THE ANALYTIC CONTINUATION OF GENERALIZED FUNCTIONS WITH RESPECT TO A PARAMETER

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Let P be a polynomial in N variables with real coefficients, and let Θ be a region in the space $\mathbf{R}^n$ such that P is nonnegative inside Θ and is equal to zero on the boundary.

Let λ be a complex number with Re $\lambda > 0$. We define a continuous function $P_\Theta(\lambda)$ by the formula $P_\Theta(\lambda)(x) = \{P(x)^\lambda$ for $x \in \Theta$ and 0 for $x \notin \Theta\}$. We shall consider $P_\Theta(\lambda)$ as a function of λ with values in the space S' of slowly increasing generalized functions. It is clear that for Re $\lambda > 0$ $P_\Theta(\lambda)$ is an analytic function of λ. In the first chapter we shall prove the following theorem.

THEOREM 1. $P_\Theta(\lambda)$ extends as a meromorphic function of λ to the entire complex plane Λ of the variable λ. The poles of this function lie on a finite number of arithmetic progressions A_i, where $A_i = \{\lambda_i - n \mid n = 0, 1, 2, \ldots\}$.

This theorem (in a stronger form) has been proved in [1] and [2] using a theorem of Hironaka on the resolution of singularities. Our proof makes no use of the resolution of singularities and is therefore considerably simpler.

We make use of the method of analytic continuation applied by Riesz in [7] for the case of quadratic polynomials. Indeed, Theorem 1 follows from the following theorem.

THEOREM 1'. There exist a differential operator $\mathscr{D}_P$ with polynomial coefficients which has polynomial dependence on λ and a nonzero polynomial d_P in λ such that for all λ

$$\mathscr{D}_P(\lambda)(P_\Theta(\lambda + 1)) = d_P(\lambda) P_\Theta(\lambda).$$

The derivation of Theorem 1 from Theorem 1' can be found in [5] (Ch. III) and in [4].

The proof of Theorem 1' is purely algebraic; it is based on the study of modules over the ring D of differential operators with polynomials coefficients.

In the second chapter we study integral transformations in the space S'.

Suppose that there is given a polynomial mapping A: $X \to Y$, where X and Y are finite-dimensional linear spaces over $\mathbf{R}$. From a generalized function $\mathscr{E} \in S'_Y$ we wish to construct the "corresponding" function $A^*\mathscr{E} \in S'_X$. Such a construction can be carried out for functions $\mathscr{E} \in S'_{Y_0}$, where the space S'_{Y_0}, which is defined in [3], consists of functions which satisfy a "large" system of differential equations with polynomial coefficients (see Definition 4.2 and Theorem 4.3). With the same methods it is possible to obtain a number of interesting corollaries which are gathered together at the end of §4. Here is one of them.

Let P be a positive polynomial in N variables which increases at infinity. We consider the integral $\int P^{-\lambda} dx_1 \cdot \ldots \cdot dx_N$. When Re λ is large it is defined and gives an analytic function $f(\lambda)$.

Proposition. The function $f(\lambda)$ extends as a meromorphic function to the entire complex plane of the variable λ and satisfies the following equation which is similar to the functional equation for the Γ-function:

$$f(\lambda) = a_1(\lambda) f(\lambda + 1) + \ldots + a_k(\lambda) f(\lambda + k),$$

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where $a_1, \ldots, a_k$ are certain rational functions of λ.

CHAPTER I

THE ANALYTIC CONTINUATION OF THE FUNCTION $P_\Theta(\lambda)$

§Modules over a Ring of Differential Operators

Let K be a field of characteristic zero, and let X be a finite-dimensional linear space over K. We denote by $R_X(K)$ the ring of polynomial functions on X and by $D_X(K)$ the ring of differential operators with polynomial coefficients on X. If $x_1, \ldots, x_N$ are coordinates on X, then $R_X(K) = K[x_1, \ldots, x_N]$, and $D_X(K)$ is an algebra over K with generators $x_1, \ldots, x_N$, $\partial/\partial x_1, \ldots, \partial/\partial x_N$ and the relations

$$[x_i, x_j] = \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0, \quad \left[\frac{\partial}{\partial x_i}, x_j\right] = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol.

If some argument is valid for any K and X or for a K and X given beforehand, then in place of $D_X(K)$ we shall write D_X, $D(K)$, $D_N (N = \dim X)$, or simply D.

In D_N we fix a filtration $D^0 \subset D^1 \subset \ldots \subset D^n \subset \ldots$, where D^n is the linear subspace of D_N consisting of polynomials of degree no greater than n in the generators x_i and $\partial/\partial x_j$.

The associated graded ring $\Sigma_N = \bigoplus_{n=0}^{\infty} \Sigma^{(n)}$ (where $\Sigma^{(n)} = D^n/D^{n-1}$) is a ring of polynomials in the generators $x_1, \ldots, x_N$, $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N} \in \Sigma^{(1)}$. The spaces $\Sigma^n = \bigoplus_{i=0}^{n} \Sigma^{(i)}$ give a natural filtration in the ring of polynomials Σ.

We shall consider modules over the ring $D_X(K)$.

If M is a D_N-module* and $f_1, \ldots, f_s$ is a system of generators, then we set $M^n = D^n(f_1, \ldots, f_s)$ and $d_M(n) = \dim M^n$.

Proposition 1.1. $d_M(n)$ is a polynomial in n for large n.

Proof. Let $\bar{M}$ be the free module with generators $g_1, \ldots, g_s$, let ρ: $\bar{M} \to M$ be the mapping given by $\rho(g_i) = f_i$, and let $L = \operatorname{Ker} \rho$. It is clear that $\bar{M}^n/L \cap M^n$, i.e., $\dim M^n = \dim \bar{M}^n - \dim (L \cap \bar{M}^n)$.

Since $\dim \bar{M}^n = s \cdot \dim D^n = s \cdot \dim \Sigma^n = s \cdot \binom{n+2N}{2N}$, it suffices to show that $\dim (L \cap \bar{M}^n)$ is a polynomial in n for large n.

We set $\bar{M}_\Sigma = \bigoplus_{n=0}^{\infty} \bar{M}^n/\bar{M}^{n-1}$ and $L_\Sigma = \bigoplus_{n=0}^{n} L \cap \bar{M}^n/L \cap \bar{M}^{n-1} \subset \bar{M}_\Sigma$.

It is easy to verify that

a) $\bar{M}_\Sigma$ is a free Σ-module and L_Σ is a Σ-submodule of $\bar{M}_\Sigma$;

b) $\dim (L_\Sigma \cap M^n_\Sigma = \dim L \cap M^n$, where $\bar{M}^n_\Sigma = \bigoplus_{i=0}^{n} \bar{M}^i/\bar{M}^{i-1}$.

Proposition 1.1 now follows easily from the following proposition.

Proposition 1.1' (see [9], Theorem 4.1). Let Σ be a ring of polynomials, let H be a free Σ-module with the natural filtration H^n, and let E be a Σ-submodule in H. Then $\dim (E \cap H^n)$ is a polynomial in n for large n.

Definition 1.1. Let M be a finitely generated D-module, and let $f_1, \ldots, f_s$ be a system of generators. We denote by d(M) the degree of the polynomial $d_M(n)$ and set $e(M) = a \cdot d(M)!$, where a is the leading coefficient of the polynomial $d_M(n)$.

LEMMA 1.2. 1) d(M) and e(M) do not depend on the choice of the system of generators $f_1, \ldots, f_s$.

*Unless otherwise specified, we assume that M is a left D_N-module. However, all definitions and results of this section go over without change to the case of right D_N-modules.

2) e(M) is a natural number.

3) If $0 \to M_1 \to M \to M_2 \to 0$ is an exact sequence of D-modules, then $d(M) = \max(d(M_1), d(M_2))$, and $e(M) = e(M_1)$ (or $e(M_2)$) if $d(M_1) > d(M_2)$ (or $d(M_2) > d(M_1)$), $e(M) = e(M_1 + e(M_2)$, if $d(M_1) = d(M_2)$.

Proof. 1) If two systems of generators and the filtrations $\{M^n\}$ and $\{\tilde{M}^n\}$, corresponding to them are given, then it is clear that $M^{n-k} \subset \tilde{M}^n \subset M^{n+k}$ for some k. Therefore, the polynomial $d_M(n)$ is defined up to polynomials of lower degree, i.e., d(M) and e(M) are uniquely defined.

2) Since $d_M(n)$ assumes integer values, it is a linear combination with integer coefficients of polynomials of the form $\binom{n}{i}$ (see [9], Ch. VII). From this it follows that e(M) is an integer. The number e(M) is positive, since $d_M(n) \geq 0$.

3) Let $f_1, \ldots, f_s$ be a system of generators in M. Then their images in M_2 form a system of generators. It is clear that $M_2^n = M^n/M_1 \cap M^n$, and therefore $d_{M_2}(n) = d_M(n) - d'_{M_1}(n)$, where $d'_{M_1}(n) = \dim(M_1 \cap M^n)$. As has been shown in [3] (Proposition 1.3), for some k we have $M_1^{n-k} \subset M_1 \cap M^n \subset M_1^{n+k}$, i.e., $d'_{M_1}(n) - d_{M_1}(n)$ has degree less than $d_{M_1}(n)$. This implies the required formulas.

The numbers d(M) and e(M) characterize the "functional dimension" of the finitely generated D-module M. We shall need similar characterizations for D-modules which are not finitely generated.

Definition 1.2. Let M be a D-module, and let $d \geq 0$, $e > 0$ be integers. A (d, e)-filtration of the module M is a system of subspaces $M^0 \subset M^1 \subset \ldots \subset M^n \subset \ldots$ in M such that

a) $D^i M^n \subset M^{n+i}$, $\bigcup_n M^n = M$,

b) $\dim M^n \leq (e/d!)n^d + o(n^d)$.

If M is a finitely generated D-module, then the standard filtration $\{M^n\}$ is a (d(M), e(M))-filtration. It is clear that if a D-module M has a (d, e)-filtration, then for any finitely generated submodule $L \subset M$ either $d(L) < d$, or $d(L) = d$ and $e(L) \leq e$ (it will be shown below that this is a sufficient condition for the existence of a (d, e)-filtration).

THEOREM 1.3. Let M be a finitely generated D_N-module. Then either $M = 0$ or $d(M) \geq N$.

The proof of this theorem will be given in §5.

COROLLARY 1.4. Suppose that a D_N-module M admits a (d, e)-filtration. Then

a) $d < N$, implies that $M = 0$,

b) if $d = N$, then the module M has finite length not exceeding e (and, in particular, the module M is finitely generated).

Proof. a) Part a) follows immediately from Theorem 1.3. We shall prove b).

In M let there be given submodules $0 = M_0 \subset M_1 \subset \ldots \subset M_k = M$, with $M_{i-1} \neq M_i$ $(i = 1, 2, \ldots, k)$.

We will show that $k \leq e$. We choose elements $f_1, \ldots, f_k$ such that $f_i \in M_i$ and $f_i \notin M_{i-1}$, and we set $L_i = D_N(f_1, \ldots, f_i)$. The module L_k admits an (N, e)-filtration (as a submodule of M), and therefore $d(L_i/L_{i-1}) \leq N$ $(i = 1, \ldots, k)$. By Theorem 1.3 $d(L_i/L_{i-1}) = N$. Since $e(L_i/L_{i-1}) \geq 1$, it follows from Lemma 1.2 that $e(L_k) \geq k$. This means that $k \leq e(L_k) \leq e$, i.e., the length of M does not exceed e.

§2. Proof of Theorem 1'

We first present a purely algebraic formulation of Theorem 1'.

Definition 2.1. Let P be a polynomial in N variables over the field **C**. We construct over the ring $D_N(\mathbf{C}(\lambda))$ (where $\mathbf{C}(\lambda)$ is the field of rational functions of the variable λ) a module M_P as follows: the elements of the module M_P are expressions of the form $Q \cdot P^{\lambda-k}$, where Q is a polynomial in $x_1, \ldots, x_N$ with coefficients in $\mathbf{C}(\lambda)$. (We identify the expressions $Q \cdot P^{\lambda-k}$ and $Q' \cdot P^{\lambda-n}$, if $Q \cdot P^n = Q' \cdot P^k$.) The action of the ring $D_N(\mathbf{C}(\lambda))$ on M_P is defined by the following formulas:

$$x_i(Q \cdot P^{\lambda-k}) = (x_i Q)\, P^{\lambda-k},$$
$$\frac{\partial}{\partial x_i}(Q \cdot P^{\lambda-k}) = \frac{\partial Q}{\partial x_i} P^{\lambda-k} + (\lambda - k)\frac{\partial P}{\partial x_i} Q \cdot P^{\lambda-k-1}.$$

Definition 2.2. We denote by S^{Λ} the space of analytic functions $\mathscr{E}(\lambda)$ of the variable $\lambda \in \Lambda$ with values in S' defined in a region $\mathrm{Re}\,\lambda > C$ (where the constant C depends on $\mathscr{E}$). We regard the functions $\mathscr{E}$ and $\mathscr{E}'$ as defining the same element in S^{Λ} if they agree in some region $\mathrm{Re}\,\lambda > C$.

S^{Λ} is equipped in a natural way with the structure of a $D(C(\lambda))$-module.

LEMMA 2.1. The mapping Ψ: $M_P \to S^{\Lambda}$, given by the formula $\Psi(Q \cdot P^{\lambda-k}) = Q \cdot P_{\oplus}(\lambda - k)$, is a mapping of $D_N(C(\lambda))$-modules.

The proof is by direct computation; use is hereby made of the fact that for $\mathrm{Re}\,\lambda > m$ $P_{\oplus}(\lambda)$ is an m times continuous differentiable function.

Lemma 2.1 reduces the proof of Theorem 1' to the study of the module M_P. Indeed, it must be shown that there exists an operator $\mathscr{D} \in D_N(C(\lambda))$ such that $\mathscr{D}(P \cdot P^{\lambda}) = P^{\lambda}$.

To this end we consider in M_P the filtration $M_P^n = \{Q \cdot P^{\lambda-k}$, where $\deg Q \leq (p+1)n\}$ (here p is the degree of the polynomial P).

It is easy to verify that the filtration $\{M_P^n\}$ is an $N\ (p+1)^N)$-filtration and therefore the $D_N(C(\lambda))$-module M_P has finite length.

In M_P we consider the increasing sequence of submodules $M_i = D_N(C(\lambda))\ (P^{\lambda-i})$. Since the module M_P has finite length, $M_{i-1} = M_i$ for some i. In other words, there exists an operator $\mathscr{D}_i \in D_N(C(\lambda))$ such that $\mathscr{D}_i(P^{\lambda-i+1}) = P^{\lambda-i}$.

If now in the coefficients of the operator $\mathscr{D}_i$ we let $\lambda - i \to \lambda$, then we obtained the required operator $\mathscr{D}$ such that $\mathscr{D}(P \cdot P^{\lambda}) = P^{\lambda}$. This completes the proof of Theorems 1 and 1'.

CHAPTER II

INTEGRAL TRANSFORMATIONS IN THE SPACE S'

§3. Algebraic Constructions

In this chapter we shall apply the methods of Chapter I to the regularization of certain integral transformations in the space S'.

We first state precisely what we mean by the space of generalized functions S' and the space of generalized forms Ω'.

Let X be an N-dimensional space over the field $\mathbf{R}$, $x_1, \ldots, x_N$ be coordinates on X. We denote by S (by Ω) the space of infinitely differentiable functions (differential forms of degree N) on X which are rapidly decreasing together with all derivatives. We provide S and Ω with the usual topology (see [5]). The form $dx_1 \ldots dx_N$ gives an isomorphism $S \to \Omega$ $(\varphi \to \varphi dx_1 \ldots d \times N)$.

The bilinear form $(\varphi, \omega) = \int \varphi\omega\ (\varphi \in S,\ \omega \in \Omega)$ provides a pairing of the spaces S and Ω.

We consider on S the natural structure of a left D_X-module (in the case of the field of real numbers we mean by D_X the ring $D_X(\mathbf{C})$). Then Ω has a unique structure of a right D_X-module such that $(\varphi, \omega\mathscr{D}) = (\mathscr{D}\varphi, \omega)$ for all $\varphi \in S$, $\omega \in \Omega$, $\mathscr{D} \in D_X$. Indeed, if $\mathscr{D}$ is a polynomial in the x_i, then $\omega\mathscr{D} = \mathscr{D} \cdot \omega$; if $\mathscr{D}$ is a vector field, then $\omega\mathscr{D} = -L_{\mathscr{D}}\omega$, where $L_{\mathscr{D}}\omega$ is the Lie derivative along the field $\mathscr{D}$ of the form ω.

We set $S'_X = \Omega^*$ and $\Omega'_X = S^*$. We define on S'_X the structure of a left D_X-module and on $\langle \mathscr{D}\mathscr{E}, \omega \rangle = \langle \mathscr{E}, \omega\mathscr{D} \rangle$, $\langle \mathscr{F}\mathscr{D}, \varphi \rangle = \langle \mathscr{F}, \mathscr{D}\varphi \rangle$, where $\mathscr{D} \in D_X$, $\mathscr{E} \in S'_X$, $\mathscr{F} \in \Omega'_X$, $\varphi \in S$ $\omega \in \Omega$, $\langle , \rangle$ —is the pairing of S' with Ω and Ω' with S. The form (φ, ω) here gives a homomorphism of D_X-modules $S \to S'_X$ and $\Omega \to \Omega'_X$.

We denote by F: $S'_X \to S'_X$ the Fourier transform (F depends on the choice of coordinates). As is known (see [5]), for any function

$$F(x_j\mathscr{E}) = -i\frac{\partial}{\partial x_j}F\mathscr{E}, \quad F\left(\frac{\partial}{\partial x_j}\mathscr{E}\right) = -ix_jF\mathscr{E}.$$

In this chapter we shall consider the following situations.

I. Let X and Y be finite-dimensional spaces over $\mathbf{R}$, and let A: $X \to Y$ be a polynomial mapping. If $\mathscr{E}$ is a continuous function on Y, then it is possible to define a function $A^*\mathscr{E}$ on X by the equation $A^*\mathscr{E}(x) = \mathscr{E}(Ax)$. We wish to define the operation A^* on functions $\mathscr{E} \in S'_Y$.

II. Let A: X → Y be a polynomial mapping as before. We wish to define the operator A_* of integration on sections. This operation must take a form of Ω'_X into a form in Ω'_Y. For example, if $\mathcal{F} \in \Omega'_X$ has compact support, then it is possible to define the form $A_*\mathcal{F} \in \Omega'_Y$ by the equation $\langle A_*\mathcal{F}, \varphi\rangle = \langle \mathcal{F}, A^*\varphi\rangle$, where $\varphi \in S_Y$.

III. We wish to define the product $\mathcal{E} = \mathcal{E}_1 \cdot \mathcal{E}_2$ of generalized functions $\mathcal{E}_1$ and $\mathcal{E}_2$ and also the product of a function $\mathcal{E} \in S'_X$ and a form $\mathcal{F} \in \Omega'_X$.

Situation I has been considered in detail in the case dim Y = 1 in [3]. We are interested in the following two questions:

1) how to define the operations A^*, A_*, and the product for a sufficiently broad class of generalized functions;

2) what equations (with polynomial coefficients) should the functions so obtained satisfy.

We first take up the second question. We shall present the algebraic constructions for the situations I, II, and III.

Definition 3.1. Let K be a field of characteristic zero, let X and Y be finite-dimensional spaces over K, and let A: X → Y be a polynomial mapping with coefficients in K. Let $x_1, \ldots, x_N$ and $y_1, \ldots, y_m$ be coordinates on X and Y, and let A_j be the expression of y_j as a polynomial in the x_i.

1. Let M be a left D_Y-module. We construct a left D_X-module A^*M as follows: as an R_X-module $A^*M = R_X \underset{R_Y}{\otimes} M$ (where in R_X the structure of an R_Y algebra is defined by means of the mapping A^*: $R_Y \to R_X$), and the operators $\partial/\partial x_i$ act as follows:

$$\frac{\partial}{\partial x_i}(Q \otimes f) = \frac{\partial Q}{\partial x_i} \otimes f + \sum_{j=1}^{m} Q \frac{\partial A_j}{\partial x_i} \otimes \frac{\partial}{\partial y_j} f \qquad (Q \in R_X,\ f \in M).$$

If $f \in M$, then we set $A^*f = 1 \otimes f \in A^*M$.

2. If L is a right D_X-module, then we define a right D_Y-module A_*L as follows:

$$A_*L = (L \underset{R_Y}{\otimes} D_Y)/L_0,$$

where L_0 is the subspace generated by the elements

$$\left\{f \frac{\partial}{\partial x_i} \otimes \mathcal{D} - \sum_{j=1}^{m} f \frac{\partial A_j}{\partial x_i} \otimes \frac{\partial}{\partial y_j}\mathcal{D}\right\} \qquad (f \in L,\ \mathcal{D} \in D_Y).$$

The structure of a D_Y-module is introduced by the equation $(f \otimes \mathcal{D})\mathcal{D}_1 = f \otimes \mathcal{D}\mathcal{D}_1$ $(f \in L,\ \mathcal{D},\ \mathcal{D}_1 \in D_Y)$. If $f \in L$, then we set $A_*f = f \otimes 1 \in A_*L$.

3. a) If M_1 and M_2 are left D_X-modules, we define the D_X-module $M_1 \boxtimes M_2$, as follows: as an R_X-module $M_1 \boxtimes M_2 = M_1 \underset{R_X}{\otimes} M_2$, and the operators $\partial/\partial x_i$ act as follows:

$$\frac{\partial}{\partial x_i}(f_1 \otimes f_2) = \frac{\partial}{\partial x_i} f_1 \otimes f_2 + f_1 \otimes \frac{\partial}{\partial x_i} f_2 \qquad (f_1 \in M_1,\ f_2 \in M_2).$$

If $f_1 \in M_1$ and $f_2 \in M_2$, then we set $f_1 \boxtimes f_2 = f_1 \otimes f_2 \in M_1 \boxtimes M_2$

b) If M is a left and L a right D_X-module, then we consider the right D_X-module, $L \boxtimes_0 M = L \underset{R_X}{\otimes} M$ in which the operators $\partial/\partial x_i$ act as follows:

$$(g \otimes f)\frac{\partial}{\partial x_i} = g \frac{\partial}{\partial x_i} \otimes f - g \otimes \frac{\partial}{\partial x_i} f \qquad (g \in L,\ f \in M).$$

Remarks. 1. Definition 3.1 (in the case K = R) agrees with the natural representations. For example, the natural mappings A^*: $S_Y \to S'_X$, A_*: $\Omega_X \to \Omega'_Y$ and $S_X \times S_X \to S_X$ extend to mappings of D-modules A^*S_Y into S'_X, $A_*\Omega_X$ into Ω'_Y and $S_X \boxtimes S_X$ into S_X.

If $\mathcal{E} \in S'_Y$, then it is natural to suppose that the function $A^*\mathcal{E} \in S'_X$ "must" satisfy the same equations as the element $A^*\mathcal{E}$ satisfies in the D_X-module $A^*(D_Y(\mathcal{E}))$ (similarly for A_*, $\boxtimes$ and $\boxtimes_0$).

2. The modules A^*M and A_*L can be obtained from a single construction. Indeed, for any right D_X-module L and left D_Y-module M we consider the linear space $\langle L, M\rangle = (L \underset{R_Y}{\otimes} M)/LM_0$, where LM_0 is the subspace generated by the elements

$$\left\{g \frac{\partial}{\partial x_i} \otimes f - \sum_{j=1}^{m} g \frac{\partial A_j}{\partial x_i} \otimes \frac{\partial}{\partial y_j} f\right\} \qquad (g \in L,\ f \in M).$$

Then $A^*M = \langle D_X, M\rangle$, $A_*L = \langle L, D_Y\rangle$. Moreover, $\langle L, M\rangle = A_*L \underset{D_Y}{\otimes} M = L \underset{D_X}{\otimes} A^*M$.

The following proposition describes the properties of the operations A^*, A_*, $\boxtimes$, and $\boxtimes_0$.

Proposition 3.1. 1) The operations A^*, A_*, $\boxtimes$ and $\boxtimes_0$ are well defined and do not depend on the choice of systems of coordinates on X and Y.

2) For a finite-dimensional space Z over a field K we denote by $\mathscr{L}_Z$ ($\mathscr{R}_Z$) the category of left (right) D_Z-modules. Then the operations A^* and A_* define functors $A^*: \mathscr{L}_Y \to \mathscr{L}_X$, $A_*: \mathscr{R}_X \to \mathscr{R}_Y$. The operations $\boxtimes$ and $\boxtimes_0$ define bifunctors $\boxtimes: (\mathscr{L}_X, \mathscr{L}_X) \to \mathscr{L}_X$, $\boxtimes_0: (\mathscr{R}_X, \mathscr{L}_X) \to \mathscr{R}_X$.

3) Suppose that polynomial mappings A: X → Y and B: Y → Z are given. Then $(BA)^* = A^*B^*$ and $(BA)_* = B_*A_*$. If $M, M' \in \mathscr{L}_Y$, $L \in \mathscr{R}_X$, then

$$A^*(M \boxtimes M') = A^*M \boxtimes A^*M' \text{ and } A_*(L \boxtimes_0 A^*M) = A_*L \boxtimes_0 M.$$

4) Let A: X → Y be an invertible polynomial mapping, and let A: $D_X \to D_Y$ be the corresponding ring isomorphism. The isomorphism A induces category isomorphisms $A_{\mathscr{L}}: \mathscr{L}_X \to \mathscr{L}_Y$ and $A_{\mathscr{R}}: \mathscr{R}_X \to \mathscr{R}_Y$. Then $A_* = A_{\mathscr{R}}$, $A^* = A_{\mathscr{L}}^{-1}$.

The proof of Proposition 3.1 consists of simple verification.

The following basic theorem describes the behavior of the numerical characteristics d and e introduced in §1 for the operations A^*, A_*, $\boxtimes$ and $\boxtimes_0$.

THEOREM 3.2. Let K be a field of characteristic zero, let X and Y be finite-dimensional spaces over K, and let A: X → Y be a polynomial mapping of degree q (if A is a mapping at a point we set q = 1). Then

1) If a left D_Y-module M admits a (d, e)-filtration, then the D_X-module A^*M admits a (d', e') filtration, where $d - \dim Y = d' - \dim X$, $e' = e \cdot q^{\dim X + \dim Y}$.

2) If a right D_X-module L admits a (d, e)-filtration, then the right D_Y-module A_*L admits a (d', e')-filtration, where $d - \dim X = d' - \dim Y$, $e' = e \cdot q^{\dim X + \dim Y}$.

3) If the left D_X-modules M_1, M_2 admit filtrations of the type (d_1, e_1) and (d_2, e_2), then the D_X-module $M_1 \boxtimes M_2$ admits a (d', e')-filtration, where $d' = d_1 + d_2 - \dim X$, $e' = e_1 \cdot e_2$.

(A similar assertion holds for the operation $\boxtimes_0$.)

Theorem 3.2 will be proved in §5.

COROLLARY 3.3. For any finite-dimensional space Z we denote by $\mathscr{L}_{Z0}$ ($\mathscr{R}_{Z0}$) the category of finitely generated left (right) D_Z-modules M for which $d(M) \leq \dim Z$.

Let A: X → Y, be a polynomial mapping as before. Then $A^*(\mathscr{L}_{Y0}) \subset \mathscr{L}_{X0}$, $A_*(\mathscr{R}_{X0}) \subset \mathscr{R}_{Y0}$, $\boxtimes(\mathscr{L}_{X0}, \mathscr{L}_{X0}) \subset \mathscr{L}_{X0}$ and $\boxtimes_0(\mathscr{R}_{X0}, \mathscr{L}_{X0}) \subset \mathscr{R}_{X0}$.

Corollary 3.3. follows immediately from Theorem 3.2 and Corollary 1.4.

§4. Regularization of Integral Transformations

We will carry out the regularization of functions of the type $A^*\mathscr{E}$ (and similarly $\mathscr{E}_1 \cdot \mathscr{E}_2$) according to the following program.

1. We first construct a smoothing family of functions $\mathscr{E}(\lambda)$ depending analytically on the parameter λ such that

a) $\mathscr{E}(\lambda) \in S_Y$, $\mathscr{E}(0) = \mathscr{E}$,

b) when Re λ is large, $\mathscr{E}(\lambda)$ is a function which is continuously differentiable many times.

2. When Re λ is large, it is possible to define a function $A^*\mathscr{E}(\lambda)$ by the equation $A^*\mathscr{E}(\lambda)(x) = \mathscr{E}(\lambda)(Ax)$. This function (as an element of the space S'_X) depends analytically on λ.

3. Under certain hypotheses on the generalized function $\mathscr{E}$ the function $A^*\mathscr{E}(\lambda)$ with values in S'_X extends analytically to a neighborhood of the point $\lambda = 0$ (possibly as a mermorphic function). It is then possible to set

$A^*\mathscr{E}=$ the zero-order term of the Laurent series at $\lambda = 0$ of the function $A^*\mathscr{E}(\lambda)$.

We first construct the smoothing family.

Definition 4.1. Let Y be a finite-dimensional space over R, and let $y_1, \ldots, y_m$ be coordinates on Y. We choose a strictly positive polynomial in $y_1, \ldots, y_m$ which is increasing at infinity (for example, $P = 1 + y_1^2 + \ldots + y_m^2$).

For any function $\mathscr{E} \in S'_Y$ and any complex number λ we set $\mathscr{E}_P(\lambda) = F^{-1}(P^{-\lambda}F\mathscr{E})$.

LEMMA 4.1. For any function $\mathscr{E} \in S'_Y$ there exist constants $\alpha, \beta > 0$ such that for Re $(\alpha\lambda - \beta) > l$ the function $F^{-1}(P^{-\lambda}\mathscr{E})$ is l times continuous differentiable.

Proof. As is shown in [5], $\mathscr{E} = \sum \mathscr{D}_i f_i$, where $\mathscr{D}_i \in D_Y$, $f_i \in L_1(Y)$. We will find α and β for each term $\mathscr{D}f$.

It is easy to check that $P^{-\lambda}\mathscr{D}f = \sum_{i=0}^{k} \tilde{\mathscr{D}}_i P^{-\lambda-i} f$, where k = deg $\mathscr{D}$, and the $\tilde{\mathscr{D}}_i$ are elements of D_Y which have polynomial dependence on λ. This means that the function $F^{-1}(P^{-\lambda}\mathscr{D}f)$ can be written in the form $\sum \mathscr{D}_i F^{-1}(P^{-\lambda-i}f)$ where the $\mathscr{D}_i$ are operators of D_Y of bounded degree which depend on λ. It therefore suffices to find the constants α and β for the function f.

From the Seidenberg-Tarski theorem it follows that $P(y) > C\|y\|^{\alpha}$ for all $y \in Y$ and some $\alpha, C > 0$. Therefore for Re $\alpha\lambda > l$ the function $P^{-\lambda}f$, multiplied by any polynomial of degree l lies in $L_1(Y)$; thus, for Re $\alpha\lambda > l$ the function $F^{-1}(P^{-\lambda}f)$ is l times continuously differentiable. This completes the proof of the lemma.

We have shown that for Re λ large $\mathscr{E}_P(\lambda)$ is a sufficiently smooth function. Therefore, if we are given a polynomial mapping A: X → Y, then (for large Re λ) it is possible to define the function $A^*\mathscr{E}_P(\lambda)$. We wish to determine under what conditions on $\mathscr{E}$ the function $A^*\mathscr{E}_p(\lambda)$ extends analytically to a neighborhood of the point $\lambda = 0$.

It turns out that for this it is sufficient that the function $\mathscr{E}$ should satisfy a "large" system of differential equations with polynomial coefficients. We give the precise definition of the space S'_0 of such functions.

Definition 4.2. Let Z be a finite-dimensional space over **R**.

1. For any function $\mathscr{E} \in S'_Z$ we denote by $D(\mathscr{E})$ the D_Z-submodule in S'_Z generated by $\mathscr{E}$ and we set $d_{\mathscr{E}}(n) = d_{D(\mathscr{E})}(n)$, $d(\mathscr{E}) = d(D(\mathscr{E}))$.

2. We denote by S'_{Z_0} the subspace in S'_Z consisting of functions $\mathscr{E}$, for which $d(\mathscr{E}) \leqslant \dim Z$.

3. Similarly, we introduce the numbers $d_{\mathscr{F}}(n)$, $d(\mathscr{F})$ for forms $\mathscr{F} \in \Omega'_Z$ and the space $\Omega'_{Z_0} \subset \Omega'_Z$.

The space S'_0 was introduced in [3] (see Definition 2.1 and Theorem 3.1). It is proved there that functions of the space S'_0 have nice analytic properties (see Theorem A).

We shall prove certain elementary properties of the space S'_0 (we will not formulate the analogous properties for the space Ω'_0).

Proposition 4.2. 1) S'_0 is a D-submodule of S'.

2) We consider the Fourier transform F: $S'_Z \to S'_Z$. Then if $\mathscr{E} \in S'_Z$, it follows that $d_{\mathscr{E}}(n) = d_{F\mathscr{E}}(n)$. In particular, $d(\mathscr{E}) = d(F\mathscr{E})$, and hence the space S'_{Z0} is invariant under Fourier transform.

3) Suppose that in a connected region $C \subset \mathbf{C}$ there is given an analytic function $\mathscr{E}(\lambda)$ with values in S'_Z. Then there exists a countable set $\Xi \subset C$ such that if $\lambda, \mu \in C \setminus \Xi$, then $d_{\mathscr{E}(\lambda)}(n) = d_{\mathscr{E}(\mu)}(n)$, and if

$\lambda\in C\setminus\Xi$, $\mu\in\Xi$, then $d_{\mathscr{E}(\lambda)}(n)\geqslant d_{\mathscr{E}(\mu)}(n)$ for all n. In particular, if $\mathscr{E}(\lambda)\in S'_{Z0}$ for all λ in some region $C_1\subset C$, then $\mathscr{E}(\lambda)\in S'_{Z0}$ for all $\lambda\in C$.

3a) If in a connected region $C\subset C$ there is given a meromorphic function $\mathscr{E}(\lambda)$ with values in S'_{Z0}, then all the coefficients of the Laurent series of the function $\mathscr{E}(\lambda)$ at any point $\lambda\in C$ lie in S'_{Z0}.

Proof. 1) If $\mathscr{E}_1,\mathscr{E}_2\in S'_0$, $\mathscr{D}_1,\mathscr{D}_2\in D$ and $\mathscr{E}=\mathscr{D}_1\mathscr{E}_1+\mathscr{D}_2\mathscr{E}_2$, then $D(\mathscr{E})\subset D(\mathscr{E}_1)+D(\mathscr{E}_2)$, and therefore $d(\mathscr{E})\leqslant\max(d(\mathscr{E}_1),d(\mathscr{E}_2))$, i.e., $\mathscr{E}\in S'_0$.

2) Let z_j be coordinates on Z. We define the isomorphism F: $D_Z\to D_Z$ by $F(z_j)=-i(\partial/\partial z_j)$, $F(\partial/\partial z_j)=-iz_j$. It is clear that if $\mathscr{E}\in S'_Z$, $\mathscr{D}\in D_Z$ then $F(\mathscr{D}\mathscr{E})=F(\mathscr{D})(F\mathscr{E})$ and $F(D^n_Z)=D^n_Z$. Therefore $FD^n_Z(\mathscr{E})=D^n_Z(F\mathscr{E})$, i.e., $d_{\mathscr{E}}(n)=d_{F\mathscr{E}}(n)$.

3) For each $\lambda\in C$ we define in the finite-dimensional space D^n_Z the system of equations $\langle\mathscr{D}\mathscr{E}(\lambda),\omega\rangle=0$, where ω run through the space Ω. Each of these equations depends analytically on λ, and hence everywhere except on a countable set Ξ_n of points λ the system has maximal rank. The rank of this system at the point λ is by definition equal to $d_{\mathscr{E}(\lambda)}(n)$. This implies the assertion of the lemma if we take $\Xi=\bigcup\Xi_n$.

3a) Multiplying $\mathscr{E}(\lambda)$ by a scalar function, it can be assumed that it is analytic. We will show that $\frac{d}{d\lambda}\mathscr{E}(\lambda)\in S'_0$ for all $\lambda\in C$. The function $\tilde{\mathscr{E}}(\lambda)=(\mathscr{E}(\lambda)-\mathscr{E}(\lambda_0))/(\lambda-\lambda_0)$ lies in S'_0 for $\lambda\neq\lambda$, and hence $\left(\frac{d}{d\lambda}\mathscr{E}\right)(\lambda_0)=\tilde{\mathscr{E}}(\lambda_0)$ lies in S'_0. Continuing this process, we find that all the derivatives of the function $\mathscr{E}(\lambda)$ lies in S'_0.

THEOREM 4.3. Let X and Y be finite-dimensional spaces over **R**, and let A: X → Y be a polynomial mapping. Let us suppose that a function $\mathscr{E}\in S'_{Y0}$ is given. Then the function $A^*\mathscr{E}_P(\lambda)$ (defined for large Re λ) extends analytically as a meromorphic function to the entire complex plane Λ of the variable λ, and moreover $A^*\mathscr{E}_P(\lambda)\in S'_{X0}$.

We will analytically extend the function $A^*\mathscr{E}_P(\lambda)$ by the same method as in §2. We first formulate the method in a general form.

Definition 4.3. 1. We denote by η the automorphism of the field $C(\lambda)$ over the field C obtained from $\lambda\to\lambda+1$; if X is a linear space over C, we denote by η the corresponding automorphism of the ring $D_X(C(\lambda))=D_X(C)\otimes C(\lambda)$.

2. An η-module is a $D(C(\lambda))$-module M in which there is defined an isomorphism η: M → M linear over C such that $\eta(\mathscr{D}f)=\eta(\mathscr{D})\eta(f)$ for all $\mathscr{D}\in D(C(\lambda))$, $f\in M$. Further, an η-morphism of η-modules is a morphism of $D(C(\lambda))$-modules which preserves the operation η.

3. In the $D(C(\lambda))$-module S^Λ we define the automorphism η by the formula $(\eta f)(\lambda)=f(\lambda+1)$.

Proposition 4.4. Suppose that there is given an η-module M which is finitely-generated as a $D(C(\lambda))$-module and an η-morphism Ψ: M → S^Λ. Then

a) for any element $f\in M$ the function Ψf extends as a meromorphic function to the entire complex plane Λ of the variable λ; the poles of the function Ψf belong to a finite number of arithmetic progressions of the form $A_i=\{\lambda_i-n\mid n=0,1,\ldots\}$.

b) The function $\Psi f(\lambda)$ satisfies the equation

$$\Psi f(\lambda)=\mathscr{D}_1(\lambda)\Psi f(\lambda+1)+\ldots+\mathscr{D}_k(\lambda)\Psi f(\lambda+k),$$

where $\mathscr{D}_1,\ldots,\mathscr{D}_k\in D(C(\lambda))$.

Proof. It is clear that a) follows from b). We will prove b).

We consider in M an increasing chain of submodules $M_i=D(C(\lambda))\times(f,\eta^{-1}f,\ldots,\eta^{-i}f)$. Since the ring $D(C(\lambda))$ is Noetherian (see [3]) and the module M is finitely generated, it follows that the sequence of modules M_i stabilizes, i.e., $M_{k-1}=M_k$ for some k.

This means that there exist operators $\tilde{\mathscr{D}}_1,\ldots,\tilde{\mathscr{D}}_k\in D(C(\lambda))$ such that $\eta^{-k}f=\tilde{\mathscr{D}}_1\eta^{-k+1}f+\ldots+\tilde{\mathscr{D}}_kf$. Applying the operator η^k to this equality, we obtain $f=\mathscr{D}_1\eta f+\ldots+\mathscr{D}_k\eta^kf$, where $\mathscr{D}_i=\eta^k\tilde{\mathscr{D}}_i$. The proof of the proposition is complete.

To prove Theorem 4.3 it remains to verify that the function $A^* \mathscr{E}_P(\lambda)$ belongs to a finitely generated η-module. For this we investigate the algebraic constructions of all the mappings.

1. We denote by F the automorphism of the ring D_Y (and the ring $D_Y(C(\lambda))$), given by the formulas $F(y_j) = -i\frac{\partial}{\partial y_j}$, $F\left(\frac{\partial}{\partial y_j}\right) = -iy_j$, where $y_1, \ldots, y_m$ are the coordinates on Y.

If L is a D_Y-module (or a $D_Y(C(\lambda))$-module) we denote by FL the D_Y-module which is constructed as follows: as a linear space the module FL is isomorphic to L and under the natural isomorphism F: L → FL we have $F(\mathscr{D}g) = F(\mathscr{D})F(g)$ for all $\mathscr{D} \in D_Y$, $g \in L$

2. We construct the $D_Y(C(\lambda))$-module M'_P. The elements of M'_P are expressions of the form $QP^{-\lambda-k}$, where $Q \in R_Y(C(\lambda))$ (here $QP^{-\lambda-k} = Q'P^{-\lambda-n}$, if $QP^n = Q'P^k$). The operators $\partial/\partial y_j$ ($j = 1, \ldots, m$) are defined by

$$\frac{\partial}{\partial y_j}(QP^{-\lambda-k}) = \frac{\partial Q}{\partial y_j}P^{-\lambda-k} - (\lambda+k)Q\frac{\partial P}{\partial y_j}P^{-\lambda-k-1}.$$

In M'_P we define the automorphism η by the equation $\eta(QP^{-\lambda-k}) = \eta(Q)P^{-\lambda-k-1}$ (we note that the module M'_P is obtained from the module M_P introduced in §2 by letting $\lambda \to -\lambda$).

As shown in §2, the module M'_P admits a $(m, (p+1)^m)$-filtration, where m = dim Y and p is the degree of the polynomial P.

3. Let $M_0 = D_Y(\mathscr{E})$, and let $M = M_0 \otimes C(\lambda)$ be a $D_Y(C(\lambda))$-module. Since $\mathscr{E} \in S'_{Y_0}$, it follows that M admits an (m, e)-filtration for some e.

4. It is easy to verify that the mapping $\mathscr{E} \to A^*\mathscr{E}_P(\lambda)$ defines a mapping Ψ $D_X(C(\lambda))$-module $\hat{M} = A^*F^{-1}(M'_P \boxtimes FM)$ into the $D_X(C(\lambda))$-module S^{Λ}_X. If the natural structure of an η-module is introduced in $\hat{M}$, then the mapping Ψ is an η-morphism.

From Corollary 3.3 it follows that the module $\hat{M}$ is finitely generated and $d(\hat{M}) \le \dim X$. Therefore, Theorem 4.3 follows from Propositions 4.4 and 4.2.

THEOREM 4.5. Suppose that there is given a polynomial mapping A: X → Y and a positive polynomial P on X which is increasing at infinity.

1) If $\mathscr{E}, \mathscr{E}' \in S'_{X_0}$, then the function $\mathscr{E}_P(\lambda) \cdot \mathscr{E}' \in S'_X$ which is defined for large Re λ extends as a meromorphic function to the whole plane Λ. Moreover, $\mathscr{E}_P(\lambda) \cdot \mathscr{E}' \in S'_{X_0}$.

2) Let $\mathscr{F} \in \Omega'_{X_0}$. For large Re λ we define the form $A_*\mathscr{F}_P(\lambda) \in \Omega'_Y$ by

$$\langle A_*\mathscr{F}_P(\lambda), \varphi\rangle = \langle P^{-\lambda}\mathscr{F}, A^*\varphi\rangle \qquad (\varphi \in S_Y).$$

Then the form $A_*\mathscr{F}_P(\lambda)$ extends as a meromorphic function of λ to the entire plane Λ. Moreover, $A_*\mathscr{F}_P(\lambda) \in \Omega'_{Y_0}$.

The proof of Theorem 4.5 is similar to that of Theorem 4.3 and is therefore omitted.

The means of constructing the function $A^*\mathscr{E}$ (and similarly $A_*\mathscr{F}$ and $\mathscr{E}_1 \cdot \mathscr{E}_2$) presented in Theorem 4.3 depends on the choice of the polynomial P. However, fixing P, we obtain a linear mapping A^*_P : $S'_{Y_0} \to S'_{X_0}$. There are hereby not always equalities which "must" hold (for example, the equality $\partial/\partial x_i$ $A^*\mathscr{E} = \sum_{j=1}^{m} \frac{\partial A_j}{\partial x_i} A^*\left(\frac{\partial}{\partial y_j}\mathscr{E}\right)$). However, they are satisfied if we go over from S'_X to the space S'_X/L, where L is the D_X-module in S'_X generated by the negative terms of the Laurent series at the point λ = 0 of the function $A^*\mathscr{E}_P(\lambda)$.

We present several interesting corollaries of Theorems 4.3 and 4.5.

COROLLARY 4.6. Let a polynomial P, a region Θ, and a function $P_\Theta(\lambda)$ be given as in the introduction, and let the function $\mathscr{E} \in S'_0$. Then the function $\mathscr{E}(\lambda) = \mathscr{E} \cdot P_\Theta(\lambda)$ which is defined in the region Re λ > C, lies in S'_0, extends as a meromorphic function to the entire plane Λ, and satisfies the equation

$$\mathscr{E}(\lambda) = \mathscr{D}_1(\lambda)\mathscr{E}(\lambda+1) + \ldots + \mathscr{D}_k(\lambda)\mathscr{E}(\lambda+k),$$

where $\mathscr{D}_1, \ldots, \mathscr{D}_k \in D(C(\lambda))$.

COROLLARY 4.7. Let P be a polynomial in N variables, $\mathscr{E} \in S_0'$, and suppose that in the region Re $\lambda > C$ the integral $f(\lambda) = \int P^{-\lambda}\cdot\mathscr{E}\cdot dx_1 \ldots dx_N$ is defined. (For example, $\mathscr{E} \equiv 1$, and P is strictly positive and increases at infinity.) Then the scalar function $f(\lambda)$ extends as a meromorphic function to the entire Λ plane and satisfies the equation

$$f(\lambda) = a_1(\lambda) f(\lambda + 1) + \ldots + a_k(\lambda) f(\lambda + k),$$

where $a_1, \ldots, a_k$ are certain rational functions of λ.

COROLLARY 4.8. Let L be a differential operator with constant coefficients on the space R^N, $\mathscr{E}_0' \in S_0'$. Then there exists a function $\mathscr{E}' \in S_0'$ such that $L\mathscr{E}' = \mathscr{E}_0'$.

Proof. Going over to the Fourier transform, we obtain the equation $Q\cdot\mathscr{E} = \mathscr{E}_0$, where $\mathscr{E}_0 \in S_9'$, and Q is a polynomial. It can be assumed that Q is nonnegative (otherwise we replace Q by the polynomial $Q\bar{Q}$). Let $P = 1 + x_1^2 + \ldots + x_N^2$.

For Re $\lambda > 0$ and large Re μ we consider the function $\mathscr{E}(\lambda, \mu) = Q^\lambda F^{-1}(P^{-\mu}F\mathscr{E}_0)$. Just as in Theorem 4.3, we prove that $\mathscr{E}(\lambda, \mu)$ extends as a meromorphic function of λ and μ to the entire space $C^2 = \{\lambda, \mu\}$, while $\mathscr{E}(\lambda, \mu) \in S_0'$.

It is clear that $Q\cdot\mathscr{E}(\lambda, \mu) = \mathscr{E}(\lambda + 1, \mu)$ and $\mathscr{E}(0, \mu) = \mathscr{E}_0(\mu) = F^{-1}(P^{-\mu}F\mathscr{E}_0)$. In particular, $\mathscr{E}_0(0) = \mathscr{E}_0$.

We define the function $\mathscr{E}_1(\mu)$ as the zero-order term of the Laurent series with respect to λ of the function $\mathscr{E}(\lambda, \mu)$ at the point $(-1, \mu)$. It is clear that $Q\cdot\mathscr{E}_1(\mu) = \mathscr{E}_0(\mu)$.

If we now denote by $\mathscr{E}$ the zero-order term of the Laurent expansion of the function $\mathscr{E}_1(\mu)$ at the point $\mu = 0$, then $\mathscr{E} \in S_0'$ and $Q\cdot\mathscr{E} = \mathscr{E}_0$. This proves the corollary.

§5. Proof of Theorems 3.2 and 1.3

If a filtration $\{M^n\}$ is given in a D-module M, then we have the sequence of numbers $a_n = \dim M^n$. We shall present several simple assertions regarding such sequences.

Definition 5.1. 1) We denote by Π the set of nondecreasing sequences $a = (a_0, a_1, \ldots, a_n, \ldots)$ of nonnegative numbers.

2) If $a, b \in \Pi$, then $a \geq b$ means that $a_n \geq b_n$ for all n.

3) If $a \in \Pi$, then we define the sequence σa by $(\sigma a)_n = a_0 + \ldots + a_n$.

4) If $a, b \in \Pi$, then we define the sequence $a * b$ by

$$(a * b)_n = a_0(b_n - b_{n-1}) + a_1(b_{n-1} - b_{n-2}) + \ldots + a_n b_0 = b_0(a_n - a_{n-1}) + \ldots + b_n a_0.$$

It is easy to verify the following assertion.

LEMMA 5.1. 1) If $a, b, c \in \Pi$, $a \geq b$, then $\sigma a \geq \sigma b$, $a * c \geq b * c$.

2) If a_n is a polynomial for large n with $a_n = (e/d!)\, n^d + o(n^d)$, then (σa) is a polynomial for large n with $(\sigma a)_n = (e/(d+1)!)n^{d+1} + o(n^{d+1})$.

3) If $a_n \leq (e/d!)n^d + o(n^d)$, $b_n \leq (k/m!)n^m + o(n^m)$, then $(a * b)_n \leq (ke/(d+m)!)n^{d+m} + o(n^{d+m})$.

We shall now prove several facts regarding filtrations of a D-module M.

Proposition 5.2. Let M be a D(K)-module, and let $d \geq 0$, $e > 0$ be integers. Then the following conditions are equivalent.

1) M has a countable basis over K and for any finitely generated submodule $L \subset M$ either $d(L) < d$, or $d(L) = d$ and $e(L) \leq e$.

2) The module M admits a (d, e)-filtration.

3) In the module M there exists a filtration $\{M^N\}$ such that

a) $D^i M^n \subset M^{n+i}$, $\bigcup_n M^n = M$,

b) if we set $a_n = \dim M^n$, then for some k we have $(\sigma^k a)_n \leq (e/(d+k)!)n^{d+k} + o(n^{d+k})$.

Proof. 2) $\Rightarrow$ 3). Obvious.

3) $\Rightarrow$ 1). Let L be a D-submodule in M, and let $f_1, \ldots, f_s \in M^m$ be its generators. Then $d_L(n) \le a_{n+m}$. If we set $b_n = d_L(n-m)$ ($b_n = 0$ for $n < m$), then $b_n \le a_n$. It is clear that b_n is a polynomial in n for large n and $b_n = (d(L)/d(L)!)n^{d(L)} + o(n^{d(L)})$. Therefore

$$(\sigma^k b)_n = (e(L)/(d(L)+k)!)\,n^{d(L)+k} + o(n^{d(L)+k}) \leqslant (\sigma^k a)_n \leqslant (e/(d+k)!)\,n^{d+k} + o(n^{d+k}).$$

Thus, $d(L) < d$ or $d(L) = d$, $e(L) \le e$.

1) $\Rightarrow$ 2). Let $f_1, f_2, \ldots$ be a basis for M. We set $r(n) = (e/d!)n^d + n^{d-1/2}$. It follows from the hypothesis that for each i there exists a natural number s(i) such that $\dim D^n(f_1, \ldots, f_i) \le r(n)$ for all $n \ge s(i)$.

We introduce in the module M the filtration $M^n = \sum_{i=1}^{\infty} D^{n-s(i)} f_i$ and show that $\dim M^n \le r(n)$ for all n. For this it suffices to show that $\dim \sum_{i=1}^{m} D^{n-s(i)} f_i \leqslant r(n)$ for all n and m.

We carry out the proof by induction on m. For $n \ge s(m)$ we have

$$\dim \sum_{i=1}^{m} D^{n-s(i)} f_i \leqslant \dim D^n(f_1, \ldots, f_m) \leqslant r(n).$$

For $n < s(m)$ we have

$$\dim \sum_{i=1}^{m} D^{n-s(i)} f_i = \dim \sum_{i=1}^{m-1} D^{n-s(i)} f_i,$$

where the right side is no greater than r(n) by the induction hypothesis. Thus, we have constructed a (d, e)-filtration of the module M, i.e., we have proved the implication 1) $\Rightarrow$ 2). This completes the proof of Proposition 5.2.

Proof of Theorem 3.2, Part 1). We decompose the mapping A: $X \to Y$ into a product of the mappings A_1: $X \to X + Y$, A_2: $X + Y \to X + Y$ and A_3: $X + Y \to Y$, where $A_1(x) = (x, 0)$, $A_2(x, y) = (x, y + Ax)$ and $A_3(x, y) = y$. It is sufficient to prove the theorem for A_1, A_2, and A_3 separately.

1. The Mapping A_1. It is possible to decompose the mapping A_1 into a composition of imbeddings of the form B: $Z \to T$, where B is a linear imbedding of codimension 1.

On T we introduce coordinates $t, z_1, \ldots, z_N$ in such a way that the equation $t = 0$ specifies the space $Z \subset T$.

Let M be a D_T-module with a (d, e)-filtration $\{M^n\}$. Then by definition 3.1 $B^*M = M/tM$.

We set $L = \{f \in M \mid \text{for some } n\ t^n f = 0\}$. L is a D_T-submodule of M, since if $\mathscr{D} \in D_T^k$, then $t^{n+k}\mathscr{D} f = \tilde{\mathscr{D}}\, t^n f = 0$.

LEMMA 5.3. Suppose there is given a D_1-module L (here $D_1 = K[t, \partial/\partial t]$) such that for each $f \in L$ $t^n f = 0$ for large n. Then $tL = L$.

Proof. Let $f \in L$. We set $L_i = D_1(t^{n-1}f, \ldots, t^{n-i}f)$, where $t^n f = 0$. Then $0 = L_0 \subset L_1 \subset \ldots \subset L_n$, and $f \in L_n$. It is sufficient to prove that for each module $\bar{L}_i = L_i/L_{i-1}$ the equality $t\bar{L}_i = \bar{L}_i$ is satisfied.

The module $\bar{L}_i$ is generated by one generator g (equal to the image of $t^{n-i}f$) such that $tg = 0$; this means that the elements $(\partial/\partial t)^j g$ form a basis in $\bar{L}_i$. Moreover, $t(\partial/\partial t)^j g = -j(\partial/\partial t)^{j-1} g$, i.e., $t\bar{L}_i = \bar{L}_i$. This completes the proof of the lemma.

We return to the proof of Theorem 3.2. We have shown that $tL = L$. Therefore, if we set $M_0 = M/L$, then $B^*M_0 = M_0/tM_0 = M/(tM + L) = B^*M$. Replacing the module M by M_0, we may assume that $tf \neq 0$ for any nonzero element $f \in M$.

In the module B^*M we introduce the filtration $B^*M^n = M^n/M^n \cap tM$ and we let $a_n = \dim B^*M^n$. Then $a_n = \dim M^n - \dim(M^n/M^n \cap tM) \le \dim M^n - \dim M^{n-1}$. This means that $(\sigma a)_n \le \dim M^n \le (e/d!)n^d + o(n^d)$. From Proposition 5.2 it follows that the module B^*M possesses a $(d-1, e)$-filtration.

2. The Mapping A_2. Let M be a D_{X+Y}-module with a (d, e)-filtration $\{M^n\}$. The module A_2^*M is isomorphic to M as a linear space. It is easy to verify that the filtration $A_2^*M^n = M^{qn}$ is a $(d, e \cdot q^{\dim X + \dim Y})$-filtration of the module A_2^*M.

We remark that for q > 1 the estimate $q^{\dim X + \dim Y}$ can be made more precise by using the special form of the mapping A_2.

3. The Mapping A_3. It is possible to decompose the mapping A_3 into a composition of projections B: $T \to Z$, where Z is a subspace of T of codimension 1.

Let t be the coordinate on T such that the equation t = 0 specifies the subspace Z. Then for any D_Z-module M $B^*M = K[t] \otimes M$. If in M there is the (d, e)-filtration $\{M^n\}$, then we set $B^*M^n = \sum_{i=0}^{n} t^i \otimes M^{n-i}$.

Then $(\dim B^*M^n) = \sigma(\dim M^n)$, i.e., B^*M^n is a (d + 1, e)-filtration of the module B^*M. This completes the proof of part 1) of Theorem 3.2.

The proof of part 2) of Theorem 3.2 is similar to that of part 1).

We now prove part 3) of Theorem 3.2. We consider the space X × X and the diagonal mapping Δ: $X \to X \times X$.

In the space $M_1 \underset{K}{\otimes} M_2$ it is possible to introduce the structure of a $D_{X \times X}$-module in a natural way. In this module we define a filtration $(M_1 \otimes M_2)^n = \sum_{i=0}^{n} M_1^i \otimes M_2^{n-i}$. Then $\dim (M_1 \otimes M_2)^n = (\dim M_1^n) * (\dim M_2^n)$, i.e., $M_1 \otimes M_2$ admits a $(d_1 + d_2, e_1 e_2)$-filtration.

It is easy to verify that $M_1 \boxtimes M_2 = \Delta^* (M_1 \otimes M_2)$. Therefore, part 3) follows from part 1).

Proof of Theorem 1.3. We will carry out the proof by induction on N; we may assume that for any module L over the ring D_{N-1} either L = 0 or $d(L) \geq N - 1$.

We assume that there exists a nonzero finitely generated $D_N(K)$-module M such that d(M) < N and arrive at a contradiction.

If $\bar{K}$ is a field containing K, then for the module $M_{\bar{K}} = M \underset{K}{\otimes} \bar{K}$ over the ring $D_N(\bar{K}) = D_N(K) \underset{K}{\otimes} \bar{K}$ we have $M_{\bar{K}} \neq 0$ and $d(M_{\bar{K}}) = d(M) < N$. Therefore, replacing the field K by $\bar{K}$ it can be assumed that the field K is uncountable and algebraically closed.

We let t denote the last coordinate x_N.

LEMMA 5.4.* The operator t in the module M has a nontrivial spectrum, i.e., for some $\alpha \in K$ the operator $(t - \alpha)$ is not invertible.

Proof. If for all $\alpha \in K$ the operator $(t - \alpha)$ is invertible, then we obtain a homomorphism of the field of rational functions K(t) into the operators on the linear space M over K. We choose $f \in M$, $f \neq 0$, and assign to each element $Q \in K(t)$ the element of $Qf \in M$.

We note that K(t) has uncountable dimension over K (since elements of the form $(t-\alpha)^{-1}$ are linearly independent). Since M has countable dimension over K, it follows that for some $Q \in K(t)$ we have $Qf = 0$. But then $f = Q^{-1}Qf = 0$ which contradicts the choice of f.

From the lemma just proved it follows that there are two possible cases.

a) For some $\alpha \in K$ $(t - \alpha) M \neq M$ and $\operatorname{Ker} (t - \alpha) = 0$.

b) For some $\alpha \in K$ $\operatorname{Ker} (t - \alpha) \neq 0$.

We consider both possibilities.

a) We consider the D_{N-1}-module $\tilde{M} = M/(t - \alpha) M$ and introduce in it the filtration $\tilde{M}^n = M^n/M^n \cap (t - \alpha) M$. Then $\dim \tilde{M}^n \leq \dim M^n - \dim M^{n-1} = a_n$. Since a_n is a polynomial in n of degree less than N − 1, it follows that for any finitely generated D_{N-1}-module $L \subset M$ we have $d(L) < N - 1$. From the induction hypothesis it follows that $\tilde{M} = 0$, i.e., $(t - \alpha) M = M$.

*The proof of this lemma coincides almost exactly with a proof of Hilbert's Nullstellensatz sent to me by M. Novodvorskii. (Hilbert's Nullstellensatz can be formulated as follows: the factor ring of the ring of polynomials $C[x_1, \ldots, x_N]$ by a maximal ideal is isomorphic to the field C.)

b) Making the change $(t - \alpha) \to t$, it can be assumed that $\operatorname{Ker} t \neq 0$. Replacing M by the submodule $L = \{f \in M \mid t^n f = 0 \text{ for large } n\}$, it can be assumed that for all $f \in M$ $t^n f = 0$ for large n.

We shall prove that the operator $(\partial/\partial t - \alpha)$ has a trivial kernel on M for any $\alpha \in K$.

Indeed, let $\left(\frac{\partial}{\partial t} - \alpha\right) f = 0$ and $t^n f = 0$. Then $\left(\frac{\partial}{\partial t} - \alpha\right) t^n f - t^n \left(\frac{\partial}{\partial t} - \alpha\right) f = nt^{n-1} f = 0$, i.e., $t^{n-1} f = 0$. Continuing this argument, we find that $t^{n-2} f = \ldots = tf = f = 0$.

Let ρ be an automorphism of the ring D_N given by $\rho(x_i) = x_i$, $\rho\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}$ $(i = 1, \ldots, N-1)$, $\rho(t) = \frac{\partial}{\partial t}$, $\rho\left(\frac{\partial}{\partial t}\right) = -t$. We consider the D_N-module M_ρ which is obtained from the module M by means of this automorphism. It is clear that $d(M_\rho) = d(M) < N$ and that in the module M_ρ $\operatorname{Ker}(t - \alpha) = 0$ for all $\alpha \in K$. By Lemma 5.4 $(t - \alpha) M_\rho \neq M_\rho$ for some $\alpha \in K$, and we again return to case a). This completes the proof of Theorem 1.3.

Remark. Theorem 1.3 is a simple consequence of the hypothesis on the "integrability of characteristics" formulated in [6]. Moreover, the method of proof is closely related to methods of [8].

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