Trace in Categories

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To Jacques Dixmier on his 65th birthday

0. Introduction

In this note we will introduce a trace morphism $tr_V : Z(g) \to Z(g)$ and give an explicit formula for it (formula (⋆) in proposition 2). This is a beautiful formula, which I think has many applications. We present one such application — namely, we reprove the description of the algebra of endomorphisms of the big projective module in the category $O$, due to W. Soergel (See [S]).

1. Definition of trace

1.1. Trace. Let $V$ be a finite-dimensional vector space over a field $k$. For every endomorphism $a \in \text{End}(V)$ we define its trace by $tr(a) = \sum a_{ii}$, where $(a_{ij})$ is the matrix of $a$ in some basis $v_i$ of $V$.

In order to see that this definition does not depend on a choice of the basis, let us write it in a more invariant form. Consider canonical morphism $\mu : V \otimes V^* \to \text{End}(V)$, $\mu(v \otimes v^*) \xi = (v^*, \xi)v$. Since $V$ is finite dimensional, this is an isomorphism, and we denote by $\nu$ the inverse morphism $\nu : \text{End}(V) \to V \otimes V^*$. We also consider natural morphisms $i : k \to \text{End}(V)$ and $p : V \otimes V^* \to k$, where $i(1) = 1_V$ and $p(v \otimes v^*) = (v^*, v)$.

Now for any endomorphism $a \in \text{End}(V)$ we define $tr(a)$ as the composition $p \circ a' \circ \nu \circ i : k \to \text{End}(V) \to V \otimes V^* \to V \otimes V^* \to k$, where $a' = a \otimes 1_{V^*}$.

1.2. Relative trace. The definition above can be immediately generalized. Namely, let $M$ be another vector space over $k$. Then we define the trace morphism $tr_V : \text{End}(V \otimes M) \to \text{End}(M)$ as follows: for every $a \in \text{End}(V \otimes M)$ we denote by $tr_V(a)$ the endomorphism of $M$ given by composition $p' \circ a' \circ \nu' \circ i' : M \to M \otimes \text{End}(V) \to M \otimes V \otimes V^* \to M \otimes V \otimes V^* \to M$, where $i' = 1_M \otimes i$ with similar formulae for $p'$, $\nu'$, and $a' = a \otimes 1_{V^*}$.

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Let \( \{ u_i \} \) be some basis of \( V \). Then we can write an endomorphism \( a \in \text{End}(V \otimes M) \) as a matrix with entries \( a_{ij} \in \text{End}(M) \). It is easy to see, that \( \text{tr}_V(a) = \sum a_{ii} \in \text{End}(M) \). Using this formula it is easy to establish functorial properties of the morphism \( \text{tr}_V \). We will need the following

**Lemma.** If \( M \) is finite-dimensional, then \( \text{tr}(\text{tr}_V(a)) = \text{tr}(a) \).

**2. Formula for trace morphism**

Let us assume that the field \( k \) is algebraically closed of characteristic 0. We fix a reductive Lie algebra \( \mathfrak{g} \) over \( k \) and denote by \( U(\mathfrak{g}) \) its universal enveloping algebra. The center of this algebra we call central algebra and denote by \( Z(\mathfrak{g}) \).

We denote by \( M(\mathfrak{g}) \) the category of \( \mathfrak{g} \)-modules. Let us fix a finite-dimensional \( \mathfrak{g} \)-module \( V \) and denote by \( F_V : M(\mathfrak{g}) \to M(\mathfrak{g}) \) the functor \( F_V(M) = V \otimes M \). Using functorial properties of morphism \( \text{tr}_V \) one can easily check that \( \text{tr}_V : \text{End}(V \otimes M) \to \text{End}(M) \) defines a morphism \( \text{tr}_V : \text{End}(F_V) \to \text{End}(\text{Id}) \) from endomorphisms of functor \( F_V \) to endomorphisms of identity functor on \( M(\mathfrak{g}) \). We would like to have an explicit description of this morphism.

In order to do this we need an explicit description of source and target of the morphism \( \text{tr}_V \). The target space \( \text{End}(\text{Id}) \) is isomorphic to \( Z(\mathfrak{g}) \). It can be explicitly described using Harish-Chandra’s theorem. Namely, fix a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \) and denote by \( L \) the dual space and by \( W \) the corresponding Weyl group. Then \( \text{End}(\text{Id}) = Z(\mathfrak{g}) \) can be identified with \( \mathcal{F}(L)^W \), where \( \mathcal{F}(L) \) is the algebra of polynomial functions on \( L \) and \( \mathcal{F}(L)^W \) its \( W \)-invariant part.

Probably there exists a similar description for \( \text{End}(F_V) \), but I do not know it. However we have a natural morphism \( Z(\mathfrak{g}) \to \text{End}(F_V) \), which sends each \( z \in Z(\mathfrak{g}) \) to an endomorphism of the functor \( F_V \), given by the action of \( z \) on \( V \otimes M \in M(\mathfrak{g}) \). Composing it with the trace map \( \text{tr}_V \), we get a trace map

\[
\text{tr}_V : Z(\mathfrak{g}) \to Z(\mathfrak{g}).
\]

We will give an explicit formula for this morphism, as a morphism of \( \mathcal{F}(L)^W \) into itself.

Let \( P(V) \) be the set of weights of \( V \) with multiplicities counted. We define a convolution \( f \mapsto P(V) * f \) on the space of functions on \( L \) by

\[
(P(V) * f)(x) = \sum f(x + \mu),
\]

the sum being taken over all weights \( \mu \in P(V) \subset L \) with multiplicities counted. We denote by \( \Lambda \) the elementary \( W \)-skew symmetric function on \( L \), that is \( \Lambda(x) = \prod_{\alpha > 0} h_{\alpha}(x) \).
Proposition. Suppose we identified \( Z(\mathfrak{g}) \) with \( F(L)^W \) using Harish-Chandra isomorphism. Then the trace morphism \( \text{tr}_V : Z(\mathfrak{g}) \rightarrow Z(\mathfrak{g}) \) satisfies the following identity

\[
\Lambda \cdot \text{tr}_V(f) = P(V) \ast (\Lambda \cdot f).
\]

This proposition gives a formula for \( \text{tr}_V \), namely

\[
\text{tr}_V(f) = \Lambda^{-1}(P(V) \ast (\Lambda \cdot f)) .
\]

Proof. For each integral dominant regular weight \( \lambda \in L \) we denote by \( V_\lambda \) an irreducible finite dimensional \( \mathfrak{g} \)-module with highest weight \( \lambda - \rho \), where \( \rho \) is half sum of positive roots. It is known that the action of an element \( z \in Z(\mathfrak{g}) \) on \( V_\lambda \) is given by multiplication on \( f(\lambda) \), where \( f \) is the function on \( L \) corresponding to \( z \). Also \( \dim (V_\lambda) = C \cdot \Lambda(\lambda) \), where \( C \) is some constant, which implies, that \( \text{tr}(z|V_\lambda) = C \cdot (\Lambda f)(\lambda) \).

To prove formula (*) it is enough to check that the functions on both sides coincide for integral dominant \( \lambda \) which are sufficiently regular. Choose such a \( \lambda \) and let us compute the trace of the operator \( \text{tr}_V(z) \) on \( V_\lambda \). On the one hand it equals \( C \cdot (\Lambda \cdot \text{tr}_V(f))(\lambda) \). On the other hand by lemma in 1.2, it equals \( \text{tr}(z|V \otimes V_\lambda) \). It is known, that \( V \otimes V_\lambda \) is isomorphic to \( \oplus V_{\lambda + \mu} \), where sum is taken over weights \( \mu \in P(V) \) with multiplicities. Hence

\[
\text{tr}(z|V \otimes V_\lambda) = \sum_{\mu} \text{tr}(z|V_{\lambda + \mu}) = C \cdot \sum_{\mu} \Lambda f(\lambda + \mu) = C \cdot (P(V) \ast (\Lambda f))(\lambda).
\]

This proves formula (*). \(\square\)

3. Trace in categories

The definition of the trace morphism is a special case of a general construction in category theory. This construction is interesting in itself, so I would like to describe it, though we will not use it explicitly in application described in section 4.

Let \( A \) and \( B \) be two categories and \( F : A \rightarrow B \) a functor. Suppose that

(a) The functor \( F \) has a left adjoint functor \( E : B \rightarrow A \) and a right adjoint functor \( G : B \rightarrow A \).

(b) We have fixed a morphism of functors \( \nu : G \rightarrow E \).

Then for all objects \( X, Y \in A \) we define morphism

\[
\text{tr} : \text{Hom}_B(F(X), F(Y)) \rightarrow \text{Hom}_A(X, Y)
\]

by

\[
\text{tr}(a) = i_Y \circ E(a) \circ \nu_{F(X)} \circ j_X : X \rightarrow GF(X) \rightarrow EF(X) \rightarrow EF(Y) \rightarrow Y,
\]

where \( j_X : X \rightarrow GF(X) \) and \( i_Y : EF(Y) \rightarrow Y \) are adjunction morphisms, \( \nu_{F(X)} : GF(X) \rightarrow EF(X) \) \( \nu \)-morphism, corresponding to the object \( F(X) \), \( a : F(X) \rightarrow F(Y) \) any morphism in \( B \) and \( E(a) : EF(X) \rightarrow EF(Y) \) the corresponding morphism in \( A \).
In particular, for any object $X \in A$ we get a morphism $tr : \text{End}(F(X)) \rightarrow \text{End}(X)$. It is easy to see that this morphism has natural functorial properties. In particular, it defines a morphism

$$tr : \text{End}(F) \rightarrow \text{End}(Id_A).$$

The case discussed in section 2 corresponds to $A = B = M(g), F = F_V, E = G = F_{V^*}$.

4. An application.

Description of the endomorphism algebra of the big projective module

Fix a maximal nilpotent subalgebra $n$ normalized by $h$ and denote by $O$ the corresponding category of highest weight modules. For every weight $\lambda \in L = h^*$ we denote by $M_\lambda$ the corresponding Verma module of highest weight $\lambda - \rho$, by $L_\lambda$ its irreducible quotient, and by $P_\lambda$ the projective cover of $L_\lambda$ in the category $O$ (see [BGG1]).

Fix a regular integral antiprimitive weight $\lambda$. Then $M_\lambda$ is irreducible and $P_\lambda$ is what we call the “big” projective module. We want to describe the algebra $\text{End}(P_\lambda)$ of its endomorphisms in category $O$.

**Theorem.** The natural morphism $\eta : Z(g) \rightarrow \text{End}(P_\lambda)$ is an epimorphism. Its kernel coincides with the ideal $J_\lambda$, described below. In particular, the algebra $\text{End}(P_\lambda)$ is isomorphic to $Z(g)/J_\lambda$ which in turn is isomorphic to the cohomology algebra of the flag variety $X$ of algebra $g$.

Let us describe the ideal $J_\lambda$. We identify $Z(g)$ with the algebra $F(L)^W$ and consider a linear functional $\nu = \nu_\lambda : F(L)^W \rightarrow k$, given by

$$\nu(f) = \sum_w T(w\lambda)(\Lambda f)/\Lambda(0) = \sum_w \epsilon(w) \cdot w(T(\lambda)(\Lambda f))/\Lambda(0).$$

Here $T(\mu)$ is a translation operator on $F(L)$, $T(\mu)(h)(x) = h(x + \mu)$, $\epsilon(w)$ is the sign of element $w \in W$. In order to see that these two expressions coincide, we use that $w(f) = f, w(\Lambda) = \epsilon(w)\Lambda$ and $T(w\lambda) = wT(\lambda)w^{-1}$.

**Remark.** In order to compute $\nu(f)$ we use the fact that $f$ is a polynomial, i.e. we computed $\nu(f)$ using some kind of limit.

In terms of the functional $\nu$ the ideal $J_\lambda$ is described as

$$J_\lambda = \{ f \in F(L)^W \mid \nu(f \cdot F(L)^W) = 0 \}.$$

In order to prove the theorem it is enough to check the following three lemmas.
Lemma 1. \( \dim(\text{End}(P_\lambda)) \leq \#(W) \).

Lemma 2. Set \( J = \text{Ann}_{Z(\mathfrak{g})}(P_\lambda) \), i.e. \( J \) is the kernel of morphism \( \eta : F(L)^W \rightarrow \text{End}(P_\lambda) \). Then \( \nu(f) = 0 \) for all \( f \in J \). In particular, \( J \subset J_\lambda \).

Lemma 3. The algebra \( F(L)^W/J_\lambda \) has dimension equal to \( \#(W) \) and is isomorphic to the cohomology algebra of the flag variety \( X \).

Remark. Lemmas 1 and 3 are more or less straightforward exercises on category \( O \) and cohomology algebra of flag variety respectively. From the point of view of this note the key statement is lemma 2.

Proofs. 1. Let \( V \) be an irreducible finite dimensional \( \mathfrak{g} \)-module with lowest weight \( \lambda \), \( \theta \) the corresponding character of \( Z(\mathfrak{g}) \). Consider Verma module \( M_0 \), corresponding to weight 0. Then \( P_\lambda \) is the direct summand of \( F_\lambda(M_0) \), corresponding to the character \( \theta \) of \( Z(\mathfrak{g}) \). This implies that \( P_\lambda \) has a composition series whose factors are isomorphic to Verma modules \( M_\mu \), with \( \mu \) of the form \( w\lambda \) for \( w \in W \); moreover \([P_\lambda : M_{w\lambda}] \leq [F_V(M_0) : M_{w\lambda}] = 1\). Thus \( \dim \text{Hom}(P_\lambda, P_\lambda) \leq \sum_w \dim \text{Hom}(P_\lambda, M_{w\lambda}) = \sum_w [M_{w\lambda} : L_\lambda] \leq \#(W) \).

2. Let \( P(V)_e \) be the set of extremal weights in \( P(V) \). Clearly \( P(V)_e = \{w\lambda | w \in W\} \). Choose a \( W \)-invariant polynomial \( p \) on \( L \) with the following properties:
   a) The polynomial \( p \) vanishes up to order \( \geq \#(W) \) at all non extremal points of \( P(V) \).
   b) The polynomial \( 1-p \) vanishes up to order \( \geq \#(W) \) at all extremal points of \( P(V) \).

Clearly, the corresponding element \( z(p) \in Z(\mathfrak{g}) \), acting on the module \( F_V(M_0) \), gives a projection onto the submodule \( P_\lambda \). This shows, that a function \( f \in F(L)^W \) lies in the ideal \( J = \text{Ann}(P_\lambda) \) iff \( z(f) \cdot z(p) = 0 \) on \( F_V(M_0) \). In this case clearly \( tr_V(z(f) \cdot z(p)) = 0 \) on the module \( M_0 \).

We claim that the action of the operator \( tr_V(z(f) \cdot z(p)) \) on \( M_0 \) is given by multiplication by \( \nu(f) \), which implies that \( \nu(f) = 0 \) for \( f \in J \).

Using formula (*) from section 2 we see that the operator \( tr_V(z(f) \cdot z(p)) \) acts on \( M_0 \) as a scalar \( \sum_{\mu} [\Lambda f(z \mu)/(\Lambda(z))(0)] \).

Using properties of \( p \) we can rewrite this sum as \( \sum_{\mu} (\Lambda f(z \mu)/(\Lambda(z))(0)) \),
where the sum is over extremal weights \( \mu \). Since \( \Lambda f \) is skew-symmetric under the action of \( W \), and extremal weights are of the form \( w\lambda \), this sum equals to \( \nu(f) \).

3. Given a commutative \( k \)-algebra \( B \) and a linear map \( \nu : B \rightarrow k \) we denote by \( J(B, \nu) \) the ideal \( J(B, \nu) = \{b \in B | \nu(bb) = 0\} \) and by
$Q(B, \nu)$ the quotient algebra $B/J(B, \nu)$. By definition $J_\lambda = J(F(L)^W, \nu)$. Our aim is to compute the algebra $Q = Q(F(L)^W, \nu)$.

Set $A = F(L)^W$. Clearly $\nu$ vanishes on some power of ideal $J_\theta \subset A$ corresponding to the character $\theta$. Hence the algebra $Q$ will not be changed if we replace $A$ by its completion $\hat{A}$ at $\theta$.

Since $\lambda$ is regular point of $L$, the algebra $\hat{A}$ is naturally isomorphic to the completion $\hat{F}_\lambda$ of $F(L)$ at point $\lambda$.

Translation operator $T(\lambda)$, $T(\lambda)f(x) = f(x + \lambda)$, identifies $\hat{F}_\lambda$ with the algebra $\hat{F}_0$–completion of $F(L)$ at 0. Let us identify $\hat{A}$ with $\hat{F}_0$ using $T(\lambda)$. Then the functional $\nu$ on $\hat{A}$ corresponds to the following functional $\nu'$ on $\hat{F}_0$

$$\nu'(f) = \nu(T^{-1}_0(\lambda)(f)) = \left(\sum_{w} \epsilon(w) w(T(\lambda)A \cdot f)/A\right)(0).$$

In other words, if we define a linear map $\tau : \hat{F}_0 \to k$ by

$$\tau(h) = [Alt(h)/A](0) = \left(\sum_{w} \epsilon(w) \cdot w(h)/A\right)(0)$$

then $\nu'(f) = \tau(T(\lambda)(\cdot f))$. Since the function $T(\lambda)(A)$ is invertible in $\hat{F}_0$, we have $Q(\hat{A}, \nu) = Q(\hat{F}_0, \nu) = Q(\hat{F}_0, \nu') = Q(F(L), \tau) = Q(F(L), \tau)$.

In order to describe this last algebra let us consider an ideal $J_+$ in $F(L)$, generated by $W$–invariant polynomials of positive degree, and denote by $H$ the quotient algebra $F(L)/J_+$. It is easy to see that $\tau(J_+) = 0$, i.e. $\tau$ can be considered as a functional on $H$, and $Q(F(L), \tau) = Q(H, \tau)$.

By well known result of A. Borel (see [BGG2] or [D]) $H$ is isomorphic to cohomology algebra of flag variety $X$ and functional $\tau$ on $H$ is given by evaluation on fundamental class of $X$. This implies, that the bilinear form $< h, f > = \tau(hf)$ on $H$ is non degenerate and hence $Q(H, \tau) = H$ (direct algebraic proof of the fact that this form is non degenerate see in [D], Prop. 4). This proves lemma 3.

**Remark.** Slightly modifying above arguments one can prove the following more general result

**Theorem.** Let $\lambda \in L$ be any antidominant weight, $L_\lambda$ an irreducible module with highest weight $\lambda - \rho$ and $P_\lambda$ its projective cover in category $O$. Then the natural morphism $\eta : Z(g) \to \text{End}(P_\lambda)$ is an epimorphism. Its image is isomorphic to $F(L)^W(\lambda)/J(W(\lambda/R))$, where $W(\lambda) = \{w \in W | w\lambda = \lambda\}$, $W(\lambda/R) = \{w \in W | w\lambda - \lambda \in \text{Root lattice} R\}$, $F(L)^W(\lambda)$ is the algebra of $W(\lambda)$–invariant polynomial functions on $L$ and $J(W(\lambda/R))$ is an ideal, generated by $W(\lambda/R)$–invariant polynomials of positive degree.

This finite–dimensional algebra can be realized as cohomology algebra of some partial flag variety.
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