INTEGRAL FORMS AND THE STOKES FORMULA ON SUPERMANIFOLDS

I. N. Bernshtein and D. A. Leites UDC 517.3+513.015.7

The concepts of "super" algebra and "super" analysis are being rapidly developed at the present time, basically in connection with the requirements of theoretical physics. Many of the concepts of classical analysis are carried over to the "super" case (see the survey [1], and [2-6]).

1. A superalgebra is defined to be a Z/2Z-graded algebra $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$; the grading is called the parity and is denoted by p. A superalgebra is said to be commutative if $ba = (-1)^{p(a)p(b)}ab$ for homogeneous elements $a, b \in \mathcal{A}$.

Example. The Grassman algebra (exterior algebra) of q variables $\xi_1, \ldots, \xi_q$, denoted by $\Lambda(\xi)$, is a commutative superalgebra with respect to the parity defined by the equation $p(\xi_i) = \bar{1}$.

An analog of the determinant is defined for matrices over a commutative superalgebra $\mathcal{A}$. More precisely, let J be an invertible matrix composed of the blocks J_{11}, J_{12}, J_{21}, J_{22}, where J_{11} and J_{22} are square matrices with elements from $\mathcal{A}_{\bar{0}}$, while the elements of J_{12} and J_{21} lie in $\mathcal{A}_{\bar{1}}$. The function $\mathrm{Ber}\, J = \det(J_{11} - J_{12} J_{22}^{-1} J_{21}) \det^{-1} J_{22}$ is called the Berezinian.† As was shown in [4], $\mathrm{Ber}\,(I \cdot J) = \mathrm{Ber}\, I \cdot \mathrm{Ber}\, J$.

2. The superspace of dimension (p, q) is defined to be the ringed space $\mathcal{R}^{p,q} = (\mathbf{R}^p, \mathfrak{A}_{p,q})$. The bundle of commutative superalgebras $\mathfrak{A}_{p,q}$ is given by its sections over the open domains $U \subset \mathbf{R}^p$: $\Gamma(U, \mathfrak{A}_{p,q}) = C^\infty(U) \otimes \Lambda(\xi_1, \ldots, \xi_q)$ (see [3]).

A superdomain of dimension (p, q) is defined to be a ringed space $\mathcal{U}^{p,q}$ (or simply $\mathcal{U}$), $\mathcal{U} = (U, \mathfrak{A}_{p,q}|_U)$, where U is an open subset in $\mathbf{R}^p$. The sections in $\Gamma(U, \mathfrak{A}_{p,q})$ are called functions on $\mathcal{U}$; the superalgebra of such functions is denoted by $C^\infty(\mathcal{U})$. If $u_1, \ldots, u_p$ are coordinates in U, then the set of functions $x = (u_1, \ldots, u_p, \xi_1, \ldots, \xi_q)$ is called a system of coordinates in $\mathcal{U}$.

3. The morphisms of superdomains are the morphisms of ringed spaces. If $\mathcal{V} = (V, \mathfrak{A}_{m,n})$ is a superdomain with coordinates $y = (v, \eta)$ and $\varphi: \mathcal{U} \to \mathcal{V}$ is a morphism of superdomains, then a homomorphism $\varphi^*: C^\infty(\mathcal{V}) \to C^\infty(\mathcal{U})$ of superalgebras is determined. This morphism φ is uniquely determined by the set of functions $\varphi^*(y)$.

If $\varphi: \mathcal{U} \to \mathcal{V}$ is an isomorphism of superdomains, then the set of functions $\varphi^*(y)$ is also called a system of coordinates in $\mathcal{U}$.

4. Let (u, ξ) be a system of coordinates in the superdomain $\mathcal{U}$. Any element $f \in C^\infty(\mathcal{U})$ can be uniquely represented in the form

$$f(u, \xi) = \sum f_\alpha(u)\, \xi^\alpha = \sum f_\alpha(u)\, \xi_1^{\alpha_1} \ldots \xi_q^{\alpha_q}, \quad f_\alpha(u) \in C^\infty(U), \quad \alpha_i = 0, 1.$$

Partial derivatives are defined on the superalgebra $C^\infty(\mathcal{U})$ [3]:

$$\partial/\partial u_i\, (f(u)\xi^\alpha) = (\partial/\partial u_i f(u))\, \xi^\alpha,$$

$$\partial/\partial \xi_i\, (f(u)\, \xi_1^{\alpha_1} \ldots \xi_q^{\alpha_q}) = \alpha_i (-1)^{\alpha_1 + \cdots + \alpha_{i-1}} f(u)\, \xi_1^{\alpha_1} \ldots \xi_i^{\alpha_i - 1} \ldots \xi_q^{\alpha_q}.$$

†Named after F. A. Berezin, who first found this formula in 1971.

Moscow State University, Karelian Branch of the Academy of Sciences of the USSR. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 11, No. 1, pp. 55-56, January-March, 1977. Original article submitted October 6, 1976.

Let $x = (u, \xi)$ and $y = (v, \eta)$ be two systems of coordinates on the superdomain $\mathcal{U}$. The matrix $J = (J_{ij})$, where $J_{ij} = \partial y_j/\partial x_i$, is called the matrix of derivatives of the coordinate change $x \to y$. The function Ber J is called the Jacobian of the coordinate change and is denoted by $D(y)/D(x)$.

5. A volume form on a superdomain $\mathcal{U}$ is defined to be an expression ρ that in the coordinates $x = (u, \xi)$ can be written in the form $\rho = fD(x)$, where $f \in C^\infty(\mathcal{U})$, $D(x)$ is simply a symbol, and the notation in different coordinates is adjusted so that $fD(x) = f(D(x)/D(y))D(y)$.

If $\rho = fD(x)$ is a finite volume form, then an integral is defined by the formula (see [2, 3, 6])

$$\int_{\mathcal{U}} \rho = \int_{\mathcal{U}} f(x)\, D(x) = \int_{U} f_{1\ldots 1}(u)\, du_1 \ldots du_p, \qquad (*)$$

where $f_{1\ldots 1}(u)$ is the coefficient of $\xi_1 \ldots \xi_q$ in the expansion in Sec. 4.

As shown in [3, 6], for a fixed orientation of U the integral (*) does not depend on the choice of the system of coordinates. If the form is not finite, then this is not true.

Example (A. N. Rudakov). Let $\mathcal{U}$ be a superdomain with coordinates $x = (u, \xi_1, \xi_2)$, where $0 < u < 1$. Let $y = (v, \eta_1, \eta_2)$, where $v = u + \xi_1\xi_2, \eta_1 = \xi_1, \eta_2 = \xi_2$ is another system of coordinates in $\mathcal{U}$. Then $\rho = vD(y) = (u + \xi_1\xi_2)D(x)$; therefore, the integral of ρ in the system x is equal to 1, while in the system y it is equal to 0.

6. The algebra of differential forms on the superdomain $\mathcal{U}$ with coordinates $x = (u, \xi)$ is defined to be the superalgebra Ω over $C^\infty(\mathcal{U})$ with generators $du_i, d\xi_i$, $p(du_i) = \bar{0}$, $p(d\xi_i) = \bar{1}$ and relations $du_i du_j = -du_j du_i$, $du_i d\xi_j = -d\xi_j du_i$, $d\xi_i d\xi_j = d\xi_j d\xi_i$, $f\omega = (-1)^{p(f)p(\omega)}\omega f$, $\omega \in \Omega$, $f \in C^\infty(\mathcal{U})$. The mapping $d: x_i \mapsto dx_i$ has a unique extension to a differentiation $d: \Omega \to \Omega$ such that $d^2 = 0$ (see [5]). We mention that $\Omega^i \neq 0$ for $q > 0$ and all $i \geqslant 0$.

7. We define the space of integral forms Σ. Let $\mathrm{Vol}(\mathcal{U})$ be the space of volume forms on $\mathcal{U}$. We set $\Sigma = \oplus \Sigma_i$, $\Sigma_{p-q-i} = \mathrm{Vol} \otimes_{C^\infty(\mathcal{U})} (\Omega^i)^*$, and $(\Omega^i)^*$ is the $C^\infty(\mathcal{U})$-module that is the left dual to the module of i-forms Ω^i. In coordinates the elements of the space Σ have the form

$$\sum f_{\alpha\beta}(x)\, D(x)\, (\delta u)^\alpha (\delta\xi)^\beta = \sum f_{\alpha\beta}(x)\, D(x)\, (\delta u_1)^{\alpha_1} \ldots (\delta u_p)^{\alpha_p} (\delta\xi_1)^{\beta_1} \ldots (\delta\xi_q)^{\beta_q},$$

where δx_i is the basis of the space $(\Omega^1)^*$ that is dual to dx_i, $\alpha_i = 0, 1$, and $\beta_i \in Z^+$. The space Σ is a module over the algebra Ω. In Σ the differential $d: fD(x)(\delta u)^\alpha(\delta\xi)^\beta \mapsto df \cdot D(x)(\delta u)^\alpha(\delta\xi)^\beta$ is defined; it can be verified that the differential d is invariant with respect to changes of coordinates.

8. A supermanifold of dimension (p, q) is defined to be a ringed space $\mathcal{M} = (M, \mathfrak{A}_{p,q})$ that is locally isomorphic to a superdomain of dimension (p, q); M is called the underlying manifold. The concepts of volume forms and differential and integral forms carry over, in view of their invariance, to supermanifolds.

A subsupermanifold of dimension (p', q') in $\mathcal{M}$ is defined to be a ringed subspace $\mathcal{N} = (N, \mathfrak{B}_{p',q'})$ that can be specified in a neighborhood of each point $n \in N$ by the equations $u_1 = \ldots = u_{p-p'} = 0$, $\xi_1 = \ldots = \xi_{q-q'} = 0$.

9. If $\mathcal{N} \subset \mathcal{M}$ is a subsupermanifold of dimension (p', q), then the restriction mapping $\Sigma_i(\mathcal{M}) \to \Sigma_i(\mathcal{N})$ is defined that commutes with the differential d and with multiplication by differential forms. We write this mapping in the case when the $(p - 1, q)$-dimensional subsupermanifold $\mathcal{N}$ is given by the local condition $u_1 = 0$:

$$fD(x)(\delta u)^\alpha(\delta\xi)^\beta \to (-1)^{p+q-1}\alpha_1 (f|_{\mathcal{N}})\, D(u_2, \ldots, \xi_q)\, (\delta u_2)^{\alpha_2} \ldots (\delta u_p)^{\alpha_p} (\delta\xi)^\beta.$$

10. The orientation of a subsupermanifold $\mathcal{N}^{p',q} \subset \mathcal{M}^{p,q}$ is defined to be the orientation of the submanifold $N \subset M$. If $\Phi \in \Sigma_{p'-q}(\mathcal{M})$, then, restricting Φ to $\mathcal{N}$ using partitions of unity, it is possible to define the integral of Φ over $\mathcal{N}$.

11. A closed superdomain $\mathcal{F}$ in the supermanifold $\mathcal{M}$ is defined to be a pair $(F, \partial\mathcal{F})$ where $F \subset M$ is a closed region with smooth boundary ∂F, and $\partial\mathcal{F}$ is a subsupermanifold of dimension $(p - 1, q)$ whose underlying manifold coincides with ∂F. We mention that, in distinction from the case in ordinary analysis, a closed superdomain does not arise from an open subdomain.

If ρ is a volume form on $\mathcal{M}$ with compact support, then it is possible to define the integral of ρ over a closed superdomain $\mathcal{F} \subset \mathcal{M}$: With the help of a partition of unity it is reduced to the case when $\mathcal{M} = \mathcal{R}^{p,q}$, F is specified by the condition $u_1 \geqslant 0$, and $\partial\mathcal{F}$ is given by the equation $u_1 = 0$; in this case the integral of the form fD(x) over the closed superdomain $\mathcal{F}$ is equal to the integral of $f_{1...1}(u)du$ over the region F.

12. THEOREM (Stokes' Formula). Let $\mathcal{F} \subset \mathcal{M}^{p,q}$ be a closed superdomain, let $\Phi \in \Sigma_{p-q-1}(\mathcal{M})$ be an integral form with compact support, and let $\mathcal{F}$ and $\partial\mathcal{F}$ be equipped with orientations that are compatible in the sense of classical analysis. Then

$$\int_{\partial\mathcal{F}} \Phi|_{\partial\mathcal{F}} = \int_{\mathcal{F}} d\Phi.$$

LITERATURE CITED

1. V. I. Ogievetskii and L. Mezinchesku, Usp. Fiz. Nauk, 117, No. 4, 637-700 (1975).
2. F. A. Berezin, Mat. Zametki, 1, No. 3, 269-276 (1967).
3. F. A. Berezin and D. A. Leites, Dokl. Akad. Nauk SSSR, 224, No. 3, 505-508 (1975).
4. D. A. Leites, Usp. Mat. Nauk, 30, No. 3, 156 (1975).
5. D. A. Leites, Funkts. Anal. Prilozhen., 9, No. 4, 75-76 (1975).
6. V. F. Pakhomov, Mat. Zametki, 16, No. 1, 65-75 (1974).

MODULES OF C^∞-ORBITAL NORMAL FORMS FOR SINGULAR POINTS OF VECTOR FIELDS ON A PLANE

R. I. Bogdanov

UDC 517.92

This note is a continuation of [1, 2]. Here we give the C^∞-orbital normal form of the germ at a singular point of a vector field of class C^∞ on a plane in which the linear part of the vector field is not identically equal to zero.

The definitions and notation given in [1, 2] will not be repeated here.

Definition 1. An r-jet of the germ $v \in V$ is said to be C^∞-sufficient if all the germs $u \in V$: $\pi_r u = \pi_r v$, C^∞ are mutually C^∞-orbitally equivalent.

Definition 2. Let $v \in V_1$ be a germ with nontrivial linear part and λ_1, λ_2 characteristic values of the matrix $\pi_1 v$. A resonance is defined to be a relation $\lambda_1(m_1 - 1) + \lambda_2(m_2 - 1) = 0$, where $m_i \in \mathbf{Z}$, $m_i \geqslant 0$, $i = 1, 2$, $m_1 + m_2 \geqslant 3$. The defining pair of the resonance is the pair of integers m, n: $\lambda_1 m + \lambda_2 n = 0$, $(m, n) = 1$.

THEOREM 1. The C^∞-sufficient 1-jets of germs in V_1 are exhausted by the following three classes of C^∞-orbital equivalences:

1) $$v = \pm x_1\partial_1 + \lambda x_2\partial_2, \tag{1}$$

the pair $(\pm 1, \lambda)$ do not have resonances,

2) $$v = (sx_1 + x_2)\,\partial_1 + sx_2\partial_2, \quad s = \pm 1, \tag{2}$$

3) $$v = x_2\partial_1 - x_1\partial_2 + c\,(x_1\partial_1 + x_2\partial_2), \quad c \neq 0, \tag{3}$$

c being a real parameter.

THEOREM 2. The germs $v \in V_1$ with finite C^∞-sufficient jets are equivalent to one of the germs of the following seven series of one-parameter families and of the two exceptional series:

Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 11, No. 1, pp. 57-58, January-March, 1977. Original article submitted April 15, 1976.