Trace in Categories

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To Jacques Dixmier on his 65th birthday

0. Introduction

In this note we will introduce a trace morphism $tr_V : \mathcal{Z}(\mathfrak{g}) \longrightarrow \mathcal{Z}(\mathfrak{g})$ and give an explicit formula for it (formula $(*)$ in proposition 2). This is a beautiful formula, which I think has many applications. We present one such application – namely, we reprove the description of the algebra of endomorphisms of the big projective module in the category $\mathcal{O}$, due to W. Soergel (See [S]).

1. Definition of trace

1.1. Trace. Let V be a finite-dimensional vector space over a field k. For every endomorphism $a \in \mathrm{End}(V)$ we define its trace by $tr(a) = \sum a_{ii}$, where (a_{ij}) is the matrix of a in some basis v_i of V.

In order to see that this definition does not depend on a choice of the basis, let us write it in a more invariant form. Consider canonical morphism $\mu : V \otimes V^* \rightarrow \mathrm{End}(V)$, $\mu(v \otimes v^*)\xi = (v^*, \xi)v$. Since V is finite dimensional, this is an isomorphism, and we denote by ν the inverse morphism $\nu : \mathrm{End}(V) \rightarrow V \otimes V^*$. We also consider natural morphisms $i : k \rightarrow \mathrm{End}(V)$ and $p : V \otimes V^* \rightarrow k$, where $i(1) = 1_V$ and $p(v \otimes v^*) = (v^*, v)$.

Now for any endomorphism $a \in \mathrm{End}(V)$ we define $tr(a)$ as the composition $p \circ a' \circ \nu \circ i : k \rightarrow \mathrm{End}(V) \rightarrow V \otimes V^* \rightarrow V \otimes V^* \rightarrow k$, where $a' = a \otimes 1_{V^*}$.

1.2. Relative trace. The definition above can be immediately generalized. Namely, let M be another vector space over k. Then we define the trace morphism $tr_V : \mathrm{End}(V \otimes M) \rightarrow \mathrm{End}(M)$ as follows: for every $a \in \mathrm{End}(V \otimes M)$ we denote by $tr_V(a)$ the endomorphism of M given by composition $p' \circ a' \circ \nu' \circ i' : M \rightarrow M \otimes \mathrm{End}(V) \rightarrow M \otimes V \otimes V^* \rightarrow M \otimes V \otimes V^* \rightarrow M$, where $i' = 1_M \otimes i$ with similar formulae for p', ν', and where $a' = a \otimes 1_{V^*}$.

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Let $\{v_i\}$ be some basis of V. Then we can write an endomorphism $a \in \text{End}(V \otimes M)$ as a matrix with entries $a_{ij} \in \text{End}(M)$. It is easy to see, that $tr_V(a) = \sum a_{ii} \in \text{End}(M)$. Using this formula it is easy to establish functorial properties of the morphism tr_V. We will need the following

LEMMA. *If M is finite-dimensional, then $tr(tr_V(a)) = tr(a)$.*

2. Formula for trace morphism

Let us assume that the field k is algebraically closed of characteristic 0. We fix a reductive Lie algebra $\mathfrak{g}$ over k and denote by $U(\mathfrak{g})$ its universal enveloping algebra. The center of this algebra we call **central algebra** and denote by $\mathcal{Z}(\mathfrak{g})$.

We denote by $M(\mathfrak{g})$ the category of $\mathfrak{g}$-modules. Let us fix a finite-dimensional $\mathfrak{g}$-module V and denote by $F_V : M(\mathfrak{g}) \to M(\mathfrak{g})$ the functor $F_V(M) = V \otimes M$. Using functorial properties of morphism tr_V one can easily check that $tr_V : \text{End}(V \otimes M) \to \text{End}(M)$ defines a morphism $tr_V : \text{End}(F_V) \to \text{End}(Id)$ from endomorphisms of functor F_V to endomorphisms of identity functor on $M(\mathfrak{g})$. We would like to have an explicit description of this morphism.

In order to do this we need an explicit description of source and target of the morphism tr_V. The target space $\text{End}(Id)$ is isomorphic to $\mathcal{Z}(\mathfrak{g})$. It can be explicitly described using Harish-Chandra's theorem. Namely, fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and denote by L the dual space and by W the corresponding Weyl group. Then $\text{End}(Id) = \mathcal{Z}(\mathfrak{g})$ can be identified with $\mathcal{F}(L)^W$, where $\mathcal{F}(L)$ is the algebra of polynomial functions on L and $\mathcal{F}(L)^W$ its W-invariant part.

Probably there exists a similar description for $\text{End}(F_V)$, but I do not know it. However we have a natural morphism $\mathcal{Z}(\mathfrak{g}) \to \text{End}(F_V)$, which sends each $z \in \mathcal{Z}(\mathfrak{g})$ to an endomorphism of the functor F_V, given by the action of z on $V \otimes M \in M(\mathfrak{g})$. Composing it with the trace map tr_V, we get a trace map

$$tr_V : \mathcal{Z}(\mathfrak{g}) \longrightarrow \mathcal{Z}(\mathfrak{g}).$$

We will give an explicit formula for this morphism, as a morphism of $F(L)^W$ into itself.

Let $P(V)$ be the set of weights of V with multiplicities counted. We define a convolution $f \mapsto P(V) * f$ on the space of functions on L by $(P(V) * f)(x) = \sum f(x + \mu)$, the sum being taken over all weights $\mu \in P(V) \subset L$ with multiplicities counted. We denote by Λ the elementary W-skew symmetric function on L, that is $\Lambda(x) = \prod_{\alpha>0} h_\alpha(x)$.

Proposition. *Suppose we identified $\mathcal{Z}(\mathfrak{g})$ with $F(L)^W$ using Harish-Chandra isomorphism. Then the trace morphism $tr_V : \mathcal{Z}(\mathfrak{g}) \to \mathcal{Z}(\mathfrak{g})$ satisfies the following identity*

$$\Lambda \cdot tr_V(f) = P(V) * (\Lambda \cdot f). \qquad (*)$$

This proposition gives a formula for tr_V, namely

$$tr_V(f) = \Lambda^{-1}(P(V) * (\Lambda \cdot f)).$$

Proof. For each integral dominant regular weight $\lambda \in L$ we denote by V_λ an irreducible finite dimensional $\mathfrak{g}$-module with highest weight $\lambda - \rho$, where ρ is half sum of positive roots. It is known that the action of an element $z \in \mathcal{Z}(\mathfrak{g})$ on V_λ is given by multiplication on $f(\lambda)$, where f is the function on L corresponding to z. Also dim $(V_\lambda) = C \cdot \Lambda(\lambda)$, where C is some constant, which implies, that $tr(z|V_\lambda) = C \cdot (\Lambda f)(\lambda)$.

To prove formula $(*)$ it is enough to check that the functions on both sides coincide for integral dominant λ which are sufficiently regular. Choose such a λ and let us compute the trace of the operator $tr_V(z)$ on V_λ. On the one hand it equals $C \cdot (\Lambda \cdot tr_V(f))(\lambda)$. On the other hand by lemma in 1.2, it equals $tr(z|V \otimes V_\lambda)$. It is known, that $V \otimes V_\lambda$ is isomorphic to $\bigoplus_\mu V_{\lambda+\mu}$, where sum is taken over weights $\mu \in P(V)$ with multiplicities. Hence $tr(z|V \otimes V_\lambda) = \sum_\mu tr(z|V_{\lambda+\mu}) = C \cdot \sum_\mu \Lambda f(\lambda + \mu) = C \cdot (P(V) * (\Lambda f))(\lambda)$. This proves formula $(*)$. □

3. Trace in categories

The definition of the trace morphism is a special case of a general construction in category theory. This construction is interesting in itself, so I would like to describe it, though we will not use it explicitly in application described in section 4.

Let A and B be two categories and $F : A \longrightarrow B$ a functor. Suppose that

(a) The functor F has a left adjoint functor $E : B \longrightarrow A$ and a right adjoint functor $G : B \longrightarrow A$.
(b) We have fixed a morphism of functors $\nu : G \longrightarrow E$.

Then for all objects $X, Y \in A$ we define morphism

$$tr : \mathrm{Hom}_B(F(X), F(Y)) \longrightarrow \mathrm{Hom}_A(X, Y) \text{ by}$$

$$tr(a) = i_Y \circ E(a) \circ \nu_{F(X)} \circ j_X : X \to GF(X) \to EF(X) \to EF(Y) \to Y,$$

where $j_X : X \to GF(X)$ and $i_Y : EF(Y) \to Y$ are adjunction morphisms, $\nu_{F(X)} : GF(X) \to EF(X)$ ν-morphism, corresponding to the object $F(X)$, $a : F(X) \to F(Y)$ any morphism in B and $E(a) : EF(X) \to EF(Y)$ the corresponding morphisms in A.

In particular, for any object $X \in A$ we get a morphism $tr : \mathrm{End}(F(X)) \to \mathrm{End}(X)$. It is easy to see that this morphism has natural functorial properties. In particular, it defines a morphism

$$tr : \mathrm{End}(F) \longrightarrow \mathrm{End}(Id_A).$$

The case discussed in section 2 corresponds to $A = B = M(\mathfrak{g}), F = F_V, E = G = F_{V^*}$.

4. An application. Description of the endomorphism algebra of the big projective module

Fix a maximal nilpotent subalgebra $\mathfrak{n}$ normalized by $\mathfrak{h}$ and denote by $\mathcal{O}$ the corresponding category of highest weight modules. For every weight $\lambda \in L = \mathfrak{h}^*$ we denote by M_λ the corresponding Verma module of highest weight $\lambda - \rho$, by L_λ its irreducible quotient, and by P_λ the projective cover of L_λ in the category $\mathcal{O}$ (see[BGG1]).

Fix a regular integral antidominant weight λ. Then M_λ is irreducible and P_λ is what we call the "big" projective module. We want to describe the algebra $\mathrm{End}(P_\lambda)$ of its endomorphisms in category $\mathcal{O}$.

Theorem. *The natural morphism $\eta : \mathcal{Z}(\mathfrak{g}) \to End(P_\lambda)$ is an epimorphism. Its kernel coincides with the ideal J_λ, described below. In particular, the algebra $End(P_\lambda)$ is isomorphic to $\mathcal{Z}(\mathfrak{g})/J_\lambda$ which in turn is isomorphic to the cohomology algebra of the flag variety X of algebra $\mathfrak{g}$.*

Let us describe the ideal J_λ. We identify $\mathcal{Z}(\mathfrak{g})$ with the algebra $F(L)^W$ and consider a linear functional $\nu = \nu_\lambda : F(L)^W \to k$, given by

$$\nu(f) = [\sum_w T(w\lambda)(\Lambda f))/\Lambda](0) = [\sum_w \epsilon(w) \cdot w(T(\lambda)(\Lambda f))/\Lambda](0).$$

Here $T(\mu)$ is a translation operator on $F(L)$, $T(\mu)(h)(x) = h(x+\mu)$, $\epsilon(w)$ is the sign of element $w \in W$. In order to see that these two expressions coincide, we use that $w(f) = f$, $w(\Lambda) = \epsilon(w)\Lambda$ and $T(w\lambda) = wT(\lambda)w^{-1}$.

Remark. In order to compute $\nu(f)$ we use the fact that f is a polynomial, i.e. we computed $\nu(f)$ using some kind of limit.

In terms of the functional ν the ideal J_λ is described as

$$J_\lambda = \{f \in F(L)^W \mid \nu(f \cdot F(L)^W) = 0\}.$$

In order to prove the theorem it is enough to check the following three lemmas.

Lemma 1. $\dim(End(P_\lambda)) \le \#(W)$.

Lemma 2. *Set $J = Ann_{\mathcal{Z}(\mathfrak{g})}(P_\lambda)$, i.e. J is the kernel of morphism $\eta : F(L)^W \longrightarrow End(P_\lambda)$. Then $\nu(f) = 0$ for all $f \in J$. In particular, $J \subset J_\lambda$.*

Lemma 3. *The algebra $F(L)^W/J_\lambda$ has dimension equal to $\#(W)$ and is isomorphic to the cohomology algebra of the flag variety X.*

Remark. Lemmas 1 and 3 are more or less straightforward exercises on category $\mathcal{O}$ and cohomology algebra of flag variety respectively. From the point of view of this note the key statement is lemma 2.

Proofs. 1. Let V be an irreducible finite dimensional $\mathfrak{g}$-module with lowest weight λ, θ the corresponding character of $\mathcal{Z}(\mathfrak{g})$. Consider Verma module M_0, corresponding to weight 0. Then P_λ is the direct summand of $F_\lambda(M_0)$, corresponding to the character θ of $\mathcal{Z}(\mathfrak{g})$. This implies that P_λ has a composition series whose factors are isomorphic to Verma modules M_μ with μ of the form $w\lambda$ for $w \in W$; moreover $[P_\lambda : M_{w\lambda}] \le [F_V(M_0) : M_{w\lambda}] = 1$. Thus $\dim \operatorname{Hom}(P_\lambda, P_\lambda) \le \sum_w \dim \operatorname{Hom}(P_\lambda, M_{w\lambda}) = \sum_w [M_{w\lambda} : L_\lambda] \le \#(W)$.

2. Let $P(V)_e$ be the set of extremal weights in $P(V)$. Clearly $P(V)_e = \{w\lambda \mid w \in W\}$. Choose a W-invariant polynomial p on L with the following properties:

a) The polynomial p vanishes up to order $\ge \#(W)$ at all non extremal points of $P(V)$.

b) The polynomial $1-p$ vanishes up to order $\ge \#(W)$ at all extremal points of $P(V)$.

Clearly, the corresponding element $z(p) \in \mathcal{Z}(\mathfrak{g})$, acting on the module $F_V(M_0)$, gives a projection onto the submodule P_λ. This shows, that a function $f \in F(L)^W$ lies in the ideal $J = \operatorname{Ann}(P_\lambda)$ iff $z(f) \cdot z(p) = 0$ on $F_V(M_0)$. In this case clearly $tr_V(z(f) \cdot z(p)) = 0$ on the module M_0.

We claim that the action of the operator $tr_V(z(f) \cdot z(p))$ on M_0 is given by multiplication by $\nu(f)$, which implies that $\nu(f) = 0$ for $f \in J$.

Using formula (*) from section 2 we see that the operator $tr_V(z(f) \cdot z(p))$ acts on M_0 as a scalar $[\sum_\mu (\Lambda f p)(x+\mu)/\Lambda(x)](0)$.

Using properties of p we can rewrite this sum as $[\sum_\mu (\Lambda f)(x+\mu)/\Lambda(x)](0)$, where the sum is over extremal weights μ. Since Λf is skew-symmetric under the action of W, and extremal weights are of the form $w\lambda$, this sum equals to $\nu(f)$.

3. Given a commutative k-algebra B and a linear map $\nu : B \to k$ we denote by $J(B,\nu)$ the ideal $J(B,\nu) = \{b \in B \mid \nu(bB) = 0\}$ and by

$Q(B,\nu)$ the quotient algebra $B/J(B,\nu)$. By definition $J_\lambda = J(F(L)^W,\nu)$. Our aim is to compute the algebra $Q = Q(F(L)^W,\nu)$.

Set $A = F(L)^W$. Clearly ν vanishes on some power of ideal $J_\theta \subset A$ corresponding to the character θ. Hence the algebra Q will not be changed if we replace A by its completion $\hat{A}$ at θ.

Since λ is regular point of L, the algebra $\hat{A}$ is naturally isomorphic to the completion $\hat{F}_\lambda$ of $F(L)$ at point λ.

Translation operator $T(\lambda)$, $T(\lambda)f(x) = f(x+\lambda)$, identifies $\hat{F}_\lambda$ with the algebra $\hat{F}_0$–completion of $F(L)$ at 0. Let us identify $\hat{A}$ with $\hat{F}_0$ using $T(\lambda)$. Then the functional ν on $\hat{A}$ corresponds to the following functional ν' on $\hat{F}_0$

$$\nu'(f) = \nu(T_\lambda^{-1}(f)) = [(\sum_w \epsilon(w)w(T(\lambda)\Lambda \cdot f))/\Lambda](0).$$

In other words, if we define a linear map $\tau : \hat{F}_0 \longrightarrow k$ by

$$\tau(h) = [\mathrm{Alt}(h)/\Lambda](0)) = [(\sum_w \varepsilon(w)\cdot w(h))/\Lambda](0)$$

then $\nu'(f) = \tau(T_\lambda(\Lambda)\cdot f)$. Since the function $T_\lambda(\Lambda)$ is invertible in $\hat{F}_0$, we have $Q(A,\nu) = Q(\hat{A},\nu) = Q(\hat{F}_0,\nu') = Q(\hat{F}_0,\tau) = Q(F(L),\tau)$.

In order to describe this last algebra let us consider an ideal J_+ in $F(L)$, generated by W-invariant polynomials of positive degree, and denote by H the quotient algebra $F(L)/J_+$. It is easy to see that $\tau(J_+) = 0$, i.e. τ can be considered as a functional on H, and $Q(F(L),\tau) = Q(H,\tau)$.

By well known result of A. Borel (see [BGG2] or [D]) H is isomorphic to cohomology algebra of flag variety X and functional τ on H is given by evaluation on fundamental class of X. This implies, that the bilinear form $< h,f >= \tau(hf)$ on H is non degenerate and hence $Q(H,\tau) = H$ (direct algebraic proof of the fact that this form is non degenerate see in [D], Prop. 4). This proves lemma 3.

Remark. Slightly modifying above arguments one can prove the following more general result

Theorem. *Let $\lambda \in L$ be any antidominant weight, L_λ an irreducible module with highest weight $\lambda - \rho$ and P_λ its projective cover in category $\mathcal{O}$. Then the natural morphism $\eta : \mathcal{Z}(\mathfrak{g}) \longrightarrow End(P_\lambda)$ is an epimorphism. Its image is isomorphic to $F(L)^{W(\lambda)}/J(W(\lambda/R))$, where $W(\lambda) = \{w \in W \mid w\lambda = \lambda\}$, $W(\lambda/R) = \{w \in W \mid w\lambda - \lambda \in$ Root lattice $R\}$, $F(L)^{W(\lambda)}$ is the algebra of $W(\lambda)$-invariant polynomial functions on L and $J(W(\lambda/R))$ is an ideal, generated by $W(\lambda/R)$-invariant polynomials of positive degree.*

This finite-dimensional algebra can be realized as cohomology algebra of some partial flag variety.

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