SMOOTH FRÉCHET GLOBALIZATIONS OF HARISH-CHANDRA MODULES

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1. **Introduction**

Let G be a linear reductive real Lie group G with Lie algebra $\mathfrak{g}$[1]. Let us fix a maximal compact subgroup K of G. The representation theory of G admits an algebraic underpinning encoded in the notion of a *Harish-Chandra module*.

By a Harish-Chandra module we shall understand a finitely generated $(\mathfrak{g}, K)$-module with finite K-multiplicities. Let us denote by $\mathcal{HC}$ the category whose objects are Harish-Chandra modules and whose morphisms are linear $(\mathfrak{g}, K)$-maps. By a *globalization* of a Harish-Chandra module V we understand a representation (π, E) of G such that the K-finite vectors of E are isomorphic to V as a $(\mathfrak{g}, K)$-module.

Let us denote by $\mathcal{SAF}$ the category whose objects are smooth admissible moderate growth Fréchet representations of G with continuous linear G-maps as morphisms. We consider the functor:

$$\mathcal{F} : \mathcal{SAF} \to \mathcal{HC}, \quad E \mapsto E^{K-\text{fin}} := \{K - \text{finite vectors of E}\} .$$

The Casselman-Wallach globalization theorem ([5], [16] and [18], Sect. 11) essentially asserts that $\mathcal{F}$ is an equivalence of categories. To phrase it differently, each Harish-Chandra module V admits an $\mathcal{SAF}$-globalization (π, V^∞) which is unique up to isomorphism. It follows that

$$V^\infty = \pi(\mathcal{S}(G))V$$

where $\mathcal{S}(G)$ is the Schwartz-algebra of rapidly decreasing functions on G, and $\pi(\mathcal{S}(G))V$ stands for the vector space spanned by $\pi(f)v$ for $f \in \mathcal{S}(G)$, $v \in V$. In particular, V is irreducible if and only if V^∞ is an algebraically simple $\mathcal{S}(G)$-module.

The objective of this paper is to present a new approach to the globalization theorem.

Our approach starts with a thorough study investigation of the topological nature of $\mathcal{SAF}$-globalizations. A norm p on a Harish-Chandra module V will be called *G-continuous* provided the completion V_p of the normed space (V, p) gives rise to a Banach-representation of G. We introduce the *Sobolev order* on the set of G-continuous norms:

$$p \prec q :\iff (\exists C > 0, k \in \mathbb{N}_0)(\forall v \in V) \qquad p(v) \leq C q_k(v)$$

where q_k refers to the k-th Sobolev norm of q. Two G-continuous norms will be called *Sobolev-equivalent* provided $p \prec q$ and $q \prec p$.

[1]Throughout this text Lie groups will be denoted by upper case Latin letters, G, K, N ... , and their corresponding Lie algebras by lower case German letters $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{n}$ etc.

Casselman's subrepresentation theorem implies that every Harish-Chandra module admits a G-continuous norm. Within our terminology the Casselman-Wallach theorem now reads:

Theorem 1.1. *Any two G-continuous norms on a Harish-Chandra module V are Sobolev-equivalent.*

On a technical level it is quite cumbersome to deal with arbitrary G-continuous norms p. However, the algebraic fact that K- multiplicities on a Harish-Chandra module are polynomially bounded, implies that the smooth vectors V_p^∞ form a nuclear Fréchet space. It implies that every G-continuous norm is Sobolev-equivalent to a K-invariant Hermitian norm (see Theorem 5.5 below).

It is convenient to introduce an auxiliary notion and call a Harish-Chandra module *good* provided it admits a unique $\mathcal{SAF}$-globalization. Note that the Casselman-Wallach Theorem is the assertion that all Harish-Chandra modules are good. Using elementary functional analysis we show in Section 7 that a Harish-Chandra module is good if and only if its associated matrix coefficients satisfy certain lower bounds which are uniform in the K-types (see Theorem 7.1).

Section 8 with Appendix A is devoted to minimal principal series representations(i.e. representations which are induced off a minimal parabolic subgroup) and their canonical Hilbert-globalizations as subspaces of $L^2(K)$. For such representations we define a Dirac-type sequence and establish essentially optimal uniform lower bounds for K-finite matrix coefficients (see Theorem 12.3 below). As a consequence we obtain that Harish-Chandra modules V of the minimal principal series are good. In addition we exhibit a generator $\xi \in V$ and an explicit linear continuous section of the quotient homomorphism

$$\mathcal{S}(G) \to V^\infty, \quad f \mapsto \pi(f)\xi \,,$$

which depends holomorphically on the representation parameter of π (see Theorem 8.1 and 12.8).

According to Casselman's subrepresentation theorem one can embed every Harish-Chandra module into a parabolically induced representation. In view of our results in Section 6 this implies that every Harish-Chandra module admits a minimal and maximal G-continuous norm with respect to the Sobolev order $\prec$ (see Cor. 8.3). Let us note that this important technical step is likewise implied by Wallach's upper bounds on matrix coefficients (see [17], Theorem 4.3.5).

In Section 9 we show that "good" is preserved by extensions, induction and tensoring with finite dimensional representations. These elementary technical results are useful in the sequel. In particular it follows that the task to show that all Harish-Chandra modules are good is reduced to irreducible modules.

In Section 10 we present Casselman's technique of holomorphic deformations of Harish-Chandra modules which, via Langlands classification, leaves us to establish that all Harish-Chandra modules of the discrete series are good. For a module V of the discrete series we proceed as follows: we embed V into a minimal principal series representation and are left to show that the unitary norm on V is Sobolev equivalent to the minimal norm. This in turn is reduced to a familiar result on meromorphic continuation of certain distributions (see [1] and Appendix B of this paper). In this context we wish to point that the fact that discrete series modules are good can be also proved using Wallach's upper bounds ([17], Theorem 4.3.5 and [18], Prop. 11.7.4).

In summary, we provide a functional analytic language for globalizations and emphasize that our main input, i.e. the estimates for minimal principal series representation in Section 8, cannot be deduced from the Casselman-Wallach theorem. In addition our bounds for matrix coefficients are locally uniform in the representation parameters and yield a Casselman-Wallach theorem for holomorphic families of Harish-Chandra modules (see Theorem 11.6). This, for instance, is useful for the theory of Eisenstein series (see Theorem 11.7 and Remark 11.8).

It was our intention to write an essentially self-contained account on the subject which is accessible to graduate students. The reading requires a good understanding of functional analysis and some basic knowledge about real reductive groups and Harish-Chandra modules as to be found in Wallach's text book [17], Sect. 1-3.

2. Basic representation theory I: growth of representations

We begin with a discussion of scale structures on Lie groups which give us an appropriate notion of size on the group. We then collect a few standards definition and facts of representation theory on topological vector spaces. After that we discuss growth issues of representations and introduce the notion of F-representation. We show that the category of smooth F-representations is isomorphic to a category of non-degenerate algebra representations of a certain Schwartz algebra.

2.1. **Scale structures on Lie groups**

Throughout this text G will denote a Lie group. It is our objective to obtain a notion of size on G which will be suitable to define growth for a representation.

By a *scale* on G we understand a function $s : G \to \mathbb{R}^+$ such that:

- s and s^{-1} are locally bounded,
- s is *submultiplicative*, i.e. $s(gh) \leq s(g)s(h)$ for all $g, h \in G$.

We introduce an ordering $\preccurlyeq$ on the space of scale functions by

$$s \preccurlyeq s' :\iff (\exists C > 0, N \in \mathbb{N})(\forall g \in G) \quad s(g) \leq Cs'(g)^N .$$

This gives us a notion of equivalence $\sim$ on scale functions:

$$s \sim s' :\iff s \preccurlyeq s' \quad \text{and} \quad s' \preccurlyeq s' .$$

By a *scale structure* on G we understand an equivalence class $[s]$ of a scale function s.

Note that every equivalence class $[s]$ admits a continuous representative. Henceforth we will only consider continuous scale functions.

Let us discuss the various natural scale structures.

2.1.1. *The maximal scale structure.* Suppose that G is connected and fix a left-invariant Riemannian metric $\mathbf{g}$ on G. Associated to $\mathbf{g}$ is the distance function $d(g, h)$, i.e. the minimum length of piecewise smooth curve joining g and h in G. Note that $d(\cdot, \cdot)$ is left G-invariant and hence can be recovered from

$$d(g) := d(g, \mathbf{1}) \qquad (g \in G) .$$

Note that $d(g)$ is *subadditive*, i.e. $d(gh) \leq d(g) + d(h)$ for all $g, h \in G$. Hence $s_{\max}(g) := e^{d(g)}$ defines a scale function on G. This scale is maximal in the sense that for any other scale s we have $s \preccurlyeq s_{\max}$ (see [7], Lemme 2). In particular, the equivalence class $[s_{\max}]$ of $s_{\max}$ is independent of the choice of the particular left invariant metric. In the sequel we will refer to $[s_{\max}]$ as the *maximal scale structure*.

2.1.2. *Algebraic scale structure.* Let G be a real algebraic group. We fix a faithful algebraic representation $\iota : G \to \mathrm{Gl}(n, \mathbb{R})$. Then

$$\|g\| := \mathrm{tr}(\iota(g)\iota(g)^t) + \mathrm{tr}(\iota(g^{-1})\iota(g)^{-t}) \qquad (g \in G)$$

defines a smooth scale function on G. If we choose another faithful algebraic representation $\iota' : G \to \mathrm{Gl}(n', \mathbb{R})$, and if $\|\cdot\|'$ is the associated scale on G, then $\|\cdot\|$ and $\|\cdot\|'$ are equivalent. The resulting scale structure on G will be referred to as the *algebraic scale structure*. We often refer to $\|\cdot\|$ as a *norm* on G – see [17], Sect. 2.A.2 for the notion of norm on a reductive group.

Lemma 2.1. *Let G be a connected real reductive group. Then the algebraic and the maximal scale structure coincide.*

Proof. Let $\iota : G \to \mathrm{Gl}(n, \mathbb{R})$ be a faithful representation and henceforth view G as a subgroup of $\mathrm{Gl}(n, \mathbb{R})$. We recall the Cartan decomposition $G = KAK$ where $K < G$ is a maximal compact sungroup and A a non-compact torus. From the definitions of the scale structures involved it is easy to see that they coincide on A. The assertion follows. □

Remark 2.2. *The maximal and algebraic scale structure of the algebraic group $G = (\mathbb{R}, +)$ are different (polynomial versus exponential growth).*

In the sequel we will understand in this paper by a Lie group G a pair $(G, [s])$ where $[s]$ is a scale structure. If G is real reductive, then $[s]$ shall be the maximal scale structure.

2.2. **Representations on topological vector spaces**

All topological vector spaces E considered in this paper are understood to be locally convex. We denote by $\mathrm{Gl}(E)$ the group of all topological linear isomorphisms of E.

Let G be a Lie group and E a topological vector space. By a *representation* (π, E) of G on E we understand a homomorphism $\pi : G \to \mathrm{Gl}(E)$ such that the resulting action $G \times E \to E$ is continuous. We emphasize that continuity is requested in both variables. For an element $v \in E$ we shall denote by

$$\gamma_v : G \to E, \quad g \mapsto \pi(g)v$$

the corresponding continuous orbit map. The following Lemma is standard (cf. [19], Sect. 4.1).

Lemma 2.3. *Let G be a Lie group, E a topological vector space, $\pi : G \to \mathrm{Gl}(E)$ a group homomorphism and $G \times E \to E$ the resulting action. Then the following statements are equivalent:*

(i) *The action $G \times E \to E$ is continuous, i.e. (π, E) is a representation.*

(ii) (a) *There exists a dense subset $E_0 \subset E$ such that for all $v \in E_0$ the orbit map $\gamma_v : G \to E$ is continuous.*
(b) *For every compact subset Q of G the set $\{\pi(g) \mid g \in Q\}$ is an equicontinuous set of linear endomorphisms of E.*

If $\pi : G \to \mathrm{Gl}(E)$ is a group homomorphism, then we say π is *locally equicontinuous* if condition (b) in the Lemma above is satisfied.

Remark 2.4. *(a) Let $\pi : G \to \mathrm{Gl}(E)$ be homomorphism. If E is a Banach space, then, in view of the uniform boundedness principle, the following statements are equivalent:*

- *The action $G \times E \to E$ is continuous, i.e. (π, E) is a representation.*
- *For all $v \in E$ the orbit map γ_v is continuous.*

In the existing literature one mostly considers representation on Banach spaces and uses the second bulleted item as a definition for representation. Let us emphasize that these two notions will be different in general.
(b) Suppose that (π, E) is a representation on a semi-normed space E. Then all operator norms of $\pi(g)$ are locally bounded in $g \in G$.

If (π, E) is a representation, then we call a continuous semi-norm p on E a *G-continuous semi-norm*, if $G \times (E, p) \to (E, p)$ is continuous. Here (E, p) stands for the vector space E endowed with the topology induced from the semi-norm p.

Remark 2.5. *Let p be a G-continuous semi-norm on a representation module E and E_p be the completion of (E, p). As $G \times (E, p) \to (E, p)$ is continuous, we obtain a representation of G on the Banach space E_p.*

Let (π, E) be a representation of G. If E is a Banach (Hilbertian, Fréchet) space, then we speak of a *Banach* (*Hilbertian*, *Fréchet*) *representation* of G.

2.3. **Growth of a representation**

In this section $G = (G, [s])$ denotes a Lie group with scale structure $[s]$.

Let (π, E) be a representation of G on a semi-normed space (E, p). Then

$$s_\pi : G \to \mathbb{R}^+, \quad g \mapsto \|\pi(g)\|$$

is a scale. We call s_π the scale associated to (π, E). We will say that (π, E) is $[s]$-*bounded* provided $s_\pi \preccurlyeq s$.

A G-continuous semi-norm p on a representation module E will be called $[s]$-*bounded* provided $G \times (E, p) \to (E, p)$ is a $[s]$-bounded representation.

Definition 2.6. *A representation (π, E) of $G = (G, [s])$ will be called an F-representation provided E is a Fréchet space whose topology is induced by a countable family of G-continuous $[s]$-bounded semi-norms $(p^n)_{n\in\mathbb{N}}$.*

Let us emphasize that Fréchet spaces are complete topological vector spaces. In the context of F-representations this will play an important role when it comes to vector valued integration.

Example 2.7. *(a) If $[s]$ is the maximal scale structure, then any representation on a semi-normed space is $[s]$-bounded.*
(b) Let $G = (\mathbb{R}, +)$ endowed with the algebraic scale structure. Then a character $\pi : G \to \mathbb{C}^*$ is $[s]$-bounded if and only if π is unitary.

Remark 2.8. (Fréchet representations versus F-representations) *Let us emphasize that a Fréchet representation is not necessarily an F-representation. Here are some examples:*
(a) Let G be non-compact connected Lie group and $E = C(G)$ be the space of continuous function on G. Then E, endowed with the topology of compact convergence, becomes a Fréchet space. Let π denote either the left or right regular action of G on E. Then (π, E) is a Fréchet but not an F-representation.
(b) Let $G = \mathrm{Sl}(2, \mathbb{R})$, $B < G$ the standard Borel subgroup and $\chi : B \to \mathbb{C}^*$ a character. Let E be the G-module of hyperfunction sections of the line bundle $G \times_B \mathbb{C}_\chi \to G/B$. As a topological vector space E is isomorphic to the hyperfunctions on the circle, hence a Fréchet space. This yields a Fréchet representation which is not an F-representation.

More generally, if (π, E) is the maximal globalization of a Harish-Chandra module (in the sense of Schmid, see [13]*), then (π, E) is a Fréchet representation but not an F-representation.*

Recall that the category of Fréchet spaces is closed under taking closed subspaces and quotients by closed subspaces. The same holds for the category of F-representations. We record this fact, but skip the very easy proof:

Lemma 2.9. *Let (π, E) be an F-representation and $H \subset E$ a closed G-invariant subspace. Then the corresponding sub and quotient representation on H, resp. E/H, are F-representations.*

2.3.1. *Representations of moderate growth.* In [5] Casselman calls a Fréchet representation (π, E) of a real reductive group G of *moderate growth* provided for any semi-norm p on E there exists a semi-norm q on E and an integer $N > 0$ such that

$$p(\pi(g)v) \leq \|g\|^N q(v) \qquad (g \in G)\,.$$

For an arbitrary Lie group $G = (G, [s])$ one thus might call a representation of *moderate growth* if for any semi-norm p on E there exists a semi-norm q on E and an integer $N > 0$ such that

$$p(\pi(g)v) \leq s(g)^N q(v) \qquad (g \in G)\,.$$

Lemma 2.10. *Let (π, E) be a Fréchet representation of the Lie group $(G, [s])$. Then the following statements are equivalent:*

(i) *(π, E) is of moderate growth.*
(ii) *(π, E) is an F-representation.*

Proof. By definition any F-representation is of moderate growth.

Conversely, assume that (π, E) is of moderate growth and let p, q and $N > 0$ be as in the definition above. Then

$$\widetilde{p}(v) := \sup_{g \in G} \frac{p(\pi(g)v)}{s(g)^N}$$

defines a semi-norm on E such that

- $p \leq \widetilde{p} \leq q$.
- $\widetilde{p}(\pi(g)v) \leq s(g)^N \widetilde{p}(v)$ for all $g \in G$.

The first bulleted item implies that the semi-norms $\widetilde{p}$ define the topology on E. The second bulleted item yields that $\widetilde{p}$ is G-continuous and $[s]$-bounded.

□

2.4. **Smooth vectors and smooth representations**

2.4.1. *Smooth vectors.*

Definition 2.11. *Let (π, E) be an F-representation of G. We call a vector $v \in E$ smooth if γ_v is a smooth map. We denote by E^∞ the vector space of all smooth vectors.*

Remark 2.12. *It is common to define smooth vectors for arbitrary representations (π, E): one says $v \in E$ smooth provided γ_v is a smooth map* [4]*. If (π, E) is not an F-representation then this leads to counterintuitive examples:*

(i) *The regular action of a compact group G on the space of distributions $E = C^{-\infty}(G)$ would be smooth. More generally, if $(\pi, \mathcal{H})$ is a Hilbert representation of G and $\mathcal{H}^{-\infty}$ the topological dual of $\mathcal{H}^\infty$, then $\mathcal{H}^{-\infty}$ would define a smooth representation.*
(ii) *Let (π, E) be a Banach representation and E^ω the space of analytic vectors with its natural inductive limit topology. The dual strong dual $E^{-\omega}$ of E^ω, the space of hyperfunction, is a Fréchet space. The induced action of G on $E^{-\omega}$ would be a smooth representation.*

Note that $\mathcal{U}(\mathfrak{g})$, the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G, acts naturally on E^∞. As customary we denote this algebra action by $d\pi$.

2.4.2. *Sobolev semi-norms.* For a continuous semi-norm p on E we wish to associate a family of Sobolev semi-norms $(p_k)_{k\in\mathbb{N}_0}$. We proceed as follows: Fix a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$. For all $k \in \mathbb{N}_0$ and $v \in E^\infty$ we set

$$p_k(v) := \Big[\sum_{m_1+\ldots+m_n\leq k} p(d\pi(X_1^{m_1} \cdot \ldots \cdot X_n^{m_n})v)^2\Big]^{\frac{1}{2}}$$

and refer to p_k as a k-th Sobolev norm of p.

Remark 2.13. *(a) The definition of p_k depends on the choice of the basis $X_1, \ldots, X_n$. However, a different basis yields an equivalent semi-norm.*
(b) If p is G-continuous (resp. Hermitian), then so is p_k for any $k \in \mathbb{N}_0$.

2.4.3. *Smooth representations.* In the sequel we view E^∞ as a topological vector space with the locally convex topology induced by all Sobolev semi-norms.

Definition 2.14. *An F-representation (π, E) is called smooth if $E = E^\infty$ holds as topological vector spaces.*

Let us denote by $C^\infty(G, E)$ the space of E-valued smooth functions. We endow $C^\infty(G, E)$ with the topology of smooth compact convergence. We let G act on $C^\infty(G, E)$ as

$$g \cdot f(x) := \pi(g)f(g^{-1}x) \qquad (g, x \in G; f \in C^\infty(G, E)$$

and note that this action is continuous. Hence the space of G-invariants $C^\infty(G, E)^G$ is a closed subspace of $C^\infty(G, E)$. The following standard fact is found in [4].

Lemma 2.15. *Let (π, E) be an F-representation of G. Then the map*

$$E^\infty \to C^\infty(G, E)^G, \quad v \mapsto \gamma_v$$

is an isomorphism of topological vector spaces. In particular, E^∞ is complete.

In the sequel we call a smooth F-representation simply SF-representation.

Corollary 2.16. *Suppose that (π, E) is an F-representation. Then (π, E^∞) is an SF-representation of G.*

2.5. **Integration of representations and algebra actions**

Let us denote by $\mathcal{M}(G)$ the Banach space of complex Borel measures on G. We recall that $\mathcal{M}(G)$ carries a natural Banach algebra structure by convolution of measures:

$$(\mu * \nu)(f) := \mu_x(\nu_y(f(yx)))$$

for $\mu, \nu \in \mathcal{M}(G)$ and $f \in C_c(G)$. We denote by $\mathcal{M}_c(G) \subset \mathcal{M}(G)$ the subalgebra of compactly supported complex measures.

Remark 2.17. *The left action of G on G induces an action of G on $\mathcal{M}(G)$ by isometries. This natural action is not continuous, i.e. does not define a representation. Call a measure μ continuous provided the orbit map*

$$G \to \mathcal{M}(G), \ g \mapsto (\lambda_g)_*\mu$$

is continuous. Here $\lambda_g(x) = gx$ for $x \in G$ is the left translate. Let us denote by $\widetilde{\mathcal{M}}(G)$ the space of continuous complex measures. If we fix a left Haar measure dg on G, then the map

$$L^1(G) \to \widetilde{\mathcal{M}}(G), \quad f \mapsto f \cdot dg$$

provides an isomorphism of Banach algebras.

If (π, E) is representation of G on a complete topological vector space, then we denote by Π the corresponding algebra representation of $\mathcal{M}_c(G)$:

$$\Pi(\mu)v = \int_G \pi(g)v \; d\mu(g) \qquad (\mu \in \mathcal{M}_c(G) \, v \in E)\,. \tag{2.1}$$

Note that the defining vector valued integral converges as E is complete.

Depending on the type of the representation (π, E) larger algebras as $\mathcal{M}(G)$ might act on E. For instance if (π, E) is a bounded Banach representation, then Π extends to a representation of $\mathcal{M}(G)$. The natural algebra acting on an F-representation is the algebra of rapidly decreasing complex measures on G.

The space of *rapidly decreasing continuous complex measures* on G is defined as

$$\mathcal{R}(G) := \{\mu \in \widetilde{\mathcal{M}}(G) \mid (\forall n \in \mathbb{N}) \ s(g)^n \in L^1(G, |\mu|)\}\,.$$

Let us emphasize that $\mathcal{R}(G)$ only depends on the scale structure $[s]$. We write $L \times R$ for the regular representation of $G \times G$ on functions on G:

$$(L \times R)(g_1, g_2)f(g) := f(g_1^{-1} g g_2)$$

for $g, g_1, g_2 \in G$ and $f \in C(G)$. The following properties of $\mathcal{R}(G)$ are easy to verify:

- $(L \times R, \mathcal{R}(G))$ is an F-representation of $G \times G$.
- $\mathcal{R}(G)$ is a Fréchet algebra under convolution.
- Any F-representation (π, E) of G integrates to a continuous algebra representation

$$\mathcal{R}(G) \times E \to E, \quad (\mu, v) \mapsto \Pi(\mu)v, \tag{2.2}$$

i.e. the E-valued integrals in (2.1) converge absolutely, the bilinear map (2.2) is continuous and $\Pi(\mu * \nu) = \Pi(\mu)\Pi(\nu)$ holds for all $\mu, \nu \in \mathcal{R}(G)$.

For $u \in \mathcal{U}(\mathfrak{g})$ we will abbreviate $L_u := dL(u)$ and likewise R_u for the derived representations. The smooth vectors of $(L \times R, \mathcal{R}(G))$ constitute the *Schwartz space*

$$\mathcal{S}(G) := \{f \cdot dg \mid f \in C^\infty(G); \forall u, v \in \mathcal{U}(\mathfrak{g}), \forall n \in \mathbb{N} \\ s(g)^n L_u R_v f \in L^1(G)\}.$$

It is clear that $\mathcal{S}(G)$ is a Fréchet subalgebra of $\mathcal{R}(G)$ (see [17], Sect. 7.1 for a discussion in a wider context if G is reductive).

Remark 2.18. *Suppose that $[s]$ is the maximal or algebraic scale structure on G. Then for a function $f \in \mathcal{R}(G)$ the following assertions are equivalent: (1) f is in $\mathcal{S}(G)$, i.e. f is $L \times R$-smooth; (2) f is R-smooth; (3) f is L-smooth. In fact, a left derivative L_u at a point $g \in G$ is the same as a right derivative $R_{\mathrm{Ad}(g)^{-1}u}$ at g. Now observe that $\|\mathrm{Ad}(g)\| \leq Cs(g)^C$ for all $g \in G$ and a fixed $C > 0$.*

The natural algebra to consider for a SF-representation is the Schwartz algebra $\mathcal{S}(G)$ (see Proposition 2.20 below).

Remark 2.19. *If (π, E) is a smooth Fréchet-representation, then $\Pi(C_c^\infty(G))E = E$ by Dixmier-Malliavin [6]. Assume in addition that (π, E) is a SF-representation. As $\mathcal{R}(G)$ acts on E and $\mathcal{R}(G) \supset \mathcal{S}(G) \supset C_c^\infty(G)$, we deduce that $\Pi(\mathcal{S}(G))E = \Pi(\mathcal{R}(G))E = E$.*

If $\mathcal{A}$ is an algebra without $\mathbf{1}$ and M is an $\mathcal{A}$-module, then we call M *non-degenerate* provided $\mathcal{A}M = M$.

Proposition 2.20. *Let G be a Lie group. Then the following categories are equivalent:*

(i) *The category of SF-representations of G.*
(ii) *The category of non-degenerate continuous algebra representations of $\mathcal{S}(G)$ on Fréchet spaces.*

Proof. We already saw that every SF-representations (π, E) gives rise to a non-degenerate continuous algebra representation (Π, E) of $\mathcal{S}(G)$. Conversely let (Π, E) by a continuous non-degenerate algebra representation of $\mathcal{S}(G)$. Let us denote by $S(G)\widehat{\otimes}_\pi E$ the projective tensor product of $\mathcal{S}(G)$ and E. Clearly $S(G)\widehat{\otimes}_\pi E$ is a Fréchet space and we define an SF-module structure for G by

$$g \cdot (f \otimes v) := L(g)f \otimes v \qquad (g \in G, f \in \mathcal{S}(G), v \in E)\,.$$

As Π is non-degenerate, E becomes a quotient of $S(G)\widehat{\otimes}_\pi E$ and Lemma 2.9 completes the proof. □

3. Basic representation theory II: Banach representations

In this section we investigate Banach representations, in particular we are interested in the fine Sobolev structure of smooth vectors. We view Banach representations as appropriate local models for F-representations and point out that the main results in this section hold for F-representations as well.

3.1. Contragredient representations

Throughout this section we let E denote a Banach space. We denote by E^* its topological dual. We fix a norm p on E and note that E^* is Banach space with respect to the dual norm

$$p^*(\lambda) := \sup_{p(v)\leq 1} |\lambda(v)| \qquad (\lambda \in E^*)\,.$$

Let (π, E) be a Banach representation of G and consider the group homomorphism

$$\pi^* : G \to \mathrm{Gl}(E^*), \quad \pi^*(g)(\lambda) := \lambda \circ \pi(g^{-1})\,.$$

From the local equicontinuity of π the local equicontinuity of π^* follows (see [15], Ch. 19). However, orbit maps for π^* might fail to be continuous as we will see in an example below.

The fact that orbit maps for π^* might fail to be continuous can be dealt with in the following way. Let us consider the subspace $\widetilde{E} \subset E^*$ consisting of those vectors $\lambda \in E^*$ for which the orbit map $\gamma_\lambda : G \to E^*$ is continuous (we call this space the continuous dual). Lemma 2.3 implies that $\widetilde{E}$ is a closed G-invariant subspace of E^*. Following [4] we restrict the action of G to this subspace and obtain a representation $(\widetilde{\pi}, \widetilde{E})$ that we call the *contragredient representation* of (π, E).

Let us denote by $\mathcal{H}(G) = C_c^\infty(G) \cdot dg$ the algebra of smooth compactly supported measures on G. The standard technique of Dirac approximation yields:

Lemma 3.1. *Let (π, E) be a Banach representation of G. Then the following assertions hold:*

(i) *For all $\mu \in \mathcal{H}(G)$ the operator $\Pi^*(\mu)$ is defined on E^* and maps E^* into $\widetilde{E}$.*
(ii) *The spaces $\Pi^*(\mathcal{H}(G))(\widetilde{E})$ and $\Pi^*(\mathcal{H}(G))E^*$ are dense in $\widetilde{E}$.*

We write $\widetilde{p}$ for the restriction of p^* to $\widetilde{E}$. Consider the natural isometric morphism $E \to E^{**}$. The inclusion $\widetilde{E} \to E^*$ yields a contractive projection $E^{**} \to (\widetilde{E})^*$ and hence a contractive map $i : E \to \widetilde{\widetilde{E}} \subset (\widetilde{E})^*$.

Proposition 3.2. *Let (π, E) be a Banach representation of G and p be a defining norm on E. Then the natural morphism $i : E \to \widetilde{\widetilde{E}}$ is an isometric embedding.*

Proof. We need to show that

$$\widetilde{p}^*(v) = p(v) \qquad (v \in E)\,. \tag{3.1}$$

As $E^{**} \to (\widetilde{E})^*$ is contractive, the inequality "$\leq$" follows. As for the reverse inequality let us fix a unit vector $v \in E$. By Hahn-Banach, we find $\lambda \in E^*$ with $p^*(\lambda) = 1$ such that $\lambda(v) = 1$. Let $\epsilon > 0$ and choose a non-negative normalized $\mu \in \mathcal{H}(G)$ such that $p(\Pi(\mu)v - v) < \epsilon$. Set $\xi := \Pi^*(\mu^\vee)(\lambda)$, where $\mu^\vee$ is the push-forward of μ under $g \mapsto g^{-1}$. Then $\xi \in \widetilde{E}$ by the previous lemma. If we choose $\mathrm{supp}(\mu)$ small enough such that $\|\xi\| \leq 1 + \epsilon$, then

$$|\xi(v)| = |1 + \lambda(\Pi(\mu)v - v)| \geq 1 - \epsilon$$

and the proof is complete. □

Let us call a Banach representation (π, E) *reflexive* if the morphism $i : E \to \widetilde{\widetilde{E}}$ is an isomorphism.

Note that Proposition 3.2 shows that (π, E) is reflexive if E is reflexive (see also [19], Cor. 4.1.2.3). The converse is not true as the following example shows.

Example 3.3. *Let G be a compact Lie group and (π, E) be the left regular representation of G on $E = L^1(G)$. Then it is easy to check that:*

(i) $\widetilde{E} = C(G)$ *while* $E^* = L^\infty(G)$.

(ii) $\widetilde{C(G)} = L^1(G)$ *while* $C(G)^*$ *is the space of Radon measures on* G.

This shows that a representation (π, E) *can be reflexive while the Banach space* E *is not reflexive.*

3.2. **Induced Sobolev norms**

Our definition of smooth vectors for a representation (π, E) was a geometric one: we said $v \in E$ is smooth if the orbit map $\gamma_v : G \to E$ was smooth. Further we put a Fréchet topology on E^∞ via the Sobolev norms p_k which turned (π, E^∞) into an SF-module for G.

In this paragraph we will introduce a new class of Sobolev norms on E^∞ which are more quantitative and easier to work with.

To begin with we associate to $\lambda \in E^*$ and $v \in E$ the matrix coefficient

$$m_{\lambda,v}(g) := \lambda(\pi(g)v) \qquad (g \in G)$$

which is a continuous function on G and smooth provided v is smooth. We fix a relatively compact neighborhood B of $\mathbf{1}$ in G and a test function ϕ on G which is supported in B and positive near $\mathbf{1}$.

We denote by $\mathcal{F}$ the space of test function on G which are supported in B. For a continuous semi-norm q on $\mathcal{F}$ we define a semi-norm p_q on E^∞, the *semi-norm induced by* q, by

$$p_q(v) := \sup_{p^*(\lambda)\leq 1} q(\phi \cdot m_{\lambda,v}) \qquad (v \in E^\infty)\,.$$

Note that the choices of both B and ϕ for the definition of Sp_s are irrelevant; other choices yield equivalent norms.

Typical examples we have in mind for q are L^p-Sobolev norms or semi-norms of the form $q(f) = \sup_{g\in C_B} |Df(g)|$ where D is a differential operator and C_B a compact subset of B. Since $m_{\lambda,v}(g) = m_{\lambda,\pi(g)v}(\mathbf{1}) = m_{\widetilde{\pi}(g^{-1})\lambda,v}(\mathbf{1})$ we conclude that the semi-norms p_q are G-continuous. In the special case where q is the L^2-Sobolev norm on $\mathcal{F}$ for order $s \in \mathbb{R}$ we write Sp_s instead of p_q.

Lemma 3.4. *Let* (π, E) *be a Banach representation of* G*. Then the following assertions hold:*

(i) *For all* $k \in \mathbb{N}_0$ *there exists a constant* $C_k > 0$ *such that*

$$Sp_k(v) \leq C_k \cdot p_k(v) \qquad (v \in E^\infty)\,.$$

(ii) *For all* $k \in \mathbb{N}_0$ *and* $s > k + \dim G/2$ *there exists a constant* $c_s > 0$ *such that*

$$p_k(v) \leq c_s \cdot Sp_s(v) \qquad (v \in E^\infty)\,.$$

Proof. The first assertion is obvious and the second is the standard Sobolev Lemma. □

3.2.1. *Laplace Sobolev norms.* For our fixed basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ we define a Laplace-element

$$\Delta := X_1^2 + \ldots + X_n^2 \in \mathcal{U}(\mathfrak{g})\,.$$

If p is a defining norm of E, then we set

$$^\Delta p_{2k}(v) := \Big(\sum_{j=0}^k p(d\pi(\Delta^j)v)^2\Big)^{\frac{1}{2}} \qquad (v \in E^\infty)\,.$$

We will refer to $^\Delta p_{2k}$ as the 2k*th Laplace-Sobolev norm* of p. Note that $^\Delta p_{2k}$ is equivalent to a norm induced from q where

$$q(f)^2 = \sum_{j=0}^k |\Delta^j f(\mathbf{1})|^2\,.$$

The next result is an immediate consequence of Lemma 3.4 and was motivated by [8], Rem. 5.6 (b).

Proposition 3.5. *Let* G *be Lie group and* (π, E) *be a Banach representation of* G*. Then for all* $k \in \mathbb{N}$ *there exists a* $C_k > 0$ *such that*

$$p_{2k}(v) \leq C_k \cdot {}^\Delta p_{2k+\dim G}(v) \qquad (v \in E^\infty)\,.$$

In particular, the topology of E^∞ *is defined by the family of Sobolev norms* $({}^\Delta p_{2k})_{k\in\mathbb{N}}$*.*

We can put this into a more general context: For any $s \in \mathbb{R}$ we let E_s be the completion of E^∞ with respect to Sp_s.

Lemma 3.6. *For a Banach representation* (π, E) *the following assertions hold true:*

(i) $E^\infty = \bigcap_{s>0} E_s$.
(ii) *For all* $s \in \mathbb{R}$ *the map*

$$d\pi(\Delta) : E_s \to E_{s-2}$$

has closed range. If it is injective, then it is an isomorphism of Banach spaces.

3.3. **Action of distributions**

In this paragraph e recall a few facts and notions about compactly supported distributions on G and their action on smooth vectors.

Recall that $C^\infty(G)$ carries a natural Fréchet topology and that its dual, in symbols $\mathcal{D}'_c(G)$, are the distributions with compact support.

We let G act on $C^\infty(G)$ by left translation in the argument. This induces an action of $\mathcal{U}(\mathfrak{g})$ on $C^\infty(G)$. If $u \mapsto u^t$ is the canonical anti-automorphism of $\mathcal{U}(\mathfrak{g})$, we then obtain action of $\mathcal{U}(\mathfrak{g})$ on distributions $\mathcal{D}'_c(G)$ by

$$(u * T)(f) := T(u^t f) \qquad (u \in \mathcal{U}(\mathfrak{g}), T \in \mathcal{D}'(G), f \in C_c^\infty(G))\,.$$

Note that $\mathcal{D}'_c(G)$ is an algebra under convolution: for $S, T \in \mathcal{D}'_c(G)$ and $\phi \in C^\infty(G)$ define

$$(S * T)(\phi) := S_x(T_y(\phi(xy)))\,.$$

By the fundamental theorem of distribution theory every $T \in \mathcal{D}'_c(G)$ can be expressed as $T = u * f$ for some $u \in \mathcal{U}(\mathfrak{g})$ and $f \in C_c(G)$. One checks that

$$\Pi(f)v := \int_G f(g)\pi(g)d\pi(u)v\; dg \qquad (v \in E^\infty)$$

does not depend on the representation $T = u * f$ and defines an algebra action of $\mathcal{D}'_c(G)$ on E^∞.

Lemma 3.7. (Elliptic regularity) *Let B be neighborhood of $\mathbf{1}$ in G. Let $m \in \mathbb{N}$ be such that $2m > \dim G$. Then*

$$\delta_{\mathbf{1}} = \Delta^m * f_1 + f_2$$

for a $C^{2m-\dim G-1}$-function f_1 and a smooth function f_2, both supported in B.

Proof. By local solvability and regularity of elliptic PDE (see [10], Th. 7.3.1 and Cor. 7.3.1) there exists a $C^{2m-\dim G-1}$-function f on G, smooth on $G\backslash\{\mathbf{1}\}$, such that $\Delta^m * f = \delta_{\mathbf{1}}$ holds in a small neighborhood $V \subset B$ of $\mathbf{1}$ in G.

Let U be a relatively compact open neighborhood of $\mathbf{1}$ with $\overline{U} \subset V$ and let ψ be a test function with $\psi\;|_U = 1$ and $\operatorname{supp}\psi \subset V$. Set $f_1 := \psi f$ and $f_2 := \Delta^m(\psi f) - \psi\Delta^m(f)$ and note that f_1 and f_2 are both supported in B with f_2 smooth. □

Corollary 3.8. *Let (π, E) is a Banach representation of G. Then for $u \in \mathcal{U}(\mathfrak{g})$ and $m \in \mathbb{N}$ as in Lemma 3.7:*

$$v = \Pi(f_1)d\pi(\Delta^m)v + \Pi(f_2)v \qquad (v \in E^\infty) \tag{3.2}$$

3.4. **Banach representations for a reductive group**

In this paragraph we assume that G is a real reductive group. We fix a maximal compact subgroup $K < G$.

Recall the notion of Laplace-Sobolev norm ${}^{\Delta}p_{2k}$ for a Banach representation (π, E). For that we fixed a basis $X_1, \ldots, X_n$ to define the Laplacian element $\Delta = \sum_{j=1}^n X_j^2$. The choice of the basis is in fact irrelevant and henceforth we will use a specific basis which is suitable for us. Such a basis is constructed as follows. Let $\mathfrak{k}$ denote the Lie algebra of K and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the associated Cartan decomposition of $\mathfrak{g}$. We fix a non-degenerate invariant bilinear form $B(X, Y)$ on $\mathfrak{g}$ such that B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. If $\theta : \mathfrak{g} \to \mathfrak{g}$ is the Cartan-involution associated to $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, then $\langle X, Y\rangle = -B(\theta(X), Y)$ is an inner product. Our choice of $X_1, \ldots, X_n$ will be such that $X_1, \ldots X_m$ forms an orthonormal basis of $\mathfrak{k}$ and $X_{m+1}, \ldots, X_n$ is an orthonormal basis of $\mathfrak{p}$.

With regard to our choice of basis we obtain a central element (Casimir element) in $\mathcal{Z}(\mathfrak{g})$ by setting

$$\mathcal{C} = -\sum_{j=1}^m X_j^2 + \sum_{j=m+1}^n X_j^2$$

and with $\Delta_{\mathfrak{k}} := \sum_{j=1}^m X_j^2$ we arrive at the familiar relation

$$\mathcal{C} = \Delta - 2\Delta_{\mathfrak{k}}\,. \tag{3.3}$$

Let $s \in \mathbb{N}_0$ and define the $2s$*-th* K*-Sobolev norm* of p by

$$p_{2s,K}(v)^2 := \sum_{j=0}^s p(\Delta_{\mathfrak{k}}^s v)^2 \qquad (v \in E^\infty)\,.$$

We claim that K-Sobolev semi-norms $p_{s,K}$ can be naturally defined for every $s \in \mathbb{R}$. Heuristically this can be seen as follows: Let $\mathfrak{t} < \mathfrak{k}$ be a Cartan subalgebra. We fix a notion of positivity on $\mathfrak{t}$ and identify irreducible representations τ of K with its highest weight in $i\mathfrak{t}^*$. For an element $\lambda \in i\mathfrak{t}^*$ we write $|\lambda|$ for its Cartan-Killing norm. Now on each K-type $E[\tau]$ of E one has

$$-\Delta_{\mathfrak{k}} v = (\underbrace{|\tau + \rho_{\mathfrak{k}}|^2 - |\rho_{\mathfrak{k}}|^2}_{=:\|\tau\|})v \qquad (v \in E[\tau])$$

and it is clear how $\Delta_{\mathfrak{k}}^s$ should be defined as an operator by breaking a vector $v \in E$ into its K-Fourier series. On a more formal level we note that action of $\Delta_{\mathfrak{k}}^s$ on E is realized by left convolution with a distribution

$\Theta_s \in C^{-\infty}(K)$ on K. Hence the fact that $C^{-\infty}(K)$ acts continuously on every SF-module yields our claim.

Proposition 3.9. *Let (π, E) be a Banach representation of a real reductive group G. Suppose that one of the following conditions is satisfied:*

(i) *The linear map $d\pi(\mathcal{C}) : E^\infty \to E^\infty$ extends to a morphism on E.*

(ii) *There exists a polynomial P such that $P(d\pi(\mathcal{C}))|_{E^\infty} \equiv 0$.*

Then the topology on the SF-module E^∞ is induced by the K-Sobolev norms norms $(p_{2n,K})_{n\in\mathbb{N}_0}$.

Proof. Assume first that $d\pi(\mathcal{C})$ extends to a continuous linear operator on E. Let $v \in E^\infty$ and note that:

$$p(d\pi(\Delta)v) = p(d\pi(C + 2\Delta_{\mathfrak{k}})v) \leq Cp(v) + 2p(d\pi(\Delta_{\mathfrak{k}})v)\,.$$

The assertion follows with Proposition 3.5.

Assume now that the second condition is staisfied and let d be the degree of the polynomial P. As $\Delta = \mathcal{C} + 2\Delta_{\mathfrak{k}}$ we get for $k \in \mathbb{N}$, $k > d$, that

$$\Delta^k = \sum_{j=0}^{m} Q_{j,k}(\mathcal{C})\Delta_{\mathfrak{k}}^j$$

with $Q_{j,k}$ polynomials of degree smaller than d. For m sufficiently large, Lemma 3.7 yields $\delta_{\mathbf{1}} = \Delta^m * f_1 + f_2$ with f_2 smooth and f_1 in $C^{2m-\dim G-1}(G)$ and both compactly supported. In particular we get for $l \in \mathbb{N}_0$

$$\begin{aligned}\Delta^l &= \Delta^{m+l} * f_1 + \Delta^l * f_2 \\ &= \sum_{j=0}^{m+l} \Delta_{\mathfrak{k}}^j * Q_{j,m+l}(\mathcal{C}) * f_1 + \Delta^l * f_2\end{aligned}$$

With $F_j := Q_{j,m+l}(\mathcal{C}) * f_1$ and $F = \Delta^l * f_2$ we thus get

$$\Delta^l = \sum_{j=0}^{m+l} \Delta_{\mathfrak{k}}^j * F_j + F\,.$$

If m is large compared to l, the F_j are continuous and F is smooth. The assertion follows if we apply this identity to a smooth vector. □

For a representation (π, E) of G let us write E_K^∞ for the space of smooth vectors for the K-representation $\pi\,|_K$.

Corollary 3.10. *Let (π, E) be a Banach representation of a real reductive group G. Suppose that one of the conditions in Proposition 3.9 holds. Then $E^\infty = E_K^\infty$.*

Proof. Set $V := E^\infty$. We claim that V is dense in the Fréchet space E_K^∞.

In order to prove the claim we first note that the Garding subspace $\Pi_K(C^\infty(K) \cdot dk)E$ is dense in E_K^∞. Let $\mu \in C^\infty(K) \cdot dk$ be a smooth measure and $v = \Pi_K(\mu)u$ for some $u \in E$. Let $(\nu_k)_{k\in\mathbb{N}} \subset C_c^\infty(G) \cdot dg$ be a Dirac sequence so that $u_k := \Pi(\nu_k)u \to u$ in E. Then $v_k := \Pi_K(\mu)u_k$ is in E^∞ and converges to v in E_K^∞, establishing our claim.

The claim implies that E_K^∞ is the completion of V with respect to $(p_{2n,K})_{n\in\mathbb{N}_0}$ and the conclusion follows with Proposition 3.9. □

4. Harish-Chandra modules I: algebraic facts

In this section we will review some central algebraic facts about Harish-Chandra modules. This is followed by a discussion of basic topological properties of their globalizations in the following section.

From now on we assume that G is a linear reductive group. Let us fix a maximal compact subgroup K of G.

By a K-module V we shall understand a complex vector space endowed with linear algebraic action of K, that is V is a union of finite dimensional algebraic K-representations.

If (π, E) is a representation of K, then we denote by $E^{K-\mathrm{fin}}$ the K-module consisting of K-finite vectors.

We call a K-module E *weakly admissible* provided for all finite dimensional representations (τ, W) of K the multiplicity space $\mathrm{Hom}_K(W, E)$ is finite dimensional.

We call a representation (π, E) of G *weakly admissible* provided $E^{K-\mathrm{fin}}$ is a weakly admissible K-module.

By a $(\mathfrak{g}, K)$-module V we understand a module for $\mathfrak{g}$ and K such that:

- The derived action of K coincides with the action of $\mathfrak{g}$ restricted to $\mathfrak{k} := \mathrm{Lie} K$.
- The actions are compatible, i.e.

$$k \cdot X \cdot v = \mathrm{Ad}(k)X \cdot k \cdot v$$

  for all $k \in K$, $X \in \mathfrak{g}$ and $v \in V$.

Remark 4.1. *(a) If (π, E) is a weakly admissible Banach representation of G, then $E^{K-\mathrm{fin}}$ consists of smooth vectors and is stable under $\mathfrak{g}$ – in other words $E^{K-\mathrm{fin}}$ is a weakly admissible $(\mathfrak{g}, K)$-module.*
(b) Let us emphasize that a weakly admissible $(\mathfrak{g}, K)$-module is not necessary finitely generated as a $\mathfrak{g}$-module. For example the tensor product of two representations of the holomorphic discrete series for $(\mathfrak{g}, K)$=$(\mathfrak{sl}(2,\mathbb{R}), \mathrm{SO}(2,\mathbb{R}))$ is admissible but not finitely generated as a $\mathfrak{g}$-module.

Let us denote by $\mathcal{Z}(\mathfrak{g})$ the center of $\mathcal{U}(\mathfrak{g})$ and by $\mathrm{spec}(\mathcal{Z}(\mathfrak{g}))$ its spectrum, i.e. the set of all algebra characters $\mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$. A $\mathcal{U}(\mathfrak{g})$-module V will be called $\mathcal{Z}(\mathfrak{g})$-finite provided there exists an ideal $I \triangleleft \mathcal{Z}(\mathfrak{g})$ of finite codimension which annihilates V, or, equivalently, provided there exists $\chi_1, \ldots, \chi_n \in \mathrm{spec}(\mathcal{Z}(\mathfrak{g}))$ and $d \in \mathbb{N}$ such that $\prod_{j=1}^{n}(\chi_j(z) - z)^d v = 0$ for all $v \in V$ and $z \in \mathcal{Z}(\mathfrak{g})$.

We denote by $\mathrm{Irr}(\mathfrak{g}, K)$ be the set of equivalence classes of irreducible $(\mathfrak{g}, K)$-modules. As customary we will not distinguish between equivalence classes and their representatives. As every $V \in \mathrm{Irr}(\mathfrak{g}, K)$ is of countable dimension, Dixmier's version of Schur's Lemma is applicable (see [17], 0.5.1 and 0.5.2) and associates to V an *infinitesimal character* $\chi_V \in \mathrm{spec}(\mathcal{Z}(\mathfrak{g}))$.

Harish-Chandra study of distributional characters of irreducible admissible representations led him to the following fundamental result:

Theorem 4.2. (Harish-Chandra)

(i) *Every $V \in \mathrm{Irr}(\mathfrak{g}, K)$ is weakly admissible.*
(ii) *The map*

$$\mathrm{Irr}(\mathfrak{g}, K) \to \mathrm{spec}(\mathcal{Z}(\mathfrak{g})), \quad V \mapsto \chi_V$$

has finite fibers.

Harish-Chandra's theorem allows us to characterize finitely generated weakly admissible modules in various useful ways:

Theorem 4.3. *For a weakly admissible $(\mathfrak{g}, K)$-module V the following assertions are equivalent:*

(i) *V is finitely generated as a $\mathfrak{g}$-module.*
(ii) *V is $\mathcal{Z}(\mathfrak{g})$-finite.*
(iii) *V is finitely generated as an $\mathfrak{n}$-module, where $\mathfrak{n}$ is a maximal unipotent subalgebra of $\mathfrak{g}$.*

Proof. (i)$\Rightarrow$(ii) follows from the fact that we can take generators of V belonging to K-types and the fact that $\mathcal{Z}(\mathfrak{g})$ preserves K-types. Harish-Chandra's Theorem 4.2 implies (ii)$\Rightarrow$(i).

The implication (iii)⇒(i) is clear. Finally, a result of Osborne ([17], Prop. 3.7) asserts that

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}) F \mathcal{Z}(\mathfrak{g}) \mathcal{U}(\mathfrak{k})$$

for a finite dimensional subspace $F \subset \mathcal{U}(\mathfrak{g})$. Thus (i) and (ii) together imply (iii). □

A $(\mathfrak{g}, K)$-module V will be called *Harish-Chandra module* or *admissible* $(\mathfrak{g}, K)$*-module* if the conditions in the theorem above are satisfied. Likewise, we will call a smooth Fréchet representation (π, E) *admissible* provided the underlying $(\mathfrak{g}, K)$-module $E^{K-\mathrm{fin}}$ is admissible.

Harish-Chandra modules form a category $\mathcal{HC}$ with morphisms the linear $(\mathfrak{g}, K)$-maps. This category has a natural duality structure which we are going to describe now. If V is a Harish-Chandra module, then we denote by V^* its algebraic dual and by $\widetilde{V} \subset V^*$ the K-finite vectors in V^*. Note that $\widetilde{V}$ is a $\mathfrak{g}$-submodule of V^*. As V is weakly admissible the same holds for $\widetilde{V}$ and we readily obtain that

$$\widetilde{\widetilde{V}} = V\,.$$

As V is $\mathcal{Z}(\mathfrak{g})$-finite, the same holds for $\widetilde{V}$ and thus $\widetilde{V}$ is again a Harish-Chandra module by Theorem 4.3 above. We refer to $\widetilde{V}$ as the *Harish-Chandra module dual to* V.

Another basic feature of Harish-Chandra modules is the Casselman embedding theorem which asserts that every Harish-Chandra module can be embedded into the K-finite vectors of some minimal principal series representation ([5]). To be more specific, let us fix an Iwasawa decomposition $G = NAK$ and write $P_{\min} = MAN$ for the associated minimal parabolic subgroup. Here $M := Z_K(A)$ is the centralizer of A in K. For a finite dimensional $P_{\min}$-module W we denote by

$$I^\infty(W) := C^\infty(W \times_{P_{\min}} G)$$

the smooth sections of the G-equivariant vector-bundle $W \times_{P_{\min}} G \to P_{\min}\backslash G$, that is the smooth functions $f : G \to W$ which satisfy $f(pg) = p \cdot f(g)$ for all $p \in P_{\min}$ and $g \in G$. We topologize $I^\infty(W)$ with the natural Fréchet topology of compact convergence of all derivatives. Note that $I^\infty(W)$ becomes an admissible SF-module for G under right-displacements in the arguments, i.e. the prescription

$$R(g)f(x) := f(xg) \qquad (g, x \in G, f \in I^\infty(W))$$

gives rise to an admissible SF-representation $(R, I^\infty(W))$. The corresponding $(\mathfrak{g}, K)$-module

$$I(W) := I^\infty(W)^{K-\mathrm{fin}}$$

is a Harish-Chandra module.

Theorem 4.4. (Casselman) *For every Harish-Chandra module V there exists a finite dimensional $P_{\min}$-module W and a $(\mathfrak{g}, K)$-embedding $V \to I(W)$.*

Proof. [17], Cor. 4.2.4. □

Let us remark that the restriction morphism

$$\mathrm{Res}_K : I^\infty(W) \to C^\infty(W \times_M K), \quad f \mapsto f|_K$$

is a K-equivariant isomorphism of Fréchet spaces. The image $C^\infty(W \times_M K)$ is often more convenient to work with as K is a compact manifold. It is important to note that the corresponding G-representation $(\pi, C^\infty(W \times_M K))$ defined by $\pi(g) := \mathrm{Res}_K \circ R(g) \circ (\mathrm{Res}_K)^{-1}$ completes to a Hilbert representation $(\pi, L^2(W \times_M K))$.

As a K-module $I(W)$ is isomorphic to $\mathbb{C}[W \times_M K]$ and this allows us to give a polynomial bound on the K-multiplicities of a Harish-Chandra module. More specifically let us denote by $\widehat{K}$ the set of equivalence classes of irreducible unitary representations of K. We will identify an equivalence class $[\tau] \in \widehat{K}$ with a representative τ. If V is a K-module, then we denote by $V[\tau]$ its τ-isotypical part. Similarly we denote for $v \in V$ by v_τ its τ-isotypical component.

Let $\mathfrak{t}$ be the Lie algebra of a maximal torus of K. We often identify τ with its highest weight in $i\mathfrak{t}^*$ (with respect to a fixed positive system). In particular, $|\tau| \geq 0$ will refer to the Cartan-Killing norm of the highest weight τ. As a consequence of Theorem 4.4 we obtain Harish-Chandra's multiplicity bound.

Theorem 4.5. (Harish-Chandra) *Let V be a Harish-Chandra module. Then there exists a $C > 0$ such that*

$$\dim V[\tau] \leq C(1 + |\tau|)^{\dim K - \dim M} \qquad (\tau \in \widehat{K}).$$

5. Harish-Chandra modules II: topological properties

This section is devoted to topological properties of representations (π, E) whose K-finite vectors form a Harish-Chandra module.

5.1. **Definition and existence of globalizations**

Given a Harish-Chandra module V we say that a representation (π, E) of G is a *globalization* of V provided the K-finite vectors $E^{K-\mathrm{fin}}$ of E are smooth and isomorphic to V as a $(\mathfrak{g}, K)$-module.

Every Harish-Chandra module V admits a Hilbert globalization $\mathcal{H}$ (and hence an SF-globalization by taking the smooth vectors in $\mathcal{H}$). In fact, by Theorem 4.4, we can embed V into some minimal principal series $I(W)$ and the closure of V in $L^2(W \times_M K)$ defines a Hilbert globalization $\mathcal{H}$ of V. Note that $\mathcal{H}^\infty$ coincide with the closure of V in $I^\infty(W)$.

Remark 5.1. *We caution the reader that there exist irreducible Banach representation (π, E) of G which are not admissible* [14], *i.e. they are not globalizations of Harish-Chandra modules. However, if $(\pi, \mathcal{H})$ happens to be a unitary irreducible representation, then Harish-Chandra has shown that π is admissible.*

Remark 5.2. *Let V be a Harish-Chandra module and (π, E) a Banach globalization. Then*

$$\Pi(\mathcal{R}(G))V = \Pi(\mathcal{S}(G))V \, .$$

Indeed, by a basic result of Harish-Chandra there exists for each $v \in V$ a $K \times K$-finite $h \in C_c^\infty(G)$ such that $\Pi(h)v = v$. As $\mathcal{R}(G) * C_c^\infty(G) \subset \mathcal{S}(G)$ the asserted equality is established.

5.2. **The contragredient of a Banach-globalization**

In the introductory section we saw that the dual of a Banach representation (π, E) might not be continuous which brought us to the notion of contragredient representation $(\widetilde{\pi}, \widetilde{E})$. Recall that the continuous dual $\widetilde{E}$ was the largest closed subspace of E^* on which the dual action is continuous. If E happens to be reflexive then $\widetilde{E} = E^*$

The case where (π, E) is a Banach globalization of a Harish-Chandra module V is of particular interest to us. Here the situation is as follows:

Lemma 5.3. *Let V be a Harish-Chandra module and $\widetilde{V}$ its dual. If (π, E) is a Banach globalization of V, then $\widetilde{V} \subset \widetilde{E}$. In particular, $(\widetilde{\pi}, \widetilde{E})$ is a Banach globalization of $\widetilde{V}$.*

Proof. Let (π, E) be a Banach globalization of V. For a K-type $\tau \in \widehat{K}$ let us consider the projection

$$\mathrm{pr}_\tau : E \to E[\tau] = V[\tau], \quad v \mapsto \dim(\tau) \int_K \overline{\chi_\tau(k)} \pi(k) v \; dk$$

on the τ-isotypical component. Here χ_τ refers to the character of τ. As pr_τ is continuous, the first assertion $\widetilde{V} \subset \widetilde{E}$ follows. Further, a K-type of $\widetilde{E}$ does not vanish on V as V is dense in E. Thus the K-finite vectors of $\widetilde{E}$ are contained in $\widetilde{V}$. □

5.3. **Nuclear structures on smooth vectors**

The goal of this section is to prove that the smooth vectors of a Banach-globalization carry the structure of a nuclear Fréchet space.

It is convenient to introduce some useful notation in this regard. Let V be a Harish-Chandra module and p be a norm on V. Then we say that p is a G-continuous norm on V provided the completion of V with respect to p gives rise to a Banach representation of G.

We introduce a preorder on the set of G-continuous norms on a Harish-Chandra module V: in symbols

$$p \prec q :\Longleftrightarrow\ (\exists k \in \mathbb{N}_0, C > 0)\ p(v) \leq C q_k(v) \qquad (v \in V)\,.$$

We say that p and q are *Sobolev-equivalent*, in symbols $p \asymp q$, provided $p \prec q$ and $q \prec p$.

To begin with we recall a familiar result from convex analysis (John's Theorem, see [12], Th. 3.3).

Lemma 5.4. *Let $(V, \|\cdot\|)$ be a finite dimensional normed vector space of dimension n. Then there exists $\lambda_1, \ldots, \lambda_n \in V^*$ with $\|\lambda_i\| = 1$, $1 \leq i \leq n$, such that the associated Hermitian form*

$$Q(x) := \sum_{j=1}^{n} |\lambda_j(x)|^2$$

satisfies

$$\|\cdot\|^2 \leq Q \leq 2n\|\cdot\|^2\,.$$

Theorem 5.5. *Let V be a Harish-Chandra module and p be a G-continuous norm. Then the following assertions hold:*

- (i) *There exists a G-continuous Hilbert-norm q such that p is Sobolev equivalent to q.*
- (ii) *There exits a $k \in \mathbb{N}$ such that inclusions $(V, p_k) \to (V, q)$ and $(V, q) \to (V, p_k)$ are nuclear.*

Proof. Let p be a G-continuous norm of E. We recall from Theorem 4.5 that there is an integer $N > 0$ such that

$$m(\tau) := \dim V[\tau] \leq (1 + |\tau|)^N \qquad (\tau \in \widehat{K})\,. \tag{5.1}$$

According to Lemma 5.4 we find for each $\tau \in \widehat{K}$ a basis $\lambda_1^\tau, \ldots, \lambda_{m(\tau)}^\tau$ of $V[\tau]^*$ such that the Hermitian form

$$Q_\tau(v) := \sum_{j=1}^{m(\tau)} |\lambda_j(v)| \qquad (v \in V[\tau])$$

satisfies

$$p^2(v) \leq Q_\tau(v) \leq 2m(\tau)p^2(v) \qquad (v \in V[\tau])\,. \tag{5.2}$$

Define a Hermitian form Q on V by

$$Q(v) := \sum_{\tau \in \widehat{K}} Q_\tau(v_\tau)$$

and let $\widetilde{q}(v)$ be the associated norm. Let now $k \in 2\mathbb{N}$ and $\widetilde{q}_{k,K}$ be the k-th K-Sobolev norm of q. For $\tau \in \widehat{K}$ we set

$$\|\tau\|_k := \Big(\sum_{j=0}^{k/2} \|\tau\|^{2j}\Big)^{\frac{1}{2}}$$

and note that $\|\tau\|_k \asymp \|\tau\|^k$. For all $v \in V[\tau]$ we have

$$\widetilde{q}_{k,K}(v) = \|\tau\|_k \widetilde{q}(v)\,.$$

Since $p(v) \leq \sum_{\tau \in \widehat{K}} p(v_\tau)$, we obtain for sufficiently large k that

$$\begin{aligned} p(v) &\leq \sum_{\tau \in \widehat{K}} \widetilde{q}(v_\tau) = \sum_{\tau \in \widehat{K}} \frac{1}{\|\tau\|_k} \|\tau\|_k \ \widetilde{q}(v_\tau) \\ &\leq \Big[\sum_{\tau \in \widehat{K}} \frac{1}{\|\tau\|_k^2}\Big]^{\frac{1}{2}} \cdot \Big[\sum_{\tau \in \widehat{K}} \|\tau\|_k^2 Q_\tau(v_\tau)\Big]^{\frac{1}{2}} \\ &\leq C\widetilde{q}_{k,K}(v) \end{aligned}$$

The second inequality in (5.2) combined with the multiplicity bound (5.1) yields a constant $C > 0$ such that

$$\widetilde{q}(v) \leq Cp_{k,K}(v) \qquad (v \in V)$$

provided $k \in \mathbb{N}$ is taken large enough.

As $\widetilde{q}$ might not be G-continuous, we have to address this issue. First note that $p_{K,k} \leq Cp_k$ and that p_k is G-continuous. Thus for sufficiently large $c > 0$ the prescription

$$q(v) := \Big(\int_G \widetilde{q}(\pi(g)v)^2 e^{-cd(g)}\ dg\Big)^{\frac{1}{2}}$$

defines a G-continuous Hilbert norm with $q \leq Cp_k$. Local Sobolev on the other hand readily yields that $\widetilde{q} \leq Cq_k$ for $k \in \mathbb{N}$ sufficiently big. This completes the proof of the theorem. $\square$

Corollary 5.6. *Let (π, E) be an SF-globalization of a Harish-Chandra module V. Then:*

(i) *E is a nuclear Fréchet space.*
(ii) *The topology on E is determined by a countable family of G-continuous K-invariant Hilbert semi-norms.*

In view of Theorem 5.5 it is no loss of generality to assume that a G-continuous norm on a Harish-Chandra module to be Hermitian. In addition we will request that all norms are K-invariant.

5.3.1. *Weighted function spaces.* This subsection is about natural realizations of Harish-Chandra modules in weighted function spaces on G.

For $m \geq 0$ we define the weighted Banach-space

$$C(G)_m := \left\{ f \in C(G) \mid p^m(f) := \sup_{g \in G} \frac{|f(g)|}{\|g\|^m} < \infty \right\}.$$

We view $C(G)_m$ as a module for G under the right regular action R. Note that this action might not be continuous in general (take $m = 0$ and G not compact). From the properties of the norm one readily shows that

$$p^m(R(g)v) \leq \|g\|^m \cdot p^m(v)$$

for all $g \in G$. Thus the action is locally equicontinuous. It follows that the smooth vectors for this action $C(G)_m^\infty$ define an SF-module for G. Note that $C(G)_m^\infty \subset C^\infty(G)$ as a consequence of the local Sobolev-Lemma.

Likewise we associate to $m \geq 0$ the weighted Hilbert-space

$$L^2(G)_m := L^2(G, \|g\|^{-m} dg).$$

Note that the right regular action of G on $L^2(G)_m$ defines a Hilbert representation of G. Let us denote by h^m the corresponding Hilbert norm.

Let $k_0 > 0$ be such that $\int_G \|g\|^{-k_0} \, dg < \infty$. Then for all $m \in \mathbb{R}$ one obtains a continuous embedding

$$C(G)_{(m-k_0)/2} \to L^2(G)_m \tag{5.3}$$

or to phrase it equivalently that there exists a constant $C > 0$ such that

$$h^m \leq C \cdot p^{(m-k_0)/2}\,. \tag{5.4}$$

To obtain inequalities of the reverse kind we shall employ the Sobolev Lemma on G. It is not hard to show that the derivatives of the norm function $\|\cdot\|$ are bounded by a multiple of $\|\cdot\|$. Hence we obtain constants $C > 0$ and $l_0 \in \mathbb{N}$ with l_0 independent from m such that

$$p^m \leq C \cdot h_{l_0}^{2m} \tag{5.5}$$

holds on $L^2(G)_m^\infty$.

If V is a Harish-Chandra module, then we denote by $\Xi \subset V$ a $\mathcal{Z}(\mathfrak{g})$-invariant set of generators of minimal dimension, say k.

Let V be a Harish-Chandra module V and $\Xi \subset \widetilde{V}$ a $\mathcal{Z}(\mathfrak{g})$-stable set of generators as above. We fix an inner product on Ξ and let $\xi_1, \ldots \xi_k$ be an orthonormal basis of Ξ. The inner product on Ξ yields an inner product on the dual space Ξ^*. Attached to Ξ we consider the G-equivariant embedding

$$\phi_\Xi : V^\infty \to C^\infty(G) \otimes \Xi^* = C^\infty(G, \Xi^*); \phi_\Xi(v)(g)(\xi) := m_{\xi,v}(g)$$

with $\xi \in \Xi$ and $m_{\xi,v}(g) = \xi(\pi(g)v)$ the corresponding matrix coefficient.

We claim that $\operatorname{im} \phi_\Xi$ lies in some $C_m(G)^\infty \otimes \Xi^*$ for m suitably large. In fact choose a Banach globalization (π, E) of V with norm q. Then

$$\max_{1\leq j\leq k} |m_{\xi_j,v}(g)| \leq C\|g\|^N q(v) \qquad (v \in V^\infty)$$

for suitable constants N and $C > 0$. Hence $\operatorname{im} \phi_\Xi \subset C_N(G)^\infty \otimes \Xi^*$.

Let now E_N be the closure of $\phi_\Xi(V^\infty)$ in $C_N(G) \otimes \Xi^*$.

Lemma 5.7. *With the notation from above, E_N defines a Banach globalization of V.*

Proof. It is clear that E_N is a Banach space. With regard to the norm on E_N the operators $\pi(g)$ are bounded by $\|g\|^N$. Hence the action is locally equicontinuous. Further, as p_N is dominated by q on V^∞ we conclude that all orbit maps $\gamma_v : G \to E_N$ are continuous. Thus $G \times E_N \to E_N$ is a representation by Lemma 2.3. □

From our construction it is clear that the smooth vectors E_N^∞ for E_N coincide with the SF-closure $\phi_\Xi(V^\infty)$ in $C_N(G)^\infty \otimes \Xi^*$. Let us denote the restriction of p^N to E_N by the same symbol.

Set $N' := 2N + k_0$. Then there is a natural G-equivariant embedding

$$\psi_\Xi : V^\infty \to L^2(G)_{N'} \otimes \Xi^*;\ \psi_\Xi(v)(g)(\xi) = m_{\xi,v}(g)$$

and the closure of the image defines a Hilbert globalization $\mathcal{H}_{N'}$ of V.

5.3.2. *The dual of an SF-globalization.* The material in this subsection is not needed in the sequel of this article. However it contains a fact worthwhile which is worthwhile to mention and thematically fits in our discussion.

Let (π, E) be an SF-globalization of a Harish-Chandra module V. In Corollary 5.6 we have shown that E is a nuclear Fréchet space. As nuclear Fréchet spaces are reflexive, it follows that the dual representation (π^*, E^*) of (π, E) exists and that the bi-dual representation (π^{**}, E^{**}) is naturally isomorphic to (π, E).

For any SF-representation (π, E) we recall that the natural action of $\mathcal{S}(G)$,

$$\mathcal{S}(G) \times E \to E, \quad (f, v) \mapsto \Pi(f)v$$

is continuous, i.e. a continuous bilinear map (Proposition 2.20).

On the dual side we obtain a dual action

$$\mathcal{S}(G) \times E^* \to E^*, \quad (f, \lambda) \mapsto \Pi^*(f)\lambda;\ \Pi^*(f)\lambda = \lambda \circ \Pi(f^*)$$

where $f^*(g) = f(g^{-1})$. For a general SF-representation on a reflexive Fréchet space the dual dual action Π^* of $\mathcal{S}(G)$ might not be continuous. For globalizations however matters behave well and we record:

Lemma 5.8. *Let (π, E) be an SF-globalization of a Harish-Chandra module V. Then the dual algebra action $\mathcal{S}(G) \times E^* \to E^*$ is continuous.*

Proof. We recall from Corollary 5.6 that $(\pi, E) = \varprojlim (\pi_n, E_n)$ is a projective limit of Hilbert representations (π_n, E_n). Thus $(\pi^*, E^*) = \varinjlim (\pi_n^*, E_n^*)$ is a direct limit of Hilbert representation. As for all $n \in \mathbb{N}$ the dual action $\mathcal{S}(G) \times E_n^* \to E_n^*$ is continuous, the assertion follows. □

6. Minimal and maximal SF-globalizations of Harish-Chandra modules

Let us introduce a preliminary notion and call a Harish-Chandra module V *good* if it admits a unique SF-globalization. Eventually it will turn out that all Harish-Chandra modules are good (Casselman-Wallach).

As we will see below there are two natural extremal SF-globalizations of a Harish-Chandra module, namely *minimal* and *maximal* SF-globalizations. Eventually they will coincide but they are useful objects towards a proof of the Casselman-Wallach theorem.

6.1. **Minimal globalizations**

An SF-globalization, say V^∞, of an Harish-Chandra module V will be called *minimal* if the following universal property holds: if (π, E) is an SF-globalization of V, then there exists a continuous G-equivariant map $V^\infty \to E$ which extends the identity morphism $V \to V$.

It is clear that minimal globalizations are unique. Let us show that they actually exist. We need to collect some facts about matrix coefficients. In this context we record the following result (see [5]):

Lemma 6.1. *Let (π, E) be a Banach globalization of a Harish-Chandra module V. Then for all $\xi \in \widetilde{V} \subset E^*$ and $v \in V$, the matrix coefficient*

$$m_{\xi,v}(g) = \xi(\pi(g)v) \qquad (g \in G)$$

is an analytic function on G. In particular $m_{\xi,v}$ is independent of the particular Banach globalization (π, E) of V.

We will now give the construction of the minimal globalization of a Harish-Chandra module V. For that let us fix a Banach globalization (π, E) of V. Let $\mathbf{v} = \{v_1, \ldots, v_k\}$ be a set of generators of V and consider the map

$$\mathcal{S}(G)^k \to E, \quad \mathbf{f} = (f_1, \ldots, f_k) \mapsto \sum_{j=1}^{k} \Pi(f_j)v_j \, .$$

This map is linear, continuous and G-equivariant (with $\mathcal{S}(G)^k$ considered as a module for G under the left regular representation). Let us write

$$\mathcal{S}(G)_{\mathbf{v}} := \{\mathbf{f} \in \mathcal{S}(G)^k \mid \sum_{j=1}^{k} \Pi(f_j)v_j = 0\}$$

for the kernel of this linear map. Note that $\mathcal{S}(G)_{\mathbf{v}}$ is a closed G-submodule of $\mathcal{S}(G)^k$. We claim that $\mathcal{S}(G)_{\mathbf{v}}$ is independent of the choice of the particular globalization (π, E) of V: In fact, for $v \in V$ and $f \in \mathcal{S}(G)$ we have $\Pi(f)v = 0$ if and only if $\xi(\Pi(f)v) = 0$ for all $\xi \in \widetilde{V}$. As $g \mapsto m_{\xi,v}(g) = \xi(\pi(g)v)$ is analytic and hence independent of π (Lemma 6.1), the claim follows.

Lemma 2.9 shows that $\mathcal{S}(G)^k/\mathcal{S}(G)_{\mathbf{v}}$ is an SF-module for G. Since $\Pi(\mathcal{S}(G)^{K\times K})V = V$ for $\mathcal{S}(G)^{K\times K}$ the $K\times K$-finite functions of $\mathcal{S}(G)$, it follows that $\mathcal{S}(G)^k/\mathcal{S}(G)_{\mathbf{v}}$ is an SF-globalization of V. By construction $\mathcal{S}(G)/\mathcal{S}(G)_{\mathbf{v}}$ is the minimal globalization V^∞.

We record the following general Lemma on quotients of Harish-Chandra modules in relation to minimal globalizations.

Lemma 6.2. *Let V be a Harish-Chandra module and V^∞ its unique minimal SF-globalization. Let $W \subset V$ be a submodule and $U := V/W$. Let $\overline{W}$ be the closure of W in V^∞. Then $U^\infty = V^\infty/\overline{W}$.*

Proof. Let us write $(\pi_U, V^\infty/\overline{W})$ for the quotient representation obtained from (π, V^∞). Then $\Pi(\mathcal{S}(G))V = V^\infty$ implies that $\Pi_U(\mathcal{S}(G))U = V^\infty/\overline{W}$ and hence the assertion. □

Remark 6.3. *Suppose that a Harish-Chandra module V admits a maximal G-continuous norm p with respect to our Sobolev ordering $\prec$. Then V^∞ coincides with the smooth vectors of the Banach completion of (V,p). However, we want to emphasize that the existence of a minimal globalization does not automatically imply the existence of a maximal norm.*

6.2. **Dual norms**

Let q be a G-continuous Hilbert norm on a Harish-Chandra module V. The associated Sobolev norms $(q_n)_{n\in\mathbb{N}}$ induce an SF-structure on V.

Recall the notion q^* of dual norm. Our discussion from Subsection 3.4, then readily yields:

Lemma 6.4. *Let q and p be G-continuous Hermitian norms on a Harish-Chandra module V. Then $q \asymp p$ if and only if $q^* \asymp p^*$.*

6.3. **Maximal Globalizations**

Let us call an SF-globalization of V, say $V^\infty_{\max}$, *maximal* if for any SF-globalization (π, E) of V there exists a continuous linear G-map $E \to V^\infty_{\max}$ sitting above the identity morphism $V \to V$.

It is clear that maximal globalizations are unique provided that they exist. Moreover, in case a maximal globalizations of a Harish-Chandra module V exists, then V is good if and only if $V^\infty = V^\infty_{\max}$.

Let us emphasize that a maximal globalization of V exists if and only if there exists a G-continuous Hilbert norm q such that $q \prec p$ for all G-continuous norms p on V. Since a module V is good if and only if $p \asymp q$ for all G-continuous norms p and q on V, we obtain from Lemma 6.4 that:

Lemma 6.5. *A Harish-Chandra module V is good if and only if its dual $\widetilde{V}$ is good.*

With the elementary tools developed so far we cannot give a construction of maximal globalizations, but we would like to emphasize that the existence of maximal globalization would be implied by the matrix coefficient bounds established in [17], Sect 4. For us the following criterion will be sufficient:

Lemma 6.6. *Let U be a good Harish-Chandra module and U^∞ its unique SF-globalization. Let $V \subset U$ be a submodule and let $\overline{V}$ be the closure of V in U^∞. Then $V^\infty_{\max} = \overline{V}$.*

Proof. Let p be a G-continuous Hilbert norm on V. Further let $\widetilde{q}$ be a G-continuous Hilbert norm on U and $q := \widetilde{q}|_V$. We have to show that $q \prec p$. Let $\pi : \widetilde{U} \to \widetilde{V}$ be the map dual to the inclusion $V \to U$. As $\widetilde{U}$ is good we get that $p^* \prec \widetilde{q}^* \circ \pi$. It follows that $q \prec p$. □

We conclude this paragraph with an observation which will be frequently used later on.

Lemma 6.7. *Let $V_1 \subset V_2 \subset V_3$ be an inclusion chain of Harish-Chandra modules. Suppose that V_2 and and V_3/V_1 are good. Then V_2/V_1 is good.*

Proof. Let $\overline{V_3}$ be an SF-globalization of V_3. Let $\overline{V}_1$, $\overline{V_2}$ be the closures of $V_{1,2}$ in $\overline{V}_3$. As V_2 is good we have $\overline{V_2} = V_2^\infty$ and thus Lemma 6.2 implies that $\overline{V_2}/\overline{V_1} = (V_2/V_1)^\infty$. Our second assumption gives $(V_3/V_1)^\infty = \overline{V_3}/\overline{V_1}$ and Lemma 6.6 yields in addition that $\overline{V_2}/\overline{V_1} = (V_2/V_1)^\infty_{\max}$. □

7. Lower bounds for matrix coefficients

The objective of this section is to show that Harish-Chandra modules are good if and only if they feature certain lower bounds for matrix coefficients which are uniform in the K-type.

As before, given a Harish-Chandra module V we fix a finite dimensional $\mathcal{Z}(\mathfrak{g})$-invariant set of generators $\Xi \subset \widetilde{V}$. We let $\xi_1, \ldots, \xi_k$ be a basis of Ξ. For $r > 0$ we define balls in G by

$$B_r := \{g \in G \mid \|g\| < r\}.$$

Set $r_0 := \min\{\|g\| \mid g \in G\} \geq 1$.

Theorem 7.1. *Let V be a Harish-Chandra module. Then V is good if and only if for all G-continuous norms q on V^∞ there exists a choice of $\Xi \subset \widetilde{V}$ and constants $c_1, c_2, c_3 > 0$ such that*

$$\Big(\sum_{j=1}^k \int_{B_r} |\xi_j(\pi(g)v)|^2 \, dg\Big)^{\frac{1}{2}} \geq \frac{c_2}{(1+|\tau|)^{c_3}} \cdot q(v)$$

for all $\tau \in \widehat{K}$, $v \in V[\tau]$ and $r > \max\{r_0, (1+|\tau|)^{c_1}\}$

In view of the local Sobolev Lemma this is equivalent to the following pointwise version.

Theorem 7.2. *Let V be a Harish-Chandra module. Then V is good if and only if for all G-continuous norms q on V^∞ there exists a choice of Ξ and constants $c_1, c_2, c_3 > 0$ such that for all $\tau \in \widehat{K}$ and $v \in V[\tau]$ there exist a $g_\tau \in G$ such that $\|g_\tau\| \leq (1+|\tau|)^{c_1}$ and*

$$\max_{1\leq j\leq k} |\xi_j(\pi(g_\tau)v)| \geq \frac{c_2}{(1+|\tau|)^{c_3}} \cdot q(v)\,.$$

Proof. [2] Suppose that V is good. We shall establish the pointwise lower bound in Theorem 7.2. By assumption there exists an $n \in \mathbb{N}$ and $C > 0$ such that

$$|\xi_j(\pi(g)v)| \leq C \cdot \|g\|^n q(v)$$

for all $v \in V^\infty$, $g \in G$ and $1 \leq j \leq k$. For $N \geq n$ we write E_N for the Banach completion of V^∞ with respect to the norm

$$p^N(v) := \max_{1\leq j\leq k} \sup_{g\in G} \frac{|\xi_j(\pi(g)v)|}{\|g\|^N} \qquad (v \in V)\,.$$

We recall that E_N is a Banach module for G (cf. Lemma 5.7).

As V is good, we obtain that

$$V^\infty = E_N^\infty = E_{N'}^\infty \tag{7.1}$$

for all $N, N' \geq n$. Now fix N and let $N' = N + l > N$. In view of Proposition 3.9 there exists an $s \in 2\mathbb{N}_0$ and $C > 0$ such that

$$p^N(v) \leq C \cdot p_{s,K}^{N'}(v) \tag{7.2}$$

for all $v \in V^\infty$.

Let us fix $\tau \in \widehat{K}$, $v \in V[\tau]$ and $g_\tau \in G$ such that $g \mapsto \max_{1\leq j\leq k} \frac{|\xi_j(\pi(g)v)|}{\|g\|^{N'}}$ becomes maximal at g_τ. We then derive from (7.2) that

[2] Throughout this paper we use the convention that capital constants $C > 0$ might vary from line to line

$$\max_{1\leq j\leq k}\frac{|\xi_j(\pi(g_\tau)v)|}{\|g_\tau\|^N}\leq C\cdot(1+\|\tau\|)^s\cdot\max_{1\leq j\leq k}\frac{|\xi_j(\pi(g_\tau)v)|}{\|g_\tau\|^{N+l}}$$

for all $v\in V[\tau]$, i.e.

$$\|g_\tau\|\leq C\cdot(1+\|\tau\|)^{\frac{s}{l}}\,.$$

Here $\rho_{\mathfrak{k}}\in i\mathfrak{t}^*$ is the usual half sum $\rho_{\mathfrak{k}}=\frac{1}{2}\operatorname{tr}\operatorname{ad}_{\mathfrak{k}}$.

On the other hand (7.1) combined with Proposition 3.9 implies likewise that there exists $C>0$ and $s'>0$ such that

$$q(v)\leq C\cdot p_{s',K}^{N'}(v)$$

for all $v\in V^\infty$. For $v\in V[\tau]$ we then get

$$|\xi(\pi(g_\tau)v)|\geq\frac{C\cdot\|g_\tau\|^{N'}}{(1+\|\tau\|)^{s'}}\cdot q(v)\,.$$

As $\|g\|\geq 1$ for all $g\in G$, the asserted lower bound is established.

Assume now that the lower bound in Theorem 7.1 holds true. Let $N>0$ be large enough so that $m_{\xi_j,v}$ is square integrable with respect to $\frac{dg}{\|g\|^N}$ and define a Hermitian norm by

$$p(v)^2:=\sum_{j=1}^k\int_G|\xi_j(\pi(g)v)|^2\,\frac{dg}{\|g\|^N}\,.$$

In view of the lower bound we readily obtain that $q\prec p$. On the other hand, for large enough N we have that $p\prec q$ and the proof is complete. □

8. Minimal principal series representations

This section is devoted to the study of minimal principal series representation of G and contains one of our main results. In particular we will show that all modules of the minimal principal series are good.

Recall the minimal parabolic subgroup $P_{\min}=MAN$ of G. For a finite dimensional $P_{\min}$-module W we considered the corresponding induced module $V:=I(W)$ with its canonical Hilbert globalization $\mathcal{H}:=L^2(W\times_M K)$. Note that $\mathcal{H}^\infty=C^\infty(W\times_M K)$. In the sequel $\|\cdot\|$ will refer to the L^2-norm on $\mathcal{H}$. We now state one of the main theorems of this paper:

Theorem 8.1. *Let $V = I(W)$ be a minimal principal series representation of G and $\mathcal{H} = L^2(W \times_{P_{\min}} K)$ its canonical Hilbert globalization. Let $\xi_1, \dots, \xi_k$ be a set of generators of V.*

(i) *Then there exists constants, $c_1, c_2, C_1, C_2 > 0$ such that for all $\tau \in \widehat{K}$ and $v_\tau \in V[\tau]$ there exists functions $f_{\tau,1}, \dots, f_{\tau,k} \in C_c^\infty(G)$ with the following properties:*
 (a) $\sum_{j=1}^k \Pi(f_{\tau,j})\xi_j = v_\tau$.
 (b) $\operatorname{supp}(f_{\tau,j}) \subset \{g \in G \mid \|g\| < C_1(1 + \|\tau\|)^{c_1}\}$, *for all* $1 \leq j \leq k$.
 (c) $\sum_{j=1}^k \|f_{\tau,j}\|_1 \leq C_2 \cdot \|v_\tau\| \cdot (1 + \|\tau\|)^{c_2}$, *where $\|\cdot\|_1$ refers to the norm in $L^1(G)$.*

(ii) *One has $C^\infty(W \times_K W) = V^\infty$ and the surjection*

$$\mathcal{S}(G)^k \mapsto V^\infty, (f_1, \dots, f_k) \mapsto \sum_{j=1}^k \Pi(f_j)\xi_j$$

admits a continuous linear section $V^\infty \to \mathcal{S}(G)^k$.

(iii) *V is good.*

Remark 8.2. *(a) Below we will deduce (ii) from (i). Further (iii) follows from (i) as $\widetilde{I(W)} \simeq I(W^*)$. In Appendix B we reduce assertion (i) to the case of spherical principal series representations and prove it in this case.*
(b) The constant c_1 can be made explicite. Certainly it depends on the particular norm $\|\cdot\|$ on G. Let us fix a specific $K \times K$-invariant norm, namely

$$\|a\| = \sum_{w \in W} a^{w\rho} \qquad (a \in A)$$

with $\rho \in \mathfrak{a}^*$ the Weyl half sum. Then our proof shows that any choice of c_1 with

$$\frac{1}{2}c_1 > \dim \mathfrak{a} = \operatorname{rank}_{\mathbb{R}}(G)$$

is possible.

The constant $c_2 > 0$ depends only on the growth rate of the $P_{\min}$-representation W.

In view of Casselman's embedding theorem we can embed every Harish-Chandra module V into a minimal principal series modules $I(W)$. As $I(W)$ is good by Theorem 8.1 (iii) we thus conclude from Lemma 6.6:

Corollary 8.3. *Every Harish-Chandra module V admits a maximal globalization $V_{\max}^{\infty}$. In particular, V admits a unique minimal and maximal G-continuous norm with respect to the Sobolev order $\prec$.*

Proof of Theorem 8.1 (ii). Assuming Theorem 8.1(i) we are going to establish (ii). For any $\tau \in \widehat{K}$, let $v_{\tau,1}, \ldots, v_{\tau,l(\tau)}$ be an orthonormal basis of the τ-isotypical component $\mathcal{H}[\tau] = V[\tau]$. Let $v \in V^\infty = \mathcal{H}^\infty$ be a smooth vector, that is $v = \sum_{\tau\in\widehat{K}} \sum_{j=1}^{l(\tau)} c_{\tau,j} v_{\tau,j} \in \mathcal{H}$ and for all $N > 0$ one has

$$\sum_{\tau\in\widehat{K}} \sum_{j=1}^{l(\tau)} |c_{\tau,j}|(1 + \|\tau\|)^N < \infty\,. \tag{8.1}$$

Given τ and $1 \leq j \leq l(\tau)$ we choose $f_{\tau,j,1}, \ldots, f_{\tau,j,k}$ as in (i), (a)-(c), that is $\sum_{i=1}^k \Pi(f_{\tau,j,i})\xi_i = v_{\tau,j}$ etc.

For all $1 \leq i \leq k$ we set $f_i := \sum_{\tau\in\widehat{K}} \sum_{j=1}^{l(\tau)} c_{\tau,j} f_{\tau,j,i}$. We first claim that $\int_G |f_i(g)| \cdot \|g\|^r \, dg < \infty$ for all $r > 0$. In fact,

$$\begin{aligned}\int_G |f_i(g)| \cdot \|g\|^r \, dg &\leq \sum_\tau \sum_j |c_{\tau,j}| \int_G |f_{\tau,j,i}(g)| \cdot \|g\|^r \, dg \\ &\leq \sum_{\tau,j} |c_{\tau,j}| \int_{\{\|g\| \leq C_1(1+\|\tau\|^{c_1})\}} |f_{\tau,j,i}(g)| \cdot \|g\|^r \, dg \\ &\leq \sum_{\tau,j} C_1^r (1 + \|\tau\|)^{rc_1} |c_{\tau,j}| \int_G |f_{\tau,j,i}(g)| dg \\ &\leq C_1^r C_2 \sum_{\tau,j} (1 + \|\tau\|)^{rc_1+c_2} |c_{\tau,j}|\end{aligned}$$

which is finite in view of (8.1).

Note that $\sum_{j=1}^k \Pi(f_j)\xi_j = v$. Now for each $1 \leq j \leq k$ we choose a $K \times K$-finite test function ϕ_j such that $\Pi(\phi_j)\xi_j = \xi_j$. This allows us to replace f_j by $f_j * \phi_j$ and thus we may assume that $f_j \in L^1(G, \|g\|^r dg)^\infty$ for all $r > 0$. Here "∞" refers the smooth vectors of the right regular representation of G on $L^1(G, \|g\|^r dg)$. In view of Remark 2.18 we obtain $f_j \in \mathcal{S}(G)$ as to be shown. $\square$

9. Reduction steps: extensions, tensoring and induction

In this section we will show that "good" is preserved by induction, tensoring with finite dimensional representations and as well by extensions. We would like to emphasize that these results are not new can be found, mostly we different proofs, for instance in [18], Sect. 11.7.

9.1. Extensions

Lemma 9.1. *Let*

$$0 \to U \to L \to V \to 0$$

be an exact sequence of Harish-Chandra modules. If U and V are good, then L is good.

Proof. Let $(\pi, \overline{L})$ be a smooth Fréchet globalization of L. Define a smooth Fréchet globalization $(\pi_U, \overline{U})$ of U by taking the closure of U in $\overline{L}$. Likewise we define a smooth Fréchet globalization $(\pi_V, \overline{V})$ of $V = L/U$ by $\overline{V} := \overline{L}/\overline{U}$. By assumption we have $\overline{U} = \Pi_U(\mathcal{S}(G))U$ and $\overline{V} = \Pi_V(\mathcal{S}(G))V$. As $0 \to \overline{U} \to \overline{L} \to \overline{V} \to 0$ is exact, we deduce that $\Pi(\mathcal{S}(G))L = \overline{L}$ as vector spaces. Finally the open mapping theorem implies that $\Pi(\mathcal{S}(G))L = \overline{L}$ as topological vector spaces, i.e. L is good. □

As Harish-Chandra modules admit finite composition series we conclude:

Corollary 9.2. *In order to show that all Harish-Chandra modules are good it is sufficient to establish that all irreducible Harish-Chandra modules are good.*

9.2. Tensoring with finite dimensional representations

This subsection is devoted to tensoring a Harish-Chandra module with a finite dimensional representation.

Let V be a Harish-Chandra module and V^∞ its minimal globalization. Let (σ, W) be a finite dimensional representation of G. Set

$$\mathbf{V} := V \otimes W$$

and note that $\mathbf{V}$ is a Harish-Chandra module as well. It is our goal to show that the minimal globalization of $\mathbf{V}$ is given by $V^\infty \otimes W$.

Let us fix a covariant inner product $\langle\cdot,\cdot\rangle$ on W. Let $w_1, \ldots, w_k$ be a corresponding orthonormal basis of W. With that we define the $C^\infty(G)$-valued $k\times k$-matrix

$$\mathfrak{S} := (\langle\sigma(g)w_i, w_j\rangle)_{1\leq i,j\leq k}$$

and record the following:

Lemma 9.3. *With the notation introduced above, the following assertions hold:*

(i) *The map*

$$\mathcal{S}(G)^k \to \mathcal{S}(G)^k,\ \mathbf{f} = (f_1, \ldots, f_k) \mapsto \mathfrak{S}(\mathbf{f})$$

is a linear isomorphism.

(ii) *The map*

$$[C_c^\infty(G)]^k \to [C_c^\infty(G)]^k,\ \mathbf{f} = (f_1, \ldots, f_k) \mapsto \mathfrak{S}(\mathbf{f})\,.$$

is a linear isomorphism.

Proof. First, we observe that the determinant of $\mathfrak{S}$ is 1 and hence $\mathfrak{S}$ is invertible. Second, all coefficients of $\mathfrak{S}$ and $\mathfrak{S}^{-1}$ are of moderate growth, i.e. dominated by a power of $\|g\|$. Both assertions follow. □

Lemma 9.4. *Let V be a Harish-Chandra module and (σ, W) be a finite dimensional representation of G. Then*

$$\mathbf{V}^\infty = V^\infty \otimes W\,.$$

Proof. We denote by $\pi_1 = \pi\otimes\sigma$ the tensor representation of G on $V^\infty\otimes W$. It is sufficient to show that $v\otimes w_j$ lies in $\Pi_1(\mathcal{S}(G))\mathbf{V}$ for all $v\in V^\infty$ and $1\leq j\leq k$.

Fix $v\in V^\infty$. It is no loss of generality to assume that $j=1$. By assumption we find $\xi\in V$ and $f\in\mathcal{S}(G)$ such that $\Pi(f)\xi = v$.

We use the previous lemma and obtain an $\mathbf{f} = (f_1, \ldots, f_k)\in\mathcal{S}(G)^k$ such that

$$\mathfrak{S}^t(\mathbf{f}) = (f, 0, \ldots, 0)\,.$$

We claim that

$$\sum_{j=1}^k \widetilde{\Pi}(f_j)(\xi\otimes w_j) = v\otimes w_1\,.$$

In fact, contracting the left hand side with $w_i^* = \langle\cdot, w_i\rangle$ we get that

$$(\mathrm{id} \otimes w_i^*) \left(\sum_{j=1}^{k} \Pi_1(f_j)(\xi \otimes w_j) \right) = \sum_{j=1}^{k} \int_G f_j(g) \langle \sigma(g) w_j, w_i \rangle \pi(g) \xi \; dg$$
$$= \delta_{1i} \cdot \int_G f(g) \pi(g) \xi \; dg = \delta_{1i} \cdot v$$

and the proof is complete. □

Proposition 9.5. *Let V be a good Harish-Chandra module and (σ, W) be a finite dimensional representation of G. Then $\mathbf{V} = V \otimes W$ is good.*

Proof. It is easy to see that maximal and minimal Sobolev norms (with respect to $\prec$) on V induce maximal and minimal Sobolev norms on $\mathbf{V}$. The assertion follows. □

9.3. **Induction**

Let $P \supset P_{\min}$ be a parabolic subgroup with Langlands decomposition

$$P = N_P A_P M_P \, .$$

Note that $A_P < A$, $M_P A_P = Z_G(A_P)$ and $N = N_P \rtimes (M_P \cap N)$. For computational purposes it is useful to recall that parabolics P above $P_{\min}$ are parameterized by subsets F of the simple roots Π in $\Sigma(\mathfrak{a}, \mathfrak{n})$. We then often write P_F instead of P, A_F instead of A_P etc. The correspondence $F \leftrightarrow P_F$ is such that

$$A_F = \{ a \in A \mid (\forall \alpha \in F) \; a^\alpha = 1 \} \, .$$

We make an emphasis on the two extreme cases for F, namely: $P_\emptyset = P_{\min}$ and $P_\Pi = G$.

In the sequel we write $\mathfrak{a}_P$, $\mathfrak{n}_P$ for the Lie algebras of A_P and N_P and denote by $\rho_P \in \mathfrak{a}_P^*$ the usual half sum. Note that $K_P := K \cap M_P$ is a maximal compact subgroup of M_P. Let V_σ be a Harish-Chandra module for M_P and $(\sigma, V_\sigma^\infty)$ its minimal SF-globalization.

For $\lambda \in (\mathfrak{a}_P)_\mathbb{C}^*$ we define as before the smooth principal series with parameter (σ, λ) by

$$E_{\sigma,\lambda} = \{ f \in C^\infty(G, V_\sigma^\infty)) \mid (\forall \; nam \in P \; \forall \; g \in G)$$
$$f(namg) = a^{\rho_P + \lambda} \sigma(m) f(g) \} \, .$$

and representation $\pi_{\sigma,\lambda}$ by right translations in the arguments of functions in $E_{\sigma,\lambda}$.

In this context we record:

Proposition 9.6. *Let $P \supseteq P_{\min}$ be a parabolic subgroup with Langlands decomposition $P = N_P A_P M_P$. Let V_σ be an irreducible good Harish-Chandra module for M_P. Then for all $\lambda \in (\mathfrak{a}_P)_{\mathbb{C}}^*$ the induced Harish-Chandra module $V_{\sigma,\lambda}$ is good. In particular, $V_{\sigma,\lambda}^\infty = E_{\sigma,\lambda}$.*

Proof. As $\widetilde{V}_{\sigma,\lambda} \simeq V_{\widetilde{\sigma},-\lambda}$ it is sufficient to show that $V_{\sigma,\lambda}$ is good.

In the first step we will show that $E_{\sigma,\lambda}$ is the maximal globalization of $V_{\sigma,\lambda}$. To begin with let $N^P := M_P \cap N$ and $A^P := M_P \cap A$. Then $Q := N^P A^P M$ is a minimal parabolic subgroup of M_P. As V_σ is irreducible, we find an embedding of V_σ into a minimal principal series module of M_P, say I_σ:

$$I_\sigma = \mathrm{Ind}_Q^{M_P}(\mathbf{1} \otimes (\mu + \rho^P) \otimes \gamma)$$

with $\mu \in (\mathfrak{a}^P)_{\mathbb{C}}^*$ and (γ, U_γ) an irreducible representation of M. Then $L^2(U_\gamma \times_M K_P)$ is a Hilbert globalization of I_σ and we denote by $\mathcal{H}_\sigma$ the closure of V_σ in $L^2(U_\gamma \times_M K_P)$. As V_σ is good, it follows that $V_\sigma^\infty = \mathcal{H}_\sigma^\infty$. With $\mathcal{H}_\sigma$ we obtain a Hilbert model for $V_{\sigma,\lambda}$ namely $\mathcal{H}_{\sigma,\lambda} = L^2(\mathcal{H}_\sigma \times_{K_P} K)$. Notice that the smooth vectors of $\mathcal{H}_{\sigma,\lambda}$ coincide with $E_{\sigma,\lambda}$.

We proceed with double induction (see [11], Ch. VII, §2 (4)) and obtain an embedding of $V_{\sigma,\lambda}$ into the minimal principal series representation $\mathbf{V} := V_{\gamma,\mu+\lambda}$. Let us endow $\mathbf{V}$ with the Hilbert structure induced from the compact model $\mathbf{H} = L^2(U_\gamma \times_M K)$. Observe that the embedding $\mathcal{H}_{\sigma,\lambda}$ to $\mathbf{H}$ is isometric. As $\mathbf{V}$ is good we get that the maximal globalization of $V_{\sigma,\lambda}$ is the closure of $V_{\sigma,\lambda}$ in $\mathbf{V}^\infty$. From our discussion it follows that this closure is $\mathcal{H}_{\sigma,\lambda}^\infty = E_{\sigma,\lambda}$.

To conclude the proof we need to show that $E_{\sigma,\lambda}$ coincides with the minimal globalization of $V_{\sigma,\lambda}$ as well. We proceed dually: start from the realization of V_σ as a quotient of a minimal principal series J_σ of M_P etc. As before we will end up with a Hilbert model $\widehat{H}_{\sigma,\lambda}$ for $V_{\sigma,\lambda}$ with $\widehat{H}_{\sigma,\lambda}^\infty = E_{\sigma,\lambda}$ and an orthogonal projection of some Hilbert globalization $\widehat{\mathbf{H}}$ of some good tensor product module onto $\widehat{H}_{\sigma,\lambda}$. Hence Lemma 6.2 implies that $E_{\sigma,\lambda}$ equals the minimal globalization. □

10. Reduction steps II: deformation theory and discrete series

We already know that every irreducible Harish-Chandra modules V can be written as a quotient U/H where U is good. Suppose that H is in fact a kernel of an intertwiner $I : U \to W$ with W good. Suppose in addition that we can deform $I : U \to W$ holomorphically (as to be made precise below). Then, provided U and W are good we will show that $\operatorname{im} I \simeq U/H$ is good. In view of the Langlands-classification, the assertion that every Harish-Chandra module is good then reduces to the case of discrete series representations.

This section is organized as follows: we recall the holomorphic deformation theory of Casselman (see [5], Sect. 9) in a slightly modified form. Then we prove that discrete series are good, and, finally, prove the Casselman-Wallach globalization theorem.

10.1. **Deformation theory**

For a complex manifold D and a Harish-Chandra module U we write $\mathcal{O}(D, U)$ for the space of maps $f : D \to U$ such that for all $\xi \in \widetilde{U}$ the contraction $\xi \circ f$ is holomorphic. Henceforth we will use D exclusively for the open unit disc.

By a holomorphic family of Harish-Chandra modules (parameterized by D) we understand a family of Harish-Chandra modules $(U_s)_{s\in D}$ such that:

(i) For all $s \in D$ one has $U_s = U_0 =: U$ as K-modules.
(ii) For all $X \in \mathfrak{g}$, $v \in U$ and $\xi \in \widetilde{U}$ the map $s \mapsto \xi(X_s \cdot v)$ is holomorphic. Here we use X_s for the action of X in U_s.

Given a holomorphic family $(U_s)_{s\in D}$ we form $\mathcal{U} := \mathcal{O}(D, U)$ and endow it with the following $(\mathfrak{g}, K)$-structure: for $X \in \mathfrak{g}$ and $f \in \mathcal{U}$ we set

$$(X \cdot f)(s) := X_s \cdot f(s)\,.$$

We emphasize that the algebra multiplication of $\mathcal{O}(D)$ on $\mathcal{U}$ commutes with the $(\mathfrak{g}, K)$-action.

Of particular interest are the Harish-Chandra modules $\mathbf{U}_k := \mathcal{U}/s^k\mathcal{U}$ for $k \in \mathbb{N}$. To get a feeling for this objects let us discuss a few examples for small k.

Example 10.1. *(a) For $k = 1$ the constant term map*

$$\mathbf{U}_1 \to U, \quad f + s\mathcal{U} \mapsto f(0)$$

is an isomorphism of $(\mathfrak{g}, K)$-modules.

(b) For $k = 2$ we observe that the map

$$\mathbf{U}_2 \to U \oplus U, \quad f + s^2\mathcal{U} \mapsto (f(0), f'(0))$$

provides an isomorphism of K-modules. The resulting $\mathfrak{g}$-action on the right hand side is twisted and given by

$$X \cdot (u_1, u_2) = (Xu_1, Xu_2 + X'u_1)$$

where

$$X'u := \frac{d}{ds}\Big|_{s=0} X_s \cdot u\,.$$

Let us remark that $X' = 0$ for all $X \in \mathfrak{k}$.

We notice that $\mathbf{U}_2$ features the submodule $s\mathcal{U}/s^2\mathcal{U}$ which corresponds to $\{0\} \oplus U$ in the above trivialization. The corresponding quotient $(\mathcal{U}/s^2\mathcal{U})/(s\mathcal{U}/s^2\mathcal{U})$ identifies with $U \oplus U/\{0\} \oplus U \simeq U$. In particular $\mathcal{U}/s^2\mathcal{U}$ is good if U is good by the extension Lemma 9.1.

From the previous discussion it follows that $\mathbf{U}_k$ is good for all $k \in \mathbb{N}_0$ provided that U is good.

Let now W be another Harish-Chandra module and $\mathcal{W}$ a holomorphic deformation of W as above. By a morphism $\mathcal{I} : \mathcal{U} \to \mathcal{W}$ we understand a family of $(\mathfrak{g}, K)$-maps $I_s : U_s \to W_s$ such that for all $u \in U$ and $\xi \in W^*$ the assignments $s \mapsto \xi(I_s(u))$ are holomorphic. Let us write I for I_0 set $I' := \frac{d}{ds}\Big|_{s=0} I_s$ etc. We set $H := \ker I$.

We now make two additional assumptions on our holomorphic family of intertwiners:

- I_s is invertible for all $s \neq 0$.
- There exists a $k \in \mathbb{N}_0$ such that $J(s) := s^k I_s^{-1}$ is holomorphic on D.

If these conditions are satisfied, then we call $I : U \to W$ *holomorphically deformable.*

For all $m \in \mathbb{N}$ we write $\mathbf{I}_m : \mathbf{U}_m \to \mathbf{W}_m$ for the intertwiner induced by $\mathcal{I}$. Likewise we define $\mathbf{J}_m$.

Example 10.2. *In order to get a feeling for the intertwiners $\mathbf{I}_m$ let us consider the example $\mathbf{I}_2 : \mathbf{U}_2 \to \mathbf{W}_2$. In trivializing coordinates this map is given by*

$$\mathbf{I}_2(u_1, u_2) = (I(u_1), I(u_2) + I'(u_1))\,.$$

We set $\mathbf{H}_m := \ker \mathbf{I}_m \subset \mathbf{U}_m$. For $m < n$ we view $\mathbf{U}_m$ as a K-submodule of $\mathbf{U}_n$ via the inclusion map

$$\mathbf{U}_m \to \mathbf{U}_n, \quad f + s^m\mathcal{U} \mapsto \sum_{j=0}^{m-1} \frac{f^{(j)}(0)}{j!} s^j + s^n\mathcal{U}\,.$$

We write $p_{n,m} : \mathbf{U}_n \to \mathbf{U}_m$ for the reverting projection (which are $(\mathfrak{g}, K)$-morphisms).

The following Lemma is related to an observation of Casselman, see [5], Prop. 9.3.

Lemma 10.3. *Suppose that $k \in \mathbb{N}_0$ is minimal such that $J(s)$ is holomorphic. Then the morphism*

$$\mathbf{I}_{2k}|_{\mathbf{H}_k + s^k\mathcal{U}/s^{2k}\mathcal{U}} : \mathbf{H}_k + s^k\mathcal{U}/s^{2k}\mathcal{U} \to s^k\mathcal{W}/s^{2k}\mathcal{W}$$

is onto. Moreover, its kernel is given by $s^k\mathbf{H}_k \subset s^k\mathcal{U}/s^{2k}\mathcal{U}$.

Proof. Clearly, $\mathbf{I}_{2k}(\mathbf{H}_k + s^k\mathcal{U}/s^{2k}\mathcal{U}) \subset s^k\mathcal{W}/s^{2k}\mathcal{W}$ and hence the map is defined. Let us check that it is onto. Let $[w] \in s^k\mathcal{W}/s^{2k}\mathcal{W}$ and $w \in s^k\mathcal{W}$ be a representative. Note that $\mathcal{I}^{-1}|_{s^k\mathcal{W}} : s^k\mathcal{W} \to \mathcal{U}$ is defined. Set $u := \mathcal{I}^{-1}(w)$ and write $[u]$ for its equivalence class in $\mathbf{U}_{2k}$. Then $u \in \mathbf{H}_k + s^k\mathcal{U}/s^{2k}\mathcal{U}$ and $\mathbf{I}_{2k}([u]) = [w]$ which shows that the map is onto.

A simple verification shows that $s^k\mathbf{H}_k$ lies in the kernel. Further, the first part of the proof shows that $p_{2k,k} \circ \mathbf{I_{2k}}^{-1} : s^k\mathcal{W}/s^{2k}\mathcal{W} \to \mathbf{H}_k$ is an injection. Thus $s^k\mathbf{H}_k$ is the kernel. □

If we set $V_3 := \mathbf{H}_k + s^k\mathcal{U}/s^{2k}\mathcal{U}$, $V_2 := s^k\mathcal{U}/s^{2k}\mathcal{U}$ and $V_1 := s^k\mathbf{H}_k$, the previous Lemma implies an inclusion chain

$$V_1 \subset V_2 \subset V_3$$

with

$$V_2/V_1 \simeq \mathbf{U}_k/\mathbf{H}_k, \qquad V_2 \simeq \mathbf{U}_k \qquad \text{and} \qquad V_3/V_1 \simeq \mathbf{W}_k\,.$$

Hence in combination with the squeezing Lemma 6.7 we obtain that $\mathbf{U}_k/\mathbf{H}_k$ is good if U and W are good.

We wish to show that U/H is good. Write $H_{k,1} := p_{k,1}(\mathbf{H}_k)$ for the projection of $\mathbf{H}_k$ to $\mathbf{U}_1 \simeq U$. Note that $H_{k,1}$ is a submodule of H. We arrive at the exact sequence

$$0 \to U/H \simeq s^{k-1}U/s^{k-1}H \to \mathbf{U}_k/\mathbf{H}_k \to U/H_{k,1} \to 0\,.$$

But U/H is a quotient of $U/H_{k,1}$. Thus putting an SF-topology on U we get one on H, $\mathbf{U}_k$, $\mathbf{H}_k$, $\mathbf{U}_k/\mathbf{H}_k$ and $U/H_{k,1}$. As a result the induced topology on U/H is both a sub and a quotient of the good topology on $\mathbf{U}_k/\mathbf{H}_k$. Hence U/H is good.

We summarize our discussion.

Proposition 10.4. *Suppose that $I : U \to W$ is an intertwiner of good Harish-Chandra modules which allows holomorphic deformations $\mathcal{I} : \mathcal{U} \to \mathcal{W}$. Then* $\operatorname{im} I$ *is good.*

10.2. **Discrete series**

The objective of this subsection is to show that every Harish-Chandra module belonging to the discrete series is good.

Let $Z < G$ be the center of G. Throughout this section V shall denote a unitarizable irreducible Harish-Chandra module, i.e. there exists a unitary irreducible globalization $(\pi, \mathcal{H})$ of V. We say that V is *square integrable* or *belongs to the discrete series* provided for all $v \in V$ and $\xi \in \widetilde{V}$ one has

$$\int_{G/Z} |m_{\xi,v}(g)|^2 \, d(gZ) < \infty .$$

In this situation, there exists a constant $d(\pi)$, the formal degree, such that for every unitary norm p on V one has

$$\frac{1}{d(\pi)} p(v)^2 p^*(\xi)^2 = \int_{G/Z} |m_{\xi,v}(g)|^2 \, d(gZ) \qquad (v \in V, \xi \in \widetilde{V}) . \tag{10.1}$$

Proposition 10.5. *Let V be a Harish-Chandra module of the discrete series. Then V is good.*

Proof. It is no loss of generality to assume that the center $Z < G$ is compact. Choose a minimal principal series $U := V_{\sigma,\lambda}$ such that V embeds into U. Let $\xi \in \widetilde{U}$ such that $\xi|_V \neq 0$. Then there exists an $s_0 > 0$ such that all matrix coefficients $m_{\xi,v}$, $v \in U$, belong to $L^2(G, \|g\|^{-s_0} dg)$. For every $s \in \mathbb{C}$ with $\operatorname{Re} s > s_0$ we define a continuous Hermitian form on U^∞ by setting

$$(v, w)_s := \int_G m_{\xi,v}(g) \overline{m_{\xi,w}(g)} \|g\|^{-s} \, dg .$$

Let us write $B(U^\infty, U^\infty)$ for the topological vector space of continuous Hermitian forms on U^∞. We obtain a holomorphic map

$$\{s \in \mathbb{C} \mid \operatorname{Re} s > s_0\} \to B(U^\infty, U^\infty), \quad s \mapsto (\cdot, \cdot)_s$$

and in Appendix B we show that it admits a meromorphic continuation to the complex plane.

Let $(\cdot, \cdot)$ be the constant part of $s \mapsto (\cdot, \cdot)_s$ at $s = 0$. Note that for $v, w \in V$ we have

$$(v, w) = \int_G m_{\xi,v}(g) \overline{m_{\xi,w}(g)} \, dg .$$

As, on the other hand $(\cdot, \cdot) \in B(U^\infty, U^\infty)$ and U^∞ is good, we conclude from (10.1) and Lemma 6.6 that the unitary norm p on V is a maximal norm on V w.r.t. $\prec$. As V is unitary, p is also minimal. □

10.3. **Proof of the Casselman-Wallach-Theorem**

Theorem 10.6. *All Harish-Chandra modules are good.*

Proof. Let V be a Harish-Chandra module. We have to show that V is good. In view of Corollary 9.2, we may assume that V is irreducible. Next we use Langland's classification (see [11],Ch. VIII, Th. 8.54) and combine it with our Propositions on deformation 10.4 and induction 9.6. This reduces to the case where V is tempered. However, the case of tempered readily reduces to square integrable ([17], Ch. 5, Prop. 5.2.5). The case of square integrable Harish-Chandra modules was established in Proposition 10.5. □

10.4. **Discussion**

In the introduction we phrased the Casselman-Wallach in several different ways. One way was the equivalence of categories $\mathcal{HC}$ and $\mathcal{SAF}$ or, equivalently, that there is only one Sobolev-equivalence class of G-continuous norms on a Harish-Chandra module.

The objective of this subsection is to show that the equivalence of categories $\mathcal{HC} \simeq \mathcal{SAF}$ can be slightly refined.

By a *marking* of a Fréchet space E we shall understand an increasing family $(p_n)_{n\in\mathbb{N}}$ of semi-norms which define the topology on E. Pairs $(E,(p_n)_n)$ will henceforth be called marked Fréchet spaces – in the literature one also finds the notion of *graded* Fréchet space (see [9]).

By a morphism of marked Fréchet spaces $(E,(p_n)_n)$ and $(F,(q_n)_n)$ we understand a linear map $T: E \to F$ with the following property: there exists a $k \in \mathbb{N}_0$ such that for all $n \in \mathbb{N}$ there exists $C_n > 0$ with:

$$q_n(T(x)) \leq C_n p_{n+k}(x) \qquad (x \in E)\,.$$

In this sense we obtain the additive category of marked Fréchet spaces, say $\mathcal{F}_{\rm mark}$.

For an F-representation (π, E) we are automatically led to the notion of a G-continuous marking. Let us define now $\mathcal{SAF}_{\rm mark} \subset \mathcal{F}_{\rm mark}$ to be the sub-category of smooth admissible F-representation with respect to a G-continuous marking. The refined Casselman-Wallach theorem asserts that

$$\mathcal{HC} \simeq \mathcal{SAF}_{\rm mark}\,. \tag{10.2}$$

Note that this immediate from the fact that V^∞ is a quotient of $\mathcal{S}(G)^k$ for some $k \in \mathbb{N}$.

11. Applications

11.1. Lifting $(\mathfrak{g}, K)$-morphisms

Let (π, E) be a representation of G on a complete topological vector space E. Let us call (π, E) an $\mathcal{S}(G)$-*representation* if the natural action of $C_c^\infty(G)$ on E extends to a separately continuous action of $\mathcal{S}(G)$ on E. Some typical examples we have in mind are smooth functions of moderate growth on certain homogeneous spaces. Let us mention a few.

Example 11.1. *(a) Let $\Gamma < G$ be a lattice, that is a discrete subgroup with cofinite volume. Reduction theory (Siegel sets) allows us to control "infinity" of the quotient $Y := \Gamma\backslash G$ and leads to a natural notion of moderate growth. For every $\alpha > 0$ there is a natural SF-module $C_\alpha^\infty(Y)$ of smooth functions on Y with growth rate at most α. The smooth functions of moderate growth $C_{\rm mod}^\infty(Y) = \lim_{\alpha\to\infty} C_\alpha^\infty(Y)$ become a complete inductive limit of the SF-spaces $C_\alpha^\infty(Y)$. Hence $\mathcal{S}(G)$ acts on $C_{\rm mod}^\infty(Y)$.*

The space of K and $\mathcal{Z}(\mathfrak{g})$-finite elements in $C_{\rm mod}^\infty(Y)$ is referred to as the space of automorphic forms on Y.
(b) Let $H < G$ be a symmetric subgroup, i.e. an open subgroup of the fixed point set of an involutive automorphism of G. We refer to $X := H\backslash G$ as a semisimple symmetric space. The Cartan-decomposition of X allows us to control growth on X and yields natural SF-modules $C_\alpha^\infty(X)$ of smooth functions with growth rate at most α. As before one obtains $C_{\rm mod}^\infty(X) = \lim_{\alpha\to\infty} C_\alpha^\infty(X)$ a natural complete $\mathcal{S}(G)$-module of functions with moderate growth.

If (π_1, E_1), (π_2, E_2) are two representations, then we denote by $\mathrm{Hom}_G(E_1, E_2)$ for the space of continuous G-equivariant linear maps from E_1 to E_2.

Proposition 11.2. *Let V be a Harish-Chandra module and V^∞ its unique SF-globalization. Then for any smooth $\mathcal{S}(G)$-representation (π, E) of G the linear map*

$$\mathrm{Hom}_G(V^\infty, E) \to \mathrm{Hom}_{(\mathfrak{g},K)}(V, E^{K-\rm fin}), \quad T^\infty \mapsto T := T^\infty|_V$$

is a linear isomorphism.

Proof. It is clear that the map is injective. To show that the map is onto let us write λ, resp. Λ, for the representation of G, resp. $\mathcal{S}(G)$, on V^∞. Let $v \in V^\infty$. Then we find $f \in \mathcal{S}(G)$ such that $v = \Lambda(f)w$ for some $w \in V$. We claim that

$$T^\infty(v) := \Pi(f)T(w)$$

defines a linear operator. In order to show that this definition makes sense we have to show that $T^\infty(v) = 0$ if $\Lambda(f)w = 0$. Let $\xi \in (E^*)^{K-\mathrm{fin}}$ and $\mu := \xi \circ T \in \widetilde{V}$. We consider two distributions on G, namely

$$\Theta_1(\phi) := \xi(\Pi(\phi)T(w)) \quad \text{and} \quad \Theta_2(\phi) := \mu(\Lambda(\phi)w) \qquad (\phi \in C_c^\infty(G)).$$

We claim that $\Theta_1 = \Theta_2$. In fact, both distributions are $\mathcal{Z}(\mathfrak{g})-$ and $K \times K$-finite. Hence they are represented by analytic functions on G and thus uniquely determined by their derivatives on K. The claim follows.

It remains to show that T is continuous. We recall the construction of the minimal SF-globalization of V, namely $V^\infty = \mathcal{S}(G)^k/\mathcal{S}(G)_{\mathbf{v}}$. As the action of $\mathcal{S}(G)$ on E is separately continuous, the continuity of T^∞ follows. □

11.2. **Automatic continuity**

For a Harish-Chandra module V we denoted by V^* its algebraic dual. Note that V^* is naturally a module for $\mathfrak{g}$.

If $\mathfrak{h} < \mathfrak{g}$ is a subalgebra, then we write $(V^*)^{\mathfrak{h}}$, resp. $(V^*)^{\mathfrak{h}-\mathrm{fin}}$, for the space of $\mathfrak{h}$-fixed, resp. $\mathfrak{h}$-finite, algebraic linear functionals on V.

We call a subalgebra $\mathfrak{h} < \mathfrak{g}$ a *(strong) automatic continuity* subalgebra ((S)AC-subalgebra for short) if for all Harish-Chandra modules V one has

$$(V^*)^{\mathfrak{h}} \subset (V^\infty)^* \quad \text{resp.} \quad (V^*)^{\mathfrak{h}-\mathrm{fin}} \subset (V^\infty)^*.$$

Problem 11.3. *(a) Is it true that $\mathfrak{h}$ is AC if and only if $\langle \exp \mathfrak{h} \rangle < G$ has an open orbit on $G/P_{\min}$?*
(b) Is it true that $\mathfrak{h}$ is SAC if $[\mathfrak{h}, \mathfrak{h}]$ is AC ?

The following examples of (S)AC-subalgebras are known:

- $\mathfrak{n}$, the Lie algebra of an Iwasawa N-subgroup, is AC and $\mathfrak{a}+\mathfrak{n}$, the Lie algebra of an Iwasawa AN-subgroup, is SAC. (Casselman).
- Symmetric subalgebras, i.e. fixed point sets of involutive automorphisms of $\mathfrak{g}$, are AC (Brylinski, Delorme, van den Ban; cf. [3], [2]).

Here we only wish to discuss Casselman's result. We recall the definition of the Casselman-Jacquet module $j(V) = \bigcup_{k \in \mathbb{N}_0} (V/\mathfrak{n}^k V)^*$ and note that $j(V) = (V^*)^{\mathfrak{a}+\mathfrak{n}-\mathrm{fin}}$.

Theorem 11.4. (Casselman) *Let $\mathfrak{n}$ be the Lie algebra of an Iwasawa N-subgroup of G and $\mathfrak{a}+\mathfrak{n}$ the Lie algebra of an Iwasawa AN-subgroup.*

Then $\mathfrak{n}$ *is an AC and* $\mathfrak{a}+\mathfrak{n}$ *is SAC. In particular, for all Harish-Chandra modules* V *one has* $j(V) \subset (V^\infty)^*$.

Proof. Let us emphasize that the proof needs only results up to Section 8, i.e. that minimal principal series are good.

We first prove that $\mathfrak{a}+\mathfrak{n}$ is SAC. Let V be a Harish-Chandra module and $0 \neq \lambda \in j(V)$. By definition there exists a $k \in \mathbb{N}$ such that $\lambda \in (V/\mathfrak{n}^k V)^*$. Write (σ, U_σ) for the finite dimensional representation of $P_{\min}$ on $V/\mathfrak{n}^k V$ and denote by I_σ the corresponding induced Harish-Chandra module. Note that $I_\sigma^\infty = C^\infty(U_\sigma \times_{P_{\min}} G)$.

Applying Frobenius reciprocity to the identity morphism $V/\mathfrak{n}^k V \to U$ yields a non-trivial $(\mathfrak{g}, K)$-morphism $T : V \to I_\sigma$ (cf. [17], 4.2.2). Now T lifts to a continuous G-map $T^\infty : V^\infty \to I_\sigma^\infty$ by Proposition 11.2. If $\mathrm{ev} : I_\sigma^\infty \to U_\sigma$ denotes the evaluation map at the identity, then $\lambda^\infty := \lambda \circ \mathrm{ev} \circ T^\infty$ provides a continuous extension of λ to V^∞.

The assertion that $\mathfrak{n}$ is AC follows from the fact that the space of $(V^*)^{\mathfrak{n}}$ is finite dimensional (Casselman), and in particular $\mathfrak{a}$-finite. $\square$

11.3. **Lifting of holomorphic families of $(\mathfrak{g}, K)$-maps**

We wish to give a version of lifting (cf. Proposition 11.2) which depends holomorphically on parameters.

To begin with we need a generlaization of Theorem 12.3 and Theorem 12.8 for principal series representations which are induced from an arbitrary parabolic subgroup.

Let $P = N_P A_P M_P$ be a parabolic above $P_{\min}$. We fix an SAF-representation $(\sigma, V_\sigma^\infty)$ of M_P and write V_σ for the corresponding Harish-Chandra module.

As K-modules we identify all $V_{\sigma,\lambda}$ with $V := \mathbb{C}[V_\sigma \times_{K_P} K]$. Note that V_σ is a K_P-quotient of some $\mathbb{C}[K_P]^m$, $m \in \mathbb{N}$. Double induction gives an identification of V as a K-quotient of $\mathbb{C}[K]^m$. Note that $C^\infty(K)^m$ induces the unique SF-topology on V^∞. For each τ we write χ_τ for its character and $\delta_{\sigma,\tau}$ for the orthogonal projection of $\underbrace{(\chi_\tau, \ldots, \chi_\tau)}_{m-times}$ to $V[\tau]$, the τ-isotypical part of V.

Theorem 11.5. *Let* $P = N_P A_P M_P$ *be a parabolic subgroup and* V_σ *an irreducible unitarizable Harish-Chandra module for* M_P. *Let* $Q \subset (\mathfrak{a}_P)^*_\mathbb{C}$ *be a compact subset and* $N > 0$. *Then there exists* $\xi \in \mathbb{C}[V_\sigma \times_{K_P} K]$ *and constants* $c_1, c_2 > 0$ *such that for all* $\tau \in \widehat{K}$, $\lambda \in Q$, *there exists* $a_\tau \in A$, *independent of* λ, *with* $\|a_\tau\| \leq (1+|\tau|)^{c_1}$ *and numbers* $b_\sigma(\lambda, \tau) \in \mathbb{C}$ *such* *that*

$$\|[\pi_{\sigma,\lambda}(a_\tau)\xi]_\tau - b_\sigma(\lambda,\tau)\delta_{\sigma,\tau}\| \leq \frac{1}{(|\tau|+1)^{N+c_2}}$$

and

$$|b_\sigma(\lambda,\tau)| \geq \frac{1}{(|\tau|+1)^{c_2}}\,.$$

Here $\|\cdot\|$ *refers to the continuous norm on* V *induced by the realization of* V *as a quotient of* $C[K]^m \subset L^2(K)^m$.

Proof. Let us first discuss the case where $P = P_{\min}$ and σ is finite dimensional. With the reduction steps explained in the beginning of the next section this becomes a simple modification of Theorem 12.3.

As for the general case we identify V_σ as a quotient of a minimal principal series for M_P. Using double induction we can write the $V_{\sigma,\lambda}$'s consistently as quotients of such minimal principal series. The assertion follows. □

As a consequence we get an extension of Theorem 12.8.

Theorem 11.6. *Let* $Q \subset (\mathfrak{a}_P)^*_\mathbb{C}$ *be a compact subset. Then there exist a continuous map*

$$Q \times C^\infty(V_\sigma^\infty \times_{K_P} K) \to \mathcal{S}(G), \quad (\lambda, v) \mapsto f(\lambda, v)$$

which is holomorphic in the first variable, linear in the second and such that

$$\Pi_{\sigma,\lambda}(f(\lambda,v))\xi = v\,.$$

As a Corollary to Theorem 11.6 we obtain the holomorphic lifting result:

Theorem 11.7. *Let* (π, E) *be a Banach representation of* G. *Within the notation of Theorem 11.6 let* $\Omega \subset (\mathfrak{a}_P)^*_\mathbb{C}$ *be an open set and*

$$T : \Omega \times \mathbb{C}[V_\sigma \times_{K_P} K] \to E^\infty$$

a map such that:

- *For every* $v \in \mathbb{C}[V_\sigma \times_{K_P} K]$ *the assignment* $\Omega \ni \lambda \mapsto T(\lambda, v) \in E^\infty$ *is holomorphic.*
- *For every* λ *the assignment*

  $$V_{\sigma,\lambda} = \mathbb{C}[V_\sigma \times_{K_P} K] \to E^\infty, \quad v \mapsto T(\lambda, v)$$

  is a $(\mathfrak{g}, K)$*-map.*

Then T *admits a holomorphic extension to a map*

$$T^\infty : \Omega \times C^\infty(V_\sigma^\infty \times_{K_P} K) \to E^\infty\,.$$

Proof. It is no loss of generality to assume that Ω is relatively compact. Within the notation of Theorem 11.6 we define

$$T^\infty(\lambda, v) := \Pi(f(\lambda, v))T(\lambda, \xi)\,.$$

$\square$

Remark 11.8. (Application to Eisenstein series) *Let $\Gamma < G$ be a lattice and $Y := \Gamma\backslash G$. Let*

$$T : \Omega \times \mathbb{C}[V_\sigma \times_{K_P} K] \to C^\infty_{\rm mod}(Y)$$

be a map which satisfies the conditions in Theorem 11.7. Then, basic automorphic theory implies for all relatively compact $\Omega \subset (\mathfrak{a}_P)^*_\mathbb{C}$ the existence of a growth index α such that $\operatorname{im} T \subset L^2_\alpha(Y)$. In particular Theorem 11.7 is applicable.

A typical application is as follows. Let us consider Eisenstein series attached to the lattice $\Gamma < G$. We assume that $P = M_P A_P N_P$ is cuspidal and set $L := M_P \cap K$. Fix a finite dimensional unitary representation (σ, U) of L and a $\Gamma \cap M_P$-invariant L^2-section ψ of the vector bundle $(\Gamma \cap M_P)\backslash M_P \times_L U \to (\Gamma \cap M_P)\backslash M_P$. Suppose that all contractions $\langle \psi, u\rangle \in L^2(\Gamma \cap M_P\backslash M_P)$, $u \in U$, generate a Harish-Chandra module for $(\mathfrak{m}_P, L)$. Let f be a smooth section of the K-equivariant vector bundle $U \times_L K \to L\backslash K$. Then one defines Eisenstein series

$$E(\lambda, \psi, f)(\Gamma g) := \sum_{\gamma \in \Gamma\cap P\backslash\Gamma} \widetilde{a}(\gamma g)^\lambda \langle \psi(\widetilde{m}(\gamma g)), f(\widetilde{k}(g))\rangle$$

where $g = \widetilde{n}(g)\widetilde{a}(g)\widetilde{m}(g)\widetilde{k}(g) \in N_P A_P M_P K$ and $\lambda \in (\mathfrak{a}_P)^*_\mathbb{C}$. Suppose you have shown that for all K-finite sections f that $E(\lambda, \psi, f)$ can be meromorphically continued to some region in the parameter space $\Omega \subset (\mathfrak{a}_P)^*_\mathbb{C}$. Then the same holds true for all smooth sections f.

12. Appendix A: Spherical principal series and the proof of Theorem 8.1(i)

We first discuss how the proof of Theorem 8.1(i) reduces to the case of spherical principal series. We use the notation from Section 8.

First it is clear that it is sufficient to establish the result for *one* fixed set of generators $\xi_1, \ldots, \xi_k$ of $I(W)$. Next let $\{0\} = W_0 \subset W_1 \subset \ldots \subset W_n = W$ be a Jordan-Hölder series of W. It induces an inclusion chain of $(\mathfrak{g}, K)$-modules

$$\{0\} = I(W_0) \subset I(W_1) \subset \ldots \subset I(W_n) = I(W)$$

with $I(W_{j+1})/I(W_j) \simeq I(W_{j+1}/W_j)$. It follows that we can and will assume that W is irreducible. In particular, the $P_{\min}$-representation W factors to $P_{\min}/N \simeq M \times A$. Let us write $\sigma \times \chi$ for this $M \times A$-representation on W. Next there exists a finite dimensional representation F of G with N-invariants F^N such that $W \hookrightarrow F^N \otimes \mathbb{C}_\chi \subset F \otimes \mathbb{C}_\chi$. Further $I(F \otimes \mathbb{C}_\chi) \simeq I(\mathbb{C}_\chi) \otimes F$ and thus we obtain an embedding $I(W) \hookrightarrow I(\mathbb{C}_\chi) \otimes W$. In view of our discussion of tensoring with finite dimensional representations (see Subsection 9.2) matters reduce to $W = \mathbb{C}_\chi$.

12.1. **Spherical principal series representations**

In this section we introduce a Dirac-type sequence for spherical principal series representations (see Subsection 12.1.4 with Theorem 12.3). This allows us to establish lower bounds for matrix-coefficients which are uniform in the K-types (cf. Corollary 12.4). These lower bounds are essentially sharp, locally uniform in the representation parameter, and stronger than the more abstract estimates in Theorem 7.1.

The lower bounds established give us a constructive method for finding Schwartz-functions representing a given smooth vector and as a side product a proof of Theorem 8.1(i) for spherical principal series (see Subsection 12.1.5).

According to the Iwasawa decomposition $G = NAK$ we decompose elements $g \in G$ as

$$g = \widetilde{n}(g)\widetilde{a}(g)\widetilde{k}(g)$$

with $\widetilde{n}(g) \in N$, $\widetilde{a}(g) \in A$ and $\widetilde{k}(g) \in K$. We recall $M = Z_K(A)$ and the minimal parabolic subgroup $P_{\min} = NAM$ of G.

The Lie algebras of A, N and K shall be denoted by $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{k}$. Complexification of Lie-algebras are indicated with a $\mathbb{C}$-subscript, i.e. $\mathfrak{g}_\mathbb{C}$ is the complexification of $\mathfrak{g}$ etc. As usually we define $\rho \in \mathfrak{a}^*$ by $\rho(Y) := \frac{1}{2}\operatorname{tr}(\operatorname{ad}_\mathfrak{n} Y)$ for $Y \in \mathfrak{a}$.

The smooth spherical principal series with parameter $\lambda \in \mathfrak{a}_\mathbb{C}^*$ is defined by

$$\begin{aligned}\mathcal{H}_\lambda^\infty := \{f \in C^\infty(G) \,|(\forall nam \in P_{\min}, \forall g \in G) \\ f(namg) = a^{\rho+\lambda} f(g)\}\end{aligned}$$

We note that R defines a smooth representation of G on $\mathcal{H}_\lambda^\infty$ which we denote henceforth by π_λ. The restriction map to K defines a K-isomorphism:

$$\mathrm{Res}_K : \mathcal{H}_\lambda^\infty \to C^\infty(K\backslash M), \quad f \mapsto f|_K .$$

The resulting action of G on $C^\infty(M\backslash K)$ is given by

$$[\pi_\lambda(g)f](Mk) = f(M\widetilde{k}(kg))\widetilde{a}(kg)^{\lambda+\rho} .$$

This action lifts to a continuous action on the Hilbert completion $\mathcal{H}_\lambda = L^2(M\backslash K)$ of $C^\infty(M\backslash K)$. We note that this representation is unitary provided that $\lambda \in i\mathfrak{a}^*$.

We denote by V_λ the K-finite vectors of π_λ and note that $V_\lambda = \mathbb{C}[M\backslash K]$ as K-module. For later reference we record that the dual representation of $(\pi_\lambda, \mathcal{H}_\lambda)$ is isomorphic to $(\pi_{-\lambda}, \mathcal{H}_{-\lambda})$ via the G-equivariant pairing

$$(12.1)\quad (\cdot,\cdot): \ \mathcal{H}_{-\lambda} \times \mathcal{H}_\lambda \to \mathbb{C}, \quad (\xi, v) := \int_{M\backslash K} \xi(Mk)v(Mk)\ d(Mk) .$$

Here G-equivariance means that

$$(\pi_{-\lambda}(g)\xi, v) = (\xi, \pi_\lambda(g^{-1})v)$$

for all $g \in G$.

12.1.1. *K-expansion of smooth vectors.* We recall $\widehat{K}$, the set of equivalence classes of irreducible unitary representations of K. If $[\tau] \in \widehat{K}$ we let (τ, U_τ) be a representative. Further we write $\widehat{K}_M$ for the subset of M-spherical equivalence classes, i.e.

$$[\tau] \in \widehat{K}_M \iff U_\tau^M := \{u \in U_\tau \mid \tau(m)u = u \ \forall m \in M\} \neq \{0\} .$$

Given a finite dimensional representation (τ, U_τ) of K we denote by (τ^*, U_τ^*) its dual representation. With each $[\tau] \in \widehat{K}_M$ comes the realization mapping

$$r_\tau : U_\tau \otimes (U_\tau^*)^M \to L^2(M\backslash K), \quad u \otimes \eta \mapsto (Mk \mapsto \eta(\tau(k)u)) .$$

Let us fix a K-invariant inner product on U_τ. This inner product induces a K-invariant inner product on U_τ^*. We obtain an inner product on $U_\tau \otimes (U_\tau^*)^M$ which is independent of the chosen inner product on U_τ. If we denote by $d(\tau)$ the dimension of U_τ, then Schur-orthogonality implies that

$$\frac{1}{d(\tau)}\|u \otimes \eta\|^2 = \|r_\tau(u \otimes \eta)\|^2_{L^2(M\backslash K)} .$$

Taking all realization maps together we arrive at a K-module isomorphism

$$\mathbb{C}[M\backslash K] = \sum_{\tau\in\widehat{K}_M} U_\tau \otimes (U_\tau^*)^M .$$

Let us fix a maximal torus $\mathfrak{t} \subset \mathfrak{k}$ and a positive chamber $\mathcal{C} \subset i\mathfrak{t}^*$. We often identify τ with its highest weight in $\mathcal{C}$ and write $|\tau|$ for the norm (with respect to the positive definite form B) of the highest weight. As $d(\tau)$ is polynomial in τ we arrive at the following characterization of the smooth functions:

$$C^\infty(M\backslash K) = \Big\{ \sum_{\tau\in\widehat{K}_M} c_\tau u_\tau \ | c_\tau \in \mathbb{C}, u_\tau \in U_\tau \otimes (U_\tau^*)^M, \|u_\tau\| = 1$$

$$(\forall N \in \mathbb{N}) \sum_{\tau\in\widehat{K}_M} |c_\tau|(1+|\tau|)^N < \infty\Big\} .$$

Let us denote by δ_{Me} the point-evaluation of $C^\infty(M\backslash K)$ at the base point Me. We decompose δ_{Me} into K-types:

$$\delta_{Me} = \sum_{\tau\in\widehat{K}_M} \delta_\tau$$

where

$$\delta_\tau = d(\tau)\sum_{i=1}^{l(\tau)} u_i \otimes u_i^*$$

with $u_1, \ldots, u_{l(\tau)}$ any basis of U_τ^M and $u_1^*, \ldots, u_{l(\tau)}^*$ its dual basis. For $1 \leq i, j \leq l(\tau)$ we set

$$\delta_\tau^{i,j} := u_i \otimes u_j^*$$

and record that $\delta_\tau = d(\tau)\sum_{i=1}^{l(\tau)} \delta_\tau^{i,i}$. Note the following properties of δ_τ and $\delta_\tau^{i,j}$:

- $\|\delta_\tau^{i,i}\|_\infty = \delta_\tau^{i,i}(Me) = 1$.
- $\delta_\tau * \delta_\tau = \delta_\tau$.
- $\delta_\tau * f = f$ for all $f \in L^2(M\backslash K)_\tau := \operatorname{im} r_\tau$.

12.1.2. *Non-compact model.* We have seen that the restriction map Res_K realizes $\mathcal{H}_\lambda^\infty$ as a function space on $M\backslash K$. Another standard realization will be useful for us. Let us denote by $\overline{N}$ the opposite of N. As $NAM\overline{N}$ is open and dense in G we obtain a faithful restriction mapping:

$$\operatorname{Res}_{\overline{N}} : \mathcal{H}_\lambda^\infty \to C^\infty(\overline{N}), \quad f \mapsto f|_{\overline{N}} .$$

Note that this map is not onto. The transfer of compact to non-compact model is given by

$$\begin{aligned}\mathrm{Res}_{\overline{N}}\circ\mathrm{Res}_K^{-1}: C^\infty(M\backslash K)\to C^\infty(\overline{N}),\\ f\mapsto F;\ F(\overline{n}):=\widetilde{a}(\overline{n})^{\lambda+\rho}f(\widetilde{k}(\overline{n}))\end{aligned}$$

The transfer of the Hilbert space structure on $\mathcal{H}_\lambda = L^2(M\backslash K)$ results in the L^2-space $L^2(\overline{N}, \widetilde{a}(\overline{n})^{-2\operatorname{Re}\lambda}d\overline{n})$ with $d\overline{n}$ an appropriately normalized Haar measure on $\overline{N}$. In the sequel we also write $\mathcal{H}_\lambda$ for $L^2(\overline{N}, \widetilde{a}(\overline{n})^{-2\operatorname{Re}\lambda}d\overline{n})$ in the understood context. The full action of G in the non-compact model is not of relevance to us, however we will often use the A-action which is much more transparent in the non-compact picture:

$$[\pi_\lambda(a)f](\overline{n}) = a^{\lambda+\rho}f(a^{-1}\overline{n}a)$$

for all $a\in A$ and $f\in L^2(\overline{N}, \widetilde{a}(\overline{n})^{-2\operatorname{Re}\lambda}d\overline{n})$.

12.1.3. *K-finite vectors with fast decay.* The fact that Res_K is an isomorphism follows from the geometric fact that $P_{\min}\backslash G\simeq M\backslash K$. Now $\overline{N}$ embeds into $P_{\min}\backslash G = M\backslash K$ as an open dense subset. In fact the complement is algebraic and we are going to describe it as the zero set of a K-finite functions f on $M\backslash K$. We will show that f can be chosen such that f restricted to $\overline{N}$ has polynomial decay of arbitrary fixed order.

Let (σ, W) be a finite dimensional faithful irreducible representation of G. We assume that W is K-spherical, i.e. W admits a non-zero K-fixed vector, say v_K. It is known that σ is K-spherical if and only if there is a real line $L\subset W$ which is fixed under $\overline{P}_{\min} = MA\overline{N}$. Let $L=\mathbb{R}v_0$ and $\mu\in\mathfrak{a}^*$ be such that $\sigma(a)v_0 = a^\mu\cdot v_0$ for all $a\in A$, in other words: v_0 is a lowest weight vector of σ and μ is the corresponding lowest weight.

Let now $\langle\cdot,\cdot\rangle$ be an inner product on W which is θ-covariant: if $g = k\exp(X)$ for $k\in K$ and $X\in\mathfrak{p}$ and $\theta(g) := k\exp(-X)$, then covariance means

$$\langle\sigma(g)v, w\rangle = \langle v, \sigma(\theta(g)^{-1})w\rangle$$

for all $v, w\in W$ and $g\in G$. Such an inner product is unique up to scalar by Schur's Lemma. Henceforth we request that v_0 is normalized and we fix v_K by $\langle v_0, v_K\rangle = 1$. Consider on G the function

$$f_\sigma(g) := \langle\sigma(g)v_0, v_0\rangle\,.$$

The restriction of f_σ to K is also denoted by f_σ.

Let now $\overline{n} \in \overline{N}$ and write $\overline{n} = \widetilde{n}(\overline{n})\widetilde{a}(\overline{n})\widetilde{k}(\overline{n})$ according to the Iwasawa decomposition. Then $\widetilde{k}(\overline{n}) = n^*\widetilde{a}(\overline{n})^{-1}\overline{n}$ for some $n^* \in N$. Consequently

$$f_\sigma(\widetilde{k}(\overline{n})) = \widetilde{a}(\overline{n})^{-\mu}\,.$$

If $(\overline{n}_j)_j$ is a sequence in $\overline{N}$ such that $\widetilde{k}(\overline{n}_j)$ converges to a point in $M\backslash K - \widetilde{k}(\overline{N}) =: M\backslash K - \overline{N}$, then $\widetilde{a}(\overline{n}_j)^{-\mu} \to 0$. Hence

$$M\backslash K - \overline{N} \subset \{Mk \in M\backslash K \mid f_\sigma(k) = 0\}\,.$$

As f_σ is non-negative one obtains for all regular σ that equality holds:

$$M\backslash K - \overline{N} = \{Mk \in M\backslash K \mid f_\sigma(k) = 0\}$$

(this reasoning is not new and goes back to Harish-Chandra). Let us fix such a σ now.

We claim that the mapping $\overline{n} \to f_\sigma(\overline{n})$ is the inverse of a polynomial mapping, i.o.w. the map

$$\overline{N} \to \mathbb{R}, \quad \overline{n} \mapsto \widetilde{a}(\overline{n})^\mu$$

is a polynomial map. But this follows from

$$\widetilde{a}(\overline{n})^\mu = \langle \sigma(\overline{n})v_K, v_0\rangle$$

by means of our normalizations.

In order to make estimates later on we introduce coordinates on $\overline{N}$. For that we first write $\overline{\mathfrak{n}}$ as semi-direct product of $\mathfrak{a}$-root vectors:

$$\overline{\mathfrak{n}} = \mathbb{R}X_1 \ltimes (\mathbb{R}X_2 \ltimes (\ldots \ltimes \mathbb{R}X_n)\ldots)\,.$$

Accordingly we write elements of $\overline{\mathfrak{n}}$ as $X := \sum_{j=1}^n x_j X_j$ with $x_i \in \mathbb{R}$. We note the following two facts:

- The map
  $$\Phi : \overline{\mathfrak{n}} \to \overline{N}, \quad X \mapsto \overline{n}(X) := \exp(x_1X_1)\cdot\ldots\cdot\exp(x_nX_n)$$
  is a diffeomorphism.
- One can normalize the Haar measure $d\overline{n}$ of $\overline{N}$ in such a way that:
  $$\Phi^*(d\overline{n}) = dx_1\cdot\ldots\cdot dx_n\,.$$

We introduce a norm on $\overline{\mathfrak{n}}$ by setting

$$\|X\|^2 := \sum_{j=1}^n |x_j|^2 \qquad (X \in \overline{\mathfrak{n}})\,.$$

Finally we set

$$f_\sigma(X) := f_\sigma(\widetilde{k}(\overline{n}(X))) = \widetilde{a}(\overline{n}(X))^{-\mu}$$

and summarize our discussion.

Lemma 12.1. *Let $m > 0$. Then there exists $C > 0$ and a finite dimensional K-spherical representation (σ, W) of G such that:*

(i) $M\backslash K - \overline{N} = \{Mk \in M\backslash K \mid f_\sigma(k) = 0\}$.
(ii) $|f_\sigma(X)| \leq C \cdot (1 + \|X\|)^{-m}$ *for all* $X \in \overline{\mathfrak{n}}$.

12.1.4. *Dirac type sequences.* Dirac sequences do not exist for Hilbert representations as they are features of an L^1-theory. However, rescaled they exist for the Hilbert representations we shall consider.

Recall our function f_σ on $M\backslash K$. We let $\xi = \xi_\sigma$ be the corresponding function transferred to $\overline{N} \simeq \overline{\mathfrak{n}}$ i.e.

$$\xi(X) := \widetilde{a}(\overline{n}(X))^{\rho+\lambda} f_\sigma(\widetilde{k}(\overline{n}(X))) = \widetilde{a}(\overline{n}(X))^{\rho+\lambda-\mu} .$$

It is clear that ξ is a K-finite vector for π_λ.

We recall that $\xi(X)$ satisfies the inequality

$$|\xi(X)| \leq C \cdot (1 + \|X\|)^{-m}$$

where we can choose m as large as we wish (provided σ is sufficiently regular and large). Record the normalization $\xi(0) = 1$.

We will chose m at least that large that ξ becomes integrable and write $\|\xi\|_1$ for the corresponding $L^1(\overline{N})$-norm.

The operators $\pi_\lambda(a)$ can be understood as scaling operators in the non-compact picture. For our purpose the scaling in one direction of A will be sufficient. To make this precise we fix an element $Y \in \mathfrak{a}$ such that $\alpha(Y) \geq 1$ for all roots $\alpha \in \Sigma(\mathfrak{a}, \mathfrak{n})$. For $t > 0$ we put

$$a_t := \exp((\log t)Y) .$$

Note that for $\eta \in \mathfrak{a}_\mathbb{C}^*$ one has

$$a_t^\eta = t^{\eta(Y)} .$$

In the sequel we will often abbreviate and simply write t^η for $t^{\eta(Y)}$.

In order to explain the idea of this section let us assume for a moment that λ is real. Then ξ is a positive function and

$$\left(\frac{a_t^{\rho-\lambda}}{\|\xi\|_1} \cdot \pi_\lambda(a_t)\xi \right)_{t>0}$$

forms a Dirac sequence for $t \to \infty$ (If λ is not real, then ξ is oscillating and we have to be slightly more careful).

In the compact picture this means

$$\lim_{t\to\infty} \frac{a_t^{\rho-\lambda}}{\|\xi\|_1} \cdot \pi_\lambda(a_t) f_\sigma = \delta_{Me} = \sum_{\tau \in \widehat{K}_M} \delta_\tau .$$

It is our goal to understand this limit in the K-types: How large do we have to choose t in dependence of τ such that the τ-isotypical part of $\frac{a_t^{\rho-\lambda}}{\|\xi\|_1}\cdot\pi_\lambda(a_t)f_\sigma$ approximates δ_τ well. It turns out that t can be chosen polynomially in τ. If we denote by D_τ the transfer of the character δ_τ to the non-compact model, the precise statement is as follows.

Theorem 12.2. *Let $\lambda\in\mathfrak{a}_\mathbb{C}^*$ and $N>0$. Then there exists a choice of σ and hence of $\xi=\xi_\sigma\in V_\lambda$, constants $c>2$, $C>0$ such that for all $\tau\in\widehat{K}_M$ one has*

$$[\pi_\lambda(a_{t(\tau)})\xi]_\tau = a_{t(\tau)}^{-\rho+\lambda}\cdot I_\xi\cdot D_\tau + R_\tau$$

where $t(\tau):=(1+|\tau|)^c$,

$$I_\xi := \int_{\overline{N}}\xi(\overline{n})\ d\overline{n}\neq 0$$

and remainder $R_\tau\in\mathcal{H}_\lambda[\tau]$ satisfying

$$\frac{\|R(\tau)\|}{|a_{t(\tau)}^{-\rho+\lambda}|}\leq\frac{C}{(1+|\tau|)^N}\,.$$

Proof. Recall the M-fixed functions $\delta_\tau^{i,j}\in L^2(M\backslash K)_\tau$, $1\leq i,j\leq l(\tau)$ for $\tau\in\widehat{K}_M$. In the sequel we abbreviate and set $d:=d(\tau)$, $l:=l(\tau)$.

Let $D_\tau^{i,j}(\overline{n})=\widetilde{a}(\overline{n})^{\rho+\lambda}\delta_\tau^{i,j}(\widetilde{k}(\overline{n}))$ the transfer of $\delta_\tau^{i,j}$ to the non-compact model. We also set $D_\tau^{i,j}(X):=D_\tau^{i,j}(\overline{n}(X))$ for $X\in\overline{\mathfrak{n}}$. Let us note that $|D_\tau^{i,j}(0)|=\delta_{ij}$.

As $\pi_\lambda(a)\xi$ is M-fixed for all $a\in A$ we conclude that

$$[\pi_\lambda(a_t)\xi]_\tau=\sum_{i,j=1}^{l} b_{i,j}(t)\cdot d\cdot D_\tau^{ij}\,.$$

If $\langle\cdot,\cdot\rangle$ denotes the Hermitian bracket on $\mathcal{H}_\lambda=L^2(\overline{N},\widetilde{a}(\overline{n})^{-2\operatorname{Re}\lambda}d\overline{n})$, then the coefficients $b_{i,j}(t)$ are obtained by the integrals

$$b_{i,j}(t)=\langle\pi_\lambda(a_t)\xi,D_\tau^{i,j}\rangle=\int_{\overline{\mathfrak{n}}}(\pi_\lambda(a_t)\xi)(X)\cdot\overline{D_\tau^{i,j}(X)}\cdot\widetilde{a}(\overline{n}(X))^{-2\operatorname{Re}\lambda}\ dX\,,$$

where we used the notation

$$dX:=dx_1\cdot\ldots\cdot dx_n$$

for $X=\sum_{j=1}^n x_jX_j$.

Fix $1\geq t_0>0$ and set $t=t_0^{-2}$.

We split the integrals for $b_{i,j}(t)$ into two parts $b_{i,j}(t)=b_{i,j}^1(t)+b_{i,j}^2(t)$ with

$$b^1_{i,j}(t) := \int_{\{\|X\|\geq t_0\}} (\pi_\lambda(a_t)\xi)(X) \cdot \overline{D_\tau(X)} \cdot \widetilde{a}(\overline{n}(X))^{-2\operatorname{Re}\lambda}\, dX\,.$$

In our first step of the proof we wish to estimate $b^1_{i,j}(t)$. For that let $C, q_1 > 0$ be such that

$$\widetilde{a}(\overline{n}(X))^{-2\operatorname{Re}\lambda} \leq C \cdot (1+\|X\|)^{q_1}\,.$$

Likewise, by the definition of $D^{i,j}_\tau$ we obtain constants $C, q_2 > 0$ which only depend on $\operatorname{Re}\lambda$ and such that

$$|D^{i,j}_\tau(X)| \leq C \cdot (1+\|X\|)^{q_2}$$

for all τ and $1 \leq i,j \leq l$. Set $q := q_1 + q_2$.

From the inequalities just stated we arrive at:

$$|b^1_{i,j}(t)| \leq C \cdot t^{\operatorname{Re}\lambda+\rho} \int_{\{\|X\|\geq t_0\}} |\xi(\operatorname{Ad}(a_t)^{-1}X)| \cdot (1+\|X\|)^q\, dX\,.$$

As $|\xi(X)| \leq C \cdot (1+\|X\|)^{-m}$ for some constants $C, m > 0$ we thus get that

$$|b^i_1(\tau,t)| \leq C \cdot t^{\operatorname{Re}\lambda+\rho} \int_{\{\|X\|\geq t_0\}} \frac{(1+\|X\|)^q}{(1+\|\operatorname{Ad}(a_t)^{-1}X\|)^m}\, dX\,.$$

By the definition of a_t we get that $\|\operatorname{Ad}(a_t)^{-1}X\| \geq t\|X\|$ and hence

$$|b^1_{i,j}(t)| \leq C \cdot t^{\operatorname{Re}\lambda+\rho} \int_{\{\|X\|\geq t_0\}} \frac{(1+\|X\|)^q}{(1+t\|X\|)^m}\, dX\,.$$

We continue this estimate by employing polar coordinates for $X \in \overline{\mathfrak{n}}$:

$$\begin{aligned}
|b^1_{i,j}(t)| &\leq C \cdot t^{\operatorname{Re}\lambda+\rho} \int_{t_0}^\infty \frac{r^n(1+r)^q}{(1+tr)^m}\, \frac{dr}{r}\\
&= C \cdot t_0^{n-2(\operatorname{Re}\lambda+\rho)} \int_1^\infty \frac{r^n(1+t_0r)^q}{(1+tt_0r)^m}\, \frac{dr}{r}\\
&= C \cdot t_0^{n-2(\operatorname{Re}\lambda+\rho)} \int_1^\infty \frac{r^n(1+t_0r)^q}{(1+t_0^{-1}r)^m}\, \frac{dr}{r}\\
&= C \cdot t_0^{n-2(\operatorname{Re}\lambda+\rho)+m} \int_1^\infty \frac{r^n(1+t_0r)^q}{(t_0+r)^m}\, \frac{dr}{r}\\
&\leq C \cdot t_0^{n-2(\operatorname{Re}\lambda+\rho)+m} \int_1^\infty r^{n+q-m}\, \frac{dr}{r}\,.
\end{aligned}$$

Henceforth we request that $m > n + q + 1$. Thus for every $m' > 0$ there exist a choice of ξ and a constant $C > 0$ such that

$$|b^1_{i,j}(t)| \leq C \cdot t^{-m'} . \tag{12.2}$$

Next we choose t in relationship to $|\tau|$. Basic finite dimensional representation theory yields that in a fixed compact neighborhood of $X = 0$ the gradient of $D^{i,j}_\tau$ is bounded by $C \cdot (1 + |\tau|)$ for a constant C independent of τ. Let $\gamma > 1$. Then for $\|X\| \leq (1 + |\tau|)^{-\gamma}$ the mean value theorem yields the following estimate

$$|D^{i,j}_\tau(X) - D^{i,j}_\tau(0)| \leq C \cdot (1 + |\tau|)^{-\gamma+1} \tag{12.3}$$

This brings us to our choice of t, namely

$$t = t(\tau) := (1 + |\tau|)^{2\gamma} .$$

Recall the definition of

$$I_\xi = \int_{\overline{\mathfrak{n}}} \xi(X)\, dX$$

Here we might face the obstacle that I_ξ might be zero. However as $\xi(X) = \widetilde{a}(n(X))^{\rho+\lambda-\mu}$ it follows $I \neq 0$ provided μ is large enough. So for any m' we find such a non-zero I_ξ.

In the following computation we will use the simple identity:

$$\int_{\overline{\mathfrak{n}}} \pi_\lambda(a_t) f(X)\, dX = t^{\lambda-\rho} \int_{\overline{\mathfrak{n}}} f(X)\, dX$$

for all integrable functions f. Now if $i \neq j$, then $D^{i,j}_\tau(0) = 0$ and we obtain from (12.2) and (12.3) that

$$\begin{aligned}
b_{i,j}(t) &= \int_{\{\|X\| \leq t_0\}} (\pi_\lambda(a_t)\xi)(X) \cdot \overline{D^{i,j}_\tau(X)}\, \widetilde{a}(\overline{n}(X))^{-2\operatorname{Re}\lambda} dX \\
&\quad + O\left(\frac{1}{(1 + |\tau|)^{2\gamma m'}}\right) \\
&= \int_{\{\|X\| \leq t_0\}} (\pi_\lambda(a_t)\xi)(X) \overline{\big(D^{i,j}_\tau(X) - D^{i,j}_\tau(0)\big)}\, \widetilde{a}(\overline{n}(X))^{-2\operatorname{Re}\lambda}\, dX \\
&\quad + O\left(\frac{1}{(1 + |\tau|)^{2\gamma m'}}\right) \\
&= t^{\lambda-\rho} \cdot \|\xi\|_1 \cdot O\left(\frac{1}{(1 + |\tau|)^{\gamma-1}}\right) + O\left(\frac{1}{(1 + |\tau|)^{2\gamma m'}}\right) .
\end{aligned}$$

For $i = j$ we have $D^{i,i}_\tau(0) = 1$ and we obtain in a similar fashion that

$$\begin{aligned} b_{i,i}(t) &= \int_{\{\|X\|\leq t_0\}} (\pi_\lambda(a_t)\xi)(X) \cdot \widetilde{a}(\overline{n}(X))^{-2\operatorname{Re}\lambda}\, dX \\ &\quad + t^{\lambda-\rho} \cdot \|\xi\|_1 \cdot O\left(\frac{1}{(1+|\tau|)^{\gamma-1}}\right) + O\left(\frac{1}{(1+|\tau|)^{2\gamma m'}}\right) \\ &= t^{\lambda-\rho} \cdot I_\xi + O\left(\frac{1}{(1+|\tau|)^{\gamma-1}}\right) + O\left(\frac{1}{(1+|\tau|)^{2\gamma m'}}\right). \end{aligned}$$

If we choose $c := 2\gamma$ and $\gamma - 1 = N$ and m' large enough, the assertion of the theorem follows. □

The proof of the theorem shows that the approximation can be made uniformly on any compact subset $Q \subset \mathfrak{a}_\mathbb{C}^*$. We further observe that $a_{t(\tau)}$ is bounded from above and below by powers of $1 + |\tau|$. If switch to the compact models $\mathcal{H}_\lambda = L^2(M\backslash K)$ and denote f_σ also by ξ, then an alternative version of the theorem is as follows:

Theorem 12.3. *Let $Q \subset \mathfrak{a}_\mathbb{C}^*$ be a compact subset and $N > 0$. Then there exists $\xi \in \mathbb{C}[M\backslash K]$ and constants $c_1, c_2 > 0$ such that for all $\tau \in \widehat{K}_M$, $\lambda \in Q$, there exists $a_\tau \in A$, independent of λ, with $\|a_\tau\| \leq (1+|\tau|)^{c_1}$ and numbers $b(\lambda, \tau) \in \mathbb{C}$ such that*

$$\|[\pi_\lambda(a_\tau)\xi]_\tau - b(\lambda,\tau)\delta_\tau\| \leq \frac{1}{(|\tau|+1)^{N+c_2}}$$

and

$$|b(\lambda,\tau)| \geq \frac{1}{(1+|\tau|)^{c_2}}\,.$$

Here $\|\cdot\|$ refers to the norm in $L^2(M\backslash K)$.

Finally we deduce the following lower bound for matrix coefficients. Recall the non-degenerate complex bilinear G-equivariant pairing $(\cdot,\cdot)$ between $\mathcal{H}_\lambda$ and $\mathcal{H}_{-\lambda}$.

Corollary 12.4. *Let $Q \subset \mathfrak{a}_\mathbb{C}^*$ be a compact subset. Then there exists $\xi \in \mathbb{C}[M\backslash K]$, constants $c_1, c_2, c_3 > 0$ such that*

$$\sup_{\substack{g\in G \\ \|g\|\leq(1+|\tau|)^{c_1}}} |(\pi_\lambda(g)\xi, v)| \geq c_2 \frac{1}{(1+|\tau|)^{c_3}} \|v\|$$

for all $\lambda \in Q$, $\tau \in \widehat{K}_M$ and $v \in V_{-\lambda}[\tau]$. Here $\|v\|$ refers to the norm on $\mathcal{H}_{-\lambda} = L^2(M\backslash K)$. In particular there exist a $s \in \mathbb{R}$ such that

$$\sup_{\substack{g\in G \\ \|g\|\leq(1+|\tau|)^{c_1}}} |(\pi_\lambda(g)\xi, v) \geq c_2 \|v\|_{s,K}$$

for all $\lambda \in Q$, $\tau \in \widehat{K}_M$ *and* $v \in V_{-\lambda}[\tau]$.

Thus Theorem 7.1 in conjunction with the above Corollary yields the Casselman-Wallach Theorem for spherical principal series:

Corollary 12.5. *Let* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ *and* V_λ *the Harish-Chandra module of the corresponding spherical principal series. Then* V_λ *admits a unique smooth Fréchet globalization.*

12.1.5. *Constructions in the Schwartz algebra.* Let us fix a relatively compact open neighborhood $Q \subset \mathfrak{a}_{\mathbb{C}}^*$. We choose the K-finite element $\xi \in \mathbb{C}[M\backslash K]$ such that the conclusion of Theorem 12.3 is satisfied.

Lemma 12.6. *Let* U *be an* $\mathrm{Ad}(K)$*-invariant neighborhood of* $\mathbf{1}$ *in* G *and* $\mathcal{F}(U)$ *the space of* $\mathrm{Ad}(K)$*-invariant test functions supported in* U*. Then there exists a holomorphic map*

$$Q \to \mathcal{F}(U), \quad \lambda \mapsto h_\lambda$$

such that $\Pi_\lambda(h_\lambda)\xi = \xi$.

Proof. Let $V_\xi \subset \mathbb{C}[M\backslash K]$ be the K-module generated by ξ. Let $n := \dim V_\xi$. Let U_0 be a $\mathrm{Ad}(K)$-invariant neighborhood of $\mathbf{1} \in G$ such that $U_0^n \subset U$.

Note that any $h \in \mathcal{F}(U_0)$ induces operators

$$T(\lambda) := \Pi_\lambda(h)|_{V_\xi} \in \mathrm{End}(V_\xi)\,.$$

The compactness of Q allows us to employ uniform Dirac-approximation: we can choose h such that

$$Q \to \mathrm{Gl}(V_\xi), \quad \lambda \mapsto T(\lambda)$$

is defined and holomorphic. Let $n := \dim V_\xi$. By Cayley-Hamilton $T(\lambda)$ is a zero of its characteristic polynomial and hence

$$\mathrm{id}_{V_\xi} = \frac{1}{\det T(\lambda)} \sum_{j=1}^{n} c_j(\lambda) T(\lambda)^j$$

with $c_j(\lambda)$ holomorphic. Set now

$$h_\lambda := \frac{1}{\det T(\lambda)} \sum_{j=1}^{n} c_j(\lambda) \underbrace{h_\lambda * \ldots * h_\lambda}_{\text{j-times}}\,.$$

Then $Q \ni \lambda \mapsto h_\lambda \in \mathcal{F}(U)$ is holomorphic and $\Pi_\lambda(h_\lambda)\xi = \xi$. $\square$

For a compactly supported measure ν on G and $f \in \mathcal{S}(G)$ we define $\nu * f \in \mathcal{S}(G)$ by

$$\nu * f(g) = \int_G f(x^{-1}g)\, d\nu(x)\,.$$

For an element $g \in G$ we denote by δ_g the Dirac delta-distribution at g. Further we view δ_τ as a compactly supported measure on G via the correspondence $\delta_\tau \leftrightarrow \delta_\tau(k)\, dk$.

For each $\tau \in \widehat{K}_M$ we define $h_{\lambda,\tau} \in \mathcal{S}(G)$ by

$$h_{\lambda,\tau} := \delta_\tau * \delta_{a_{t(\tau)}} * h_\lambda\,. \tag{12.4}$$

Call a sequence $(c_\tau)_{\tau\in\widehat{K}_M}$ rapidly decreasing if

$$\sup_\tau |c_\tau|(1+|\tau|)^R < \infty$$

for all $R > 0$.

Lemma 12.7. *Let $(c_\tau)_\tau$ be a rapidly decreasing sequence $(c_\tau)_\tau$ and $h_{\lambda,\tau}$ defined as in (12.4). Then*

$$H_\lambda := \sum_{\tau\in\widehat{K}_M} c_\tau \cdot h_{\lambda,\tau}$$

is in $\mathcal{S}(G)$ and the assignment $Q \ni \lambda \to H_\lambda \in \mathcal{S}(G)$ is holomorphic.

Proof. Fix $\lambda \in Q$. For simplicity set $H = H_\lambda$, $h_{\lambda,\tau} = h_\tau$.

It is clear that the convergence of H is uniform on compacta and hence $H \in C(G)$. For $u \in \mathcal{U}(\mathfrak{g})$ we record

$$R_u(h_\tau) = \delta_\tau * \delta_{a_{t(\tau)}} * R_u(h)$$

and as a result $H \in C^\infty(G)$. So we do not have to worry about right derivatives. To show that $H \in \mathcal{S}(G)$ we employ Remark 2.18: it remains to show that $H \in \mathcal{R}(G)$, i.e.

$$\sup_{g\in g} \|g\|^r \cdot |H(g)| < \infty \tag{12.5}$$

for all $r > 0$. Fix $r > 0$. Write $g = k_1 a k_2$ for some $a \in A$, $k_1, k_2 \in K$. Then

$$\|g\|^r |h_\tau(g)| \leq \|a\|^r \cdot \sup_{k,k'\in K} |h(a_t^{-1} k a k')|\,.$$

Let $Q \subset A$ be a compact set with $\log Q$ convex and $\mathcal{W}$-invariant and such that $\operatorname{supp} h \subset KQK$. We have to determine those $a \in A$ with

$$a_t^{-1} K a \cap KQK \neq \emptyset \,. \tag{12.6}$$

Define $Q_t \subset A$ through $\log Q_t$ being the convex hull of $\mathcal{W}(\log a_t + \log Q)$. Then (12.6) implies that

$$a \in Q_t \,.$$

But this means that $\|a\| << |\tau|^c$ for some $c > 0$, independent of τ. Hence (12.5) is verified and H is indeed in $\mathcal{S}(G)$.

Finally the fact that the assignment $\lambda \mapsto H_\lambda$ is holomorphic follows from the previous Lemma. □

Theorem 12.8. *Let $Q \subset \mathfrak{a}_\mathbb{C}^*$ be a compact subset. Then there exist a continuous map*

$$Q \times C^\infty(M\backslash K) \to \mathcal{S}(G), \quad (\lambda, v) \mapsto f(\lambda, v)$$

which is holomorphic in the first variable, linear in the second and such that

$$\Pi_\lambda(f(\lambda, v))\xi = v \,.$$

In particular, $\Pi_\lambda(\mathcal{S}(G))V_\lambda = \mathcal{H}_\lambda^\infty$ for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$.

Proof. Let $v \in \mathcal{H}_\lambda^\infty$. Then $v = \sum_\tau c_\tau v_\tau$ with v_τ normalized and $(c_\tau)_\tau$ rapidly decreasing. As $\mathcal{S}(G)$ is stable under left convolution with $C^{-\infty}(K)$ we readily reduce to the case where $v_\tau = \frac{1}{\sqrt{d(\tau)}}\delta_\tau$.

In order to explain the idea of the proof let us first treat the case where the Harish-Chandra module is a multiplicity free K-module. This is for instance satisfied when $G = \mathrm{Sl}(2, \mathbb{R})$.

Recall the numbers $b(\lambda, \tau)$ from Theorem 12.3 and define

$$H_\lambda := \sum_\tau \frac{c_\tau}{\sqrt{d(\tau)} \cdot b(\lambda, \tau)} h_{\lambda,\tau} \,.$$

It follows from Theorem 12.3 and the Lemma above that $Q \ni \lambda \to H_\lambda \in \mathcal{S}(G)$ is defined and holomorphic. By multiplicity one we get that

$$\Pi_\lambda(H_\lambda)\xi = \sum_\tau c_\tau v_\tau \,.$$

and the assertion follows for the multiplicity free case.

Let us move to the general case. For that we employ the more general approximation in Theorem 12.3 and set

$$H'_\lambda = \sum_{\tau \in \widehat{K}_M} \frac{c_\tau}{\sqrt{d(\tau)} \cdot b(\lambda, \tau)} h_{\lambda,\tau} \,.$$

Then

$$\Pi_\lambda(H'_\lambda)\xi = \sum_{\tau\in\widehat{K}_M} c_\tau v_\tau + R$$

where, given $k > 0$, we can assume that $\|R_\tau\| \leq |c_\tau| \cdot (|\tau| + 1)^{-k}$ for all τ (choose N in Theorem 12.3 big enough). Finally we remove the remainder R_τ by left convolution with $C^{-\infty}(K)$ (use the Neumann series $(\mathrm{id} + R)^{-1}$). □

13. Appendix B: On the meromorphic extension of certain distributions on $G \times G$

Let X and Y be real smooth affine varieties. We may view X, resp. Y, as Zariski closed subsets in $\mathbb{R}^n$, resp. $\mathbb{R}^m$. Let $\mathcal{S}'(X \times Y)$ the space of tempered distributions on $X \times Y$. Further let $p : Y \to \mathbb{R}^+$ be a positive polynomial function such that $p(y) > \|y\| \quad (y \in Y)$ for some norm $\|\cdot\|$ on $\mathbb{R}^m$. We consider the canonical projections $\pi : X \times Y \to X$ and $\rho : X \times Y \to Y$. Let $\mathcal{E} \in \mathcal{S}'(X \times Y)$. Since $\mathcal{E}$ is tempered, there exists an $r_0 > 0$ such that for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > r_0$ the prescription

$$\mathcal{I}(\lambda) := \pi_*(p^{-\lambda}\mathcal{E})$$

defines a distribution on X. Furthermore, if $D'(X)$ denotes the topological vector space of distribution on X, then the assignment

$$\{z \in \mathbb{C} \mid \mathrm{Re}\, z > r_0\} \ni \lambda \to \mathcal{I}(\lambda) \in D'(X)\,.$$

is meromorphic. Note the formula

$$\mathcal{I}(\lambda)(\phi) = \mathcal{E}(p^{-\lambda} \circ \rho \cdot \phi \circ \pi) \qquad (\phi \in C_c^\infty(X))\,.$$

Let $\mathcal{D} = \mathcal{D}(X \times Y)$ be the ring of differential operators on $X \times Y$ with polynomial coefficients.

Suppose now that $\mathcal{E}$ is holonomic, that is the $\mathcal{D}$-module generated by $\mathcal{E}$ in $\mathcal{S}'(X \times Y)$ is holonomic. A slight modification of the main result in [1] then yields differential operators $d_1(\lambda), \ldots, d_k(\lambda) \in \mathcal{D}(X)$, polynomially depending on λ, as well as a polynomial function $b(\lambda)$ such that

$$b(\lambda)\mathcal{I}(\lambda + k) = d_1(\lambda)\mathcal{I}(\lambda + k - 1) + \ldots + d_k(\lambda)\mathcal{I}(\lambda)\,.$$

In particular, $\mathcal{I}(\lambda)$ admits a meromorphic continuation as a distribution on X.

In the sequel we will use this result for $X = G \times G$ and $Y = G$. Let V be a Harish-Chandra module of a discrete series representations. Let $I = \mathrm{Ind}_{P_{\min}}^G(\sigma_1)$ be a minimal principal series representation with respect to an irreducible representation σ_1 of $P_{\min}/N = M \times A$ such that $V \hookrightarrow I$. Likewise we choose a minimal principal series $J := \mathrm{Ind}_{P_{\min}}^G(\sigma_2)$ such that $\widetilde{V} \hookrightarrow J$. Write W_1, W_2 for the representation module for σ_1, resp. σ_2. Let $\nu_{1,2} \in W_{1.2}$ be fixed non-zero elements. Let us consider the continuous surjections:

$$\Phi : C_c^\infty(G) \to I^\infty, \quad \phi \mapsto \left(g \mapsto \int_{P_{\min}} \phi(pg)\sigma_1(p)^{-1}\nu_1 \, dp \right)$$

and

$$\Psi : C_c^\infty(G) \to J^\infty, \quad \psi \mapsto \left(g \mapsto \int_{P_{\min}} \psi(pg)\sigma_2(p)^{-1}\nu_2 \, dp \right).$$

Further we let $\xi \in \widetilde{I}$ and $\eta \in \widetilde{J}$ be such that the maps

$$V \to C^\infty(G), \quad v \mapsto m_{\xi,v}$$
$$\widetilde{V} \to C^\infty(G), \quad w \mapsto m_{\eta,w}$$

are injective.

Now for $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda$ sufficiently big, the prescription

$$\mathcal{I}(\lambda)(\phi, \psi) := \int_G m_{\xi,\Phi(\phi)}(g) m_{\eta,\Psi(\psi)}(g) \|g\|^{-\lambda} \, dg \qquad (\phi, \psi \in C_c^\infty(G))$$

defines a distribution on $G \times G$. We claim that $\mathcal{I}(\lambda)$ admits a meromorphic continuation to the complex plane. In fact

$$\mathcal{E}(g, h_1, h_2) := \xi(h_1 g^{-1})(\nu_1) \cdot \eta(h_2 g^{-1})(\nu_2)$$

defines a moderately growing function on $G \times G \times G$, hence a tempered distribution. With respect to the first variable projection $\pi : G \times G \times G \to G$ we readily verify the identity

$$\mathcal{I}(\lambda) = \pi_*(\| \cdot \|^{-\lambda} \mathcal{E}).$$

The fact that $\mathcal{E}$ is holonomic implies the claim.

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