M_{i-1}$ for any i and $r \in R$ (i.e., on $\operatorname{gr}_. M$ the left and right R-module structures coincide). For example, one has the D-filtration $M_.^\vee$ defined by induction: $M_i^\vee = \{m \in M: \operatorname{ad} r(m) \in M_{i-1} \text{ for any } r \in R\}$ for $i \geqslant 0$; this D-filtration is maximal in an obvious sense. We call $M^\vee = \bigcup_i M_i^\vee$ the *differential part* of M; our M is a *differential bimodule* if $M^\vee = M_.$.

Another way to spell this out is to consider M as an $(R \otimes R)$-module; then $M_i^\vee = \{m \in M: I^{i+1} m = 0\}$, where I is the kernel of the multiplication map $R \otimes R \to R$ (i.e., the ideal of functions vanishing on the diagonal $\operatorname{Spec} R \hookrightarrow \operatorname{Spec} R \times \operatorname{Spec} R$). This description shows that our objects localize nicely: if $f \in R$, then for the localized R_f-bimodule $M_f := R_f \underset{R}{\otimes} M \underset{R}{\otimes} R_f$ one has $(M_f)_i^\vee = (M_i^\vee)_f = R_f \underset{R}{\otimes} M_i^\vee = M_i^\vee \underset{R}{\otimes} R_f$.

1.1.1. Remark. Let $M_.$ be a D-filtration on M. We have a canonical morphism of R-modules $\delta = \delta_i: \operatorname{gr}_i M \to \operatorname{Hom}_R(\Omega^1 R, \operatorname{gr}_{i-1} M)$, defined by $\delta(\overline{m})(adb) = (abm - amb) \bmod M_{i-2}$, where $a, b \in R$, $m \in M_i$, and $\overline{m}$ is the image of m in $\operatorname{gr}_i M$.

Using these morphisms we construct a complex

$$\operatorname{gr}_i M \to \operatorname{Hom}_R(\Omega^1 R, \operatorname{gr}_{i-1} M) \to \cdots \to \operatorname{Hom}_R(\Omega^j R, \operatorname{gr}_{i-j} M) \to \cdots$$

where the differential δ^i: $\operatorname{Hom}_R(\Omega^j R, \operatorname{gr}_{i-j} M) \to \operatorname{Hom}_R(\Omega^{j+1} R, \operatorname{gr}_{i-j-1} M)$

is given by

$$\delta^i(\varphi)(a_0da_1\wedge\cdots\wedge da_{j+1})=\sum(-1)^l\delta_{i-j}(\varphi(a_0da_1\wedge\cdots\wedge\widehat{da_l}\wedge\cdots\wedge da_{j+1}))(da_l).$$

Note that $M_.$ is maximal iff $\bigcup M_i = M^\vee$ and each map δ_i is injective for $i>0$.

1.1.2. Let A be an associative algebra equipped with a morphism of algebras $i\colon R\to A$. An increasing filtration $A_.$ on A is called a *D-ring filtration* if it is a ring filtration (i.e., $A_iA_j\subset A_{i+j}$), $A_{-1}=0$, $i(R)\subset A_0$, and $i(R)$ lies in the center of the associated graded algebra $\mathrm{gr}_.A$. We can consider A as an R-bimodule; then such $A_.$ is a D-filtration. Note that $A_.^\vee$ is a D-ring filtration, so it is a maximal D-ring filtration. We will say that A is an *R-differential algebra* if $A=A^\vee$ (i.e., if A is a differential R-bimodule).

1.1.3. The above definitions easily globalize. Namely, let X be a scheme. A *differential $\mathcal{O}_X$-bimodule* M is a quasicoherent sheaf on $X\times X$ supported on the diagonal $X\subset X\times X$. We will consider M as a sheaf of $\mathcal{O}_X$-bimodules on X. It has the following properties:

(i) For any open $U\subset X$, $M(U)$ is a differential $\mathcal{O}(U)$-bimodule.
(ii) If U is affine, $U=\operatorname{Spec}R$, and $f\in R$, then $M(U_f)=M(U)_f$.

Conversely, any sheaf of $\mathcal{O}_X$-bimodules M with properties (i) and (ii) is a differential $\mathcal{O}_X$-bimodule.

Differential $\mathcal{O}_X$-bimodules are Zariski local objects: they form a stack on the Zariski topology of X. If X is affine, $X=\operatorname{Spec}R$, then differential $\mathcal{O}_X$-bimodules are the same as differential R-bimodules. Note that if M, N are differential $\mathcal{O}_X$-bimodules, then so is $M\underset{\mathcal{O}_X}{\otimes}N$.

1.1.4. An *$\mathcal{O}_X$-differential algebra*, or simply a *D-algebra* on X, is a sheaf of associative algebras on the Zariski topology of X equipped with a morphism of algebras $i\colon\mathcal{O}_X\to\mathcal{A}$ such that $\mathcal{A}$ is a differential $\mathcal{O}_X$-bimodule. One defines morphisms of D-algebras in an obvious manner. The D-algebras form a stack on the Zariski topology of X. If X is affine, $X=\operatorname{Spec}R$, then $\mathcal{O}_X$-differential algebras are the same as R-differential algebras.

1.1.5. For a D-algebra $\mathcal{A}$ on X an $\mathcal{A}$-module M is, by definition, a sheaf of $\mathcal{A}$-modules which is quasicoherent as an $\mathcal{O}_X$-module. Usually we will use left $\mathcal{A}$-modules, and call them simply $\mathcal{A}$-modules. They form an abelian category $\mathcal{M}(X,\mathcal{A})=\mathcal{M}(\mathcal{A})$. The category of right $\mathcal{A}$-modules will be denoted ${}^r\mathcal{M}(\mathcal{A})$.

If X is affine and $A:=\mathcal{A}(X)$, then $\mathcal{A}$-modules are the same as A-modules, since one has canonical equivalence of categories $\mathcal{M}(\mathcal{A})=\mathcal{M}(A)$.

The $\mathcal{A}$-modules are local objects: if $j\colon U\hookrightarrow X$ is an open embedding then $\mathcal{A}_U$ is an $\mathcal{O}_U$-differential algebra and we have the obvious adjoint functors $\mathcal{M}(\mathcal{A})\underset{j_*}{\overset{j^*}{\rightleftarrows}}\mathcal{M}(\mathcal{A}_U)$; the categories $\mathcal{M}(\mathcal{A}_U)$ form a stack over the Zariski topology of X.

If N is any (quasicoherent) $\mathscr{O}_X$-module then an $\mathscr{A}$-*action* on N is a structure of $\mathscr{A}$-modules on N compatible with the $\mathscr{O}$-module structure (i.e., $i(f)m = fm$ for $f \in \mathscr{O}_X$, $m \in N$).

1.1.6. FIRST EXAMPLES. (i) Let M, N be (quasicoherent) $\mathscr{O}_X$-modules. A $\mathbb{C}$-linear morphism $f: M \to N$ is called a *differential operator* if for any affine $U \subset X$ the corresponding morphism $f_U: M(U) \to N(U)$ lies in the differential part of the $\mathscr{O}(U)$-bimodule $\mathrm{Hom}_{\mathbb{C}}(M(U), N(U))$.

The differential operators form a sheaf of $\mathscr{O}_X$-bimodules $\mathrm{Diff}(M, N) \subset \mathrm{Hom}_{\mathbb{C}}(M, N)$. If M is coherent then $\mathrm{Diff}(M, N)$ is a differential $\mathscr{O}_X$-bimodule. In particular, for a coherent sheaf M we have an $\mathscr{O}_X$-differential algebra $D_M := \mathrm{Diff}(M, M)$. We put $D_X := D_{\mathscr{O}_X}$.

(ii) If $\mathscr{A}$, B are D-algebras on schemes X, Y respectively, then we have a D-algebra $\mathscr{A} \boxtimes B$ on $X \times Y$ such that for affine $U \subset X$, $V \subset Y$ one has $(\mathscr{A} \boxtimes B)(U \times V) = \mathscr{A}(U) \underset{\mathbb{C}}{\otimes} B(V)$.

1.2. Lie algebroids. For a scheme X a *Lie algebroid* L on X is a (quasicoherent) $\mathscr{O}_X$-module equipped with a morphism of $\mathscr{O}_X$-modules $\sigma: L \to \mathscr{T}_X$ ($:= \mathrm{Der}\,\mathscr{O}_X =$ tangent sheaf of X) and a $\mathbb{C}$-linear pairing $[\cdot, \cdot]: L \underset{\mathbb{C}}{\otimes} L \to L$ such that

- $[\cdot, \cdot]$ is a Lie algebra bracket and σ commutes with brackets,
- for $l_1, l_2 \in L$, $f \in \mathscr{O}_X$ one has $[l_1, fl_2] = f[l_1, l_2] + \sigma(l_1)(f)l_2$.

For a Lie algebroid L we set $L^{(0)} := \mathrm{Ker}\,\sigma$. This is an $\mathscr{O}_X$-Lie algebra.

A Lie algebroid is called *smooth* if it is a locally free $\mathscr{O}$-module of finite rank. Lie algebroids form a category in an obvious way; this is a stack on the Zariski topology of X.

A *connection* on a Lie algebroid L is an $\mathscr{O}_X$-linear section $\nabla: \mathscr{T}_X \to L$ of σ (so $\sigma\nabla = \mathrm{id}_{\mathscr{T}_X}$). Such a ∇ is integrable if it commutes with brackets. For a connection ∇, its *curvature* $C(\nabla) \in \mathrm{Hom}_{\mathscr{O}_X}(\bigwedge^2 \mathscr{T}_X, L^{(0)})$ is defined by $C(\nabla)(\tau_1 \wedge \tau_2) = [\nabla(\tau_1), \nabla(\tau_2)]$. The connections on L form a $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{T}_X, L^{(0)})$-torsor $\mathscr{C}(L)$.

Let us describe some examples of Lie algebroids.

1.2.1. The tangent sheaf $\mathscr{T}_X$ is a Lie algebroid (with $\sigma = \mathrm{id}_{\mathscr{T}_X}$). For any Lie algebroid L there is a unique morphism $L \to \mathscr{T}_X$ and a morphism $\mathscr{T}_X \to L$ is the same as an integrable connection on L.

1.2.2. Assume that a Lie algebra $\mathfrak{g}$ acts on X, i.e., we have a morphism of Lie algebras $\alpha: \mathfrak{g} \to \mathscr{T}_X$. Then $\tilde{\mathfrak{g}}_X = \mathscr{O}_X \underset{\mathbb{C}}{\otimes} \mathfrak{g}$ becomes a Lie algebroid in a natural way: the map $\sigma: \mathscr{O}_X \underset{\mathbb{C}}{\otimes} \mathfrak{g} \to \mathscr{T}_X$ is $\sigma(f \otimes \gamma) = f\alpha(\gamma)$ and the bracket on $\mathscr{O}_X \underset{\mathbb{C}}{\otimes} \mathfrak{g}$ is defined by $[f_1 \otimes \gamma_1, f_2 \otimes \gamma_2] = f_1 f_2 \otimes [\gamma_1, \gamma_2] + f_1\alpha(\gamma_1)(f_2) \otimes \gamma_2 - f_2\alpha(\gamma_2)(f_1) \otimes \gamma_1$.

1.2.3. Let G be an algebraic group, $\mathfrak{g} = \mathrm{Lie}\,G$, and F be a G-torsor over X. We have the Lie algebroid $\widetilde{\mathscr{T}_F}$ of infinitesimal symmetries of (X, F):

a section of $\widetilde{\mathcal{T}_F}$ is a pair $(\tau, \tilde{\tau})$, where τ is a vector field on X and $\tilde{\tau}$ is a G-invariant vector field on F that lifts τ. We have $\sigma(\tau, \tilde{\tau}) = \tau$, $[(\tau_1, \tilde{\tau}_1), (\tau_2, \tilde{\tau}_2)] = ([\tau_1, \tau_2], [\tilde{\tau}_1, \tilde{\tau}_2])$. Note that $\mathfrak{g}_F = \widetilde{\mathcal{T}}_F^{(0)}$ coincides with the F-twist of $\mathcal{O}_X \otimes \mathfrak{g}$ (with the usual $\mathcal{O}_X$-linear bracket) with respect to the adjoint action of G.

One can consider Lie algebroids of infinitesimal symmetries of any geometric object over X of a local nature. For example, for a vector bundle $\mathcal{E}$, we have a Lie algebroid $\widetilde{\mathcal{T}_{\mathcal{E}}}$, which coincides with the Lie algebroid of the corresponding GL-torsor.

1.2.4. Assume we have a smooth groupoid acting on X, i.e., we have a scheme Y equipped with two smooth projections $\pi_1, \pi_2 \colon Y \to X$, an embedding $e \colon X \hookrightarrow Y$, and a composition law $Y \underset{\pi_2, \pi_1}{\times} Y \to Y$ that satisfies the usual axioms (see, e.g., [D4]). The Lie algebroid L of our groupoid is the normal bundle for the embedding e; one defines the Lie bracket and projection σ by the usual formulas. If our groupoid is an ordinary group G acting on X (so $Y = G \times X$), then L coincides with the Lie algebroid from example 1.2.2 for $\mathfrak{g} = \operatorname{Lie} G$, α being the corresponding infinitesimal action of $\mathfrak{g}$.

1.2.5. Let $\mathcal{A}$ be a D-algebra on X. Put $\operatorname{Lie}\mathcal{A} := \{(\tau, a) \in \mathcal{T}_X \times \mathcal{A} : i\tau(f) = ai(f) - i(f)a$ for any $f \in \mathcal{O}_X\}$. This is a Lie algebroid on X in an obvious manner (one has $\sigma(\tau, a) = \tau$ and $[(\tau_1, a_1), (\tau_2, a_2)] = ([\tau_1, \tau_2], [a_1, a_2])$); we will call $\operatorname{Lie}\mathcal{A}$ the *Lie algebroid of* $\mathcal{A}$.

Clearly $\mathcal{A} \rightsquigarrow \operatorname{Lie}\mathcal{A}$ is a functor from the category of D-algebras to the category of Lie algebroids. It has a left adjoint functor that assigns to a Lie algebroid L its *universal enveloping D-algebra* $\mathcal{U}(L)$. Explicitly, $\mathcal{U}(L)$ is a sheaf of algebras equipped with the morphisms of sheaves $i \colon \mathcal{O}_X \to \mathcal{U}(L)$, $i_L \colon L \to \mathcal{U}(L)$; it is generated, as an algebra, by the images of these morphisms and the only relations are

(i) i is a morphism of algebras;
(ii) i_L is a morphism of Lie algebras;
(iii) for $f \in \mathcal{O}_X$, $l \in L$ one has $i_L(fl) = i(f)i_L(l)$, $[i_L(l), i(f)] = i(\sigma(l)f)$.

One checks easily that $\mathcal{U}(L)$ is actually a D-algebra.

Note that a $\mathcal{U}(L)$-module is the same as an $\mathcal{O}$-module with an L-action (i.e., an $\mathcal{O}_X$-module M equipped with a Lie algebra map $L \to \operatorname{End}_{\mathbb{C}} M$ such that $l(fm) = \sigma(l)(f)m + (fl)m$, $(fl)m = f(lm)$ for $f \in \mathcal{O}_X$, $l \in L$, $m \in M$). We will call $\mathcal{U}(L)$-modules simply L-modules.

1.3. Étale localization. All the above definitions are actually étale local. To be precise, let $\varphi \colon Y \to X$ be an étale morphism. Then the Y-diagonal $Y \hookrightarrow Y \times Y$ is a component (i.e., an open and closed subscheme) of the preimage of the X-diagonal $(\varphi \times \varphi)^{-1}(X) \hookrightarrow Y \times Y$. For a differential $\mathcal{O}_X$-bimodule M (which is an $\mathcal{O}_{X \times X}$-module supported on the diagonal) we define its pull-back

$M_Y = \varphi^* M$ as the restriction of $(\varphi \times \varphi)^* M = \mathscr{O}_{Y\times Y} \underset{(\varphi\times\varphi)^{-1}\mathscr{O}_{X\times X}}{\otimes} (\varphi \times \varphi)^{-1} M$ to the Y-diagonal. This is a differential $\mathscr{O}_Y$-bimodule.

We have a canonical embedding $\varphi^{-1}M \hookrightarrow \varphi^* M$ of $\varphi^{-1}\mathscr{O}_X$-bimodules that induces isomorphisms $\mathscr{O}_Y \underset{\varphi^{-1}\mathscr{O}_X}{\otimes} \varphi^{-1}M \xrightarrow{\sim} \varphi^* M \xleftarrow{\sim} \varphi^{-1}M \underset{\varphi^{-1}\mathscr{O}_X}{\otimes} \mathscr{O}_Y$. Therefore, the differential $\mathscr{O}_X$-bimodules are sheaves on the étale topology $X_{\text{ét}}$ of X; they form a stack on $X_{\text{ét}}$.

If M, N are differential $\mathscr{O}_X$-bimodules then $\varphi^*(M \underset{\mathscr{O}_X}{\otimes} N) = \varphi^* M \underset{\mathscr{O}_Y}{\otimes} \varphi^* N$. Therefore, if $\mathscr{A}$ is an $\mathscr{O}_X$-differential algebra, then $\mathscr{A}_Y = \varphi^*\mathscr{A}$ is an $\mathscr{O}_Y$-differential algebra (the product is the composition $\varphi^*\mathscr{A} \underset{\mathscr{O}_Y}{\otimes} \varphi^*\mathscr{A} \xrightarrow{\sim} \varphi^*(\mathscr{A}\otimes\mathscr{A}) \xrightarrow{\varphi^*(\cdot)} \varphi^*\mathscr{A}$); the embedding $\varphi^{-1}\mathscr{A} \hookrightarrow \varphi^*\mathscr{A}$ is a ring morphism.

We see that the D-algebras are also sheaves on $X_{\text{ét}}$, and they form a stack over $X_{\text{ét}}$. If M is an $\mathscr{A}$-module then $M_Y = \varphi^* M := \mathscr{O}_Y \underset{\varphi^{-1}\mathscr{O}_X}{\otimes} \varphi^{-1}M$ is an $\mathscr{A}_Y$-module. So we can consider M as a sheaf of $\mathscr{A}$-modules on $X_{\text{ét}}$; the categories $\mathscr{M}(\mathscr{A}_Y)$ form a stack over $X_{\text{ét}}$.

1.4. Functoriality. Let $\varphi: Y \to X$ be a morphism of schemes, and $\mathscr{A}$, $\mathscr{B}$ be D-algebras on X, Y respectively. Consider the sheaf $\varphi^*\mathscr{A} = \mathscr{O}_Y \underset{\varphi^{-1}\mathscr{O}_X}{\otimes} \varphi^{-1}\mathscr{A}$. This is a (quasicoherent) $\mathscr{O}_Y$-module equipped with a right $\varphi^{-1}\mathscr{A}$-action.

1.4.1. Definition. A *φ-morphism* $\lambda: \mathscr{B} \to \mathscr{A}$ is an action of D-algebra $\mathscr{B}$ on $\varphi^*\mathscr{A}$ that commutes with the right $\mathscr{A}$-action.

In other words, it is a morphism of algebras $\lambda: \mathscr{B} \to \operatorname{End}_{\varphi^{-1}\mathscr{A}}(\varphi^*\mathscr{A})$ such that $\lambda(i(f))a = fa$ for $f \in \mathscr{O}_Y$, $a \in \varphi^*\mathscr{A}$.

Remark. If φ is the identity morphism, then $\varphi^*\mathscr{A} = \mathscr{A}$, and λ is a usual morphism of D-algebras $\mathscr{B} \to \mathscr{A}$.

Note that a φ-morphism λ defines for any $\mathscr{A}$-module M a $\mathscr{B}$-action on $\varphi^* M := \mathscr{O}_Y \underset{\varphi^{-1}\mathscr{O}_X}{\otimes} \varphi^{-1}M$: one has $\varphi^* M = \varphi^*\mathscr{A} \underset{\varphi^{-1}\mathscr{A}}{\otimes} \varphi^{-1}M$, and we put $b(a\otimes m) := \lambda(b)a \otimes m$ for $b \in \mathscr{B}$, $a \in \varphi^*\mathscr{A}$, $m \in \varphi^{-1}M$. We get a canonical functor $(\varphi, \lambda)^*: \mathscr{M}(\mathscr{A}) \to \mathscr{M}(\mathscr{B})$.

If $\psi: Z \to Y$ is another morphism of schemes, $\mathscr{C}$ is a D-algebra on Z, and $\mu: \mathscr{C} \to \mathscr{B}$ is a ψ-morphism, we define the composition $\lambda\cdot\mu: \mathscr{C} \to \mathscr{A}$ as the $(\varphi\cdot\psi)$-morphism which is the action of $\mathscr{C}$ on $(\varphi\cdot\psi)^*\mathscr{A} = \psi^*(\varphi^*\mathscr{A})$ considered as $(\psi, \mu)^*(\varphi, \lambda)^*\mathscr{A}$. The functors $(\varphi\cdot\psi, \lambda\cdot\mu)^*$, $(\psi, \mu)^* \cdot (\varphi, \lambda)^*: \mathscr{M}(\mathscr{A}) \to \mathscr{M}(\mathscr{C})$ coincide.

1.4.2. A *D-scheme*, or *D-variety*, is a pair $(X, \mathscr{A})$, where X is a scheme, and $\mathscr{A}$ is a D-algebra on X. A morphism of D-schemes $(Y, \mathscr{B}) \to (X, \mathscr{A})$ is a pair (φ, λ) as above. We see that D-schemes form a category which we denote D-Sch; one has a canonical projection $D\text{-Sch} \to \text{Sch}$, $(X, \mathscr{A}) \mapsto X$, whose fiber over X is the category of D-algebras on X.

In fact this projection makes D-Sch a prefibered category over Sch (see [SGA1] for the terminology). This means that for any morphism of schemes $\varphi: Y \to X$ and any D-algebra $\mathscr{A}$ on X, there is a (canonical) D-algebra $\varphi^{\cdot}\mathscr{A}$ on Y equipped with φ-morphisms $\varphi^{\cdot}\mathscr{A} \to \mathscr{A}$ such that for a D-algebra $\mathscr{B}$ on Y the φ-morphisms $\mathscr{B} \to \mathscr{A}$ are the same as morphisms $\mathscr{B} \to \varphi^{\cdot}\mathscr{A}$. Indeed, $\varphi^{\cdot}\mathscr{A}$ is the algebra $\mathrm{Diff}_{\varphi^{-1}(\mathscr{A})}(\varphi^{*}\mathscr{A}, \varphi^{*}\mathscr{A})$ of all $(\mathscr{O}_Y)$-differential operators on $\varphi^{*}\mathscr{A}$ that commute with the right $\varphi^{-1}\mathscr{A}$-action. The universality property implies that $\varphi^{\cdot}$ is a functor from the category of D-algebras on X to that on Y. It is compatible with Zariski localization; if $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$ are affine schemes and $A = \Gamma(X, \mathscr{A})$ then $\Gamma(Y, \varphi^{\cdot}\mathscr{A})$ is the ring of all S-differential operators acting on $S \underset{R}{\otimes} A$ that commute with the right A-action.

1.4.3. REMARK. For a differential bimodule M on X we define its *pull-back* $\varphi^{\cdot}M$ as $\mathrm{Diff}_{\varphi^{-1}\mathscr{O}_X}(\mathscr{O}_Y, \varphi^{*}M) = \{l \in \mathrm{Diff}(\mathscr{O}_Y, \mathscr{O}_Y \underset{\varphi^{-1}\mathscr{O}_X}{\otimes} \varphi^{-1}M) : l(f\varphi^{*}(g)) = l(f)g \text{ for } f \in \mathscr{O}_Y, g \in \mathscr{O}_X\}$; this is a differential $\mathscr{O}_Y$-bimodule. This notation is compatible with the earlier one: if $\mathscr{A}$ is a D-algebra on X then we have an isomorphism of $\mathscr{O}_Y$-bimodules

$$\mathrm{Diff}_{\varphi^{-1}\mathscr{A}}(\varphi^{*}\mathscr{A}, \varphi^{*}\mathscr{A}) \xrightarrow{\sim} \mathrm{Diff}_{\varphi^{-1}\mathscr{O}_X}(\varphi^{*}\mathscr{O}_Y, \varphi^{*}\mathscr{A}), \; n \mapsto n \circ \varphi^{*}(i).$$

Note that the functor $\varphi^{\cdot}$ is, in general, neither left nor right exact (being the composition of the right exact functor φ^{*} and the left exact one $\mathrm{Diff}_{\varphi^{-1}\mathscr{O}_X}(\mathscr{O}_Y, \cdot)$), and it behaves badly with respect to composition of φ's. To recover the φ-functoriality one should, perhaps, work with the derived categories from the very beginning.

1.4.4. Let $Z \xrightarrow{\psi} Y$ be another morphism of schemes. For a D-algebra $\mathscr{A}$ on X the composition of the canonical ψ- and φ-morphisms $\psi^{\cdot}\varphi^{\cdot}\mathscr{A} \to \varphi^{\cdot}\mathscr{A} \to \mathscr{A}$ defines, by the universality property, a canonical morphism $c_{\varphi,\psi}: \psi^{\cdot}\varphi^{\cdot}\mathscr{A} \to (\varphi\psi)^{\cdot}\mathscr{A}$. If $\chi: W \to Z$ is the third morphism of schemes then the compositions $c_{\varphi\psi,\chi}\chi^{\cdot}(c_{\varphi,\psi})$, $c_{\varphi,\psi\chi}c_{\psi,\chi}: \chi^{\cdot}\psi^{\cdot}\varphi^{\cdot}\mathscr{A} \to (\varphi\psi\chi)^{\cdot}\mathscr{A}$ coincide. In general, $c_{\varphi,\psi}$ are not isomorphisms (a possible remedy would be to change the definition of $\varphi^{\cdot}$ so that the right $\varphi^{\cdot}$ would send the D-algebras on X to the differential graded D-algebras on Y; we will not pursue this line further).

1.4.5. In certain cases one can describe $\varphi^{\cdot}$ quite explicitly.

(i) φ is étale. Then, by 1.3, $\varphi^{*}\mathscr{A}$ is an $\mathscr{O}_Y$-differential algebra. One has a canonical isomorphism $\varphi^{*}\mathscr{A} \xrightarrow{\sim} \varphi^{\cdot}\mathscr{A}$ of D-algebras that assigns to $\alpha \in \varphi^{*}\mathscr{A}$ the operator of the left multiplication by α on $\varphi^{*}\mathscr{A}$. The inverse map $\varphi^{\cdot}\mathscr{A} \to \varphi^{*}\mathscr{A}$ is $l \mapsto l(1 \otimes 1)$.

(ii) φ is a projection $Y = T \times X \to X$. Then $\varphi^*\mathscr{A} = \mathscr{O}_T \boxtimes \mathscr{A}$, and one has an isomorphism of D-algebras $D_T \boxtimes \mathscr{A} \xrightarrow{\sim} \varphi^{\cdot}\mathscr{A}$ that sends $\partial \otimes a \in D_T \boxtimes \mathscr{A}$ to the operator $t \otimes b \mapsto \partial(t) \otimes ab$.

(iii) $\varphi \colon Y \hookrightarrow X$ is a closed embedding defined by an ideal $I \subset \mathscr{O}_X$. Then $\varphi^*\mathscr{A} = \mathscr{A}/I\mathscr{A}$. Let $N(I\mathscr{A}) := \{\alpha \in \mathscr{A} : \alpha I \subset I\mathscr{A}\}$ be the normalizer of the ideal $I\mathscr{A}$ in the algebra $\mathscr{A}$. One has an isomorphism of D-algebras $N(I\mathscr{A})/I\mathscr{A} \xrightarrow{\sim} \varphi^{\cdot}\mathscr{A}$ that sends $\alpha \in N(I\mathscr{A})$ to the operator of left multiplication by α on $\mathscr{A}/I\mathscr{A}$; the inverse map is $\gamma \mapsto \gamma(1 \otimes 1)$.

1.4.6. Let $\varphi \colon Y \to X$ be a morphism of schemes, and L, N be Lie algebroids on X, Y respectively. A *φ-morphism* $\gamma \colon N \to L$ is an $\mathscr{O}_Y$-linear map $\gamma \colon N \to \varphi^*L$ which satisfies the following conditions. Take $n \in N$; write $\gamma(n) = \sum f^i \otimes l^i$, $f^i \in \mathscr{O}_Y$, $l^i \in L$. Then one requires that $\gamma([n_1, n_2]) = \sum f_1^i f_2^j \otimes [l_1^i, l_2^j] + \sigma(n_1)(f_2^j) \otimes l_2^j - \sigma(n_2)(f_1^i) \otimes l_1^i$, and $\sigma(n)(\varphi^* g) = \sum f^i \varphi^*(\sigma(l^i)(g))$ for $g \in \mathscr{O}_X$.

If $\psi \colon Z \to Y$ is another morphism of schemes, K is a Lie algebroid on Z, and $\delta \colon K \to N$ is a ψ-morphism, then the composition $\gamma\delta := \psi^*(\gamma)\delta$ is a $\varphi\psi$-morphism $K \to L$.

We will call a pair (X, L), where X is a scheme and L is a Lie algebroid on X, a *D-Lie scheme.* We see that D-Lie schemes form a category D-Lie (the morphisms are pairs φ, γ as above).

The projection $(X, L) \mapsto X$ makes D-Lie a prefibered category over the category of schemes (one easily constructs the pull-back functor $\varphi^{\cdot}$: (Lie algebroids on X) $\to$ (Lie algebroids on Y)).

If, in the above situation, P is an L-module on X, then φ^*P is naturally an N-module: the N-action is $n(f \otimes p) := \sigma(n)(f) \otimes p + \sum f f^i \otimes l^i p$, where $\gamma(n) = \sum f^i \otimes l^i$. Therefore, we have the pull-back functor $(\varphi, \gamma)^*$: (L-modules) $\to$ (N-modules), $(\varphi, \gamma)^*P = \varphi^*P$; clearly $(\varphi\psi, \gamma\delta)^* = (\psi, \delta)^*(\varphi, \gamma)^*$. In particular, $\varphi^*\mathscr{U}(L)$ is an N-module, hence γ defines a φ-morphism of D-algebras $\mathscr{U}(N) \to \mathscr{U}(L)$. Therefore, we have the universal enveloping algebra functor D-Lie $\to$ D-Sch, $(X, L) \mapsto (X, \mathscr{U}(L))$.

1.4.7. Let $(\varphi, \lambda) \colon (Y, \mathscr{B}) \to (X, \mathscr{A})$ be a morphism of D-schemes. It defines the *push-forward functor* between the derived categories of right modules $R(\varphi, \lambda)_* \colon D^{-r}\mathscr{M}(Y, \mathscr{B}) \to D^{-r}\mathscr{M}(X, \mathscr{A})$, $R(\varphi, \lambda)_*(P) := R\varphi_*(P \overset{L}{\underset{\mathscr{B}}{\otimes}} \varphi^*\mathscr{A})$. Here we consider $\varphi^*\mathscr{A}$ as a $(\mathscr{B}, \varphi^{-1}\mathscr{A})$-bimodule (so $P \underset{\mathscr{B}}{\otimes} \varphi^*\mathscr{A}$ is a right $\varphi^{-1}\mathscr{A}$-module) and φ_* is the sheaf theoretic direct image.

If $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$ are affine schemes, $A = \Gamma(X, \mathscr{A})$, and $B = \Gamma(Y, \mathscr{B})$, then the functor $R(\varphi, \lambda)_*$ sends a right B-module P to the complex of A-modules $P \overset{L}{\underset{B}{\otimes}} (S \underset{R}{\otimes} A)$.

1.5. Smooth localization. Assume we are in the situation of 1.4.4.

1.5.1. LEMMA. *If either φ or ψ is a smooth morphism of schemes, then $c_{\varphi,\psi}$ is an isomorphism.*

PROOF. The statement is Zariski local. We assume that $X = \operatorname{Spec} R$ is affine; put $A = \Gamma(X, \mathscr{A})$. Since locally any smooth morphism $U \to V$ is a composition $U \xrightarrow{\alpha} V \times \mathbb{C}^n \xrightarrow{\pi} V$, where α is étale and π is a projection, it suffices to check our lemma when the smooth morphism in question is either étale or the projection.

(i) ψ is étale. Then $\psi^{\cdot}\varphi^{\cdot}\mathscr{A} = \psi^*\varphi^{\cdot}\mathscr{A}$ by 1.4.5(i) and $c_{\varphi,\psi}$ is the action of $\psi^*\varphi^{\cdot}\mathscr{A}$ on $\psi^*\varphi^*\mathscr{A} = \mathscr{O}_Z \otimes_R A$. Since $\psi^* \operatorname{Diff}(\mathscr{O}_Y, \varphi^*\mathscr{A}) = \operatorname{Diff}(\mathscr{O}_Z, \psi^*\varphi^*\mathscr{A})$ the flatness of ψ implies that $c_{\varphi,\psi}$ is an isomorphism.

(ii) φ is étale. One has $\varphi^{\cdot}\mathscr{A} = \varphi^*\mathscr{A}$, $\psi^*\varphi^{\cdot}\mathscr{A} = (\varphi\psi)^*\mathscr{A}$, and $\psi^{\cdot}\varphi^{\cdot}\mathscr{A}$, respectively $(\varphi\psi)^{\cdot}\mathscr{A}$, is the sheaf of all operators $l \in \operatorname{Diff}(\mathscr{O}_Z, \mathscr{O}_Z \otimes_R A)$ that commute with the right action of $\psi^{-1}\mathscr{O}_Y$, respectively R. Since for any $f \in \mathscr{O}_Z$ the map $\mathscr{O}_Y \to \mathscr{O}_Z \otimes_R A = \psi^*\varphi^{\cdot}\mathscr{A}$, $g \mapsto l(f\psi^*(g)) - l(f)i(g)$, is a differential operator, it vanishes if and only if it vanishes on R. Hence $\psi^{\cdot}\varphi^{\cdot}\mathscr{A} = (\varphi\psi)^{\cdot}\mathscr{A}$.

(iii) $\psi\colon Z = \mathbb{C}^n \times Y \to Y$ is a projection. One has (see 1.4.5(ii)) $\psi^{\cdot}\varphi^{\cdot}\mathscr{A} = D_{\mathbb{C}^n} \boxtimes \varphi^{\cdot}\mathscr{A} = \{l \in \operatorname{Diff}(\mathscr{O}_Z, \mathscr{O}_Z \otimes_R A)\colon l(f(\varphi\psi)^*(g)) = l(f)i(g)$ for $f \in \mathscr{O}_Z, g \in R\} = (\varphi\psi)^{\cdot}\mathscr{A}$.

(iv) $\varphi\colon Y = \mathbb{C}^n \times X \to X$ is a projection. Let t_j, $j = 1, \dots, n$, be the coordinates on $\mathbb{C}^n$, $\theta = \varphi\psi$, $q_j = \psi^*(t_j) \in \mathscr{O}_Z$, so

$$\psi = (q; \theta) = (q_1, \dots, q_n; \theta).$$

One has $(\varphi\psi)^{\cdot}\mathscr{A} = \{l \in \operatorname{Diff}(\mathscr{O}_Z, \mathscr{O}_Z \otimes_R A)\colon l(fr) = l(f)i(r)$ for $f \in \mathscr{O}_Z$, $r \in R\}$, $\varphi^{\cdot}\mathscr{A} = D_{\mathbb{C}^n} \boxtimes A$, $\psi^{\cdot}\varphi^{\cdot}\mathscr{A} = \{m \in \operatorname{Diff}(\mathscr{O}_Z, q^*D_{\mathbb{C}^n} \otimes_R A)\colon m(fr) = m(f)i(r), m(fq_j) = m(f)i(t_j)$ for $f \in \mathscr{O}_Z$, $r \in R$, $j = 1, \dots, n\}$.

Write $D_{\mathbb{C}^n} = \bigoplus \mathscr{O}_{\mathbb{C}^n}\partial_{t_1}^{i_1} \cdots \partial_{t_n}^{i_n}$, so that in the above formula $m = \sum m_{i_1 \cdots i_n} \partial_{t_1}^{i_1} \cdots \partial_{t_n}^{i_n}$, where $m_{i_1 \cdots i_n} \in \operatorname{Diff}(\mathscr{O}_Z, \mathscr{O}_Z \otimes_R A)$. Conditions on $m \in \psi^{\cdot}\varphi^{\cdot}\mathscr{A}$ imply $m_{i_1 \cdots i_n} \in (\varphi\psi)^{\cdot}\mathscr{A}$ and

$$m_{i_1 \cdots i_n}(fq_j) = q_j m_{i_1 \cdots i_n}(f) + m_{i_1, \dots, i_j+1, \dots, i_n}(f).$$

The map $c_{\varphi,\psi}$ sends m to $m_{0 \dots 0}$; clearly this is an isomorphism.

1.5.2. Let $\pi_P\colon P \to X$ be a smooth X-scheme (this means that π_P is a smooth morphism). For a D-algebra $\mathscr{A}$ on X we will call $\mathscr{A}_P := \pi^{\cdot}\mathscr{A}$ the *(smooth) localization of* $\mathscr{A}$ *at* P. The above lemma claims that the pull-back functors are compatible with smooth localizations: for a commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{\widetilde{\varphi}} & P \\ {\scriptstyle \pi_Q}\downarrow & & {\scriptstyle \pi_P}\downarrow \\ Y & \xrightarrow{\varphi} & X \end{array}$$

where π_Q is also smooth, one has a canonical isomorphism $\widetilde{\varphi}^{\cdot}\mathscr{A}_P = (\varphi^{\cdot}\mathscr{A})_Q$ (both equal to $(\varphi\pi_Q)^{\cdot}\mathscr{A}$). In particular, for any morphism $\alpha\colon P' \to P$ of smooth X-schemes one has $\mathscr{A}_{P'} = \alpha^{\cdot}\mathscr{A}$; i.e., $\mathscr{A}_P$ forms a Cartesian section of D-Sch over the category X_{sm} of smooth X-schemes.

1.5.3. The category X_{sm} has a Grothendieck topology structure (smooth site of X; a covering in X_{sm} is a smooth surjective morphism of X-schemes). Note that a morphism between D-algebras on X is an isomorphism if and only if it is an isomorphism locally in smooth topology X_{sm} (use 1.4.5(i), (ii)). In fact D-algebras themselves are local objects with respect to the smooth topology: they satisfy the smooth descent property.

To be precise, let $\pi_P\colon P \to X$ be a smooth covering (i.e., a smooth surjective morphism). Consider the corresponding Čech system

$$\mathscr{P} := \cdots P \underset{X}{\times} P \underset{X}{\times} P \overset{\pi_{12},\pi_{23},\pi_{13}}{\Rrightarrow} P \underset{X}{\times} P \overset{\pi_1,\pi_2}{\rightrightarrows} P \xrightarrow{\pi_P} X.$$

For a D-algebra $\mathscr{A}$ on X denote by $g_{\mathscr{A}}$ the composition $\pi_1^{\cdot}\mathscr{A}_P \xrightarrow{\sim} \mathscr{A}_{P\underset{X}{\times}P} \xrightarrow{\sim} \pi_2^{\cdot}\mathscr{A}_P$.

A *D-algebra on $\mathscr{P}$* is a pair $(\mathscr{B}, g)$, where $\mathscr{B}$ is a D-algebra on P and $g\colon \pi_1^{\cdot}\mathscr{B} \xrightarrow{\sim} \pi_2^{\cdot}\mathscr{B}$ is an isomorphism that satisfies the cocycle property $\pi_{13}^{\cdot}(g) = \pi_{23}^{\cdot}(g)\pi_{12}^{\cdot}(g)$. The D-algebras on $\mathscr{P}$ form a category in an obvious way, and we have a canonical $\mathscr{P}$-localization functor (D-algebras on X) $\to$ (D-algebras on $\mathscr{P}$), $\mathscr{A} \mapsto (\mathscr{A}_P, g_A)$.

1.5.4. LEMMA. *This functor is an equivalence of categories.*

PROOF. Let us construct the inverse functor. Denote by $X_{\mathrm{sm}}^{(P)} \subset X_{\mathrm{sm}}$ the full subcategory of P-small objects (those T for which $\mathrm{Hom}_{X_{\mathrm{sm}}}(T, P)$ is not empty).

A D-algebra $(\mathscr{B}, g)$ on $\mathscr{P}$ defines a Cartesian section $T \mapsto \mathscr{B}_T$ of D-Sch over $X_{\mathrm{sm}}^{(P)}$ as follows. For a P-small T and a morphism $\gamma\colon T \to P$ put $\mathscr{B}_{T\gamma} := \gamma^{\cdot}\mathscr{B}$. If $\gamma'\colon T \to P$ is another morphism then, by 1.5.1, $\mathscr{B}_{T\gamma} = (\gamma, \gamma')^{\cdot}\pi_1^{\cdot}\mathscr{B}$ and $\mathscr{B}_{T\gamma'} = (\gamma, \gamma')^{\cdot}\pi_2^{\cdot}\mathscr{B}$, so we have a canonical isomorphism $g_{\gamma,\gamma'} = (\gamma, \gamma')^{\cdot}(g)\colon \mathscr{B}_{T\gamma} \xrightarrow{\sim} \mathscr{B}_{T\gamma'}$; one has $g_{\gamma,\gamma''} = g_{\gamma'\gamma''}g_{\gamma\gamma'}$ by the cocycle property of g. Therefore $\mathscr{B}_{T\gamma}$ does not depend on γ: this is our $\mathscr{B}_T$. If $T' \xrightarrow{\alpha} T$ is a morphism in $X_{\mathrm{sm}}^{(P)}$, then we have a canonical morphism

$c_\alpha\colon \alpha^{\cdot}\mathscr{B}_T \to \mathscr{B}_{T'}$ defined as a composition $\alpha^{\cdot}\mathscr{B}_{T\gamma} = \alpha^{\cdot}\gamma^{\cdot}\mathscr{B} \overset{c_{\gamma,\alpha}}{\to} (\gamma\alpha)^{\cdot}\mathscr{B} = \mathscr{B}_{T'\gamma\alpha}$ (this definition does not depend on the choice of $\gamma \in \mathrm{Hom}(T, P)$).

This c_α is actually an isomorphism (this is clear from 1.5.1 if either α is smooth or there exists a smooth $\gamma\colon T \to P$. The general case reduces to this, since c_α is compatible with smooth localizations: replace T and T' by $T \underset{X}{\times} P$ and $T' \underset{X}{\times} P$ respectively).

We can replace $X_{\mathrm{sm}}^{(P)}$ by the subcategory $X_{\text{ét}}^{(P)}$ that consists of P-small T's étale over X. By the étale descent (see 1.3) $\mathscr{B}_T$ defines a D-algebra $\mathscr{A}$ on X. Clearly $\mathscr{A}$ depends on $(\mathscr{B}, g)$ in a functorial way. One checks immediately that this functor (D-algebras on $\mathscr{P}$) $\to$ (D-algebras on X), $(\mathscr{B}, g) \mapsto \mathscr{A}$, is inverse to the $\mathscr{P}$-localization functor.

1.5.5. For a D-algebra $\mathscr{A}$ on X the categories $\mathscr{M}(\mathscr{A}_P)$ form a fibered category $\mathscr{M}(X_{\mathrm{sm}})$ over X_{sm} (with respect to the pull-back functor φ^*). The usual flat descent property for $\mathscr{O}$-modules implies that $\mathscr{M}(X_{\mathrm{sm}})$ is a stack over X_{sm}.

1.5.6. One easily checks that the analogs of 1.5.1 and 1.5.4 are valid in the Lie algebroid situation (see 1.4.6).

1.6. Affine D-schemes. Let $(X, \mathscr{A})$ be a D-scheme. Put $A := \Gamma(X, \mathscr{A})$. We have a pair (Δ, Γ) of adjoint functors $\mathscr{M}(X, \mathscr{A}) \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} \mathscr{M}(A)$; $\Gamma(M) = \Gamma(X, M)$ is the global sections functor, $\Delta(N) := \mathscr{A} \underset{A}{\otimes} N$.

1.6.1. LEMMA. *The following conditions on $(X, \mathscr{A})$ are equivalent:*

(i) *For any $M \in \mathscr{M}(X, \mathscr{A})$ one has $H^i(X, M) = 0$ for $i > 0$ and M is generated, as an $\mathscr{A}$-module, by its global sections.*

(ii) *The functors Γ, Δ are (mutually inverse) equivalences of categories.*

If the conditions of 1.6.1 hold, then we will call $(X, \mathscr{A})$ an *affine D-scheme*, and X an *$\mathscr{A}$-affine variety*.

EXAMPLES. (i) If X itself is an affine scheme, then it is $\mathscr{A}$-affine for any $\mathscr{A}$.

(ii) Let X be any quasiprojective variety. Then there exists a smooth surjective morphism $\pi\colon \widetilde{X} \to X$ such that $\widetilde{X}$ is affine (actually there exists π which is a torsor with respect to an action of some vector bundle over X). Then X is $\pi_*\mathscr{O}_{\widetilde{X}}$-affine. For a nontrivial noncommutative example, see Section 3.

1.7. D-stacks. The smooth descent property shows that we can repeat all the above notions and constructions in the context of algebraic stacks. Here is a brief sketch of the first definitions.

Let $\mathscr{X}$ be an algebraic stack (for the smooth topology, see [Lau]). A *D-algebra $\mathscr{A}$ on $\mathscr{X}$* consists of the following data:

(i) For any scheme X and a smooth 1-morphism $\pi\colon X \to \mathscr{X}$ one has a D-algebra $\mathscr{A}_{(X,\pi)}$ on X.

(ii) For any (X, π), (X', π') as above, a morphism $\alpha: X' \to X$, and a 2-morphism $\pi' \xrightarrow{\widetilde{\alpha}} \pi\alpha$, one has an isomorphism of D-algebras

$$\widetilde{\alpha}_{\mathscr{A}}: \mathscr{A}_{(X', \pi')} \to \alpha^{\cdot}\mathscr{A}_{(X, \pi)}.$$

We demand that $\widetilde{\alpha}_{\mathscr{A}}$ behave naturally with respect to compositions of $(\alpha, \widetilde{\alpha})$'s.

1.7.1. EXAMPLE. If $\mathscr{X}$ is smooth then we have the D-algebra of differential operators $D_{\mathscr{X}}$ on $\mathscr{X}$. Namely, for any (X, π) as in (i) above the scheme X is regular; we put $(D_{\mathscr{X}})_{(X, \pi)} := D_X$, and take for $\widetilde{\alpha}$ in (ii) the canonical isomorphism $D_{X'} = \alpha^{\cdot} D_X$.

1.7.2. D-algebras on $\mathscr{X}$ are local objects: one can describe them as follows. Choose a smooth covering $\pi: X \to \mathscr{X}$ (so π is a smooth surjective 1-morphism); put $Y := X \underset{\mathscr{X}}{\times} X$. Then Y is a smooth groupoid acting on X, and $\mathscr{X}$ coincides with the quotient stack $Y \setminus X$. A D-algebra $\mathscr{A}$ on $\mathscr{X}$ yields a D-algebra $\mathscr{A}_X = \mathscr{A}_{(X, \pi)}$ on X equipped with the Y-action (which is an isomorphism $g_{\mathscr{A}}: \pi_1^{\cdot}\mathscr{A}_X \xrightarrow{\sim} \pi_2^{\cdot}\mathscr{A}_X$ that satisfies the cocycle condition). As follows from 1.5.3 the functor $\mathscr{A} \mapsto (\mathscr{A}_X, g_{\mathscr{A}})$ from the category of D-algebras on $\mathscr{X}$ to the category of Y-equivariant D-algebras on X is an equivalence of categories.

For a D-algebra $\mathscr{A}$ on $\mathscr{X}$ an *$\mathscr{A}$-module* M is a collection of $\mathscr{A}_{(X, \pi)}$-modules $M_{(X, \pi)}$ together with $\widetilde{\alpha}_{\mathscr{A}}$-isomorphisms $M_{(X', \pi')} \xrightarrow{\sim} \alpha^* M_{(X, \pi)}$ compatible with the composition of $(\alpha, \widetilde{\alpha})$'s for (X, π), $(\alpha, \widetilde{\alpha})$ as above. In terms of a smooth covering $X \xrightarrow{\pi} \mathscr{X}$ such an M is the same as a Y-equivariant $\mathscr{A}_X$-module (i.e., an $\mathscr{A}_X$-module equipped with a $g_{\mathscr{A}}$-isomorphism $g_M: \pi_1^* M \xrightarrow{\sim} \pi_2^* M$ that satisfies the cocycle condition). The $\mathscr{A}$-modules form an abelian category $\mathscr{M}(\mathscr{X}, \mathscr{A})$.

We leave to the reader further translations of 1.1–1.6 to the stack setting. The only delicate point here is the construction of the derived category of $\mathscr{A}$-modules (and, consequently, the construction of the push-forward functors from 1.4.7). It turns out that for the usual derived category $D\mathscr{M}(\mathscr{X}, \mathscr{A})$ the local to global spectral sequence for Ext's no longer holds in the stack case (the first example: take $\mathscr{X} = G_m \setminus \text{point}$, $\mathscr{A} = D_{\mathscr{X}}$). One must replace it by a certain canonical t-category $D(\mathscr{X}, \mathscr{A})$ with the heart $\mathscr{M}(\mathscr{X}, \mathscr{A})$; we hope to present a construction of $D(\mathscr{X}, \mathscr{A})$ elsewhere.

1.8. Equivariant setting. Let G be an algebraic group and X be a G-variety, i.e., a scheme equipped with a G-action $\mu: G \times X \to X$. In this subsection we will give an explicit description of D-algebras on the quotient stack $G\backslash X$ and of corresponding modules.

1.8.1. Let $\mathfrak{g}$ be the Lie algebra of G. Our μ defines the infinitesimal action $\alpha: \mathfrak{g} \to \mathscr{T}_X$; by 1.2.2 we have the Lie algebroid $\widetilde{\mathfrak{g}}_X$. For a (quasicoherent) $\mathscr{O}_X$-module P a G-action on P (which is an isomorphism $\mu^* P \xrightarrow{\sim} p_X^* P$ that satisfies the cocycle condition) defines a $\mathfrak{g}$-action α_P on P that lifts α (i.e., α_P is a Lie algebra morphism $\mathfrak{g} \to \operatorname{End}_{\mathbb{C}} P$ such that $\alpha_P(\gamma)(fp) = f\alpha_P(\gamma)(p) + \alpha(\gamma)(f)p$, $\gamma \in \mathfrak{g}$, $f \in \mathscr{O}_X$, $p \in P$; equivalently, we have a morphism of Lie algebroids $\alpha_P \colon \tilde{\mathfrak{g}}_X \to \widetilde{\mathscr{T}_P}$ (see 1.2.2–1.2.3)).

1.8.2. Let M be an $\mathscr{O}_X$-differential bimodule. The group $G \times G$ acts on $X \times X$, and the diagonal subgroup $G \hookrightarrow G \times G$ preserves the diagonal. Let $G^\wedge$ be the formal completion of $G \times G$ along the diagonal G; this is a formal subgroup of $G \times G$ that preserves the formal neighborhood $X^\wedge$ of the diagonal. We define a *G-action on M* as a $G^\wedge$-action on M considered as an $\mathscr{O}_{X \times X}$-module supported on the diagonal.

One can spell out this definition without explicitly mentioning $G^\wedge$ as follows. Consider the "complex Harish-Chandra pair" $(\mathfrak{g} \times \mathfrak{g}, G)$. It acts on $X \times X$, and a G-action on an $\mathscr{O}_X$-differential bimodule is the lifting of this action to M considered as an $\mathscr{O}_{X \times X}$-module. Explicitly, a $(\mathfrak{g} \times \mathfrak{g}, G)$-action on M is a pair (μ_M, α_M) where μ_M is a G-action on M considered as an $\mathscr{O}_{X \times X}$-module (G acts on $X \times X$ diagonally), and α_M is a $(\mathfrak{g} \times \mathfrak{g})$-action on M (in other words α_M is a Lie algebra map $\mathfrak{g} \times \mathfrak{g} \to \operatorname{End}_{\mathbb{C}} M$ such that one has $\alpha_M(\gamma_1, \gamma_2)(f_1 m f_2) = \alpha(\gamma_1)(f_1) m f_2 + f_1(\alpha_M(\gamma_1, \gamma_2)m)f_2 + f_1 m \alpha(\gamma_2)(f_2)$ for $\gamma_1, \gamma_2 \in \mathfrak{g}$, $f_1, f_2 \in \mathscr{O}_X$, $m \in M$). This pair should be compatible in the sense that the action of $\mathfrak{g}$ that comes from μ_M coincides with the α_M-action of the diagonal $\mathfrak{g} \subset \mathfrak{g} \times \mathfrak{g}$, and one has $g^*(\alpha_M(\gamma_1, \gamma_2)m) = \alpha_M(\operatorname{ad}_g \gamma_1, \operatorname{ad}_g \gamma_2) g^*(m)$ for $g \in G$, $m \in M$, $\gamma_1, \gamma_2 \in \mathfrak{g}$. Here g^* stands for the μ_M-action of g on M.

REMARK. A G-action on M as an $\mathscr{O}_{X \times X}$-module defines (and is completely determined by) a G-action on M as an $\mathscr{O}_X$-module (with respect to either only left or only right $\mathscr{O}_X$-action). We can formally weaken the above definition by demanding only that in the pair (μ_M, α_M) our μ_M should be the G-action on M as on a left (or right) $\mathscr{O}_X$-module. It is easy to check that nothing changes: the compatibility with α_M immediately implies that μ_M is actually a G-action on M as an $\mathscr{O}_{X \times X}$-module.

1.8.3. Let $\mathscr{A}$ be a D-algebra on X. A *G-action on $\mathscr{A}$* is a G-action $(\mu_{\mathscr{A}}, \alpha_{\mathscr{A}})$ on $\mathscr{A}$ as an $\mathscr{O}_X$-differential bimodule such that

(i) $\mu_{\mathscr{A}}$ is compatible with the ring structure on $\mathscr{A}$, i.e., $g^*(a_1 a_2) = (g^* a_1)(g^* a_2)$, $g^*(1) = 1$ for $g \in G$, $a_1, a_2 \in \mathscr{A}$.

(ii) For $\gamma \in \mathfrak{g}$, $a_1, a_2 \in \mathscr{A}$ one has $\alpha_{\mathscr{A}}(\gamma, 0)(a_1 a_2) = (\alpha_{\mathscr{A}}(\gamma, 0)a_1)a_2$, $\alpha_{\mathscr{A}}(0, \gamma)(a_1 a_2) = a_1 \alpha_{\mathscr{A}}(0, \gamma)(a_2)$.

Note that (i) implies that the structure morphism $i \colon \mathscr{O}_X \to \mathscr{A}$ commutes with the G-action. For $\gamma \in \mathfrak{g}$ put $i_{\mathfrak{g}}(\gamma) := \alpha_{\mathscr{A}}(\gamma, 0)(1) \in \mathscr{A}$; then $\alpha_{\mathscr{A}}(\gamma_1, \gamma_2)a = i_{\mathfrak{g}}(\gamma_1)a - a i_{\mathfrak{g}}(\gamma_2)$. We can rewrite the above definition in terms of $i_{\mathfrak{g}}$. Namely, a G-action on $\mathscr{A}$ is the same as a pair $(\mu_{\mathscr{A}}, i_{\mathfrak{g}})$, where $\mu_{\mathscr{A}}$ is a G-action on $\mathscr{A}$ as on a left (or right) $\mathscr{O}_X$-module and $i_{\mathfrak{g}} \colon \mathfrak{g} \to \mathscr{A}$ is a

Lie algebra map such that

(i) $\mu_{\mathscr{A}}$ is compatible with the ring structure on $\mathscr{A}$.
(ii) $i_{\mathfrak{g}}$ commutes with the G-action (where G acts on $\mathfrak{g}$ by the adjoint representation).
(iii) The $\mathfrak{g}$-action on $\mathscr{A}$ that comes from $\mu_{\mathscr{A}}$ coincides with $\mathrm{ad}_{i_{\mathfrak{g}}}$.

We call a D-algebra equipped with a G-action a *Harish-Chandra algebra* or an *$(\mathscr{O}_X, G)$-differential algebra.*

EXAMPLE. If G acts on a coherent $\mathscr{O}_X$-module P then D_P is a Harish-Chandra algebra in an obvious way (here $i_{\mathfrak{g}} = \alpha_P$).

1.8.4. Now let P be a Lie algebroid on X. A *G-action on P* is a G-action μ_P on P as an $\mathscr{O}_X$-module together with a morphism of Lie algebras $i_{\mathfrak{g}}: \mathfrak{g} \to P$ such that

(i) One has $g^*[\zeta_1, \zeta_2] = [g^*\zeta_1, g^*\zeta_2]$, $g^*(\sigma\zeta) = \sigma(g^*\zeta)$ for $g \in G$, $\zeta_i \in P$.
(ii) $i_{\mathfrak{g}}(\mathrm{ad}_g \gamma) = g^* i_{\mathfrak{g}} \gamma$ for $g \in G$, $\gamma \in \mathfrak{g}$.
(iii) The $\mathfrak{g}$-action on P that comes from μ_P coincides with $\mathrm{ad}_{i_{\mathfrak{g}}}$.

We call a Lie algebroid equipped with a G-action a *Harish-Chandra Lie algebroid*. If X is a point, then this is the same as a Harish-Chandra pair.

For a Harish-Chandra Lie algebroid P its universal enveloping algebra $\mathscr{U}(P)$ is a Harish-Chandra algebra in an obvious manner. If $\mathscr{A}$ is a Harish-Chandra algebra on X, then $\mathrm{Lie}\mathscr{A}$ (see 1.2.5) is a Harish-Chandra Lie algebroid.

One defines morphisms of Harish-Chandra algebras and Lie algebroids in an obvious way.

EXAMPLE. $\widetilde{\mathfrak{g}}_X$ is a Harish-Chandra algebroid in an obvious manner (the G-action on $\widetilde{\mathfrak{g}}_X$ comes from the adjoint action on $\mathfrak{g} \subset \widetilde{\mathfrak{g}}_X$). For any Harish-Chandra Lie algebroid P there exists a unique morphism $\widetilde{\mathfrak{g}}_X \to P$. We have an obvious ring homomorphism $\mathscr{U}(\mathfrak{g}) \to \mathscr{U}(\widetilde{\mathfrak{g}}_X)$; the corresponding morphism of $\mathscr{O}_X$-modules $\mathscr{O}_X \otimes \mathscr{U}(\mathfrak{g}) \to \mathscr{U}(\widetilde{\mathfrak{g}}_X)$ is an isomorphism.

1.8.5. Let $\mathscr{A}$ be a Harish-Chandra algebra on X. A *weak $(\mathscr{A}, G)$-module* is an $\mathscr{A}$-module M equipped with a G-action (as an $\mathscr{O}_X$-module) compatible with the G-action on $\mathscr{A}$ (i.e., for $g \in G$, $a \in \mathscr{A}$, $m \in M$ one has $g^*(am) = g^*(a)g^*(m)$). An *$(\mathscr{A}, G)$-module* is a weak $(\mathscr{A}, G)$-module M such that the action of $\mathfrak{g}$ on M that comes from the G-action coincides with the one that comes from $i_{\mathfrak{g}}: \mathfrak{g} \to \mathscr{A}$ and the $\mathscr{A}$-module structure on M. We will also call $(\mathscr{A}, G)$-modules *Harish-Chandra modules.*

To see the difference between weak $(\mathscr{A}, G)$-modules and Harish-Chandra modules consider the tensor product algebra $\mathscr{A} \underset{\mathbb{C}}{\otimes} \mathscr{U}(\mathfrak{g})$. It carries the Harish-Chandra structure: the G-action is $g(a \otimes u) = g(a) \otimes \mathrm{Ad}_g(u)$, and the $i_{\mathfrak{g}}$-map is $i_{\mathfrak{g}}(\gamma) = i_{\mathfrak{g}}^A(\gamma) \otimes 1 + 1 \otimes \gamma$.

1.8.6. LEMMA. *Weak* $(\mathscr{A}, G)$-*modules are the same as* $(\mathscr{A} \underset{\mathbb{C}}{\otimes} \mathscr{U}(\mathfrak{g}), G)$-*modules.*

PROOF. An $(\mathscr{A} \underset{\mathbb{C}}{\otimes} \mathscr{U}(\mathfrak{g}), G)$-module is a weak $(\mathscr{A}, G)$-module (where $\mathscr{A}$ acts via $\mathscr{A} \hookrightarrow \mathscr{A} \otimes \mathscr{U}(\mathfrak{g})$, $a \mapsto a \otimes 1$). Conversely, let M be a weak $(\mathscr{A}, G)$-module. For $\gamma \in \mathfrak{g}$ consider the $\mathbb{C}$-linear endomorphism $w(\gamma)$ of M, $w(\gamma)m := \gamma^{(1)}m - \gamma^{(2)}m$, where $\gamma^{(1)}$ is the action of γ that comes from the action of G on M (as on $\mathscr{O}_X$-module) and $\gamma^{(2)}$ is the action of $i_{\mathfrak{g}}(\gamma) \in \mathscr{A}$. One checks that $w(\gamma)$ commutes with $\mathscr{A}$-action and the map $w: \mathfrak{g} \to \operatorname{End}_{\mathscr{A}} M$ is a Lie algebra homomorphism. Therefore the $\mathscr{A}$-module structure and w define a $(\mathscr{A} \underset{\mathbb{C}}{\otimes} \mathscr{U}(\mathfrak{g}))$-module structure on M, and the G-action makes M a Harish-Chandra $(\mathscr{A} \underset{\mathbb{C}}{\otimes} \mathscr{U}(\mathfrak{g}))$-module.

The $(\mathscr{A}, G)$-modules form an abelian category $\mathscr{M}(\mathscr{A}, G)$, the weak ones form an abelian category $\mathscr{M}(\mathscr{A}, G)_{\text{weak}}$. Clearly $\mathscr{M}(\mathscr{A}, G)$ is a full subcategory of $\mathscr{M}(\mathscr{A}, G)_{\text{weak}}$, which is closed under subquotients. We have the faithful forgetting of the G-action functor $o: \mathscr{M}(\mathscr{A}, G)_{\text{weak}} \to \mathscr{M}(\mathscr{A})$.

EXAMPLES. (i) If $\mathscr{P}$ is a G-equivariant Lie algebroid on X then an $\mathscr{U}(\mathscr{P})$-Harish-Chandra module M is a $\mathscr{P}$-module equipped with a G-action (as an $\mathscr{O}_X$-module) such that the corresponding infinitesimal action of $\mathfrak{g}$ coincides with the action defined by the morphism $i_{\mathfrak{g}}: \mathfrak{g} \to \mathscr{P}$. If X is a point we recover the usual notion of a $(\mathscr{P}, G)$-module.

(ii) For any Harish-Chandra algebra $\mathscr{A}$ the free left $\mathscr{A}$-module $\mathscr{A}$ is a weak $(\mathscr{A}, G)$-module in an obvious manner. The corresponding $\mathfrak{g}$-action w is $w(\zeta)a = -a i_{\mathfrak{g}}(\zeta)$.

1.8.7. LEMMA. (i) *Harish-Chandra algebras on* X *are the same as* D-*algebras on the quotient stack* $G\backslash X$.

(ii) *For a Harish-Chandra algebra* $\mathscr{A}$, $(\mathscr{A}, G)$-*modules are the same as modules over corresponding algebras on* $G\backslash X$.

PROOF. (i) Denote by $p_G, p_X: G \times X \to G, X$ the projections. Consider a D-algebra on $G\backslash X$. According to 1.7 this is the same as a pair $(\mathscr{A}, g)$, where $\mathscr{A}$ is a D-algebra on X and $g: \mu^{\cdot}\mathscr{A} \xrightarrow{\sim} p_X^{\cdot}\mathscr{A}$ is an isomorphism of D-algebras that satisfies the cocycle condition. Let us define on $\mathscr{A}$ a Harish-Chandra structure $(\mu_{\mathscr{A}}, i_{\mathfrak{g}})$, i.e., a G-action on $\mathscr{A}$ in the sense of 1.8.3.

Recall that $p_X^{\cdot}\mathscr{A} = D_G \boxtimes \mathscr{A}$ (see 1.4.5(ii)), hence $p_X^{*}\mathscr{A} = \mathscr{O}_G \boxtimes \mathscr{A}$ is the centralizer of $p_G^{-1}\mathscr{O}_G \subset \mathscr{O}_{G\times X}$ in $p_X^{\cdot}\mathscr{A}$. Since $\mu^{\cdot}\mathscr{A}$ is the pull-back of $p_X^{\cdot}\mathscr{A}$ via the isomorphism $(p_G, \mu): G \times X \xrightarrow{\sim} G \times X$ we see that $\mu^{*}\mathscr{A}$ coincides with the centralizer of $p_G^{-1}\mathscr{O}_G$ in $\mu^{\cdot}\mathscr{A}$. Therefore our $g: \mu^{\cdot}\mathscr{A} \to p_X^{\cdot}\mathscr{A}$ defines an isomorphism between the $p_G^{-1}\mathscr{O}_G$-centralizers $\mu_{\mathscr{A}}: \mu^{*}\mathscr{A} \xrightarrow{\sim} p_X^{*}\mathscr{A}$. This is the desired action of G on $\mathscr{A}$ as an $\mathscr{O}_X$-module. To define $i_{\mathfrak{g}}$ consider the embedding $\nu: \mathfrak{g} \to D_G \subset p_X^{\cdot}\mathscr{A}$ that sends $\gamma \in \mathfrak{g}$ to the corresponding left invariant vector field. For $\gamma \in \mathfrak{g}$ put $i_g(\gamma) := \nu(\gamma) - g(p_G, \mu)^*(\nu(\gamma)) \in p_X^{\cdot}\mathscr{A}$; this element commutes with $p^{-1}\mathscr{O}_G$ and $\nu(\mathfrak{g})$ and, therefore, lies in $p_X^{-1}\mathscr{A} \subset p_X^{\cdot}\mathscr{A}$. We get $i_{\mathfrak{g}}: \mathfrak{g} \to \mathscr{A}$; it is easy to check that $(\mu_{\mathscr{A}}, i_{\mathfrak{g}})$ is a Harish-Chandra structure on $\mathscr{A}$.

Conversely, for a Harish-Chandra algebra $\mathscr{A}$ we define the isomorphism $g: \mu^{\cdot}\mathscr{A} \xrightarrow{\sim} p_X^{\cdot}\mathscr{A}$ as $\mu_{\mathscr{A}}$ on the centralizers of $p_G^{-1}\mathscr{O}_G$ and extend it to the entire algebra using the above formula relating $i_{\mathfrak{g}}$ and ν. One easily checks the cocycle property for g.

(ii) Clear.

1.8.8. Here is a simple corollary of 1.8.7. Assume we have an embedding of algebraic groups $G \subset G'$. We have the induced G'-variety $X' = G' \underset{G}{\times} X = G' \times X/\{(g', x) = (g'g^{-1}, gx)\}$. The embedding $X \hookrightarrow X'$ induces an isomorphism between the quotient stacks $G\backslash X \xrightarrow{\sim} G'\backslash X'$, hence the pull-back to X is an equivalence between the categories of G'-Harish-Chandra algebras on X' and G-Harish-Chandra algebras on X (and similarly for the corresponding Harish-Chandra modules). We leave to the reader an explicit construction of the inverse functors.

1.8.9. Assume that G acts on X in a free way, so we have a morphism of schemes $\pi: X \to Z$ that identifies Z with $G\backslash X$ (i.e., X is a G-torsor over Z). According to 1.8.7 the functor $\pi^{\cdot}$: (D-algebras on Z) $\to$ (Harish-Chandra algebras on X) is an equivalence of categories. Here is an explicit construction of the inverse functor. Denote by $\mathfrak{g}_Z^{\sim}$ the X-twist of $\mathscr{O}_Z \otimes \mathfrak{g}$; explicitly, $\mathfrak{g}_Z^{\sim} = (\pi_*\mathscr{O}_X \otimes \mathfrak{g})^G$, G acts on $\mathfrak{g}$ by the adjoint action. For a Harish-Chandra algebra $\mathscr{A}$ on X put $\mathscr{A}_Z^{\sim} := (\pi_*\mathscr{A})^G$; this is a D-algebra on Z. The map $i_{\mathfrak{g}}$ sends $\mathfrak{g}_Z^{\sim}$ to $\mathscr{A}_Z^{\sim}$. One can see that $\mathscr{A}_Z^{\sim} \cdot i_{\mathfrak{g}}(\mathfrak{g}_Z^{\sim})$ is actually a 2-sided ideal in $\mathscr{A}_Z^{\sim}$; put $\mathscr{A}_Z := \mathscr{A}_Z^{\sim}/\mathscr{A}_Z^{\sim} i_{\mathfrak{g}}(\mathfrak{g}_Z^{\sim})$. The left action of $\mathscr{A}$ on $\pi^*\mathscr{A}_Z = \pi^*\mathscr{A}_Z^{\sim}/\pi^*(\mathscr{A}_Z^{\sim})i_{\mathfrak{g}}(\mathfrak{g}_Z^{\sim}) = \mathscr{A}/\mathscr{A}\, i_{\mathfrak{g}}(\mathfrak{g})$ defines the isomorphism $\mathscr{A} = \pi^{\cdot}\mathscr{A}_Z$.

For any D-algebra $\mathscr{A}_Z$ on Z consider the corresponding algebras $\mathscr{A} = \pi^{\cdot}\mathscr{A}_Z$, $\mathscr{A}_Z^{\sim}$, so $\mathscr{A}_Z = \mathscr{A}_Z^{\sim}/\mathscr{A}_Z^{\sim} i_{\mathfrak{g}}(\mathfrak{g}_Z^{\sim})$. Note that $\pi_*\mathscr{A} = \pi_*\mathscr{O}_X \underset{\mathscr{O}_Z}{\otimes} \mathscr{A}_Z^{\sim}$ and any (local) section of π (i.e., any G-isomorphism $X = G \times Z$) induces an isomorphism $\mathscr{A}_Z^{\sim} = \mathscr{A}_Z \underset{\mathbb{C}}{\otimes} \mathscr{U}(\mathfrak{g})$. Assume that G is an affine group. Then the functor $\pi_*: \mathscr{M}(X, \mathscr{A}) \to \mathscr{M}(Z, \pi_*\mathscr{A})$ is an equivalence of categories. We have the adjoint functors $\mathscr{M}(\mathscr{A}, G)_{\text{weak}} \underset{\pi^{\cdot}}{\overset{\pi_{\cdot}}{\rightleftarrows}} \mathscr{M}(Z, \mathscr{A}_Z^{\sim})$,

$$\pi_{\cdot}M = (\pi_* M)^G,$$

$$\pi^{\cdot}N = \pi^{-1}\left(\pi_*\mathscr{A} \underset{\mathscr{A}_Z^{\sim}}{\otimes} N\right) = \mathscr{A} \underset{\pi^{-1}}{\otimes} \mathscr{A}_Z^{\sim}\pi^{-1}N = \mathscr{O}_X \underset{\pi^{-1}}{\otimes} \mathscr{O}_Y \pi^{-1}N.$$

1.8.10. LEMMA. (i) *The functors* $\pi^{\cdot}$, $\pi_{\cdot}$ *are mutually inverse equivalences of categories.*

(ii) $\pi_{\cdot}$ *identifies* $\mathcal{M}(\mathcal{A}, G)$ *with* $\mathcal{M}(Z, \mathcal{A}_Z)$, $\pi^{\cdot}$ *coincides on* $\mathcal{M}(Z, \mathcal{A}_Z)$ *with* $\pi_{\cdot}^{*}$.

§2. Twisted differential operators

2.1. First definitions and equivalences. ([1]) Let X be a smooth algebraic or analytic variety over $\mathbb{C}$.

2.1.1. DEFINITION. An algebra of twisted differential operators, or simply a tdo, on X is a sheaf of associative algebras D on X equipped with a morphism of algebras $i: \mathcal{O}_X \to D$ such that there exists an increasing filtration $D_{\cdot}$ on D with the following properties:

(i) $D_{\cdot}$ is a ring filtration (i.e., $D_i D_j \subset D_{i+j}$) such that the associated graded algebra $\mathrm{gr}_{\cdot} D$ is commutative; one has $D_{-1} = 0$, $\bigcup D_i = D$.

(ii) The morphism i identifies $\mathcal{O}_X$ with D_0, and the obvious morphism of the symmetric algebra $S^{\cdot}(D_1/D_0)$ into $\mathrm{gr}_{\cdot} D$ is an isomorphism of $\mathcal{O}_X$-algebras.

(iii) The morphism $\sigma: D_1/D_0 \to \mathcal{T}_X$, $\sigma(\partial)(f) := \partial f - f\partial$, where $\partial \in D_1$, $f \in \mathcal{O}_X = D_0$, is an isomorphism.

EXAMPLE. If $\mathcal{L}$ is a line bundle on X then $D_{\mathcal{L}}$ is a tdo.

REMARK. Let D be a tdo; according to (ii) and (iii) above we have a canonical isomorphism of $\mathcal{O}_X$-algebras $\mathrm{gr}_{\cdot} D = S^{\cdot}(\mathcal{T}_X) =$ functions on the cotangent bundle to X. The algebra $\mathrm{gr}_{\cdot} D$ carries a standard Poisson bracket $\{\cdot, \cdot\}: \mathrm{gr}_a D \times \mathrm{gr}_b D \to \mathrm{gr}_{a+b-1} D$, $\{f, g\} = fg - gf \bmod D_{a+b-2}$. The above isomorphism identifies $\{\cdot, \cdot\}$ with the Poisson bracket that comes from the standard symplectic structure on the cotangent bundle.

For a tdo D the filtration $D_{\cdot}$ is a D-filtration on D. Therefore D is an $\mathcal{O}_X$-differential algebra. In fact one checks immediately that $D_{\cdot}$ is the maximal D-filtration $D_{\cdot}^{\vee}$; in particular it is uniquely determined.

It is easy to see that the canonical complex from 1.1.1 for $M = D$ is the Koszul complex. To be precise, one has the following characterization of tdo's.

2.1.2. LEMMA. *An* $\mathcal{O}_X$*-differential algebra* $\mathcal{A}$ *is a tdo iff for the maximal D-filtration* $\mathcal{A}_{\cdot} = \mathcal{A}_{\cdot}^{\vee}$ *the following conditions hold*:

(i) $i: \mathcal{O}_X \to \mathcal{A}_0$ *is an isomorphism.*

(ii) *The morphism* $\sigma: \mathcal{A}_1/\mathcal{A}_0 \to \mathrm{Der}\,\mathcal{A}_0 = \mathcal{T}_X$, $\sigma(\partial)(f) = \partial f - f\partial$, *is surjective.*

PROOF. The above remarks show that any tdo satisfies the conditions in 2.1.2. Conversely, assume $\mathcal{A}$ satisfies the conditions 2.1.2. Since $\mathcal{A}_0$ is

([1]) We borrowed 2.1–2.4 below from an unpublished manuscript of A. Beilinson and D. Kazhdan.

the centralizer of $i(\mathscr{O}_X)$, we see that σ is injective, and hence σ is an isomorphism. The term $\mathscr{A}_1$ is closed under commutators (for $\partial_1, \partial_2 \in \mathscr{A}_1$, $f \in \mathscr{O}_X$ one has $\operatorname{ad} f([\partial_1, \partial_2]) = -\{[\partial_1, \partial_2], f\} = [\operatorname{ad}\partial_1, \operatorname{ad}\partial_2](f) = -[\sigma(\partial_1), \sigma(\partial_2)](f)$), therefore the $\mathscr{O}_X$-subalgebra of $\operatorname{gr}.\mathscr{A}$ generated by $\mathscr{T}_X = \mathscr{A}_1/\mathscr{A}_0$ is commutative. It remains to show that the corresponding map $S^i(\mathscr{T}_X) \to \operatorname{gr}_i\mathscr{A}$ is an isomorphism. This follows by induction on i. Namely, assume that we know this for any $i < j$. It is easy to check that for $i < j$ the maps $\delta_i\colon \operatorname{gr}_i\mathscr{A} = S^i(\mathscr{T}_X) \to \operatorname{Hom}(\Omega^1_X, \operatorname{gr}_{i-1}\mathscr{A}) = \mathscr{T}_X \otimes S^{i-1}(\mathscr{T}_X)$ from 1.1.1 are Koszul differentials, as well as the composition $S^j(\mathscr{T}_X) \to \operatorname{gr}_j\mathscr{A} \xrightarrow{\delta_j} \operatorname{Hom}(\Omega^1_X, \operatorname{gr}_{j-1}\mathscr{A}) = \mathscr{T}_X \otimes S^{j-1}(\mathscr{T}_X)$. Since δ_j is injective (see 1.1.1) the exactness of the Koszul complex implies that $S^j(\mathscr{T}_X) \xrightarrow{\sim} \operatorname{gr}_j\mathscr{A}$.

It is easy to see that any morphism between tdo's is an isomorphism, i.e., tdo's form a groupoid $\mathscr{TDO}(X)$. Below we give several equivalent descriptions of this groupoid.

2.1.3. Definition. A Picard Lie, or simply Picard, algebroid on X is a Lie algebroid $\widetilde{\mathscr{T}}$ equipped with a central section $1_{\widetilde{\mathscr{T}}}$ of $\widetilde{\mathscr{T}}^{(0)} = \operatorname{Ker}\sigma$, such that the sequence $0 \to \mathscr{O}_X \xrightarrow{i} \widetilde{\mathscr{T}} \xrightarrow{\sigma} \mathscr{T}_X \to 0$, $i(f) = f_{\widetilde{\mathscr{T}}} := f1_{\widetilde{\mathscr{T}}}$, is exact.

Recall that σ is the standard morphism of Lie algebroids defined in 1.2; we will identify $\mathscr{O}_X$ with $\widetilde{\mathscr{T}}^0$ using the isomorphism i.

Let $\widetilde{\mathscr{T}}$ be a Picard algebroid. Since $\widetilde{\mathscr{T}}^{(0)} = \mathscr{O}_X$, the sheaf $\mathscr{C}(\widetilde{\mathscr{T}})$ of connections is an Ω^1_X-torsor. The curvature map $c\colon \mathscr{C}(\widetilde{\mathscr{T}}) \to \Omega^2_X$ has the following property: $c(\nu + \nabla) = d\nu + c(\nabla)$ for $\nu \in \Omega^1_X$, $\nabla \in \mathscr{C}(\widetilde{\mathscr{T}})$ (see 1.2).

A morphism of Picard algebroids is a morphism of Lie algebroids that preserves $1_{\widetilde{\mathscr{T}}}$'s. Picard algebroids form a groupoid $\mathscr{PA}(X)$. The Baer sum construction defines on $\mathscr{PA}(X)$ a structure of "$\mathbb{C}$-vector space in categories". Namely, for $\widetilde{\mathscr{T}}_i \in \mathscr{PA}(X)$, $\lambda_i \in \mathbb{C}$ the linear combination $\widetilde{\mathscr{T}} = \lambda_1\widetilde{\mathscr{T}}_1 + \lambda_2\widetilde{\mathscr{T}}_2$ is a Picard algebroid $\widetilde{\mathscr{T}}$ equipped with a morphism of Lie algebroids $s_{\lambda_1,\lambda_2}\colon \widetilde{\mathscr{T}}_1 \times_{\mathscr{T}_X} \widetilde{\mathscr{T}}_2 \to \widetilde{\mathscr{T}}$ such that $s_{\lambda_1,\lambda_2}(f_1, f_2) = (\lambda_1 f_1 + \lambda_2 f_2)$.

For a tdo D consider the Lie algebroid $\widetilde{\mathscr{T}}_D := \operatorname{Lie} D$ (see 1.2.5). Clearly $\widetilde{\mathscr{T}}_D = D_1$, and $1_{\widetilde{\mathscr{T}}_D} = 1 \in \mathscr{O}_X \subset D_1$ defines on $\widetilde{\mathscr{T}}_D$ the structure of a Picard algebroid. Conversely, for a Picard algebroid $\widetilde{\mathscr{T}}$ denote by $D_{\widetilde{\mathscr{T}}}$ the quotient of $\mathscr{U}(\widetilde{\mathscr{T}})$ modulo the ideal generated by the central element $1 - 1_{\widetilde{\mathscr{T}}}$. One checks immediately that $D_{\widetilde{\mathscr{T}}}$ is a tdo.

2.1.4. Lemma. *The functors $\mathscr{T}D\mathscr{O}(X) \rightleftarrows \mathscr{PA}(X)$, $D \mapsto \widetilde{\mathscr{T}}_D$, $\widetilde{\mathscr{T}} \mapsto D_{\widetilde{\mathscr{T}}}$, are mutually inverse equivalences of categories.*

2.1.5. Let $d\colon A^n \to A^{n+1}$ be a morphism of sheaves of abelian groups on X, considered as a length 2 complex $A^{\cdot}$ supported in degrees n and $n+1$.

An $A^{\cdot}$*-torsor* is a pair $(\mathcal{F}, c)$, where $\mathcal{F}$ is an A^n-torsor and $c: \mathcal{F} \to A^{n+1}$ is a map such that $c(a+\varphi) = d(a)+c(\varphi)$ for $a \in A^n$, $\varphi \in \mathcal{F}$ (in other words, c is a trivialization of the induced A^{n+1}-torsor $d(\mathcal{F})$). The $A^{\cdot}$-torsors form a groupoid $A^{\cdot}$-tors. One has $\operatorname{Aut}\mathcal{F} = \Gamma(X, \operatorname{Ker} d) = H^n(X, A^{\cdot})$, and isomorphism classes of $A^{\cdot}$-torsors are in a natural 1-1 correspondence with $H^{n+1}(X, A^{\cdot})$.

REMARK. $A^{\cdot}$-tors is a stack in Picard categories on X; if $A^{\cdot}$ is a complex of $\mathbb{C}$-vector spaces, then $A^{\cdot}$-tors is a $\mathbb{C}$-vector space in categories (one forms $\mathbb{C}$-linear combinations of torsors in an obvious way). If D is surjective, then $A^{\cdot}$-tors $=$ $(\operatorname{Ker} d)$-tors.

Consider now the truncated de Rham complex $\Omega_X^{\geqslant 1} := (\Omega_X^1 \to \Omega_X^{2\,\mathrm{cl}})$, where $\Omega_X^{2\,\mathrm{cl}}$ are closed 2-forms. By 2.1.3 we have the functor $\mathscr{C}: \mathscr{P}\mathscr{A}(X) \to \Omega_X^{\geqslant 1}$-tors, $\widetilde{\mathcal{T}} \mapsto (\mathscr{C}(\widetilde{\mathcal{T}}), c)$.

2.1.6. LEMMA. *$\mathscr{C}: \mathscr{P}\mathscr{A}(X) \to \Omega_X^{\geqslant 1}$-tors is an equivalence of $\mathbb{C}$-vector spaces in categories.*

By 2.1.5 we can identify the set of isomorphism classes of tdo's with $H^2(X, \Omega_X^{\geqslant 1})$. For a tdo D we will denote by $c_1(D) \in H^2(X, \Omega_X^{\geqslant 1})$ the corresponding cohomology class.

2.1.7. For a tdo D a *connection* ∇ on D is a connection on the corresponding Picard algebroid $\widetilde{\mathcal{T}}_D$. Note that pairs (D, ∇), ∇ is a connection on a tdo D, are rigid: the only automorphism of D that preserves ∇ is the identity. The pairs (D, ∇) are in 1-1 correspondence with closed 2-forms; for $\omega \in \Omega^{2\,\mathrm{cl}}(X)$ we will denote by $(D_\omega, \nabla_\omega)$ the (unique up to a canonical isomorphism) tdo with $c(\nabla) = \omega$. A corresponding $\Omega_X^{\geqslant 1}$-torsor $(\mathcal{F}_\omega, c_\omega)$ is $\mathcal{F}_\omega = \Omega_X^1$, $c_\omega(\nu) = d\nu + \omega$.

2.1.8. Now consider the cotangent bundle $T^* = T^*(X) \xrightarrow{\pi} X$. This is a vector bundle over X; also T^* carries a canonical symplectic 2-form ω such that π is a polarization (which is a smooth projection with Lagrangian fibers). If ν is a 1-form on X and $t_\nu: T^* \to T^*$, $t_\nu(a) = a + \nu_{\pi(a)}$, is the translation by ν, then $t_\nu^*(\omega) = \pi^*(d\nu) + \omega$.

DEFINITION. A twisted cotangent bundle is a T^*-torsor $\psi \xrightarrow{\pi_\psi} X$ (i.e., π_ψ is a fibration equipped with a simple transitive action of T^* along the fibers) together with a symplectic form ω_ψ on ψ such that π_ψ is a polarization for ω_ψ, and for any 1-form ν one has $t_\nu^*(\omega_\psi) = \pi_\psi^* d\nu + \omega$.

For a twisted cotangent bundle ψ we will denote by A_ψ the $\mathcal{O}_X$-algebra $\pi_{\psi *}\mathcal{O}_\psi$. Then A_ψ carries a Poisson bracket $\{\cdot, \cdot\}$ (defined by ω_ψ) and a filtration $A_{\psi i} =$ functions of degree $\leqslant i$ along the fibers of π_ψ. Clearly one has $A_{\psi i} = \{a \in A_\psi : \{a, \mathcal{O}_X\} \subset A_{\psi(i-1)}\}$ and the associated graded algebra $\operatorname{gr.} A_\psi$ is naturally isomorphic to the symmetric algebra $A_{T^*} = S^{\cdot}\mathcal{T}_X$ as a Poisson algebra.

2.1.9. REMARKS. (i) A T^*-torsor structure on ψ is uniquely determined by the symplectic structure ω_ψ and the polarization π_ψ (since the infinitesimal action of a 1-form $\nu \in \Omega^1(X)$ is given by the vector field $\zeta(\nu)$ defined by $\zeta(\nu) \lrcorner \omega_\varphi = \pi_\psi^*(\nu)$).

(ii) Twisted cotangent bundles over X form a groupoid $\mathcal{TCB}(X)$. According to (i), $\mathcal{TCB}(X)$ is a full subcategory of the category of triples (Y, ω_Y, π_Y) where (Y, ω_Y) is a symplectic manifold and $\pi_Y: Y \to X$ is a polarization (for the symplectic structure).

2.1.10. LEMMA. *One has a canonical equivalence of categories* $\Gamma: \mathcal{TCB}(X) \xrightarrow{\sim} \Omega_X^{\geqslant 1}$*-tors.*

PROOF. Put $\Gamma(\psi) = \Omega^1$-torsor of a section of ψ; the map $c: \Gamma(\psi) \to \Omega_X^{2\,\mathrm{cl}}$ is $c(\gamma) := \gamma^*(\omega_\psi)$. Note that the corresponding Picard algebroid $\widetilde{\mathcal{T}}_\psi$ is $A_{\psi 1}$ equipped with the bracket $\{\cdot, \cdot\}$.

The inverse functor Γ^{-1} sends an $\Omega_X^{\geqslant 1}$-torsor $(\mathcal{F}, c)$ to $(\psi, \pi_\psi, \omega_\psi)$, where $\pi_\psi: \psi \to X$ is the space of the torsor $\mathcal{F}$, and the symplectic form ω_ψ is the unique form such that for a section $\gamma \in \mathcal{F}$ of π_ψ the corresponding isomorphism $T^*X \xrightarrow{\sim} \psi$, $0 \mapsto \gamma$, identifies ω_ψ with $\omega + \pi^* c(\gamma)$.

2.1.11. REMARK. Let D be a tdo, and ψ be the corresponding twisted cotangent bundle. Then D is a "quantization" of ψ in the sense that D is a deformation of a commutative algebra A_ψ. To be precise, one has a canonical family $\mathbb{D} = \{D_t\}$ of sheaves of filtered rings on X parametrized by points $t \in \mathbb{P}^1$ (i.e., $\mathbb{D}$ is a flat $\mathcal{O}_{\mathbb{P}^1}$-algebra) such that:

(i) For $t \neq \infty$ one has $D_t = D_{t\widetilde{\mathcal{T}}}$ (here $\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}_D$; in particular, $D_1 = D$, $D_0 = D_{\mathcal{O}_X}$).

(ii) $D_\infty = A_\psi$, and the ω_ψ-Poisson bracket on A_ψ is given by the usual formula $\{\varphi_1, \varphi_2\} = [t(\widetilde{\varphi}_1\widetilde{\varphi}_2 - \widetilde{\varphi}_2\widetilde{\varphi}_1)] \bmod t^{-1}$ (here $\varphi_i \in D_\infty$, and $\widetilde{\varphi}_i$ are arbitrary local sections of D at $t = \infty$ such that $\widetilde{\varphi}_i(\infty) = \varphi_i$).

(iii) $\mathrm{gr}_a D = (S^a \mathcal{T}_X)(-a)$.

2.1.12. Let us see what the above constructions mean in the case $D = D_{\mathcal{L}}$, where $\mathcal{L}$ is a line bundle. The corresponding Picard algebroid $\widetilde{\mathcal{T}}_{\mathcal{L}} = \widetilde{\mathcal{T}}_{D_{\mathcal{L}}}$ is the Lie algebroid of infinitesimal symmetries of $(X, \mathcal{L})$, see 1.2.3. The $\Omega_X^{\geqslant 1}$-torsor $(\mathcal{F}_{\mathcal{L}}, c_{\mathcal{L}}) := \mathcal{C}(\widetilde{\mathcal{T}}_{\mathcal{L}})$ is the sheaf of connections on $\mathcal{L}$, and $c_{\mathcal{L}}$ is the usual curvature. Note that the functor $\mathcal{O}_X^*$-tors $\to \Omega_X^{\geqslant 1}$-tors is precisely the push-out functor for the morphism $d\log: \mathcal{O}_X^* \to \Omega_X^{1\,\mathrm{cl}} \subset \Omega_X^{\geqslant 1}[1]$. In particular it transforms $\otimes$ to the sum of torsors. One has $c_1(D_{\mathcal{L}}) = c_1(\mathcal{L}) \in H^2(X, \Omega_X^{\geqslant 1})$.

2.1.13. A tdo D is called locally trivial if locally it is isomorphic to $D_X = D_{\mathcal{O}_X}$; according to 2.1.6, locally trivial tdo's are the same as $\Omega_X^{1\,\mathrm{cl}}$-torsors. Note that in analytic situations every tdo is locally trivial. In algebraic situations this is not true in general. For example, let X be a compact algebraic variety. The space of isomorphism classes of tdo's $H^2(X, \Omega_X^{\geq 1})$ coincides with the Hodge filtration subspace $F^1 H^2_{DR}$, and the locally trivial tdo's correspond to those classes that vanish on some Zariski open subspace of X, i.e., precisely to $\mathbb{C}$-linear combinations of the algebraic cycles classes.

2.2. Functoriality. Let $\varphi: Y \to X$ be a morphism of smooth varieties. We have the corresponding morphism $\varphi^*: \varphi^{-1}\Omega_X^{\cdot} \to \Omega_Y^{\cdot}$ between the de Rham complexes, hence the morphism of groupoids $\varphi^{\cdot}: \Omega_X^{\geq 1}$-tors $\to \Omega_Y^{\geq 1}$-tors. If $\psi: Z \to Y$ is another morphism of varieties then we have a canonical isomorphism of functors $\psi^{\cdot}\varphi^{\cdot} = (\varphi\psi)^{\cdot}: \Omega_X^{\geq 1}$-tors $\to \Omega_Z^{\geq 1}$-tors, therefore $\Omega^{\geq 1}$-tors form a fibered category over the category of schemes. It is easy to see that $\varphi^{\cdot}$ satisfies the descent property if φ is a smooth surjective map, i.e., $\Omega^{\geq 1}$-tors form a stack on the category of smooth schemes equipped with the smooth topology.

The equivalent versions $\mathcal{TDO}$, $\mathcal{PA}$, and $\mathcal{TCB}$ of $\Omega^{\geq 1}$-tors therefore are also stacks. The pull-back functors for tdo's and Picard algebroids coincide with those defined in 1.4.

To be precise, for a Picard algebroid $\widetilde{\mathcal{T}_X}$ on X the Lie algebroid $\varphi^{\cdot}\widetilde{\mathcal{T}_X}$ (see 1.4.6) is a Picard algebroid. Namely, as an $\mathcal{O}_Y$-module $\varphi^{\cdot}\widetilde{\mathcal{T}_X}$ coincides with the fibered product $\varphi^*\widetilde{\mathcal{T}_X} \underset{\varphi^*\mathcal{T}_X}{\times} \mathcal{T}_Y$ (with respect to the projections $\varphi^*\widetilde{\mathcal{T}_X} \xrightarrow{\varphi^*(\sigma)} \varphi^*\mathcal{T}_X \xleftarrow{d\varphi} \mathcal{T}_Y$); and we put $1_{\varphi^{\cdot}\widetilde{\mathcal{T}_X}} = (\varphi^* 1_{\widetilde{\mathcal{T}_X}}, 0)$. One defines the pull-back for connections in an obvious way; this identifies $\mathcal{C}(\varphi^{\cdot}\widetilde{\mathcal{T}_X})$ with $\varphi^{\cdot}\mathcal{C}(\widetilde{\mathcal{T}_X})$.

Let us turn to tdo's:

LEMMA. (i) *Let $\varphi: Y \to X$ be a morphism of smooth schemes and D be a tdo on X. Then the $\mathcal{O}_Y$-differential algebra $\varphi^{\cdot}D$ (defined in* (1.4.1)) *is a tdo.*

(ii) *One has a canonical isomorphism of Picard algebroids $\widetilde{\mathcal{T}}_{\varphi^{\cdot}D} = \varphi^{\cdot}\widetilde{\mathcal{T}_D}$.*

PROOF. (i) The morphism φ can be written as a composition of a closed imbedding and a projection. By Lemma 1.5.1 it is enough to prove the statement for these two cases. The case of projection immediately follows from 1.4.5(ii). In the case of a closed imbedding one proves the statement by direct local computations using 1.4.5(iii).

(ii) easily follows from (i).

Note that for $\psi: Z \to Y$ the canonical morphism is compatible with the corresponding isomorphism of Picard algebroids.

According to 1.4 we have the pull-back functors $\varphi^*: \mathcal{M}(D) \to \mathcal{M}(\varphi^{\cdot}D)$ between the categories of D-modules that make the $\mathcal{M}(D)$'s a stack.

2.3. Twisted D-modules and projective connections. Let D be a tdo. We say that a D-module M is *lissé* if it is coherent as an $\mathcal{O}_X$-module.

2.3.1. LEMMA. *Let M be a lissé D-module. Then*
(i) *M is a vector bundle;*
(ii) *one has a canonical isomorphism of tdo's $D \xrightarrow{\sim} D_{(\det M)^{1/d}}$ where $d = \operatorname{rk} M$. In particular, D is locally trivial.*

PROOF. (i) Repeat a proof for ordinary D-modules (see [B]).

(ii) This canonical isomorphism comes from the morphism of Picard algebroids $\mathcal{T}_D \to \mathcal{T}_{\det M}$ that sends $1_{\mathcal{T}_D}$ to $d1_{\mathcal{T}_{\det M}}$. Formula: $\tilde{\tau} \mapsto \tilde{\tau}_{\det M}$, $\tilde{\tau}_{\det M}(m_1 \wedge \cdots \wedge m_d) = \tilde{\tau}(m_1) \wedge m_2 \wedge \cdots \wedge m_d + \cdots + m_1 \wedge \cdots \wedge \tilde{\tau}(m_d)$.

Let $\mathcal{E}$ be a coherent $\mathcal{O}_X$-module; consider the Lie algebroid $\widetilde{\mathcal{T}_{\mathcal{E}}}$ from 1.2.3. Note that $\mathcal{O}_X \cdot \mathrm{id}_{\mathcal{E}}$ is an ideal in $\widetilde{\mathcal{T}_{\mathcal{E}}}$; put $\overline{\mathcal{T}}_{\mathcal{E}} := \widetilde{\mathcal{T}_{\mathcal{E}}}/\mathcal{O}_X \cdot \mathrm{id}_{\mathcal{E}}$. This is a Lie algebroid. A *projective connection* on $\mathcal{E}$ is a connection on $\overline{\mathcal{T}}_{\mathcal{E}}$.

For a tdo D a D-action on $\mathcal{E}$ is the same as a morphism of Lie algebroids $\alpha: \mathcal{T}_D \to \widetilde{\mathcal{T}_{\mathcal{E}}}$ that sends $1_{\widetilde{\mathcal{T}}_D}$ to $\mathrm{id}_{\mathcal{E}}$. Such an α defines an integrable projective connection $\overline{\nabla}_\alpha$ on $\mathcal{E}$ by the formula $\overline{\nabla}_\alpha(\sigma(\tau)) = \alpha(\tau) \bmod \mathcal{O}_X \cdot \mathrm{id}_{\mathcal{E}}$.

2.3.2. LEMMA. *Assume that the map $\mathcal{O}_X \to \operatorname{End}\mathcal{E}$, $f \mapsto f\,\mathrm{id}_{\mathcal{E}}$, is injective. Then the above map $\alpha \mapsto \overline{\nabla}_\alpha$ from the set of pairs (D, α), where D is a tdo and α is a D-action on $\mathcal{E}$, to the set of projective integrable connections on $\mathcal{E}$ is bijective.*

PROOF. One constructs the inverse map as follows. Let $\overline{\nabla}: \mathcal{T}_X \to \overline{\mathcal{T}}_{\mathcal{E}}$ be an integrable projective connection. Then $\widetilde{\mathcal{T}_{\overline{\nabla}}} := \mathcal{T}_X \underset{\widetilde{\mathcal{T}_{\mathcal{E}}}}{\times} \widetilde{\mathcal{T}_{\mathcal{E}}}$ is a Picard algebroid, and the projection $\alpha_{\overline{\nabla}}: \widetilde{\mathcal{T}_{\overline{\nabla}}} \to \widetilde{\mathcal{T}_{\mathcal{E}}}$ defines the $D_{\widetilde{\mathcal{T}_{\overline{\nabla}}}}$-action on $\mathcal{E}$.

2.4. Subprincipal symbols.(2) Let $\Omega = \det \Omega^1_X$ be the sheaf of volume forms on X, and $\widetilde{\mathcal{T}_\Omega}$ be the corresponding Picard algebroid. One has a canonical section $\ell: \mathcal{T}_X \to \widetilde{\mathcal{T}_\Omega}$ which assigns to $\partial \in \mathcal{T}_X$ its Lie derivative $\ell(\partial)$. Clearly ℓ commutes with the Lie bracket and for $f \in \mathcal{O}_X$ one has $f\ell(\partial) = \ell(f\partial) - \partial(f)$.

2.4.1. Let D be any tdo on X. The *dual* to the tdo D is a tdo D° equipped with an isomorphism $*: D \xrightarrow{\sim} D^\circ$ of filtered sheaves such that $*(ab) = *(a) * (b)$, $*f = f$ for any sections $a, b \in D$, $f \in \mathcal{O}_X = D_0 = D_0^\circ$. Clearly one has $\mathrm{gr}_i(*) = (-1)^i \mathrm{id}_{S^i\mathcal{T}_X}$.

Consider the corresponding Picard algebroids $\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}_D}$, $\widetilde{\mathcal{T}^\circ} = \widetilde{\mathcal{T}_{D^\circ}}$: one has an isomorphism of sheaves $*: \widetilde{\mathcal{T}} \xrightarrow{\sim} \widetilde{\mathcal{T}^\circ}$ such that $*([\partial_1, \partial_2]) = -[*\partial_1, *\partial_2]$, $*(f\partial) = f*\partial - \sigma(\partial)(f)$, $*i_{\widetilde{\mathcal{T}}}(f) = i_{\widetilde{\mathcal{T}^\circ}}(f)$, and $\sigma(*\partial) = -\sigma(\partial)$ for $\partial_1, \partial_2 \in \widetilde{\mathcal{T}}$, $f \in \mathcal{O}_X$. The Picard algebroid $\widetilde{\mathcal{T}^\circ}$ is canonically isomorphic to $\widetilde{\mathcal{T}_\Omega} - \widetilde{\mathcal{T}}$. Namely, $*\partial \mapsto s_{1,-1}(-\ell\sigma(\partial), -\partial)$ (see 2.1.3 for notation).

(2) We are grateful to V. Drinfeld whose remarks made the exposition of this section much clearer.

2.4.2. EXAMPLE. Consider the tdo $D_{\Omega^{1/2}}$. The above isomorphism identifies $\widetilde{\mathscr{T}}^{\circ}_{\Omega^{1/2}}$ with $\widetilde{\mathscr{T}_{\Omega}} - \widetilde{\mathscr{T}_{\Omega^{1/2}}} = \widetilde{\mathscr{T}_{\Omega^{1/2}}}$, therefore $D_{\Omega^{1/2}}$ is a self-dual tdo (i.e., we have the involution $*\colon D_{\Omega^{1/2}} \to D_{\Omega^{1/2}}$, $*^2 = \mathrm{id}$, $*(ab) = *(b) * (a)$). Denote by $D^{\pm}_{\Omega^{1/2}}$ the ± 1 eigenspaces of $*$ on $D_{\Omega^{1/2}}$. One has $\operatorname{gr} D^{+}_{\Omega^{1/2}} = \bigoplus S^{2i}\mathscr{T}_X$, $\operatorname{gr} D^{-}_{\Omega^{1/2}} = \bigoplus S^{2i+1}\mathscr{T}_X$. The $\pm$-grading is not compatible with the product, but it is compatible (in different ways) with the brackets $[\partial_1, \partial_2] = \partial_1\partial_2 - \partial_2\partial_1$ and the symmetrized product $\partial_1 \bullet \partial_2 = (1/2)(\partial_1\partial_2 + \partial_2\partial_1)$.

2.4.3. For a tdo D consider the graded $\mathbb{C}[t]$-algebra $\widehat{D}_{\cdot} := \bigoplus D_i$ (for $\partial_1 \in \widehat{D}_i = D_i$ and $\partial_2 \in \widehat{D}_j = D_j$ their product is $\partial_1\partial_2 \in D_{i+j}$, and $t\partial_1 = \partial_1 \in D_{i+1} = \widehat{D}_{i+1}$). For any nonzero $\lambda \in \mathbb{C}$ we have the isomorphism $\widehat{D}/(t-\lambda)\widehat{D} \xrightarrow{\sim} D$, $\partial \mapsto \lambda^i \partial$ for $\partial \in \widehat{D}_i = D_i$, and $\widehat{D}/t\widehat{D} = \operatorname{gr} D = S^{\cdot}\mathscr{T}_X$.

Consider the operations $\bullet$, $\{\cdot, \cdot\}$ on $\widehat{D}$ defined by the formulas $\partial_1 \bullet \partial_2 := (1/2)(\partial_1\partial_2 + \partial_2\partial_1)$, $\{\partial_1, \partial_2\} := (1/t)(\partial_1\partial_2 - \partial_2\partial_1)$. Then $\{\cdot, \cdot\}$ is a Lie algebra bracket, $\bullet$ is a commutative (nonassociative) product, and one has $\{\partial_1, \partial_2 \bullet \partial_3\} = \{\partial_1, \partial_2\} \bullet \partial_3 + \partial_2 \bullet \{\partial_1, \partial_3\}$. Put $\widetilde{\operatorname{gr}} D = \widehat{D}/t^2\widehat{D} = \bigoplus D_i/D_{i-2}$; the operations $\bullet$, $\{\cdot, \cdot\}$ induce the corresponding operations on the quotient $\widetilde{\operatorname{gr}} D$.

2.4.4. LEMMA. *The operation $\bullet$ on $\widetilde{\operatorname{gr}} D$ is associative. Therefore $\widetilde{\operatorname{gr}} D$ is a Poisson $\mathbb{C}[t]/t^2$-algebra (with respect to the product $\bullet$ and the Poisson bracket $\{\cdot, \cdot\}$).*

PROOF. For $\partial_j \in D_{i_j}$, $j = 1, 2, 3$, one has $(\partial_1 \bullet \partial_2) \bullet \partial_3 - \partial_1 \bullet (\partial_2 \bullet \partial_3) = (1/4)[\partial_2, [\partial_1, \partial_3]] \in D_{i_1+i_2+i_3-2}$.

We can describe this Poisson algebra as follows. Consider the Picard algebroid $\mathscr{T}^{\vee}_D := \widetilde{\mathscr{T}_D} - \widetilde{\mathscr{T}_{\Omega^{1/2}}}$. Let $(\psi, \pi_\psi, \omega_\psi)$ be its twisted cotangent bundle, and $A_{\cdot} := \mathscr{O}_\psi$ be the corresponding filtered commutative algebra with Poisson bracket $\{\cdot, \cdot\}$ (see 2.1.8). As above we have the graded Poisson $\mathbb{C}[t]$-algebra $\widehat{A} := \bigoplus A_i$ (one has $\{A_i, A_j\} \subset A_{i+j-1}$), and the quotient $\widetilde{\operatorname{gr}} A = \widehat{A}/t^2\widehat{A}$ which is a $\mathbb{C}[t]/t^2$-Poisson algebra.

2.4.5. PROPOSITION. *One has a canonical isomorphism of graded Poisson $\mathbb{C}[t]/t^2$-algebras $\tilde{\sigma}\colon \widetilde{\operatorname{gr}} D \xrightarrow{\sim} \widetilde{\operatorname{gr}} A$ that lifts the isomorphism $\sigma\colon \widetilde{\operatorname{gr}} D/t\widetilde{\operatorname{gr}} D = \operatorname{gr} D \xrightarrow{\sim} S^{\cdot}\mathscr{T}_X = \widetilde{\operatorname{gr}} A/t\widetilde{\operatorname{gr}} A$.*

PROOF. For $\tau \in \mathscr{T}_X$ denote by τ^- the unique element of $\widetilde{\mathscr{T}_{\Omega^{1/2}}} = (D^{-}_{\Omega^{1/2}})_1$ such that $\sigma(\tau^-) = \tau$; for $f \in \mathscr{O}_X$ one has $(f\tau)^- = f\tau^- + \frac{\tau(f)}{2}$. Consider the isomorphism of sheaves $\theta_1\colon \widetilde{\mathscr{T}_D} \xrightarrow{\sim} \mathscr{T}^{\vee}_D$, $\theta_1(\partial) = s_{1,-1}(\partial, \sigma(\partial)^-)$ (see 2.1.3 for $s_{1,-1}$). Then θ_1 is an isomorphism of extensions of $\mathscr{T}_X$ by $\mathscr{O}_X$, it commutes with brackets and for $f \in \mathscr{O}_X$, $\partial \in \widetilde{\mathscr{T}_D}$ one has $\theta_1(f \bullet \partial) = \theta_1(f\partial + 1/2\sigma(\partial)(f)) = f\theta_1(\partial)$. Therefore, for any $i \geqslant 0$ we have the isomorphism $\theta_1 = S^i_\bullet(\theta_1)\colon \mathscr{S}^i_\bullet\widetilde{\mathscr{T}_D} \xrightarrow{\sim} A_i$ where $S^i_\bullet\widetilde{\mathscr{T}_D}$ is the symmetric power with respect to the $\mathscr{O}_X$-module structure on $\widetilde{\mathscr{T}_D}$, $(f, \partial) \mapsto f \bullet \partial$. The graded ring $\widehat{\mathscr{S}} := \bigoplus S^i_\bullet\widetilde{\mathscr{T}_D}$ is a graded $\mathbb{C}[t]$-algebra in the usual manner (the multiplication by t is the multiplication by $1 \in \mathscr{O}_X \subset \widetilde{\mathscr{T}_D}$, which is an embedding $S^i_\bullet\widetilde{\mathscr{T}_D} \subset S^{i+1}_\bullet\widetilde{\mathscr{T}_D}$), and the bracket on $\widetilde{\mathscr{T}_D}$ defines the Poisson structure on $\mathscr{S}$. Clearly $\theta = \bigoplus \theta_i\colon \widehat{\mathscr{S}} \to \widehat{A}$ is an isomorphism of the graded Poisson $\mathbb{C}[t]$-algebras. On the other hand, the identity map $\mathrm{id}_{\widetilde{\mathscr{T}_D}}$ extends to a morphism $\psi\colon \widehat{\mathscr{S}} \to \widetilde{\mathrm{gr}}\, D$ of graded $\mathbb{C}[t]$-algebras. It commutes with Poisson brackets, and the induced map $\widehat{\mathscr{S}}/t^2\widehat{\mathscr{S}} \to \widetilde{\mathrm{gr}}\, D$ is an isomorphism of $\mathbb{C}[t]/t^2$-Poisson algebras. We put $\widetilde{\sigma} := \theta\psi^{-1}\colon \widetilde{\mathrm{gr}}\, D \to \widehat{A}/t^2\widehat{A} = \widetilde{\mathrm{gr}}\, A$: this is the desired isomorphism.

REMARK. An explicit formula for the inverse isomorphism is $\widetilde{\sigma}^{-1}\colon \widetilde{\mathrm{gr}}\, A \xrightarrow{\sim} \widetilde{\mathrm{gr}}\, D$. We have $\widetilde{\sigma}_0^{-1} = \mathrm{id}_{\mathscr{O}_X}$, $\widetilde{\sigma}_1^{-1}(a) = s_{1,1}(a, \sigma(a)^-)$ (here $a \in \widetilde{\mathrm{gr}}_1 A = A_1 = \widetilde{\mathscr{T}}^\vee_D$, and we identify $\widetilde{\mathrm{gr}}_1 D = D_1 = \widetilde{\mathscr{T}_D}$ with $\widetilde{\mathscr{T}}^\vee_D + \widetilde{\mathscr{T}}_{\Omega^{1/2}}$).

If $a = a_1 \cdots a_i \in \widetilde{\mathrm{gr}}_i A = A_i/A_{i-2}$, where $a_j \in A_1$, one has $\widetilde{\sigma}_1^{-1}(a) = 1/i! \sum_g \widetilde{\sigma}_1^{-1}(a_{g(1)}) \cdots \widetilde{\sigma}_1^{-1}(a_{g(i)})$, where the sum is taken over all permutations g of i indices.

2.4.6. COROLLARY. *The boundary map*

$$\delta_D\colon H^i(X, S^i\mathscr{T}_X) \to H^{i+1}(X, S^{j-1}\mathscr{T}_X)$$

for the short exact sequence $0 \to S^{j-1}\mathscr{T}_X \to D_j/D_{j-2} \to S^j\mathscr{T}_X \to 0$ *is convolution with the class* $c_1(D) - \frac{1}{2}c_1(\Omega) \in H^1(X, \Omega^1_X)$.

2.5. Monodromic D-modules. Let H be a torus (i.e., an algebraic group isomorphic to a product of $\mathbb{C}^*$'s), $\mathfrak{h} = \mathrm{Lie}\, H$. For a variety X we will call an H-torsor $\pi\colon \widetilde{X} \to X$ over X an *H-monodromic structure* on X; we call a pair $(X, \widetilde{X})$ an *H-monodromic variety.*

2.5.1. Assume that X is smooth. An H-monodromic structure $\widetilde{X}$ on X defines, by 1.2.3, a Lie algebroid $\widetilde{\mathscr{T}} := \widetilde{\mathscr{T}_{\widetilde{X}}}$ on X. The $\mathscr{O}_X$-Lie algebra $\widetilde{\mathscr{T}}^{(0)}$ coincides with $\widetilde{\mathfrak{h}} = \mathscr{O}_X \otimes \mathfrak{h}$ identified with the (commutative) Lie algebra of vertical H-invariant vector fields; we have the short exact sequence $0 \to \widetilde{\mathfrak{h}} \to \widetilde{\mathscr{T}} \xrightarrow{\alpha} \mathscr{T}_X \to 0$. The group H acts on $D_{\widetilde{X}}$ in a usual way. On X we have the D-algebras $\pi_* D_{\widetilde{X}} \supset \widetilde{D} := [\pi_* D_{\widetilde{X}}]^H$; clearly $\pi_* D_{\widetilde{X}} = \pi_*\mathscr{O}_{\widetilde{X}} \underset{\mathscr{O}_X}{\otimes} \widetilde{D}$ as an $\mathscr{O}_X$-module.

The embedding $\widetilde{\mathscr{T}} \subset \widetilde{D}$ induces an isomorphism $\mathscr{U}(\widetilde{\mathscr{T}}) \xrightarrow{\sim} \widetilde{D}$; in particular $S(\mathfrak{h})$ coincides with the center of $\widetilde{D}$ and $\widetilde{D}/\mathfrak{h}\widetilde{D} = D_X$ (since any (local) section of π, i.e., an isomorphism $\widetilde{X} = H \times X$, identifies $\widetilde{D}$ with $D_X \otimes \mathscr{U}(\mathfrak{h}) = D_X \otimes S(\mathfrak{h})$). For $\chi \in \mathfrak{h}^*$ denote by m_χ the corresponding maximal ideal of $S(\mathfrak{h})$. Then $D_\chi := \widetilde{D}/m_\chi\widetilde{D}$ is a (locally trivial) tdo on X; clearly D_χ depends on χ in a $\mathbb{C}$-linear way. Therefore, one can consider $\widetilde{D}$ as a "linear" family of tdo's on X parametrized by $\mathfrak{h}^*$.

2.5.2. A *monodromic D-module on* X is a weak $(D_{\widetilde{X}}, H)$-module (see 1.8); denote by $\widetilde{\mathcal{M}}(X) = \mathcal{M}(D_{\widetilde{X}}, H)_{\text{weak}}$ the corresponding category. According to 1.8.9, we have the mutually inverse equivalences of categories $\widetilde{\mathcal{M}}(X) \underset{\pi}{\overset{\pi}{\rightleftarrows}} \mathcal{M}(\widetilde{D})$ compatible with $\mathcal{O}$-tensor products of modules.

Let M be a $\widetilde{D}$-module. For an ideal $I \subset S(\mathfrak{h})$ put $M^I := \{m \in M\colon Im = 0\}$; this is a $\widetilde{D}$-submodule of M. In particular for $\chi \in \mathfrak{h}^*$ we have the submodules $M_\chi := M^{m_\chi} \subset M_{\widetilde{\chi}} := \bigcup_{n\geqslant 1} M^{m_\chi^n}$; let M_{fin} be the union of the M^I's, where the I's are ideals of finite codimension in $S(\mathfrak{h})$. Clearly $M_{\text{fin}} = \bigoplus_{\chi\in\mathfrak{h}^*} M_{\widetilde{\chi}}$. Denote by $\mathcal{M}(\widetilde{D})_{\text{fin}} \subset \mathcal{M}(\widetilde{D})$ the full subcategory of $\mathfrak{h}$-*finite* modules, i.e., such M that $M_{\text{fin}} = M$; we also have the full subcategories $\mathcal{M}(X)_\chi := \mathcal{M}(D_\chi) = \{M \in \mathcal{M}(\widetilde{D})\colon M_\chi = M\} \subset \mathcal{M}(X)_{\widetilde{\chi}} := \{M \in \mathcal{M}(\widetilde{D})\colon M_{\widetilde{\chi}} = M\}$, so $\mathcal{M}(\widetilde{D})_{\text{fin}} = \prod_{\chi\in\mathfrak{h}^*} \mathcal{M}(X)_{\widetilde{\chi}}$. The equivalence $\pi^{\cdot}$ sends $\mathcal{M}(\widetilde{D})_{\text{fin}}$ to the subcategory $\widetilde{\mathcal{M}}(X)_{\text{fin}} \subset \widetilde{\mathcal{M}}(X)$; we will identify $\mathcal{M}(X)_\chi$, $\mathcal{M}(X)_{\widetilde{\chi}}$ with the corresponding subcategories of $\mathcal{M}(X)$.

2.5.3. Consider now the category $\mathcal{M}(\widetilde{X})$ of $D_{\widetilde{X}}$-modules. The projection π is affine, therefore $\pi_*\colon \mathcal{M}(\widetilde{X}) \to \mathcal{M}(\pi_* D_{\widetilde{X}}) = \mathcal{M}(X, \pi_* D_{\widetilde{X}})$ is an equivalence of categories.

Let $\mathfrak{h}^*_{\mathbb{Z}} = \operatorname{Hom}(H, \mathbb{C}^*) \subset \mathfrak{h}^*$ be the lattice of integral weights; for $\chi \in \mathfrak{h}^*$ denote by χ^* the corresponding translation automorphism of $S(\mathfrak{h})$, $\chi^*(h) = h + \chi(h)$, $h \in \mathfrak{h}$. For an ideal $I \subset S(\mathfrak{h})$ and a $\pi_* D_{\widetilde{X}}$-module N we have the $\widetilde{D}$-submodule $N^I \subset N$. Note that $N^{\overline{I}} := \sum_{\chi\in\mathfrak{h}^*_{\mathbb{Z}}} N^{\chi^* I}$ is actually a $\pi_* D_{\widetilde{X}}$-submodule. In particular, for $\overline{\chi} \in \mathfrak{h}^*/\mathfrak{h}^*_{\mathbb{Z}}$ we have the $\pi_* D_{\widetilde{X}}$-submodules $N_{\widetilde{\chi}} := N^{\overline{m}_\chi} \subset N_{\widetilde{\overline{\chi}}} := \bigcup_n N^{\overline{m}_\chi^n}$, where $\chi \in \mathfrak{h}^*$ is any lifting of $\overline{\chi}$, and also the submodule $N_{\text{fin}} := \bigcup N^{\overline{I}}$, $I \subset S(\mathfrak{h})$, has finite codimension.

Clearly $N_{\text{fin}} = \bigoplus_{\overline{\chi}\in\mathfrak{h}^*/\mathfrak{h}^*_{\mathbb{Z}}} N_{\widetilde{\overline{\chi}}}$. Again, we say that N is an $\mathfrak{h}$-*finite* module if $N_{\text{fin}} = N$. We have the corresponding full subcategories $\mathcal{M}(\pi_* D_{\widetilde{X}})_{\overline{\chi}} \subset \mathcal{M}(\pi_* D_{\widetilde{X}})_{\widetilde{\overline{\chi}}} \subset \mathcal{M}(\pi_* D_{\widetilde{X}})_{\text{fin}} \subset \mathcal{M}(\pi_* D_{\widetilde{X}})$ and the decomposition $\mathcal{M}(\pi_* D_{\widetilde{X}})_{\text{fin}} = \prod_{\overline{\chi}\in\mathfrak{h}^*/\mathfrak{h}^*_{\mathbb{Z}}} \mathcal{M}(\pi_* D_{\widetilde{X}})_{\widetilde{\overline{\chi}}}$. The above equivalence $(\pi_*)^{-1}$ sends these subcategories to $\mathcal{M}(\widetilde{X})_{\overline{\chi}} \subset \mathcal{M}(\widetilde{X})_{\widetilde{\overline{\chi}}} \subset \mathcal{M}(\widetilde{X})_{\text{fin}} \subset \mathcal{M}(\widetilde{X})$.

Consider the forgetting of the H-action functor $o\colon \widetilde{\mathcal{M}}(X) \to \mathcal{M}(\widetilde{X})$. The equivalences $\pi_\cdot$, π_* identify o with the induction functor $\mathcal{M}(\widetilde{D}) \to \mathcal{M}(\pi_* D_{\widetilde{X}})$, $M \mapsto \pi_* D_{\widetilde{X}} \underset{\widetilde{D}}{\otimes} M = \pi_* \mathcal{O}_{\widetilde{X}} \underset{\mathcal{O}_X}{\otimes} M$. Clearly o sends $\mathfrak{h}$-finite modules to $\mathfrak{h}$-finite ones, $\mathcal{M}(X)_\chi$ to $\mathcal{M}(\widetilde{X})_{\overline{\chi}}$, and $\mathcal{M}(X)_{\widetilde{\chi}}$ to $\mathcal{M}(\widetilde{X})_{\widetilde{\overline{\chi}}}$, where $\overline{\chi} = \chi \bmod \mathfrak{h}^*_{\mathbb{Z}}$.

2.5.4. LEMMA. (i) *The functors* $o\colon \mathcal{M}(X)_\chi \to \mathcal{M}(\widetilde{X})_{\overline{\chi}}$, $\mathcal{M}(X)_{\widetilde{\chi}} \to \mathcal{M}(\widetilde{X})_{\widetilde{\overline{\chi}}}$ *are equivalences of categories.*

(ii) *The corresponding functor between the derived categories* $D^b\mathcal{M}(X)_{\widetilde{\chi}} \to D^b\mathcal{M}(\widetilde{X})$ *is fully faithful. It identifies* $D^b\mathcal{M}(X)_{\widetilde{\chi}}$ *with the full subcategory of those complexes that have cohomology in* $\mathcal{M}(\widetilde{X})_{\widetilde{\overline{\chi}}}$.

PROOF. (i) The inverse functor to the induction $\mathcal{M}(X)_{\widetilde{\chi}} \to \mathcal{M}(\pi_* D_{\widetilde{X}})_{\widetilde{\overline{\chi}}}$ sends a $\pi_* D_{\widetilde{X}}$-module N to the $\widetilde{D}$-module $N_{\widetilde{\chi}} := \bigcup N^{m^n_\chi}$.

(ii) Since our functor is t-exact it suffices to check the first statement (i.e., to show that o induces isomorphism on Ext's). A standard Čech resolvent argument shows that the problem is X-local, hence we can assume that X is affine, $\widetilde{X} = H \times X$. It suffices to verify that Ext's are the same for a family of generators; take one formed by the modules $\pi^\cdot(D_X \otimes V)$, V is an $S(\mathfrak{h})$-module killed by some m^n_χ. The Künneth formula reduces the problem to the case X = point, $H = \mathbb{C}^*$, where it is obvious.

2.5.5. REMARKS. (i) For $\varphi \in \mathfrak{h}^*_{\mathbf{Z}} = \mathrm{Hom}(H, \mathbb{C}^*)$ put $\mathcal{O}_\varphi := \{f \in \pi_*\mathcal{O}_{\widetilde{X}}\colon f(h\widetilde{x}) = \varphi(h)f(\widetilde{x}),\, h \in H,\, \widetilde{x} \in \widetilde{X}\}$; this is a line bundle on X which is a $\widetilde{D}$-module in an obvious manner. Note that $\pi_*\mathcal{O}_{\widetilde{X}} = \bigoplus_{\varphi\in\mathfrak{h}^*_{\mathbf{Z}}} \mathcal{O}_\varphi$. We have an autoequivalence $T_\varphi\colon \mathcal{M}(\widetilde{D}) \xrightarrow{\sim} \mathcal{M}(\widetilde{D})$, $T_\varphi(M) = \mathcal{O}_\varphi \underset{\mathcal{O}_X}{\otimes} M$, which preserves $\mathcal{M}(\widetilde{D})_{\mathrm{fin}}$ and sends $\mathcal{M}(X)_{\widetilde{\chi}}$ to $\mathcal{M}(X)_{\widetilde{\chi+\varphi}}$. Clearly $oT_\varphi = o$.

(ii) For a section $\mathfrak{s}\colon \mathfrak{h}^*/\mathfrak{h}^*_{\mathbf{Z}} \to \mathfrak{h}^*$ put $\mathcal{M}(X)_{\widetilde{\mathfrak{s}}} = \pi_{\overline{\chi}\in\mathfrak{h}^*/\mathfrak{h}^*_{\mathbf{Z}}}\mathcal{M}(X)_{\widetilde{\mathfrak{s}(\overline{\chi})}}$. Then $o\colon \mathcal{M}(X)_{\widetilde{\mathfrak{s}}} \to \mathcal{M}(\widetilde{X})_{\mathrm{fin}}$ is an equivalence of categories. One can say that $\mathcal{M}(\widetilde{X})_{\mathrm{fin}}$ is a quotient of $\mathcal{M}(\widetilde{D})_{\mathrm{fin}}$ with respect to the action of $\mathfrak{h}^*_{\mathbf{Z}}$ by T_φ's.

(iii) $\mathfrak{h}^*/\mathfrak{h}^*_{\mathbf{Z}}$ is the character group of the fundamental group of H; the exponential map identifies $h^*/h^*_{\mathbf{Z}}$ with the dual torus $H^\vee$. The Riemann-Hilbert correspondence identifies tame $D_{\widetilde{X}}$-modules from $\mathcal{M}(\widetilde{X})_{\widetilde{\overline{\chi}}}$ with the $\widetilde{\overline{\chi}}$-monodromic perverse sheaves on $\widetilde{X}$ (those perverse sheaves that are lissé along the fibers and have $\overline{\chi}$ as eigenvalues of fiberwise monodromy).

(iv) We will use 2.5.4 to transmit the standard results about D-modules to the monodromic $\mathfrak{h}$-finite situation (without repeating the proofs). For example, the derived category $D^b\mathcal{M}(\widetilde{X})^{\mathrm{coh}}_{\mathrm{fin}} \subset D^b\mathcal{M}(\widetilde{X})^{\mathrm{coh}}$, where "coh" means "complexes with coherent cohomology", is stable with respect to the Verdier duality which sends $D^b\mathcal{M}(\widetilde{X})^{\mathrm{coh}}_{\widetilde{\overline{\chi}}}$ to $D^b\mathcal{M}(\widetilde{X})^{\mathrm{coh}}_{\widetilde{-\overline{\chi}}}$. Therefore, we have a duality on $D^b\mathcal{M}(\widetilde{D})^{\mathrm{coh}}_{\mathrm{fin}}$ which sends $D^b\mathcal{M}(X)^{\mathrm{coh}}_{\widetilde{\chi}}$ to $D^b\mathcal{M}(X)^{\mathrm{coh}}_{\widetilde{-\chi}}$; it induces a duality on the abelian category of holonomic modules $\mathcal{M}(\widetilde{D})^{\mathrm{hol}}_{\mathrm{fin}}$ which sends $\mathcal{M}(X)^{\mathrm{hol}}_{\widetilde{\chi}}$ to $\mathcal{M}(X)^{\mathrm{hol}}_{\widetilde{-\chi}}$. The same thing happens with the $\mathcal{O}$-tensor product.

If $Y \subset X$ is a smooth subvariety with the closure $\overline{Y}$, then Y carries the induced monodromic structure $\widetilde{Y} = \pi^{-1}(Y)$, and we have the Kashiwara equivalence between $\mathscr{M}(Y)_{\widetilde{\chi}}$ and the quotient of the subcategory of $\mathscr{M}(X)_{\widetilde{\chi}}$ which consists of modules supported on $\overline{Y}$ modulo those supported on $\overline{Y}\backslash Y$. We leave to the reader the general functoriality with respect to morphisms between the monodromic varieties.

(v) The monodromic category $\widetilde{\mathscr{M}}(X)$ depends only on H modulo isogeny (if $Q \subset H$ is a finite subgroup, then the $\widetilde{D}$ algebras for the H- and H/Q-monodromic structures $\widetilde{X}$, respectively $Q \setminus \widetilde{X}$, coincide).

2.5.6. Let us discuss briefly the equivariant setting. Let G be an algebraic group, and $\kappa: G \to \operatorname{Aut} H$ be an action of G on H. Since $\operatorname{Aut} H$ is discrete, κ is trivial on the connected component G^o of G and $\kappa(G)$ is finite. An *H-monodromic G-variety* is an H-monodromic variety $(X, \widetilde{X})$ together with a G-action $\mu: G \times \widetilde{X} \to \widetilde{X}$ such that $gh\widetilde{x} = \kappa(g)(h)g\widetilde{x}$ for $g \in G$, $h \in H$, $\widetilde{x} \in \widetilde{X}$; in particular $\widetilde{\mu}$ descends to an action $\mu: G \times X \to X$. Equivalently, this is a variety $\widetilde{X}$ with an action of the κ-semidirect product $G \ltimes H$ such that H acts on $\widetilde{X}$ in a free way. The infinitesimal action $\widetilde{\alpha}: \mathfrak{g} \to \mathscr{T}_{\widetilde{X}}$ sends $\mathfrak{g}$ to $\widetilde{\mathscr{T}} \subset \pi_* \mathscr{T}_{\widetilde{X}}$, therefore G acts on the Lie algebroid $\widetilde{\mathscr{T}}$ and the D-algebra $\widetilde{D}$. Note that the induced action on $\mathfrak{h} \subset \widetilde{\mathscr{T}}$ coincides with κ. A *G-equivariant monodromic module* on X is a $D_{\widetilde{X}}$-module equipped with a weak $(G \ltimes H)$-action that is strong along G.

Such modules form a category $\widetilde{\mathscr{M}}(X, G)$; we have the following equivalence of categories $\pi_\cdot : \widetilde{\mathscr{M}}(X, G) \xrightarrow{\sim} \widetilde{\mathscr{M}}(\widetilde{D}, G)$. The $\mathfrak{h}$-finite G-equivariant $D_{\widetilde{X}}$-modules form a full subcategory $\widetilde{\mathscr{M}}(X, G)_{\text{fin}} \subset \widetilde{\mathscr{M}}(X, G)$; for a κ-orbit $\chi^{\cdot} \in \kappa(G) \setminus \mathfrak{h}^*$ we have the corresponding subcategories $\mathscr{M}(X, G)_{\chi^{\cdot}} \subset \mathscr{M}(X, G)_{\widetilde{\chi}^{\cdot}} \subset \widetilde{\mathscr{M}}(X, G)_{\text{fin}}$; this provides a decomposition $\widetilde{\mathscr{M}}(X, G)_{\text{fin}} = \prod \mathscr{M}(X, G)_{\widetilde{\chi}^{\cdot}}$.

The following easy lemma shows that it suffices to consider only one-element orbits.

2.5.7. Lemma. *For $\chi \in \mathfrak{h}^*$ let G_χ be the stabilizer of χ with respect to the κ-action, and let $\chi^{\cdot}$ be the $\kappa(G)$-orbit of χ. Then the functor $\mathscr{M}(X, G)_{\chi^{\cdot}} \to \mathscr{M}(X, G_\chi)_{\widetilde{\chi}}$, $M \mapsto M_{\widetilde{\chi}}$ is an equivalence of categories.*

2.5.8. Consider now the category $\mathscr{M}(\widetilde{X}, G)$ of G-equivariant $D_{\widetilde{X}}$-modules; we have the corresponding full subcategories $\mathscr{M}(\widetilde{X}, G)_{\overline{\chi}^{\cdot}} \subset \mathscr{M}(\widetilde{X}, G)_{\widetilde{\overline{\chi}}^{\cdot}} \subset \mathscr{M}(\widetilde{X}, G)_{\text{fin}} \subset \mathscr{M}(\widetilde{X}, G)$, where $\overline{\chi}^{\cdot} \in \kappa(G)\backslash(\mathfrak{h}^*/\mathfrak{h}^*_{\mathbf{Z}})$, such that $\mathscr{M}(\widetilde{X}, G)_{\text{fin}} = \prod \mathscr{M}(\widetilde{X}, G)_{\widetilde{\overline{\chi}}^{\cdot}}$. The functor $\widetilde{\mathscr{M}}(X, G) \to \mathscr{M}(\widetilde{X}, G)$ of forgetting of the H-action sends $\mathscr{M}(X, G)_{\widetilde{\chi}^{\cdot}}$ to $\mathscr{M}(\widetilde{X}, G)_{\widetilde{\overline{\chi}}^{\cdot}}$. If $\chi \in \mathfrak{h}^*$ is a weight such that the $\kappa(G)$-stabilizers of χ and $\overline{\chi}$ coincide (e.g., if χ is fixed by $\kappa(G)$-action), then this functor is an equivalence of categories.

As in 2.5.5(iv), the elementary functoriality of G-equivariant D-modules translates immediately to the monodromic setting. For example, we have the duality on the subcategory $\widetilde{\mathcal{M}}(X, G)^{\mathrm{hol}}_{\mathrm{fin}} \subset \widetilde{\mathcal{M}}(X, G)_{\mathrm{fin}}$ of holonomic modules that sends $\mathcal{M}(X, G)_{\widetilde{\chi}}$ to $\mathcal{M}(X, G)_{-\widetilde{\chi}}$. For a smooth G-subvariety $Y \subset X$ we have the Kashiwara theorem that identifies the category $\mathcal{M}(Y, G)_{\widetilde{\chi}}$ with the subquotient category of $\mathcal{M}(X, G)_{\widetilde{\chi}}$. If $G' \supset G$ is a larger group and κ extends to $\kappa' \colon G' \to \operatorname{Aut} H$, then we have the induced H-monodromic G'-variety $(X, \widetilde{X}') = (G' \underset{G}{\times} X, G' \underset{G}{\times} \widetilde{X})$ and the pull-back functor for $\widetilde{X} \hookrightarrow \widetilde{X}'$ provides an equivalences of categories $\widetilde{\mathcal{M}}(X', G') \xrightarrow{\sim} \widetilde{\mathcal{M}}(X, G)$, $\mathcal{M}(\widetilde{X}', G') \xrightarrow{\sim} \mathcal{M}(\widetilde{X}, G)$ that identifies the corresponding categories of $\mathfrak{h}$-finite modules.

2.6. Langlands classification. Let $(X, \widetilde{X})$ be an H-monodromic G-variety such that X has only finitely many G-orbits. We present an explicit classification of the irreducible objects in $\widetilde{\mathcal{M}}(X, G)_{\mathrm{fin}} = \mathcal{M}(\widetilde{D}, G)_{\mathrm{fin}}$.

2.6.1. For $x \in X$ consider the action of the stabilizer $G_x \subset G$ on the H-torsor $\widetilde{X}_x = \pi^{-1}(x)$. Since the ($G^\kappa := \operatorname{Ker}\kappa$)- and H-actions commute we see that $G^\kappa_x := G^\kappa \cap G_x$ acts on $\widetilde{X}_x$ via the morphism $\varphi_x \colon G^\kappa_x \to H$. Note that the connected component $(\operatorname{Ker}\varphi_x)^o$ coincides with the connected component $G^o_{\widetilde{x}}$ of the stabilizer $G_{\widetilde{x}}$ of any $\widetilde{x} \in \widetilde{X}_x$. This is a normal subgroup of G_x; put $G_{(x)} := G_x / G^o_{\widetilde{x}}$. Both the G-actions on $\widetilde{X}_x$ and κ factorize through $G_{(x)}$, and φ_x defines the embedding $G^o_{(x)} \hookrightarrow H$. In particular, we have $\mathfrak{g}_{(x)} := \operatorname{Lie} G_{(x)} \hookrightarrow \mathfrak{h}$; together with κ, this defines a Harish-Chandra pair $(G_{(x)}, \mathfrak{h})$.

2.6.2. Assume that our X is a single orbit, so $X = Gx$. Then we can identify $(X, \widetilde{X})$ with an H-monodromic G-variety induced from the H-monodromic G_x-variety $(x, \widetilde{X}_x)$. Hence we have canonical equivalences of categories $\widetilde{\mathcal{M}}(X, G) = \widetilde{\mathcal{M}}(x, G_x) = \mathcal{M}(S(\mathfrak{h}), G_{(x)}) = \mathcal{M}(\mathfrak{h}, G_{(x)})$ (the category of Harish-Chandra modules for the pair $(\mathfrak{h}, G_{(x)})$). We see that $\widetilde{\mathcal{M}}(X, G)_{\mathrm{fin}} = \prod \mathcal{M}(\mathfrak{h}, G_{(x)})_{\widetilde{\chi}}$ has finitely many irreducibles. By 2.5.7 we can identify $\mathcal{M}(\mathfrak{h}, G_{(x)})_{\widetilde{\chi}}$ with $\mathcal{M}(\mathfrak{h}, G_{(x)\chi})_{\widetilde{\chi}}$. The categories $\widetilde{\mathcal{M}}(X, G)_\chi = \mathcal{M}(\mathfrak{h}, G_{(x)})_\chi$ are semisimple. We have a similar description of $\mathcal{M}(\widetilde{X}, G)_{\mathrm{fin}}$ (see 2.5.8). Note that any coherent $M \in \mathcal{M}(\widetilde{X}, G)_{\mathrm{fin}}$ is tame and lissé (i.e., RS holonomic $\mathcal{O}$-coherent in the terminology of [Bo]); any $M \in \mathcal{M}(\widetilde{X}, G)_{\widetilde{\chi}}$ for $\widetilde{\chi} \in \mathfrak{h}^*_{\mathbb{Q}}/\mathfrak{h}^*_{\mathbb{Z}} = \mathbb{Q}/\mathbb{Z} \otimes \mathfrak{h}^*_{\mathbb{Z}} = H^\vee_{\mathrm{tors}}$ has finite monodromy and hence is of geometric origin (see [BBD, (6.2.4)]).

2.6.3. Now consider the general case. Let I be the set of G-orbits on X; this is a finite partially ordered set. For $i \in I$ let Q_i be the corresponding orbit; then $i_1 \leqslant i_2$ means that $Q_{i_1} \subset \overline{Q}_{i_2}$. We will say that a subset $J \subset I$

is *closed* if for $j \in J$ any $j' \leqslant j$ lies in J; e.g., for $i \in I$ the set $\bar{\imath} := \{j \in I, j \leqslant i\}$ is closed. The closed subsets of I form a lattice $A(I)$; we can identify $A(I)$ with the lattice of all closed G-invariant subsets of X sending J to $Q_J = \bigcup_{j \in J} Q_j$ (then $Q_{\bar{\imath}} = \overline{Q}_i$). For $J \in A(I)$ denote by $\widetilde{\mathcal{M}}(X, G)_J \subset \widetilde{\mathcal{M}}(X, G)$, $\mathcal{M}(\widetilde{X}, G)_J \subset \mathcal{M}(\widetilde{X}, G)$ the full subcategories of modules supported on $\widetilde{X}_J$.

Let us fix this kind of frame. Below I could be any finite partially ordered set.

2.6.4. Definition. Let C be an abelian category. An *I-stratification on C* is a collection C_J, $J \in A(I)$, of Serre subcategories of C such that for any $J_1, J_2 \subset A(I)$ one has:

(i) $C_{J_1 \cap J_2} = C_{J_1} \cap C_{J_2}$ and $C_{J_1 \cup J_2}$ is the smallest Serre subcategory that contains C_{J_1}, C_{J_2}. In particular if $J_1 \subset J_2$ then $C_{J_1} \subset C_{J_2}$.

(ii) The embeddings

$$\begin{array}{ccccc} & & C_{J_1} & & \\ & \nearrow & & \searrow & \\ C_{J_1 \cap J_2} & & & & C_{J_1 \cup J_2} \\ & \searrow & & \nearrow & \\ & & C_{J_2} & & \end{array}$$

induce equivalences of categories

$$(C_{J_1}/C_{J_1 \cap J_2}) \times (C_{J_2}/C_{J_1 \cap J_2}) \to C_{J_1 \cup J_2}/C_{J_1 \cap J_2}.$$

(iii) For $J_1 \subset J_2$ the projection $C_{J_2} \to C_{J_2}/C_{J_1}$ has left and right adjoints denoted $j_{J_2 \setminus J_1 !}$ and $j_{J_2 \setminus J_1 *}$.

Our categories $\widetilde{\mathcal{M}}(X, G) \supset \widetilde{\mathcal{M}}(X, G)_{\mathrm{fin}} \supset \mathcal{M}(X, G)_{\widetilde{\chi}} \supset \mathcal{M}(X, G)_{\chi}$, $\mathcal{M}(\widetilde{X}, G) \supset \mathcal{M}(\widetilde{X}, G)_{\mathrm{fin}} \supset \cdots$ are I-stratified.

In any I-stratified category one has the standard devissage pattern. Namely, for $i \in I$ put $C_i := C_{\bar{\imath}}/C_{\bar{\imath} \setminus \{i\}}$; we will call C_i the *i-stratum* of C. We have the functors $j_{i!}, j_{i*} \colon C_i \to C_{\bar{\imath}}$ left and right adjoint to the projection $j_i^* \colon C_{\bar{\imath}} \to C_i$. Since $j_i^* j_{i*} = j_i^* j_{i!} = \mathrm{Id}_{C_i}$ we have a natural morphism $j_{i!} \to j_{i*}$. Put $j_{i!*} := \mathrm{Im}(j_{i!} \to j_{i*})$; this functor transforms irreducible objects to irreducible ones.

We say that C is finite if any object of C has finite length. The devissage shows that this is equivalent to the property that objects in C_i have finite length. In this situation any irreducible object of C is isomorphic to some $j_{i!*}(F_i)$, where F_i is an irreducible object in C_i, and the pair (i, F_i) is uniquely determined.

Let us summarize our discussion.

2.6.5. Lemma. (i) *The categories $\mathcal{M}(\widetilde{X}, G)$, $\widetilde{\mathcal{M}}(X, G)$ and their standard subcategories are I-stratified. The i strata coincide with the corresponding* *categories for the orbits* X_i. *For* $x \in X_i$ *we have the canonical equivalence* $F_x: \widetilde{\mathscr{M}}(\widetilde{X}, G)_i \xrightarrow{\sim} \mathscr{M}(\mathfrak{h}, G_{(x)})$.

(ii) *The categories* $\widetilde{\mathscr{M}}(X, G)^{\text{coh}}_{\text{fin}}$, $\mathscr{M}(\widetilde{X}, G)^{\text{coh}}_{\text{fin}}$ *are finite. The isomorphism classes of irreducible objects of* $\widetilde{\mathscr{M}}(X, G)_{\text{fin}}$ *are in* 1-1 *correspondence with pairs* (i, V), $i \in I$, V *is an isomorphism class of irreducible* $(\mathfrak{h}, G_{(x)})$-*modules* (*where* $x \in X_i$).

(iii) *Any* $M \in \mathscr{M}(\widetilde{X}, G)^{\text{coh}}_{\text{fin}}$ *is tame. If* $\chi^{\boldsymbol{\cdot}} \in \kappa(G) \setminus H^{\vee}_{\text{tors}} = \kappa(G) \setminus (\mathfrak{h}^*_{\mathbf{Q}}/\mathfrak{h}^*_{\mathbf{Z}})$ *then any irreducible* $M \in \mathscr{M}(\widetilde{X}, G)_{\chi^{\boldsymbol{\cdot}}}$ *is of geometric origin.*

2.6.6. Remarks. (i) The modules $j_{i!}(V)$, $j_{i*}(V)$, where $V \in \mathscr{M}(\mathfrak{h}, G_{(x)}) = \mathscr{M}(Q_i, G)$, are called !-, $*$-*standard modules* respectively.

(ii) If the embedding $j_i: Q_i \hookrightarrow X$ is affine, then the functors $j_{i!}$, j_{i*} are exact.

(iii) If C is an I-stratified category then the dual category C^o is I-stratified by C^o_J's. The duality interchanges the functors $j_{i!}$ and j_{i*}.

(iv) Let C_a, $a = 1, 2$, be I_a-stratified categories, $\varphi: I_1 \to I_2$ be a morphism of partially ordered sets, and $F: C_1 \to C_2$ be an exact functor. We say that F is a φ-*stratified functor* if $F(C_{1\bar{i}}) \subset C_{2\overline{\varphi(i)}}$ for $i \in I$. Such an F induces the exact functors $F_i: C_{1i} \to C_{2\varphi(i)}$ called the strata of F. Our functor is a *stratified equivalence* if φ is an isomorphism; F is an equivalence of categories and $F(C_{1i}) = C_{2i}$ for any $i \in I_1$. Any stratified equivalence commutes with the $j_{i!}$'s and the j_{i*}'s.

§3. Localization of representations; the structure of K-orbits on the flag variety

3.1. $\mathfrak{g}$-modules. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Denote by G the algebraic group of automorphisms of $\mathfrak{g}$, so G^o is the adjoint group, and $\mathfrak{g} = \operatorname{Lie} G$; the action of $g \in G$ on $\mathfrak{g}$ will be denoted Ad_g. Let $\mathscr{U}(\mathfrak{g})$ be the universal enveloping algebra and $\mathscr{Z} \subset \mathscr{U}(\mathfrak{g})$ be its center.

Let $\mathfrak{h}$ be the Cartan algebra of $\mathfrak{g}$, $\Delta \subset \mathfrak{h}^*$ the root system, Δ^+ the set of positive roots, $\Sigma \subset \Delta^+$ the set of simple roots, W the Weyl group, and $\rho := 1/2 \sum_{\nu \in \Delta^+} \nu$; for $\alpha \in \Delta$ let h_α be the corresponding co-root and $\sigma_\alpha \in W$ the corresponding reflection. So for any Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and $\mathfrak{n} = \mathfrak{n}_{\mathfrak{b}} := [\mathfrak{b}, \mathfrak{b}]$ we have canonical identification $\mathfrak{h} = \mathfrak{b}/\mathfrak{n}$ invariant under G^o-conjugation, and Δ^+ are weights of $\mathfrak{h}$-action on $\mathfrak{g}/\mathfrak{b} \simeq \mathfrak{n}^*$. ([3])

We will think of W as the group of affine transformations of $\mathfrak{h}^*$ that leave $-\rho$ fixed; this defines an action of W on the algebra $S(\mathfrak{h})$. One has the Harish-Chandra isomorphism $\gamma: \mathscr{Z} \xrightarrow{\sim} S(\mathfrak{h})^W$; let $\gamma^{\boldsymbol{\cdot}}: \mathfrak{h}^* = \operatorname{Spec} S(\mathfrak{h}) \to \operatorname{Spec} \mathscr{Z}$ be the corresponding W-sheeted map of spectra. For $\chi \in \mathfrak{h}^*$ we denote by $m_\chi \subset S(\mathfrak{h})$, $m_{\gamma(\chi)} \subset \mathscr{Z}$ the corresponding maximal ideals.

([3]) People often use the opposite ordering of Δ; we choose the one for which dominant weights correspond to positive line bundles on the flag space.

Denote by $U := \mathscr{U}(\mathfrak{g}) \underset{\mathscr{Z}}{\otimes} S(\mathfrak{h})$ the extended universal enveloping algebra; then $S(\mathfrak{h})$ is the center of U, the group W acts on U (via $S(\mathfrak{h})$), and $\mathscr{U}(\mathfrak{g}) = U^W$. The algebras $\mathscr{U}(\mathfrak{g})$, $S(\mathfrak{h})$, U carry canonical involutions (anti-automorphisms of order 2), denoted by $x \mapsto {}^t x$, compatible with the standard embeddings: ${}^t\xi = -\xi$, ${}^t h = -2\rho(h) - h$ for $\xi \in \mathfrak{g} \subset \mathscr{U}(\mathfrak{g}) \subset U$, $h \in \mathfrak{h} \subset S(\mathfrak{h}) \subset U$; clearly ^{t} commutes with the W-action. Denote by $S(\mathfrak{h})^{\mathrm{reg}}$ the localization of $S(\mathfrak{h})$ off the nonregular hyperplanes for the W-action (so $\mathbb{C}$-points of $S(\mathfrak{h})^{\mathrm{reg}}$ are regular weights); if A is any $S(\mathfrak{h})$-algebra put $A^{\mathrm{reg}} := S(\mathfrak{h})^{\mathrm{reg}} \underset{S(\mathfrak{h})}{\otimes} A$. In particular, we have the algebra U^{reg} and ^{t} extends to U^{reg}. The group G acts on all the above objects in a compatible way; the action on $\mathfrak{h}$, Δ, and W factors through the finite quotient G/G^o. The action of G/G^o on $\mathfrak{h}$ is faithful; we will denote it by κ.

Let $\mathscr{M}(\mathfrak{g})$ and $\mathscr{M}(U)$ be the categories of left $\mathscr{U}(\mathfrak{g})$- and U-modules and let $\mathscr{M}(U)^{\mathrm{f.g.}} \subset \mathscr{M}(U)$ be the subcategory of finitely generated ones. The embedding $\mathscr{U}(\mathfrak{g}) \subset U$ defines an obvious functor $\mathscr{M}(U) \to \mathscr{M}(\mathfrak{g})$. We also consider the categories ${}^r\mathscr{M}$ of the right modules; we will identify ${}^r\mathscr{M}$ with M in a canonical way using ^{t}.

For any ideals $I \subset \mathscr{Z}$, $J \subset S(\mathfrak{h})$ let $\mathscr{M}(\mathfrak{g})^I = \mathscr{M}(\mathscr{U}(\mathfrak{g})/I\mathscr{U}(\mathfrak{g}))$, $\mathscr{M}(U)^J = \mathscr{M}(U/JU)$ be the categories of $\mathfrak{g}$- and U-modules killed by I, J respectively. For $\chi \in \mathfrak{h}^*$ put $U_\chi := U/m_\chi U$; we have the categories $\mathscr{M}(U)_\chi := \mathscr{M}(U_\chi) = \mathscr{M}(U)^{m_\chi}$ and $\mathscr{M}(U)_{\widetilde{\chi}} := \{M \in \mathscr{M}(U)\colon$ any $m \in M$ is killed by some power of $m_\chi\} \subset \mathscr{M}(U)$; we also have the corresponding quotient $\mathscr{U}(\mathfrak{g})_{\gamma(\chi)} = \mathscr{U}(\mathfrak{g})/m_{\gamma(\chi)}\mathscr{U}(\mathfrak{g})$ and the full subcategories $\mathscr{M}(\mathfrak{g})_{\gamma(\chi)} \subset \mathscr{M}(\mathfrak{g})_{\widetilde{\gamma(\chi)}} \subset \mathscr{M}(\mathfrak{g})$. Note that the embedding $\mathscr{U}(\mathfrak{g}) \hookrightarrow U$ induces an isomorphism $\mathscr{U}(\mathfrak{g})_{\gamma(\chi)} \to U_\chi$ for any $\chi \in \mathfrak{h}^*$; if χ is regular then $\mathscr{U}(\mathfrak{g})/m^n_{\gamma(\chi)}\mathscr{U}(\mathfrak{g}) = U/m^n_\chi U$ for any n. The above functor sends $\mathscr{M}(U)_\chi$, $\mathscr{M}(U)_{\widetilde{\chi}}$ to $\mathscr{M}(\mathfrak{g})_{\gamma(\chi)}$, $\mathscr{M}(\mathfrak{g})_{\widetilde{\gamma(\chi)}}$; the functor $\mathscr{M}(U)_\chi \to \mathscr{M}(\mathfrak{g})_{\gamma(\chi)}$ is always an equivalence of categories; the functor $\mathscr{M}(U)_{\widetilde{\chi}} \to \mathscr{M}(\mathfrak{g})_{\widetilde{\gamma(\chi)}}$ is an equivalence if and only if χ is a regular weight.

3.2. The flag variety. Let $X = X_\mathfrak{g}$ be the flag variety of $\mathfrak{g}$; points of X are Borel subalgebras of $\mathfrak{g}$. For $x \in X$ let $\mathfrak{b}_x$ be the corresponding Borel subalgebra, $B_x \subset G^o$ the corresponding Borel subgroup, and $N_x \subset B_x$ the maximal nilpotent subgroup. Then $\operatorname{Lie} B_x = \mathfrak{b}_x$, $\operatorname{Lie} N_x = \mathfrak{n}_x := [\mathfrak{b}_x, \mathfrak{b}_x]$, and $\mathfrak{h} = \mathfrak{b}_x/\mathfrak{n}_x$. Put $H := B_x/N_x$. This torus (the Cartan group of G) does not depend on the choice of x by the same reason as $\mathfrak{h}$ did not; one has $\operatorname{Lie} H = \mathfrak{h}$. The group G acts on X and on H and these actions are compatible with the above actions on Lie algebras. The action of G^o on X is transitive with the stabilizer of $x \in X$ equal to B_x, so $X = G^o/B_x$.

Let $\widetilde{X} = \widetilde{X}_\mathfrak{g}$ be the enhanced flag variety (or "base affine space") of G: its point $\widetilde{x}$ is a pair $(\mathfrak{b}_x, \{a^\alpha_{\widetilde{x}}\})$, where $\mathfrak{b}_x \subset \mathfrak{g}$ is a Borel subalgebra, and $a^\alpha_{\widetilde{x}}$,

$\alpha \in \Sigma$, is a generator for the α-root subspace in $\mathfrak{g}/\mathfrak{b}_x$. The groups G and H act on $\widetilde{X}$ from the left according to formulas $g\widetilde{x} := (\mathrm{Ad}_g(\mathfrak{b}_x), \{\mathrm{Ad}_g(a^\alpha)\})$, $h\widetilde{x} = (\mathfrak{b}_x, \{\exp\alpha(h)\cdot a^\alpha\})$. One has $ghx = \kappa(g)(h)gx$; in particular, G^o commutes with H. The H-action is free, $H\setminus\widetilde{X} = X$, and the G^o-action is transitive. For $\widetilde{x} \in \widetilde{X}$ the stabilizer $G^o_{\widetilde{x}}$ equals N_x; hence we have the isomorphism $G^o/N_x \xrightarrow{\sim} \widetilde{X}$, $gN_x \mapsto g\widetilde{x}$. Note that G^o/N_x carries the H-action $h(gN_x) := gh^{-1}N_x$ (here $H = B_x/N_x$), and the above isomorphism is H-equivariant.

We will consider $\widetilde{X}$ as an H-monodromic G-variety (with the compatibility morphism κ). By 2.5 we get the D-algebra $\widetilde{D} = \mathscr{U}(\widetilde{\mathscr{T}})$ on X equipped with a G-action. The Lie algebra map $\mathfrak{g}\times\mathfrak{h} \to \widetilde{\mathscr{T}}$ defines the morphism of the universal enveloping algebras $\mathscr{U}(\mathfrak{g}\times\mathfrak{h}) = \mathscr{U}(\mathfrak{g})\otimes S(\mathfrak{h}) \xrightarrow{\widetilde{\delta}} \widetilde{D}$. It is easy to see that $\widetilde{\delta}(z\otimes 1) = \widetilde{\delta}(1\otimes\gamma(z))$ for $z \in \mathscr{Z}$, hence $\widetilde{\delta}$ factors through a morphism $\widetilde{\delta}\colon U \to \widetilde{D}$ of $S(\mathfrak{h})$-algebras. It induces the morphism $\delta_\chi\colon U_\chi \to D_\chi$ between the m_χ-quotients. Note that $U_0 = \mathscr{U}(\mathfrak{g})_{\gamma(0)}$, $D_0 = D_X$, and δ_0 comes from the infinitesimal $\mathfrak{g}$-action on X.

The above Lie algebra morphism defines a morphism $\widetilde{\mathfrak{g}\times\mathfrak{h}} \to \widetilde{\mathscr{T}}$ of Lie algebroids on X (see 1.2). This morphism is surjective; its kernel is an $\mathscr{O}_X$-Lie algebra $\widetilde{\mathfrak{b}} := \{\gamma \in \mathscr{O}_X\otimes\mathfrak{g}\colon \gamma(x)\in\mathfrak{b}_x \text{ for } x\in X\}$ embedded in $\widetilde{\mathfrak{g}\times\mathfrak{h}}$ by $\gamma\mapsto(\gamma, \gamma \bmod \widetilde{\mathfrak{n}})$. Here $\widetilde{\mathfrak{n}} = [\widetilde{\mathfrak{b}}, \widetilde{\mathfrak{b}}] = \{\gamma\in\mathscr{O}_X\otimes\mathfrak{g}\colon \gamma(x)\in\mathfrak{n}_x \text{ for } x\in X\}$, so $\widetilde{\mathfrak{b}}/\widetilde{\mathfrak{n}} = \widetilde{\mathfrak{h}}$. We see that the induced morphism $\widetilde{\mathfrak{g}} \to \widetilde{\mathscr{T}}$ is also surjective with kernel $\widetilde{\mathfrak{n}}$. Therefore $\widetilde{\mathscr{T}} = \widetilde{\mathfrak{g}\times\mathfrak{h}}/\widetilde{\mathfrak{b}} = \widetilde{\mathfrak{g}}/\widetilde{\mathfrak{n}}$, hence $\widetilde{D} = \mathscr{U}(\widetilde{\mathfrak{g}\times\mathfrak{h}}/\widetilde{\mathfrak{b}}) = \mathscr{U}(\widetilde{\mathfrak{g}}/\widetilde{\mathfrak{n}})$.

REMARK. We see that $\widetilde{\mathfrak{b}}$ and $\widetilde{\mathfrak{n}}$ are normal subalgebras in $\widetilde{\mathfrak{g}\times\mathfrak{h}}$ and $\widetilde{\mathfrak{g}}$.

3.2.1. LEMMA. *The D-algebra $\widetilde{D}$ carries a unique involution ^{t} such that the canonical morphism $\widetilde{\delta}\colon U \to \widetilde{D}$ commutes with ^{t}'s.*

Note that ^{t} induces the duality $D^o_\chi \xrightarrow{\sim} D_{-2\rho-\chi}$. It is easy to check that the morphism $\widetilde{\mathfrak{g}} \to \widetilde{\mathscr{T}_{\Omega_X}}$ (that comes from the action of G on Ω_X) induces an isomorphism of tdo's $D_{-2\rho} \xrightarrow{\sim} D_{\Omega_X}$, and, with respect to this isomorphism, the above duality coincides with the canonical one from 2.4.1.

3.2.2. LEMMA. *The morphism $\widetilde{\delta}\colon U \to \Gamma(X, \widetilde{D})$ is an isomorphism. For all $\chi \in \mathfrak{h}^*$ the corresponding morphisms $\delta_\chi\colon U_\chi \to \Gamma(X, D_\chi)$ are also isomorphisms.*

For a proof see, e.g., [S1].

3.3. Localization. According to 1.6, $\widetilde{\delta}$ defines adjoint functors

$$\mathscr{M}(U) \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \mathscr{M}(\widetilde{D}) = \mathscr{M}(X)$$

with $\Gamma(M) := \Gamma(X, M)$, $\Delta(N) := N \underset{U}{\otimes} \widetilde{D}$. These functors are $S(\mathfrak{h})$-linear, hence for any ideal $I \subset S(\mathfrak{h})$ they preserve the subcategories of modules killed by I. Also they commute with (directed) inductive limits. In particular, Γ and Δ induce adjoint functors between the full subcategories

$$\mathscr{M}(U)_\chi \underset{\Gamma_\chi}{\overset{\Delta_\chi}{\rightleftarrows}} \mathscr{M}(X)_\chi, \qquad \mathscr{M}(U)_{\widetilde{\chi}} \underset{\Gamma_{\widetilde{\chi}}}{\overset{\Delta_{\widetilde{\chi}}}{\rightleftarrows}} \mathscr{M}(X)_{\widetilde{\chi}}.$$

Recall that a weight χ is *dominant* if $(\chi+\rho)(h_\gamma) \notin \{-1, -2, \dots\}$ for any positive co-root $h_\gamma \in \mathfrak{h}$. We have the basic

3.3.1. THEOREM. *If χ is a regular dominant weight, then X is D_χ-affine (see 1.6.1), so $(\Gamma_\chi, \Delta_\chi)$ are mutually inverse equivalences of categories. The functors $(\Gamma_{\widetilde{\chi}}, \Delta_{\widetilde{\chi}})$ are also equivalences of categories.*

For a proof of the first statement see [BB1]. An easy devissage then shows that actually X is $\widetilde{D}/m_\chi^n\widetilde{D}$-affine for any $n \geqslant 1$, which implies the second statement.

For the case of nondominant or nonregular χ see [BB2, KL1].

3.3.2. Now assume we have a Harish-Chandra pair $(\mathfrak{g}, K)$, so K is an algebraic group equipped with a morphism $\mathrm{Ad}\colon K \to G$ and a Lie algebra embedding $i_{\mathfrak{k}}\colon \mathfrak{k} := \mathrm{Lie}\, K \hookrightarrow \mathfrak{g}$ which are compatible in an obvious sense. We have the corresponding categories of Harish-Chandra modules $\mathscr{M}(\mathfrak{g}, K) = \mathscr{M}(U(\mathfrak{g}), K)$, $\mathscr{M}(U, K)$, $\mathscr{M}(\widetilde{D}, K)$; for $\chi^{\cdot} \in \kappa(K) \setminus \mathfrak{h}^*$ we have the corresponding standard subcategories $\mathscr{M}(U, K)_{\chi^{\cdot}} \subset \mathscr{M}(U, K)_{\widetilde{\chi}^{\cdot}} \subset \mathscr{M}(U, K)$, etc. (see 2.5.6). As above we have an obvious functor $\mathscr{M}(U, K) \to \mathscr{M}(\mathfrak{g}, K)$ which induces the equivalences $\mathscr{M}(U, K)_{\chi^{\cdot}} \xrightarrow{\sim} \mathscr{M}(\mathfrak{g}, K)_{\gamma(\chi)^{\cdot}}$; if χ is regular and the stabilizers of χ and $\gamma(\chi)$ in $\kappa(K)$ coincide (e.g., if $\kappa(K)\chi = \chi$), then $\mathscr{M}(U, K)_{\widetilde{\gamma\chi}^{\cdot}} \to \mathscr{M}(\mathfrak{g}, K)_{\widetilde{\gamma(\chi)}^{\cdot}}$ is also an equivalence of categories.

The functors Γ and Δ send, in an obvious way, the K-equivariant modules to K-equivariant ones, therefore we have the adjoint functors

$$\mathscr{M}(U, K) \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \mathscr{M}(\widetilde{D}, K) = \widetilde{\mathscr{M}}(X, K)$$

which induce the functors between the full subcategories

$$\mathscr{M}(U, K)_{\chi^{\cdot}} \underset{\Gamma_\chi}{\overset{\Delta_\chi}{\rightleftarrows}} \mathscr{M}(D_{\chi^{\cdot}}, K) = \mathscr{M}(X, K)_{\chi^{\cdot}},$$

$$\mathscr{M}(U, K)_{\widetilde{\chi}^{\cdot}} \underset{\Gamma_{\widetilde{\chi}}}{\overset{\Delta_{\widetilde{\chi}}}{\rightleftarrows}} \mathscr{M}(\widetilde{D}, K)_{\widetilde{\chi}^{\cdot}} = \widetilde{\mathscr{M}}(X, K)_{\widetilde{\chi}^{\cdot}}.$$

3.3.3. COROLLARY. *If χ is a regular dominant weight, then $(\Gamma_{\chi^{\cdot}}, \Delta_{\chi^{\cdot}})$, $(\Gamma_{\widetilde{\chi}^{\cdot}}, \Delta_{\widetilde{\chi}^{\cdot}})$ are equivalences of categories.*

According to 2.6.5 these equivalences define an I-stratification on the categories $\mathscr{M}(U, K)_{\chi^{\cdot}}$, $\mathscr{M}(U, K)_{\widetilde{\chi}}$, where I is the set of K-orbits on X.

3.4. Admissible orbits. In the rest of this section we will collect some geometric information about K-orbits on X that will be used in the construction of the geometric Jantzen filtration. In this section our $(X, \widetilde{X})$ is any H-monodromic K-variety.

For $x \in X$ consider the pair $(\mathfrak{h}, K_{(x)})$ defined in 2.6.1. Put $\mathfrak{h}^*(x) := \{\varphi \in \mathfrak{h}^* : \kappa(K_{(x)})\varphi = \varphi,\ \varphi(i(\mathfrak{k}_{(x)})) = 0\}$ (this is the set of morphisms from $(\mathfrak{h}, K_{(x)})$ to the trivial Harish-Chandra pair $(\mathbb{C}, \{1\})$). Also put $\mathfrak{h}^*_{\mathbf{Z}}(x) := \mathfrak{h}^*_{\mathbf{Z}} \cap \mathfrak{h}^*(x)$. Since $\mathfrak{h}^*(kx) = \kappa(k)\mathfrak{h}^*(x)$, for a fixed K-orbit $Q \subset X$ the spaces $\mathfrak{h}^*(x)$ and $\mathfrak{h}^*_{\mathbf{Z}}(x)$ for $x \in Q$ are canonically identified; we denote them by $\mathfrak{h}^*(Q)$ and $\mathfrak{h}^*_{\mathbf{Z}}(Q)$.

It is easy to see that a weight $\varphi \in \mathfrak{h}^*_{\mathbf{Z}}$ belongs to $\mathfrak{h}^*_{\mathbf{Z}}(Q)$ if and only if there exists a nonzero K-invariant function f_φ on $\widetilde{Q} = \pi^{-1}(Q) \subset \widetilde{X}$ such that $f_\varphi(h\widetilde{x}) = (\exp\varphi)(h) f_\varphi(\widetilde{x})$ for $\widetilde{x} \in \widetilde{Q}$. Such a function f_φ is determined by φ uniquely up to multiplication by a nonzero constant.

Let $\overline{Q}$ be the closure of $\widetilde{Q}$ in $\widetilde{X}$. We say that f_φ is *$\overline{Q}$-regular* if $f_\varphi \in \mathscr{O}(\overline{Q}) \subset \mathscr{O}(\widetilde{Q})$; f_φ is *$\overline{Q}$-invertible* if $f_\varphi \in \mathscr{O}^*(\overline{Q})$; and f_φ is *$\overline{Q}$-positive* if f_φ is $\overline{Q}$-regular and $f_\varphi^{-1}(0) = \overline{Q} \setminus \widetilde{Q}$. Put $\mathfrak{h}^{*o}_{\mathbf{Z}}(Q) = \{\varphi \in \mathfrak{h}^*_{\mathbf{Z}}(Q) : f_\varphi$ is $\overline{Q}$-invertible$\}$, $\mathfrak{h}^{*+}_{\mathbf{Z}}(Q) = \{\varphi \in \mathfrak{h}^*_{\mathbf{Z}}(Q) : f_\varphi$ is $\overline{Q}$-positive$\}$.

3.4.1. DEFINITION. (i) An orbit Q is *admissible* if $\mathfrak{h}^{*+}_{\mathbf{Z}}(Q)$ is not empty.

(ii) The K-action on $(X, \widetilde{X})$ is *admissible* if it has finitely many orbits on X and every orbit is admissible.

3.4.2. LEMMA. (i) *For any admissible orbit Q the embeddings $Q \hookrightarrow X$, $\widetilde{Q} \hookrightarrow \widetilde{X}$ are affine.*

(ii) *$\mathfrak{h}^{*o}_{\mathbf{Z}}(Q)$ is the subgroup of $\mathfrak{h}^*_{\mathbf{Z}}(Q)$, and $\mathfrak{h}^*_{\mathbf{Z}}(Q)/\mathfrak{h}^{*o}_{\mathbf{Z}}(Q)$ has no torsion.*

(iii) *If Q is admissible, then $\mathfrak{h}^{*+}_{\mathbf{Z}}(Q)$ is a subsemigroup of $\mathfrak{h}^*_{\mathbf{Z}}(Q)$ that generates $\mathfrak{h}^*_{\mathbf{Z}}(Q)$ and is invariant under $\mathfrak{h}^{*o}_{\mathbf{Z}}(Q)$-translations. If $\varphi \in \mathfrak{h}^*_{\mathbf{Z}}(Q)$ and $n\varphi \in \mathfrak{h}^{*+}_{\mathbf{Z}}(Q)$ for some $n > 0$, then $\varphi \in \mathfrak{h}^{*+}_{\mathbf{Z}}(Q)$. The quotient $\mathfrak{h}^{*+}_{\mathbf{Z}}(Q)/\mathfrak{h}^{*o}_{\mathbf{Z}}(Q)$ is isomorphic to $\mathbb{Z}^a_+$ for some a.*

(iv) *An orbit is admissible if and only if (some, or any of) its connected components is admissible with respect to the action of the connected component K^o. Hence a K-action is admissible iff its restriction to K^o is admissible.*

(v) *Assume we have a larger group $K' \supset K$ and an extension $\kappa' : K' \to \operatorname{Aut} H$ of κ; let $(X', \widetilde{X}')$ be the induced K'-variety (see 2.5.8). For a K-orbit Q on X let $Q' = K'Q$ be the corresponding K'-orbit on X'. Then $\mathfrak{h}^*_{\mathbf{Z}}(Q) = \mathfrak{h}^*_{\mathbf{Z}}(Q')$, and the same for $\mathfrak{h}^{*o}$, $\mathfrak{h}^{*+}$.*

Hence $(X', \widetilde{X}')$ is an admissible K'-variety if and only if $(X, \widetilde{X})$ is an admissible K-variety.

3.5. Admissible orbits on the flag variety. Let $(\mathfrak{g}, K)$, $(X, \widetilde{X})$ be as in 3.3.2. We will say that our Harish-Chandra pair is *admissible* if the K-action on $(X, \widetilde{X})$ is admissible.

3.5.1. LEMMA. *The pair* $(\mathfrak{g}, N)$, *where* N *is a maximal nilpotent subgroup, is admissible.*

PROOF. Consider a Schubert cell Q_w, where $w \in W$. Let $\mathfrak{h}_{\mathbf{Z}}^{*+}$ be the cone of positive regular integral characters. We will see that for any $w \in W$ one has $\mathfrak{h}_{\mathbf{Z}}^{*+}(Q_w) \supset \rho + \mathfrak{h}_{\mathbf{Z}}^{*+}$, hence Q_w is admissible. For $\chi \in \rho + \mathfrak{h}_{\mathbf{Z}}^{*+}$ take an irreducible G^o-module V with highest weight χ; let $v \in V^N \backslash \{0\}$ be a lowest weight vector. Consider the map $\mathfrak{q}_v : \widetilde{X} = G^o/N \to V\backslash\{0\}$, $\mathfrak{q}_v(g) = gv$. It is clear that if $l \in V^*$ is a linear function on V, then $l\mathfrak{q}_v$ is a χ-homogeneous function on $\widetilde{X}$. Choose $l \in V^*$ such that $l(wv) \neq 0$, $l(\mathfrak{n}wv) = 0$ (here $\mathfrak{n} = \operatorname{Lie} N$). One has $wv \in \mathfrak{q}_v(\widetilde{Q}_w)$, the image $\mathfrak{q}_v(\widetilde{Q}_w)$ lies in the linear N-invariant subspace generated by wv, and $\mathfrak{q}_v(\overline{Q}_w - \widetilde{Q}_w) \subset \mathfrak{n}wv$, where $\overline{Q}_w$ is the closure of $\widetilde{Q}_w$. Hence $l\mathfrak{q}_v$ is the desired χ-homogenous N-invariant function that vanishes on $\overline{Q}_w \setminus \widetilde{Q}_w$.

REMARK. The B-action on X is *not* admissible.

Now assume that $(\mathfrak{g}, K)$ is a symmetric pair, which means that $\mathfrak{k} = \mathfrak{g}^\theta$ for some involution θ of $\mathfrak{g}$. Note that θ is uniquely determined by $\mathfrak{k}$ (its -1 eigenspace coincides with the Killing orthogonal complement to $\mathfrak{k}$); in particular θ commutes with $\operatorname{Ad} K$. For $x \in X$ denote by $\mu_\theta(x) \in W$ the relative position of $(\mathfrak{b}_x, \theta\mathfrak{b}_x)$. Clearly $\mu_\theta(kx) = \kappa(k)(\mu_\theta(x))$ for $k \in K$. In particular μ_θ is constant along the connected components of K-orbits; if Q^o is such a component we will write $\mu_\theta(Q^o) = \mu_\theta(x)$, $x \in Q^o$.

REMARK. The following properties are equivalent:

(i) An orbit Q is closed.
(ii) $\mu(Q) = 1$.
(iii) For $x \in Q$ one has $\dim K \cap N_x = \dim Q$.
(iv) $\dim Q = \dim X_{\mathfrak{k}}$.

3.5.2. LEMMA. *Any symmetric pair is admissible.*

PROOF. According to 3.4.2(iv) we can assume that K is connected.

(i) Let us consider the special case: $\mathfrak{g} = \mathfrak{k} \times \mathfrak{k}$, $i : \mathfrak{k} \to \mathfrak{g}$ is the diagonal embedding. Then θ is the transposition and $\widetilde{X}_{\mathfrak{g}} = \widetilde{X}_{\mathfrak{k}} \times \widetilde{X}_{\mathfrak{k}}$. If $\widetilde{x} \in \widetilde{X}_{\mathfrak{k}}$, then the K-space $\widetilde{X}_{\mathfrak{g}}$ is induced from N_x-space $\widetilde{X}_{\mathfrak{k}} = \widetilde{X}_{\mathfrak{k}} \times \{\widetilde{x}\} \hookrightarrow \widetilde{X}_{\mathfrak{g}}$, and the K-orbits on $X_{\mathfrak{g}}$ are the same as N_x-orbits on $X_{\mathfrak{k}}$: these are $Y_w = K(Q_w \times x)$, $w \in W_{\mathfrak{k}}$. One has $\mathfrak{h}^*_{\mathfrak{g}\mathbf{Z}}(Y_w) = \{(\chi, -w\chi), \chi \in \mathfrak{h}^*_{\mathfrak{k}\mathbf{Z}}\} \subset \mathfrak{h}^*_{\mathfrak{k}\mathbf{Z}} \times \mathfrak{h}^*_{\mathfrak{k}\mathbf{Z}} = \mathfrak{h}^*_{\mathfrak{g}\mathbf{Z}}$, since $K \setminus \widetilde{Y}_w$ is isomorphic to $H_{\mathfrak{k}}$, with the $(H_{\mathfrak{g}} = H_{\mathfrak{k}} \times H_{\mathfrak{k}})$-action given by the formula $(h_1, h_2)h = h_1 w(h_2^{-1})h$. Then clearly $\mathfrak{h}^{*+}_{\mathfrak{g}\mathbf{Z}}(Y_w) = \{(\chi, -w\chi), \chi \in \mathfrak{h}^{*+}_{\mathfrak{k}\mathbf{Z}}(Q_w)\}$, since $f_{(\chi, -w\chi)}|_{\widetilde{X}_w \times \{\widetilde{x}\}} = f_\chi$. Hence we are done by 3.5.1.

(ii) The general case. Consider the embeddings $m_\theta : X \hookrightarrow X \times X$, $\widetilde{m}_\theta : \widetilde{X} \hookrightarrow \widetilde{X} \times \widetilde{X}$, defined by formulas $m_\theta(x) = (x, \theta(x))$ and the same for $\widetilde{m}_\theta$. These maps are equivariant with respect to the K-action on X and the diagonal action of K on $X \times X$ (via $\operatorname{Ad}: K \to G$); one has $m_\theta(x) \in Y_{\mu_\theta(x)}$. For

$w \in W$ consider the locally closed K-invariant subvariety $X_{\theta w} := \mu_\theta^{-1}(w) = m_\theta^{-1}(Y_w) \subset X$. The number of K-orbits on X is finite. This follows from

$$\text{every } K\text{-orbit } Q \subset X_{\theta w} \text{ is open in } X_{\theta w}. \qquad (*)$$

This follows from the corresponding infinitesimal statement for any $x \in X_{\theta w}$:

$$\text{the tangent space } T_{Kx,x} \text{ to the } K\text{-orbit coincides with } dm_\theta^{-1}(T_{Y_w, m_\theta(x)}). \qquad (**)$$

We give a proof: one has $T_{Y_w, m_\theta(x)} = \{(\xi \bmod \mathfrak{b}_x, \xi \bmod \theta\mathfrak{b}_x), \xi \in \mathfrak{g}\} = \{(\xi \bmod \mathfrak{b}_x, \theta(\theta\xi \bmod \mathfrak{b}_x))\}$, $dm_\theta(T_{X,x}) = \{(\xi \bmod \mathfrak{b}_x, \theta(\xi \bmod \mathfrak{b}_x))\}$; hence $\mathscr{T}_{Y_w, m_\theta(x)} \cap dm_\theta(\mathscr{T}_{X,x}) = \{(\xi \bmod \mathfrak{b}_x, \theta(\xi \bmod \mathfrak{b}_x)) : \xi - \theta(\xi) \in \mathfrak{b}_x\} = \{(\eta \bmod \mathfrak{b}_x, \theta(\eta \bmod \mathfrak{b}_x))\}$, where $\eta = (\xi + \theta(\xi))/2 \in \mathfrak{g}^\theta$, which proves $(**)$.

In particular, $(*)$ implies that for any K-orbit Q on X one has $\mu_\theta(Q) \notin \mu_\theta(\overline{Q} \setminus Q)$. Thus for any homogeneous G^o-invariant function f on $\widetilde{Y}_{\mu_\theta(Q)}$ the function $f \circ \widetilde{m}_\theta$ is a homogeneous K-invariant function on $\widetilde{Q}$, and if f is positive, then $f \circ m_\theta$ is also positive. Now the statement (i) above finishes the proof.

3.6. Contravariant duality for standard modules. If $(\mathfrak{g}, K)$ is a finite pair, then we have the Verdier duality on the category of $S(\mathfrak{h})$-finite coherent $(\widetilde{D}, K)$-modules (see 2.5.5(iv)). This duality is local with respect to X and transforms to the Verdier duality on perverse constructible sheaves via the Riemann-Hilbert correspondence. On the other hand, if $(\mathfrak{g}, K)$ is a symmetric pair, or if $K = N$, then one has the usual contravariant duality for $(\mathfrak{g}, K)$-modules. It is an interesting problem to find a geometric (D-modules) description of this duality. At the moment one knows how the contravariant duality acts on the irreducible $(\mathfrak{g}, K)$-modules in terms of their geometric Langlands parameters. We recall this description below.

3.6.1. Consider the involution c on U, $c(u) = w_{\max}{}^t u$, where $w_{\max} \in W$ is the element of maximal length, which acts only on $S(\mathfrak{h})$. It coincides with -1 on $\mathfrak{g}$ and induces on $S(\mathfrak{h})$ the involution $c(\chi) = w_{\max}(-2\rho - \chi)$, $\chi \in \mathfrak{h}^* = \operatorname{Spec} S(\mathfrak{h})$; one has $c(\Delta^+) = \Delta^+$. For a left U-module V let V^o be the dual vector space to V considered as a left U-module via c: for $u \in U$, $v \in V$, $v^* \in V^o$ one has $\langle uv^*, v\rangle = \langle v^*, c(u)v\rangle$. As a $\mathfrak{g}$-module V^o is just the module dual to V; we use c instead of ^{t} since it transforms (regular) dominant weights to (regular) dominant ones, which is handy for localization.

3.6.2. Let us first define the contravariant duality in the case $K = N$. Choose a complementary maximal nilpotent subgroup N', $N' \cap N = \{1\}$. For a dominant regular χ consider the subcategory $\mathscr{M}(U, N)^f_{\widetilde{\chi}} \subset \mathscr{M}(U, N)$

of finitely generated modules. For $V \in \mathcal{M}(U, N)^f_{\chi}$ denote by V^c the subspace of those vectors in V^o on which $\mathfrak{n}' = \operatorname{Lie} N'$ acts in a locally nilpotent way. Then V^c is a U-submodule of V^o and the action of $\mathfrak{n}'$ on V^c integrates to an algebraic action of N', so $V^c \in \mathcal{M}(U, N')_{c(\chi)}$. One knows that actually V^c is finitely generated and $V^{cc} = V$, so the contravariant duality $c\colon (\mathcal{M}(U, N)^f_{\chi})^o \to \mathcal{M}(U, N')^f_{c(\chi)}$ is an equivalence of categories.

The Bruhat decomposition identifies the set of N-orbits on X with the set W equipped with the Bruhat order. By 3.3.3, $\mathcal{M}(U, N)^f_{\chi}$ is a W-stratified category. For $w \in W$ we have a single irreducible L_w in the corresponding stratum. The corresponding standard modules $j_{w!}(L_w)$ are the Verma modules.

Let c_W be the involution on W, $c_W(w) = w_{\max} w w_{\max}^{-1}$. One has an easy

3.6.3. Lemma. *One has* $L^c_w \approx L_{c_W(w)}$.

Using the fact that the involution c_W preserves the Bruhat order, it is easy to show that the duality c is a c_W-stratified equivalence of categories. In particular it sends !-standard modules to $*$-standard ones (see 2.6.6(iii), (iv)).

3.6.4. Consider now the case of a symmetric subgroup K. Then K is reductive and for a finitely generated $V \in \mathcal{M}(U, K)^f_{\chi}$ one knows that any irreducible representation of K occurs in V with finite multiplicity. The group K acts on V^o as an abstract group; denote by V^c the maximal subspace on which K acts algebraically. It is easy to see that $V^c = \bigoplus V^*_{\alpha} \subset \mathscr{P} i V^*_{\alpha} = V^o$, where $V = \bigoplus V_{\alpha}$ is K-isotypic decomposition of V. Clearly V^c is a U-submodule of V^o, hence $V^c \in \mathcal{M}(U, K)_{c(\chi)}$. One knows that V^c is also finitely generated and $V^{cc} = V$, so we have the contravariant duality $c\colon (\mathcal{M}(U, K)^f_{\chi})^o \xrightarrow{\sim} \mathcal{M}(U, K)^f_{\widetilde{c(\chi)}}$.

Let us describe how c acts on the Langlands parameters. Consider, as in 2.6.3, the ordered set I of K-orbits on X. Then $\mathcal{M}(U, K)^f_{\chi} = \mathcal{M}(X, K)^{\mathrm{coh}}_{\chi}$ is an I-stratified category. It turns out that c is an I-stratified functor. Let us define explicitly the corresponding involution c_I of I. For $i \in I$ consider the corresponding orbit Q_i. Put

$$\widehat{Q}_i := \{(x_1, x_2) \in X \times X : x_1 \in Q_1,\ b_{x_1} \cap b_{x_2} \text{ is a } \theta\text{-stable Cartan subalgebra}\}.$$

One knows (see, e.g, [Mil, (A2.3)]) that for $x \in Q_i$ the fiber over x of the first projection $\widehat{Q}_i \to Q_i$, $(x_1, x_2) \mapsto x_1$, is a nonempty $(K \cap N_x)$-torsor. Hence the second projection $\widehat{Q}_i \to X$, $(x_1, x_2) \mapsto x_2$, maps $\widehat{Q}_i$ onto a single K-orbit $Q_{c_I(i)}$. Clearly $c_I\colon I \to I$, $i \mapsto c_I(i)$, is an involution.

3.6.5. Lemma. *The involution c_I preserves the order on the set I and the functor $c\colon \mathcal{M}(U, K)^f_{\chi} \to \mathcal{M}(U, K)^f_{\widetilde{c(\chi)}}$ is a c_I-stratified involution.*

For a proof see [HMSW2].

Let us describe the action of c on the strata $\mathcal{M}(U, K)^{f}_{\tilde{\chi} i}$. By 2.6.5(i) for $x \in Q_i$ we have the canonical equivalence $F_x: \mathcal{M}(U, K)^{f}_{\tilde{\chi} i} = \mathcal{M}(X, K)^{\mathrm{coh}}_{\tilde{\chi} i} \xrightarrow{\sim} \mathcal{M}(\mathfrak{h}, K_{(x)})^{f}_{\tilde{\chi}}$.

Take $(x, x') \in \widehat{Q}_i$, so $x' \in Q_{c_I(i)}$. It is easy to see that the projections $K_{(x)} \leftarrow K_x \cap K_{x'} \rightarrow K_{(x')}$ are surjective and have the same kernel; therefore, they define an isomorphism $\alpha: K_{(x)} \xrightarrow{\sim} K_{(x')}$. This isomorphism extends to an isomorphism of the Harish-Chandra pairs $\alpha: (\mathfrak{h}, K_{(x)}) \rightarrow (\mathfrak{h}, K_{(x')})$ that acts on $\mathfrak{h}$ as $w_{\max}$. Denote by $c_{(x,x')}: \mathcal{M}(\mathfrak{h}, K_{(x)})^{fo}_{\tilde{\chi}} \xrightarrow{\sim} \mathcal{M}(\mathfrak{h}, K_{(x')})^{f}_{\widetilde{c(\chi)}}$ the duality functor $c_{(x,x')}(V) := \alpha_*(V^0 \otimes \varphi)$. Here $\alpha_*: \mathcal{M}(\mathfrak{h}, K_{(x)}) \rightarrow \mathcal{M}(\mathfrak{h}, K_{(x')})$ is the equivalence defined by α, V^o is the dual module, and φ is the $(\mathfrak{h}, K_{(x)})$-module $\det \mathfrak{n}_x$.

3.6.6. LEMMA. *The equivalences F_x, $F_{x'}$ identify the i-stratum of the involution $c: (\mathcal{M}(U, K)^{f}_{\tilde{\chi} i})^o \xrightarrow{\sim} \mathcal{M}(U, K)^{f}_{\widetilde{c(\chi)} c_I(i)}$ with $c_{(x,x')}$.*

For a proof see [HMSW2]. This lemma describes how c acts on the irreducible representations in terms of their Langlands parameters. By 3.6.4 and 2.6.6(iii), (iv) c interchanges !- and $*$-standard modules.

§4. The Jantzen filtration

In this section we will define the Jantzen filtration on standard modules; the main point is its relation with the monodromy filtration on nearby cycles.

4.1. The monodromy filtration. We will need a tiny complement to [D2, 1.6]. For an object Q of an abelian category and a nilpotent endomorphism $s \in \operatorname{End} Q$ let $\mu_\cdot = \mu^Q_\cdot$ denote the monodromy filtration on Q (see [D2, (1.6.1)]). Let $P^Q_i = P_i := \operatorname{Ker}(Gr^\mu_i \longrightarrow Gr^\mu_{i-2})$ be the primitive part of $Gr^\mu_\cdot$ [D2, (1.6.3)]; one has the primitive decomposition [D2, (1.6.4)]—a canonical isomorphism of graded $\mathbb{Z}[s]$-modules $Gr^\mu_\cdot \simeq \bigoplus_{j \leqslant 0} P_j \otimes \mathbb{Z}[s]/s^{-j}$, $\deg s = -2$, $\deg P_j = -j$.. Consider the following increasing filtration on $\operatorname{Ker} s$:

$$J_{!i} := \operatorname{Ker} s \cap \operatorname{Im} s^{-i} \quad \text{for } i \leqslant 0, \qquad J_{!i} = \operatorname{Ker} s \quad \text{for } i \geqslant 0.$$

Dually we define an increasing filtration

$$J_{*i} := (\operatorname{Ker} s^i + \operatorname{Im} s)/\operatorname{Im} s$$

on $\operatorname{Coker} s$. We call $J_{!\cdot}$, $J_{*\cdot}$ the *Jantzen filtrations.* Filtrations $J_{!\cdot}$, $J_{*\cdot}$ coincide with the filtrations induced by $\mu_\cdot$ on $\operatorname{Ker} s$, $\operatorname{Coker} s$; one has $Gr^{J_!}_i = P_i$, $Gr^{J_*}_i = P_{-i}$ [D2, (1.6.6)]. Consider now $\overline{Q} := Q/\operatorname{Ker} s$ together with the nilpotent endomorphism $\bar{s}$ induced by s, and let $\bar{\mu}_\cdot$ be the corresponding monodromy filtration.

4.1.1. LEMMA. (i) *The exact sequences*

$$0 \to (\operatorname{Ker} s\,,\, J_{!\cdot}) \to (Q\,,\, \mu_\cdot) \to (\overline{Q}\,,\, \overline{\mu}_{\cdot-1}) \to 0\,,$$
$$0 \to (\overline{Q}\,,\, \overline{\mu}_{\cdot+1}) \xrightarrow{s} (Q\,,\, \mu_\cdot) \to (\operatorname{Coker} s\,,\, J_{*\cdot}) \to 0$$

are strictly compatible with filtrations.

(ii) *Conversely,* $\mu_\cdot$ *is the unique increasing filtration on* Q *such that* $s\mu_\cdot \subset \mu_{\cdot-2}$ *and either one of the above two sequences is strictly compatible with filtrations.*

PROOF. (i) is [D2, 1.6.5].

(ii) Let $\mu'_\cdot$ be another such filtration strictly compatible with, say, the first exact sequence. It suffices to show that $\mu'_i \supset \mu_i$. But $\mu'_i = \mu_i$ for $i \geqslant 0$ (since $J_{!0} = \operatorname{Ker} s$). For $i \leqslant 0$ we have $\mu'_i \supset J_{!i} + s(\mu'_{i-2})$, and $s(\mu'_{i-2}) = s(\mu_{i-2})$, and we are done by downard induction on i.

Assume now that our categories are over a field k of characteristic 0, and let $\boxtimes$ be an exact k-bilinear bifunctor. Let $(R\,,\, t)$ be another object with nilpotent endomorphism, and $\mu^R_\cdot$ be its monodromy filtration. Consider the tensor product filtration $\mu^{Q,R}_i := \sum_{a+b=i} \mu^Q_a \boxtimes \mu^R_b$ on $Q \boxtimes R$. We have $Gr^{\mu^{Q,R}}_\cdot = Gr^{\mu^Q}_\cdot \boxtimes Gr^{\mu^R}_\cdot$, and the primitive decomposition together with [D2, (1.6.11, 1.6.12)] implies

4.1.2. LEMMA. (i) $\mu^{Q,R}_\cdot$ *is the monodromy filtration with respect to* $s \boxtimes \mathrm{id}_R + \mathrm{id}_Q \boxtimes t$.

(ii) *One has an "almost canonical" isomorphism* $P^{Q\boxtimes R}_{-j} \simeq \bigoplus P^Q_{-j'} \boxtimes P^R_{-j''}$, *where* $(j'\,,\, j'')$ *run through the set of pairs* $\{(j'\,,\, j'')\colon |j' - j''| \leqslant j \leqslant |j' + j''|\,,\ j \equiv j' + j'' \bmod 2\}$.

4.2. The Jantzen and the monodromy filtration in a geometric setting. Recall the construction of nearby cycles for D-modules [B, K, M, V2]; we follow mainly [B]. Let Y be a smooth variety, $f\colon Y \to \mathbb{A}^1$ be a function, and $Z := f^{-1}(0) \overset{i}{\hookrightarrow} Y \overset{j}{\hookleftarrow} U := f^{-1}(\mathbb{A}^1 - \{0\})$. For $n > 0$ consider a lissé $D_{\mathbb{A}^1-\{0\}}$-module $I^{(n)}$ with a $\mathbb{C}[s]/s^n$-action, which is a free rank 1 $(\mathcal{O}_{\mathbb{A}^1-\{0\}} \otimes \mathbb{C}[s]/s^n)$-module with generator "$t^s$" such that $t\partial_t(t^s) = st^s$ (here t is the parameter on $\mathbb{A}^1$); we have the obvious projections $I^{(n)} \to I^{(n)}/s^{n-1} = I^{(n-1)}$.

For a D_U-module M_U put $f^s M^{(n)}_U := f^* I^{(n)} \underset{O_U}{\otimes} M_U$: this is a $(D_U \otimes \mathbb{C}[s]/s^n)$-module, $f^s M^{(1)}_U = M_U$, and $f^s M^{(a)}_U = f^s M^{(n)}_U / s^a$ for $a \leqslant n$.

Assume now that M_U is holonomic. Fix some $a \geqslant 0$. Consider the morphism $s^a(n)\colon j_! f^s M^{(n)}_U \to j_* f^s M^{(n)}_U$ of $(D_Y \otimes \mathbb{C}[s]/s^n)$-modules that coincides with s^a on U; one has $s^a(n) \bmod s^{n-1} = s^a(n-1)$. The lemma about b-functions implies that the projective system $\operatorname{Coker} s^a(n)$ stabilizes, so we can put $\pi^a_f(M_U) := \operatorname{Coker} s^a(n)$ for $n \gg 0$. This is a holonomic

D_Y-module with a nilpotent endomorphism s; the restriction to U of $\pi_f^a(M_U)$ is equal to $f^s M_U^{(a)}$.

The most important π's are $\pi_f^0 =: \Psi_f^{\mathrm{un}}$—the part of the nearby cycles functor with unipotent action of monodromy (one has $\Psi_f^{\mathrm{un}}(M_U)|_U = 0$), and $\pi_f^1 =: \Xi_f$—the maximal extension functor (one has $\Xi_f(M_U)|_U = M_U$). We give a list of properties of π_f^a (see [B]):

4.2.1. Lemma. (i) *$\pi_f^a \colon M(U)_{\mathrm{hol}} \to M(Y)_{\mathrm{hol}}$ is an exact functor.*

(ii) *For $a, b \geqslant 0$ one has canonical exact sequences*

$$0 \to j_!(f^s M_U^{(a)}) \to \pi_f^{a+b}(M_U) \to \pi_f^b(M_U) \to 0,$$
$$0 \to \pi_f^b(M_U) \to \pi^{a+b}(M_U) \to j_*(f^s M_U^{(a)}) \to 0,$$

and

$$\mathrm{Im}(s^a \colon \pi_f^{a+b} \to \pi_f^{a+b}) = \pi_f^b.$$

(ii)$'$ *In particular one has exact sequences*

$$0 \to j_!(M_U) \to \Xi_f(M_U) \to \Psi_f^{\mathrm{un}}(M_U) \to 0,$$
$$0 \to \Psi_f^{\mathrm{un}}(M_U) \to \Xi_f(M_U) \to j_*(M_U) \to 0$$

with $j_! = \mathrm{Ker}(s \colon \Xi_f \to \Xi_f)$, $j_* = \mathrm{Coker}(s \colon \Xi_f \to \Xi_f)$.

(iii) *π_f^a commutes with the duality.*

Now 4.1 gives us the monodromy filtration $\mu_\cdot^{(a)}$ on π_f^a. On U the term $\mu_i^{(a)}$ coincides with $s^{[(a-i)/2]} \cdot f^s M_U^{(a)}$ (here $[\cdot] :=$ integral part). In particular, we have the monodromy filtrations on Ψ_f^{un} and Ξ_f and the Jantzen filtrations $J_{f!\cdot}$, $J_{f*\cdot}$ on $j_!$, j_* (via Ξ_f and (ii)$'$ above).

4.2.2. Remarks. (i) 4.1 implies that, up to a shift, we will get the same Jantzen filtration if we use the isomorphisms 4.2.1(ii), $j_! \simeq \mathrm{Ker}(s \colon \pi^a \to \pi^a)$ for any $a \geqslant 1$; the same holds for j_*.

(ii) One has $J_{f!0} = j_!$, $J_{f!-1} = \mathrm{Ker}(j_! \to j_{!*})$. The embedding $\Psi_f^{\mathrm{un}} \to \Xi_f$ identifies $\mathrm{Ker}(s \colon \Psi_f^{\mathrm{un}} \to \Psi_f^{\mathrm{un}})$ with $J_{f!-1}$; this isomorphism shifts the corresponding Jantzen filtration by one.

Dually, $J_{f*-1} = 0$, $J_{f*0} = j_{!*} \subset j_*$, etc.

(iii) Let $Q_U \subset U$ be a closed subvariety, and Q be the closure of Q_U in Y. Let $\mathscr{M}(Q) \subset \mathscr{M}(Y)$, $\mathscr{M}(Q_U) \subset \mathscr{M}(U)$ be the subcategories of D-modules supported on Q. The above functors π_f^a transform $\mathscr{M}(Q_U)$ to $\mathscr{M}(Q)$, and being restricted to $\mathscr{M}(Q_U)$ they depend on $f|_Q$ only. Since everything is local, we get the functors $\pi_f^a \colon \mathscr{M}(Q_U) \to \mathscr{M}(Q)$ etc., for any regular function f on Q with $Q_U = Q \setminus f^{-1}(0)$.

(iv) The above functors will not change if we multiply f by a nonzero constant $c \in \mathbb{C}$, since one has an isomorphism of $(D_{\mathbb{A}-\{0\}}[s]/s^n)$-modules

$I^n \simeq c^* I^n$, $t^s \mapsto (ct)^s$ (here $c: t \longmapsto ct$ is considered as an automorphism of $\mathbb{A}^1 - \{0\}$).

(v) The above constructions have an obvious counterpart for constructible perverse sheaves compatible with the Riemann-Hilbert correspondence (see [Bo]). One identifies canonically Ψ_f^{un} with the part of the nearby cycles functor $R\Psi_{\overline{\eta}}[-1]$ on which the geometric monodromy acts unipotently, s corresponds to the logarithm of monodromy; here $\overline{\eta}$ is the generic geometric point $\operatorname{Spec}(\bigcup_N \mathbb{C}((t^{1/N})))$ of $\mathbb{C}((t))$.

4.3. The case of standard modules. Assume we are in situation 3.1. Let $Q \subset X$ be an admissible orbit. For $\varphi \in \mathfrak{h}_{\mathbb{Z}}^{*+}(Q)$ consider the corresponding functors $\pi_{f_\varphi}^a : \mathscr{M}(\widetilde{Q}) \to \mathscr{M}(\overline{\widetilde{Q}})$, see 4.2.2 (iii); since, by 4.2.2(iv), they depend on φ only, we will write $\pi_\varphi^a := \pi_{f_\varphi}^a$.

These functors preserve K-equivariance and monodromicity (by construction). Therefore, we have the functors $\pi_\varphi^a : \mathscr{M}(\mathfrak{h}, K_{(x)})_{\widetilde{\chi}}^{f.g.} = \mathscr{M}(\widetilde{Q}, K)_{\widetilde{\chi}}^{\mathrm{hol}} \to \mathscr{M}(\overline{\widetilde{Q}}, K)_{\widetilde{\chi}}^{\mathrm{hol}} \to \mathscr{M}(\widetilde{X}, K)_{\widetilde{\chi}}$ (here $x \in Q$, see 2.6.2) and the Jantzen filtrations $J_{!\cdot}$, $J_{*\cdot}$ on the functors $j_{Q!}, j_{Q*} : \mathscr{M}(\mathfrak{h}, K_{(x)})_{\widetilde{\chi}} \to \mathscr{M}(\widetilde{X}, K)_{\widetilde{\chi}}$. In particular, we have the Jantzen filtrations on standard modules $j_{Q!}(V)$, $j_{Q*}(V)$, where V is an irreducible $(\mathfrak{h}, K_{(x)})_{\widetilde{\chi}}$-module. A priori these filtrations depend on the choice of weight $\varphi \in \mathfrak{h}_{\mathbb{Z}}^{*+}(Q)$.

Note that these constructions can be done directly in terms of the I-stratification pattern (see 2.6.4). Namely, for an orbit Q_α, a point $x \in Q_\alpha$ and a weght $\varphi \in \mathfrak{h}_{\mathbb{Z}}^{*+}(Q_\alpha)$ let $I_\varphi^{(n)}$ be the $(\mathfrak{h}, K_{(x)})$-module $\mathbb{C}[s]/s^n$ such that $h \in \mathfrak{h}$ acts as $\varphi(h)s$ and $K_{(x)}$ acts trivially. The equivalence of categories $F_x : \widetilde{\mathscr{M}}(Q_\alpha, K) \xrightarrow{\sim} \mathscr{M}(\mathfrak{h}, K_{(x)})$ (see 2.6.2) identifies $\mathscr{M} \otimes f_\varphi^*(I^{(n)})$ with $F_x(\mathscr{M}) \otimes I_\varphi^{(n)}$, and we can repeat the constructions of 4.2 using the functors $j_{\alpha!}$, $j_{\alpha*}$, and $\bigotimes I_\varphi^{(n)}$.

If $(\mathfrak{g}, K)$ is an admissible Harish-Chandra pair, we get the Jantzen filtrations on !- and $*$-standard $(U, K)_\chi$-modules ($\chi \in \mathfrak{h}^*$ is a dominant regular weight) using the equivalence 3.3.2. If $K = N$ or K is a symmetric subgroup then $j_!$-extension is contravariant conjugate to j_*-extension (see 3.6); hence the morphism $j_{\alpha!}(V \otimes I_\varphi^{(n)}) \to j_{\alpha*}(V \otimes I_\varphi^{(n)})$ is just the contravariant form. This shows that our definition of $J_{!*}$ coincides with the original Jantzen filtration.

In the Verma modules case one can define the Jantzen filtration by using the deformations of the central character in an arbitrary nondegenerate direction φ, not necessarily in the positive one. According to Barbasch [Ba], the result does not depend on the choice of φ. In the geometric situation we can repeat, in principle, the same constructions and consider for any nonzero meromorprhic function f on X the morphism $j_!(\mathscr{M}_U \otimes f^*(I^{(n)})) \to$

$j_*(\mathscr{M}_U \otimes f^*(I^{(n)}))$, where $U := X \setminus \operatorname{div}(f)$. To define vanishing cycles one needs the stabilization of cokernels when $n \to \infty$. It would be very nice if this fact were true for any f, just as in the case when f (or f^{-1}) is regular on X, but we have no idea how to prove it.

§5. Weight filtrations

5.1. Weights of nearby cycles. Gabber's theorem, which is our main tool, seems not to be published yet. ([4]) Below we reproduce the proof following Gabber's report at IHES in the spring of 1981.

Let us start with the Künneth formula for nearby cycles. Let S be the spectrum of a strictly local Henselian ring; o and η be the closed and the generic points of S; and $\overline{\eta}$ be a geometric point localized at η.

Let $X \to S$ be an S-scheme and $X_0 \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} X_\eta \overset{k}{\longleftarrow} X_{\overline{\eta}}$ be the corresponding fibers. In what follows $D^b(Y)$ will denote either the bounded derived category of étale constructible $\mathbb{Z}/\ell^n$-sheaves on Y (where ℓ is prime to char o) or its $\mathbb{Q}_l$-counterpart [D1].

By [D1, (3.2)], there exist nearby cycles functors $\Psi_{\overline{\eta}} = \Psi_{\overline{\eta}X}\colon D^b(X_\eta) \to D^b(X_0)$, $\Psi_{\overline{\eta}X} := i^*Rj_*k_*k^*$. Let $Y \to S$ be another S-scheme, and $Z = X \underset{S}{\times} Y \to S$ be the fiber product. Then for $F \in D^b(X_\eta)$, $G \in D^b(Y_\eta)$ one has a canonical morphism in $D^b(Z_0)$:

$$\Psi_{\overline{\eta}X}(F) \boxtimes \Psi_{\overline{\eta}Y}(G) \to \Psi_{\overline{\eta}Z}(F \boxtimes G). \tag{$*$}$$

5.1.1. LEMMA. $(*)$ *is an isomorphism.*

REMARK. The transcendental version (hence, the characteristic 0 case) is almost obvious by the ordinary Künneth formula applied to local varieties of vanishing cycles. This, together with the Riemann-Hilbert correspondence, implies a similar fact for tame D-modules. To obtain a similar formula for arbitrary holonomic D-modules one must use the total nearby cycles functor of Deligne [D2].

PROOF. We can assume that the coefficients are $\mathbb{Z}/\ell$ (the $\mathbb{Z}/\ell^n$ and $\mathbb{Q}_\ell$ version follow in a moment). Put $m = \dim X$, $n = \dim Y$. The proof goes by simultaneous induction in m and in n.

Let C be the cone of $(*)$. The induction assumption, together with the trick of Deligne [D1, (3.3)], shows that the cohomology sheaves of C are supported at a finite set of points. So the statement $C = 0$ is equivalent to the statement $R\Gamma(C) = 0$. The problem is local, hence we can assume X, Y to be affine. Then replacing X and Y by their closures we can assume that they are projective S-schemes. In this case $R\Gamma(C) = 0$, since $R\Gamma\Psi_{\overline{\eta}}(F) = R\Gamma(F_{\overline{\eta}})$ in the projective case.

([4]) Added in 1992: a proof appeared recently in a paper of Morihiko Saito.

Now we can pass to Gabber's theorem. Assume we are in a mixed situation, so we consider the schemes over a finite field $\mathbb{F}_q$. Let $\mathscr{M}(X)_{\text{mixed}} \subset D^b(X)_{\text{mixed}}$ be the category of mixed perverse sheaves on X and the corresponding derived category.

Let T be a curve, $o \in T$ be a closed point, and $U := T \setminus \{o\}$. Let S be a strict localization of T at o and $\bar{\eta}$ be the generic geometric point of S. For a T-scheme $f: X \to T$ put $X_0 = f^{-1}(o)$, $X_U = f^{-1}(U)$. One has the nearby cycles functor $\Psi_{\bar{\eta}x}: D^b(X_U)_{\text{mixed}} \to D^b(X_o)_{\text{mixed}}$ (see [D2]). It is convenient to use the twisted functor $\Psi_f := \Psi_{\bar{\eta}x}[-1]$. This functor is t-exact, i.e., $\Psi_f(\mathscr{M}(X_U)) \subset \mathscr{M}(X_o)$, and commutes with the Verdier duality as follows: $\Psi_f D = D\Psi_f(1)$ (here (1) is the Tate twist). The monodromy group acts on Ψ_f; for a perverse sheaf M let $s \in \operatorname{End}\Psi_f(M_U)$ be the logarithm of the unipotent part of geometric monodromy, and $\mu_.$ be the corresponding monodromy filtration on $\Psi_f(M_U)$.

5.1.2. THEOREM. *If M_U is pure of weight w, then $\mu_{\cdot+w-1}$ coincides with the weight filtration W on $\Psi_f(M_U)$.*

PROOF. The case when f is the identity (or a finite map) is Deligne's theorem [D2, (1.8.4)]. The proof in the general case follows similar lines:

(i) We can assume that M_U is irreducible.

(ii) Replacing T by a finite cover, we can assume that the geometric monodromy is unipotent.

(iii) The weights on $\operatorname{Ker} s$ (= invariants of monodromy action) are $\leqslant w-1$.

PROOF OF (iii). Consider the canonical isomorphism $\operatorname{Ker} s = \operatorname{Ker}(j_! M_U \to j_* M_U)$. Since $j_!$ does not increase weights, the weights of $j_! M_U$ are $\leqslant w$. This implies that the perverse sheaf $j_! M_U / W_{w-1}(j_! M_U)$ is pure of weight w, and hence is semisimple. But the only irreducible quotient of $j_! M_U$ is $j_{!*} M_U$, hence $\operatorname{Ker} s \subset W_{w-1}(j_! M_U)$.

Dually, the weights of $\operatorname{Coker} s$ are $\geqslant w-1$ (since $\operatorname{Coker} s = \operatorname{Coker}(j_! M_U \to j_* M_U)(1)$).

(iv) Since the weight of s is -2, to prove the theorem it suffices to show that the primitive part P_{-i} is pure of weight $w-1-i$. We have $Gr_i^{J_!} = P_i$, $Gr_i^{J_*} = P_{-i}(-i)$ (see 4.1), so (iii) implies the inequalities for weights $\{w_i\}$ of P_i: $w_i \leqslant w-1$, $w_i + 2i \geqslant w-1$, i.e., $w-1-2i \leqslant w_i \leqslant w-1$. In particular, for $i = 0$ we are done.

(v) Consider the fiber square $M_U^{\boxtimes 2}[-1]$: this is a perverse sheaf on $X \underset{T}{\times} X$ (at least over the generic point of T—the only thing we need) of weight $2w-1$. Since $\Psi_{f\times f}(M_U^{\boxtimes 2}[-1]) = \Psi_f(M_U)^{\boxtimes 2}$ by 5.1.1, Lemma 4.1.2(ii) implies that $P_{-i} \boxtimes P_{-i}(-i)$ occurs in $P_0(\Psi_{f\times f} M_U^{\boxtimes 2}[-1])$. Hence, by (iv), one has $2w_i + 2i = 2w-2$, or $w_i = w-i-1$.

5.1.3. Assume we have a parameter t at o, $t \in \mathscr{O}_T(T)$. We can define the functors of 4.2 in the mixed situation (see [B]): namely, one has the functors $\pi_f^a\colon M(U)_{\text{mixed}} \to M(X)_{\text{mixed}}$, $(\pi_f^a M_U)|_U$ is a consecutive extension of twists M_U, $M_U(1), \dots, M_U(a-1)$. Now 5.1.2 together with 4.1.1, 4.2.1, and 4.2.2(v) gives

COROLLARY. (i) *If M_U is pure of weight w then the filtrations $J_{f!}$ and $W_{\cdot+w}$ on $j_{U!}(M_U)$ coincide. The same for the filtrations J_{f*} and $W_{\cdot+w}$ on $j_{U*}(M_U)$.*

(ii) *The monodromy filtration $\mu_.$ on $\pi_f^a(M_U)$ coincides with $W_{\cdot+w+a-1}$. In particular, for $\Xi_f(M_U)$ one has $\mu_. = W_{\cdot+w}$.*

5.2. Pointwise purity and the socle property of the weight filtration. A mixed complex $F^{\cdot}$ on X is $*$-pointwise pure of weight w if for any closed point $x \in X$ the complex $i_x^*(F^{\cdot})$ is pure of weight w (i.e., $H^i i_x^* F^{\cdot}$ is pure of weight $i+w$; here i_x is the embedding $x \hookrightarrow X$). One defines !-pointwise purity similarly using $i_x^!$ instead of i_x^*; the Verdier duality interchanges $*$- and !-purity. Note that if a pure perverse sheaf is $*$-pointwise pure of weight w, then w coincides with its weight.

Now let $(X, \widetilde{X})$ be a finite H-monodromic K-variety. Recall that any pure monodromic sheaf M has finite geometric monodromy along the fibers of $\widetilde{X} \to X$, hence, if M is geometrically irreducible, it lies in $\mathscr{M}(X, K)_\chi$ for some $\chi \in H^\vee_{\text{tors}}$ (by the local monodromy theorem; note that the restriction of any monodromic sheaf to any fiber of the map $\widetilde{X} \to X$ is tame by [V1]). We will say that $\widetilde{X}$ is $(K, \overline{\chi})$-pointwise pure for $\overline{\chi} \in H^\vee_{\text{tors}}$ if any pure $M \in \mathscr{M}(X, K)_{\overline{\chi}}$ is $*$- and !-pointwise pure, and $\widetilde{X}$ is K-pointwise pure if this holds for any $\overline{\chi}$, i.e., any pure K-equivariant monodromic sheaf is $*$- and !-pure.

5.2.1. EXAMPLES. (i) Here is a simple sufficient condition for $*$-pointwise purity. Let M be a pure perverse sheaf. Assume that for any $x \in X$ there exists an ètale neighborhood U of x such that the canonical map $H^{\cdot}(U, M) \to H^{\cdot} i_x^* M$ is surjective. Then M is $*$-pointwise pure (since the weights on $H^{\cdot} i_x^* M$ are $\leqslant \cdot + w$ by definition). In particular, this implies that "toric" irreducible perverse sheaves on a toric variety are pointwise pure, which leads to an explicit formula for Goresky-MacPherson Betti numbers of toric varieties (J. Bernstein, 1981, unpublished).

(ii) According to Kazhdan-Lusztig [KL2] and Lusztig [L, Chapter 1] the flag variety $\widetilde{X}_{\mathfrak{g}}$ is N-pointwise pure. Lusztig and Vogan [LV] have shown that $X_{\mathfrak{g}}$ is K-pointwise pure if K is a fixed point subgroup of an involution; it seems that their method, together with the decomposition theorem, should prove the K-pointwise purity of $\widetilde{X}_{\mathfrak{g}}$ for any symmetric pair $(\mathfrak{g}, K)$.

Recall that one defines the socle filtration $S_.(M)$ on an object M of an abelian category by induction: $S_{-1} = 0$, $S_0 :=$ maximal semisimple subobject of M, $S_i(M)/S_{i-1}(M) := S_0(M/S_{i-1}(M))$. One defines the cosocle filtration $M = C_0(M) \supset C_{-1}(M) \supset \cdots$ in a dual manner.

If M is a mixed perverse sheaf, then $S.(M)$ and $C.(M)$ will denote the socle and cosocle filtrations on M considered as a geometric sheaf (Frobenius forgotten). Clearly both $S.$ and $C.$ are (being functorial) Frobenius invariant, hence $S.(M)$ and $C.(M)$ are mixed subsheaves of M.

5.2.2. LEMMA. *Let $i_Y\colon Y \hookrightarrow X$ be a locally closed subscheme, M_Y a pure perverse sheaf on Y of weight w, and $N \subset {}^pH^0 i_{Y*}M_Y$ a mixed subsheaf such that any irreducible subquotient of N is !-pointwise pure. Then $S.(N) = W_{\cdot+w}(N)$.*

PROOF. We have $S_{-1}(N) = 0 = W_{w-1}(N)$ (since i_{Y*} increases weights), $S_0(N) = W_w(N) = i_{Y!*}(i_Y^* N)$ (since, by the adjunction property of i_{Y*}, $S_0({}^pHi_{Y*}M_Y) = i_{!*}M_Y$). Since Gr^w is geometrically semisimple, one has $S_i(N) \supset W_{i+w}(N)$, so it remains to prove that $S_i(N) \subset W_{i+w}(N)$ for $i \geqslant 1$. We will do this by double induction: first in $\dim Y$, then in i. So assume that 5.2.2 is known for any $(Y', M_{Y'}, N')$ with $\dim Y' < \dim Y$, and that $S_j(N) = W_{j+w}(N)$ for $j < i$. Suppose that $S_i(N) \not\subset W_{i+w}(N)$. Then $S_i/S_{i-1} = S_i/W_{w+i-1}$ contains a pure geometrically irreducible subsheaf A of weight $a > i+w$ (possibly, after a finite extension of the finite base field). Note that $\operatorname{Supp} A \subset \overline{Y} \setminus Y$.

(i) Assume that A is supported at a closed point x. Consider the extension $0 \to W_{w+i-1}(N)/W_{w+i-2}(N) \to B \to A \to 0$ defined by N. Since $B \not\subset S_{i-1}(N)$, this extension is geometrically nontrivial, hence it corresponds to a nonzero element in $\operatorname{Hom}_{\text{Frob}}(A, H^1 i_x^! W_{w+i-1}(N)/W_{w+i-2}(N))$. By the !-pointwise purity condition $H^1 i_x^!(W_{w+i-1}(N)/W_{w+i-2}(N))$ has weight $w+i$; but $a > w+i$, hence a contradiction.

(ii) If $\dim \operatorname{supp} A > 0$ we will use induction in $\dim Y$. The conditions of the lemma are local, so we can assume that X is affine. Choose a "generic" hyperplane section $Z \subset X$, namely such that for any irreducible subquotient L of ${}^pH^{\cdot} i_{Y*}M_Y$ a canonical morphism $i_Z^! L(1)[2] \to i_Z^* L$ is an isomorphism. Then $M_{Y\cap Z} := i_{Y\cap Z}^!(M_Y)[1]$ is a pure perverse sheaf of weight $w+1$ on $Y \cap Z$. Consider the complex $i_{Y\cap Z*}(M_{Y\cap Z}) = i_Z^![1] i_{Y*}M_Y$; one has $W_a {}^pH^{\cdot} i_{Y\cap Z*}M_{Y\cap Z} = i_Z^![1]W_{a-1}H^l i_{Y*}M_Y$. A subsheaf $N_Z := i_Z^![1](N)$ of ${}^pH^0 i_{Y\cap Z*}M_{Y\cap Z}$ satifies the conditions of the lemma, hence, by the induction hypothesis, $i_Z^![1](A)$ has weight $i+1$. Since $i_Z^![1](A) \neq 0$ (since $\dim \operatorname{supp} A > 0$) our A has weight i, and we are done.

5.2.3. COROLLARY. *Let M_1, M_2 be pure perverse sheaves of weights w_1, w_2 that are both $*$- and !-pointwise pure. Suppose that $\operatorname{Ext}^1_{\mathscr{M}_{\text{mixed}}}(M_1, M_2) \neq 0$. Then exactly one of the following conditions holds (here $Y_i := \operatorname{supp} M_i$):*

(i) $Y_1 \subset Y_2$, $Y_1 \neq Y_2$, $w_1 = w_2 + 1$.
(ii) $Y_2 \subset Y_1$, $Y_1 \neq Y_2$, $w_1 = w_2 + 1$.
(iii) $Y_1 = Y_2$.

PROOF. Clearly either $Y_1 \subset Y_2$ or $Y_2 \subset Y_1$ (otherwise $\mathrm{Ext}^1 = 0$). Let $0 \to M_2 \to N \to M_1 \to 0$ be a nonsplit mixed extension. If $Y_1 \neq Y_2$ and $Y_1 \subset Y_2$, then a canonical morphism $N \to {}^pH^0 i_{(Y_2\setminus Y_1)*} i^*_{Y_2\setminus Y_1} N = {}^pH^0 i_{(Y_2\setminus Y_1)*}(M_2|_{Y_2\setminus Y_1})$ is injective (since N is nonsplit), so we are in situation (i) by 5.2.2. If $Y_1 \neq Y_2$ and $Y_2 \subset Y_1$, then N is a quotient of $i_{(Y_1\setminus Y_2)!}(M_1|_{Y_1\setminus Y_2})$, and (ii) follows from a statement Verdier dual to 5.2.2.

5.2.4. Let $\widetilde{X}$ be a finite monodromic K-variety. Note that if M_1, M_2 are irreducible objects in $\mathscr{M}(\widetilde{X}, K)_{\widetilde{\chi}}$ such that $\mathrm{Ext}^1_{M(\widetilde{Q}, K)_{\widetilde{\chi}}}(M_1, M_2) \neq 0$, then $\operatorname{supp} M_1 \neq \operatorname{supp} M_2$ (this follows, using the functor $i_{!*}$, from the fact that the category $\mathscr{M}(\widetilde{Q}, K)_{\widetilde{\chi},1}$ is semisimple if Q is a single orbit).

COROLLARY. *Assume that $\widetilde{X}$ is K-pointwise pure. Let M be an object in $\mathscr{M}(\widetilde{X}, K)_{\widetilde{\chi}\,\mathrm{mixed}}$ such that $W_{a-1}(M) = 0$ and $W_a(M) = S_0(M)$. Then $W_{a+i}(M) = S_i(M)$ for any i.*

PROOF. This follows by induction in i, using 5.2.2, the previous remark, and also the fact that any subquotient of M is K^0-equivariant.

5.2.5. EXAMPLE. Consider an irreducible $M \in \mathscr{M}(X, K)_{\widetilde{\chi}}$, $\widetilde{\chi} \in H^\vee_{\mathrm{tors}}$. Let $I(M)$ be an injective envelope of M in $\mathscr{M}(X, K)_{\widetilde{\chi}}$. Then $I(M)$ admits a mixed structure (possibly after a finite extension of the base field), and for any such structure the weight filtration coincides with the socle filtration up to a shift.

PROOF. The only problem is the existence of a mixed structure. But M clearly has one (being a middle extension of a lissé sheaf with finite monodromy). Any extension of the Frobenius action $M \to \mathrm{Frob}^* M$ to $I(M) \to \mathrm{Frob}^* I(M)$ defines some mixed structure on $I(M)$ (since any irreducible subquotient of $I(M)$ admits a mixed structure, and any Frobenius action on an irreducible perverse sheaf is unique up to a twist).

5.3. Jantzen conjectures. Let us apply the above considerations to $(\mathfrak{g}, K)$-modules. Let $(\mathfrak{g}, K)$ be an admissible Harish-Chandra pair (see 3.2), and $\chi \in \mathfrak{h}^*_{\mathbb{Q}}$ be a fixed rational dominant regular weight. The irreducible objects of $M(\widetilde{X}, K)_{\widetilde{\chi}}$ are of geometric origin (see 2.6.5(iii)), hence the corresponding standard objects each carry a weight filtration defined up to a shift. According to 5.1.3(i) it coincides with the Jantzen filtration. So an array of weight filtration properties also holds for its Jantzen counterpart via the equivalence $\mathscr{M}(U, K)_{\widetilde{\chi}} \xrightarrow{\widetilde{\Delta}_\chi} \mathscr{M}(X, K)_\chi$ (below we use freely the road from $\mathbb{F}$ to $\mathbb{C}$, see [BBD, Section 6]).

5.3.1. COROLLARY. *The Jantzen filtration on standard $(U, K)_\chi$-modules has semisimple consecutive quotients and does not depend on the choice of a positive deformation direction φ (see 4.3).*

5.3.2. COROLLARY. *Assume that* $\widetilde{X}$ *is* $(K, \overline{\chi})$*-pointwise pure (see* 5.2).

(i) *The Jantzen filtration* $J_{*\cdot}$ *on a* $*$*-standard* $(U, K)_\chi$*-module coincides with the socle filtration; the Jantzen filtration* $J_{!\cdot}$ *on a* $!$*-standard module coincides with the cosocle filtration.*

(ii) *If* $K = N$, *then* $J_{*\cdot}$ *also coincides, up to a shift, with the cosocle filtration, and* $J_{!\cdot}$ *coincides with the socle one.*

PROOF. (i) follows from 5.2.2 plus the Verdier dual statement. (ii) follows from 5.2.4 and the fact that any Verma module contains a unique irreducible submodule.

5.3.3. REMARKS. (i) The statement (ii) above was proved in [Ba] by purely algebraic methods. One can conjecture that it remains valid in the case of an arbitrary symmetric pair.

(ii) In fact, in [Ba] the socle property of $J_{!\cdot}$ for Verma modules was proved for Jantzen filtration defined by means of deformations of the central character in arbitrary nondegenerate directions, and we (in Section 4) used only those deformations in the positive directions. We do not know whether one can use such arbitrary deformations in the definition of $J_{!\cdot}$ for any symmetric pair.

For a regular $\chi \in \mathfrak{h}^*$ put $\Delta^{(\chi)} := \{\alpha \in \Delta : \chi(h_\alpha) \in \mathbb{Z}\}$. It is well known that $\Delta^{(\chi)}$ is a root system with the Weyl group $W^{(\chi)} = \{w \in W : w\chi - \chi \in \mathfrak{h}^*_{\mathbb{Z}}\}$ (recall that $\mathfrak{h}^*_{\mathbb{Z}} = \mathbb{Z}\Delta$). The orbit $W^{(\chi)}\chi$ contains a unique dominant weight, and for $\chi' \notin W^{(\chi)}\chi$ one has $\mathrm{Hom}(M_{\chi'}, M_\chi) = 0$ and $[M_\chi : L_{\chi'}] = 0$ (here $M_\chi \in \mathscr{M}(U, N)$ is the Verma module, L_χ is its irreducible quotient).

Let $\chi_1, \chi_2 \in \mathfrak{h}^*_{\mathbb{Q}}$ be regular weights such that $M_{\chi_1} \subset M_{\chi_2}$. Then for some (unique) dominant weight χ one has $\chi_i = w_i\chi$, where $w_i \in W^{(\chi)}$ and $w_1 \leqslant w_2$ with respect to the usual order on $W^{(\chi)}$.

5.3.4. COROLLARY. *One has* $J_{!i}(M_{\chi_1}) = M_{\chi_1} \cap J_{!i+\ell(w_2)-\ell(w_1)}(M_{\chi_2})$ *(here* ℓ *is the length function on* $W^{(\chi)}$*).*

PROOF. Since $\dim \mathrm{Hom}(M_{\chi_1}, M_{\chi_2}) = 1$, the embedding of the corresponding standard mixed sheaves is pure of certain weight a. Turning back to representations we see that $J_{!\cdot}(M_{\chi_1}) = M_{\chi_1} \cap J_{!\cdot+a}(M_{\chi_2})$. It remains to show that $a = \ell(w_2) - \ell(w_1)$. We can assume that $\ell(w_2) - \ell(w_1) = 1$ (if not, choose a chain $M_{\chi_1} \subset M_{\psi_1} \subset \cdots \subset M_{\psi_{\ell(w_2)-\ell(w_1)-1}} \subset M_{\chi_2}$ of Verma submodules such that each consecutive M has this property, and descend along it). Then Shapovalov's formula for the determinant of contravariant form implies that the vacuum vector of M_{χ_1} lies in $J_{!-1}(M_{\chi_2})/J_{!-2}(M_{\chi_2})$. Hence $a = 1$.

Let $\chi \in \mathfrak{h}^*_{\mathbb{Q}}$ be a dominant regular weight, $w_1, w_2 \in W^{(\chi)}$. Put

$$P_{w_1, w_2} := \sum_i [Gr^{J_!}_{-i}(M_{w_2\chi}) : L_{w_1\chi}]t^i.$$

5.3.5. COROLLARY. *This polynomial equals the Kazhdan-Lusztig polynomial for the group* $W^{(\chi)}$.

PROOF. According to [L, Chapter 1], Kazhdan-Lusztig polynomials are the matrix coefficients of the matrix that transforms the basis $j_{w!}(\mathbb{Q}_\ell)$ of the K-group of the category $\mathscr{M}(\widetilde{X}, N)_{\overline{\chi}\,\mathrm{mixed}}$ to the basis $j_{w!*}(\mathbb{Q}_\ell)$. Since the Jantzen filtration coincides with the weight filtration, our polynomials correspond to the entries of the inverse matrix. Since these matrices coincide up to standard changes of signs of the coefficients [KL1], we are done.

5.3.6. REMARKS. (i) Corollary 5.3.4 is Jantzen's Conjecture [J, (5.18)], see also [GJ1, (4.2)]. Corollary 5.3.5 was conjectured in [GJ1, GM]; in [GJ1] it was shown that 5.3.4 implies 5.3.5 by purely algebraic arguments.

(ii) It would be nice to get the analogs of 5.3.4 and 5.3.5 for arbitrary symmetric pairs. The only problem is to compute the weights in the space of Hom's between standard modules. Certainly one would like to know the weights in all the Ext's; we are ignorant of this even in the Verma modules case.

(iii) For a regular $\chi \in \mathfrak{h}^*$ let $\mathfrak{g}^{(\chi)}$ be a semisimple Lie algebra with the root system $\Delta^{(\chi)}$, $U^{(\chi)}$ its extended universal enveloping algebra, and $\mathfrak{h}^{(\chi)}$ its Cartan algebra. Then $\mathfrak{h}^{(\chi)}$ is (canonically) a direct summand of $\mathfrak{h}$; let $\chi_{\mathbb{Z}}$ be the $\mathfrak{h}^{(\chi)}$-component of χ, so $\chi_{\mathbb{Z}} \in \mathfrak{h}_{\mathbb{Z}}^{(\chi)*}$. One knows (see [So1] for a stronger statement) that $\mathscr{M}(U, N)_\chi$ is equivalent to a product of several copies of $\mathscr{M}(U^{(\chi)}, N^{(\chi)})_\chi$; this equivalence preserves the Verma modules. This immediately implies that in all the above results about $\mathscr{M}(U, N)_\chi$ we can drop the rationality assumption $\chi \in \mathfrak{h}_{\mathbb{Q}}^*$. Moreover, it sufficies to provide the proof for integral χ's only, which is the same as $\chi = 0$.

(iv) For a treatment of mixed categories of representations and the Koszul and Langlands dualities in this framework see [So1, So2, BGSo].

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