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Invariant Differential Operators and Irreducible Representations of Lie Superalgebras of Vector Fields

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Introduction

1. Let M be a connected n-dimensional manifold over $\mathbf{R}$ and ρ be the representation of $GL(n, \mathbf{R})$ in a finite-dimensional space V. Denote by $T(\rho)$ or $T(V)$ the space of tensor fields of type ρ. (For a precise definition see [6], [7], [4].) On $T(\rho)$ the group diff M of diffeomorphisms of M acts naturally.

The operator $c: T(\rho_1) \to T(\rho_2)$ is *invariant* if it commutes with the diff M action.

The problem of classification of invariant operators was probably first posed explicitly by O. Veblen [19] (see reviews [13], [7] for details of the history). It turns out that if we confine ourselves to irreducible representations of $GL(n, \mathbf{R})$ then, essentially, there is a *unique* invariant differential operator—it is the exterior differential d for differential forms. (This was first proved by A. N. Rudakov [11]. The result was rediscovered later by Chuu-Lian Terng [7] and A. A. Kirillov [6].)

A. N. Rudakov considered in [11] the formal analogue of the above smooth problem. We will show (see Appendix) that these problems are practically equivalent. Therefore it is reasonable to solve only the formal variant because the elementary methods of representation theory are applicable to that variant.

2. Supermanifolds and Lie superalgebras have become a topic of interest primarily because of their physical applications (see [10], [17], where the successes and the prospects for success of supermanifolds are reviewed). Here we generalize the above problem to supermanifolds.

We were interested by this problem when we tried to obtain an integration theory for supermanifolds containing an analogue of the Stokes formula [1], [2]. At the time only the usual differential forms were known, but they are impossible to integrate. After we had invented *integrable* forms (i.e., those that could be integrated) we wanted to be sure that there were no other tensor objects that could be integrated.

We remark that the Stokes formula on a manifold exists insofar as the space of differential forms possesses an invariant operator. The uniqueness of the integration theory then follows from the above result by A. N. Rudakov. So we have generalized the method by Rudakov and have described all differential operators in tensor fields on supermanifolds. As one would expect, only invariant operators act on differential and integrable forms. This *proves that an integration theory on supermanifolds containing an analogue of the Stokes formula can only be constructed with integrable forms.* (However see [2], where pseudodifferential forms on supermanifolds μ are integrated. They are not tensor fields on $\mathfrak{M}$ but they are tensor fields on the supermanifold $\hat{\mathfrak{M}}$.)

3. Let us sketch the contents of the paper. Let $\mathcal{L} = W(n,m)$ be the Lie superalgebra of formal vector fields in n even and m odd variables, $L_0 = gl(n,m)$ the subsuperalgebra of linear vector fields. We assign to any finite-dimensional representation ρ of L_0 an $\mathcal{L}$-module $T(\rho)$ of formal tensor fields of type ρ.

Differential and *integrable* forms are examples of tensor fields. We also define the invariant operators $d:\Omega^i \to \Omega^{i+1}$ and $d:\Sigma_j \to \Sigma_{j+1}$ on differential and integrable forms and in their generalization $\Phi = \oplus_{\lambda \in k}\Phi^\lambda$. The main results of the paper are the following:

1. An analogue of the Poincaré lemma exists, i.e., the cohomology of d vanishes everywhere except Ω^0 and Σ_{n-m}. Here these cohomologies are one-dimensional.
2. We describe all $\mathcal{L}$-invariant operators $c: T(\rho_1) \to T(\rho_2)$ for irreducible representations ρ_1 and ρ_2 of L_0. There is an essentially unique operator, namely d. For $n = 0$ there is one more invariant operator, i.e., the Berezin integral
$$\int : \Sigma_{-m} \to \mathbf{C} \hookrightarrow \Omega^0.$$
3. From 1 and 2 we deduce a description of the irreducible continuous representations of the Lie superalgebra $\mathcal{L}$. All of them are subquotients in $T(\rho)$ for an irreducible representation ρ of L_0. The $\mathcal{L}$-module $T(\rho)$ is irreducible if it does not coincide with any of the Ω^i or the Σ_i (and for $m = 1$ with Φ^λ). Otherwise the modules $\operatorname{Ker} d \cap \Omega^i$, $\operatorname{Im} d \cap \Sigma_i$ and $\operatorname{Im} d \cap \Phi^\lambda$ are irreducible. This constitutes the complete list of irreducible continuous $\mathcal{L}$-modules.
4. In the special case $m = 0$ we classify finite dimensional irreducible $\mathcal{L}$-modules. We also give their geometrical interpretation and compute their characters.
5. When there are no even coordinates, i.e., for $n = 0$, all operators acting on tensor fields are differential. In comparison with the purely even case ($m = 0$) we have only one extra invariant operator, namely

the integral. When $n \neq 0$, the integral is a nonlocal operator. It is natural to conjecture that the integral is the unique invariant nonlocal operator. In fact this is true—see the Appendix.

Remark. The problem solved in this paper can be generalized in several directions. For example $W(n,m)$ could be replaced by another Lie superalgebra of similar structure (see [8]), or we could consider operators in several variables (see [4], [5], [15]).

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1. Preliminaries

In this section we recall some facts on linear superspaces and Lie superalgebras that are necessary to make the paper independent. For details see [9], [18].

1. Linear algebra in superspaces. All spaces are considered over a field k of characteristic zero.

A *superspace* is a $\mathbb{Z}_2$-graded space $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. A vector $v \in V_i$, $i \in \mathbb{Z}_2$, is said to be *homogeneous*, and we say that it has *parity* equal to i, and we write $p(v) = i$. By $\Pi(V)$ we denote the superspace defined by $\Pi(V)_{\bar{0}} = V_{\bar{1}}$, $\Pi(V)_{\bar{1}} = V_{\bar{0}}$. The dimension of a superspace V is the element $\dim V = \dim V_{\bar{0}} + \epsilon \dim V_{\bar{1}}$ of the algebra $\mathbb{Z}[\epsilon]/(\epsilon^2 - 1)$. Evidently $\dim V \otimes W = \dim V \cdot \dim W$.

A *superalgebra* is a superspace A with an even morphism $A \otimes A \to A$. On A, define brackets, putting $[a,b] = ab - (-1)^{p(a)p(b)}ba$ on homogeneous elements and extending this formula by linearity on arbitrary ones. A *commutative superalgebra* is an associative superalgebra with unit with trivial brackets.

A *Lie superalgebra* is a superalgebra $\mathfrak{L}$ with operation $x \otimes y \to [x,y]$ such that

$$[x,y] = -(-1)^{p(x)p(y)}[y,x],$$
$$[x,[y,z]] = [[x,y],z] + (-1)^{p(x)p(y)}[y,[x,z]].$$

Evidently modules over associative and Lie superalgebras are well-defined.

An A-module homomorphism is a homomorphism $\varphi: M \to N$ such that $\varphi(am) = (-1)^{p(a)p(\varphi)}a\varphi(m)$ for any $a \in A$, $m \in M$. Define an A-module $\Pi(M) = \{\pi(m), \textit{where } m \in M\}$ by the formula

$$a\pi(m) = (-1)^{p(a)}\pi(am).$$

2. *Representations of $gl(n,m)$.* On the space of matrices of rank $n+m$ we define a $\mathbf{Z}_2$-grading, putting

$$p\begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{D} \end{pmatrix} = \bar{0}, \qquad p\begin{pmatrix} 0 & \mathcal{B} \\ \mathcal{C} & 0 \end{pmatrix} = \bar{1}.$$

Here $\mathcal{A}$ and $\mathcal{D}$ are of rank n and m respectively. Define the bracket by the formula

$$[X,Y] = XY - (-1)^{p(X)p(Y)}YX, \qquad \text{where} \quad X, Y \in \mathrm{Mat}_{n+m}.$$

This bracket transforms the space of matrices into a Lie superalgebra denoted by $gl(n,m)$.

In $gl(n,m)$ we distinguish two subalgebras.

a) The Cartan (diagonal) subalgebra $\mathfrak{h}$ with basis $\{h_i = E_{ii}$ for $i = 1, 2, \ldots, n+m\}$.

b) The nilpotent subalgebra n_+ with basis $\{E_{ij}, i<j\}$.

3. Let V be a $gl(n,m)$-module, $v \in V$ an eigenvector with respect to $\mathfrak{h}$, of weight λ. We will call v the *weight vector* of the weight $\lambda = (\lambda_1, \ldots, \lambda_{n+m})$ where $\lambda_i \in k$, $\lambda_i = \lambda(h_i)$. The *highest weight vector* of V is a nonzero vector $v \in V$ that is an eigenvector with respect to $\mathfrak{h}$ and $n_+ v = 0$.

Theorem (*on highest weight*). *A finite-dimensional $gl(n,m)$-module V has a highest weight vector v_h. If V is irreducible, then v_h is unique up to a multiple. Its weight λ and parity $p(v_h)$ define V up to isomorphism.*

The weight λ is called the *highest weight* of the irreducible module $V = V(\lambda_1, \ldots, \lambda_n; \lambda_{n+m}; p(v_h))$. The highest weight λ of a finite-dimensional module cannot have an arbitrary set of numerical values $(\lambda_1, \ldots, \lambda_{n+m})$ but only those that satisfy $\lambda_i - \lambda_j \in \mathbf{Z}_+$ for $i<j \leqslant n$ and for $n<i<j$.

The proof of this follows from Theorem 8 in [18].

Examples. a) $V(0,\bar{0})$ is the trivial $gl(n,m)$-module of the $(1,0)$-dimensional superspace.

b) $V(1,0,\ldots,0;\bar{0})$ is the standard (identity) module in the (n,m)-dimensional superspace of column-vectors; $V(0,\ldots,0,-1;\nu)$ is the dual $gl(n,m)$-module consisting of row-vectors (here $\nu = \bar{0}$, for $m=0$ and $\nu = \bar{1}$ for $m>0$).

c) The function

$$\mathrm{str} : \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \mapsto \mathrm{tr}\,\mathcal{A} - \mathrm{tr}\,\mathcal{D}$$

is $gl(n,m)$-invariant and is called the supertrace. It defines a $gl(n,m)$-module $V(1,\ldots,1;-1,\ldots,-1;(n-m) \bmod 2)$.

4. *Lie superalgebras of vector fields.* Denote by $\mathfrak{F}$ the commutative superalgebra $k[[x]]$ of formal power series in x, where $x=(x_1,\ldots,x_{n+m}) = (u_1,\ldots,u_n,\xi_1,\ldots,\xi_m)$ so that $p(u_i)=\bar{0}$ and $p(\xi_j)=\bar{1}$. Denote by (x) the maximal ideal in $\mathfrak{F}$ generated by $\{x_i\}$. Define a topology on $\mathfrak{F}$ so that ideals $(x)^r$, $r=0,1,2,\ldots$ are neighborhoods of zero. We have that $\mathfrak{F}$ is complete with respect to this topology.

Denote by $W(n,m)$ the Lie superalgebra of formal vector fields, i.e., of continuous derivations of $k[[x]]$. Define partial derivatives $\partial_i=\partial/\partial x_i \in W(n,m)$ putting $\partial_i(x_j)=\delta_{ij}$. Clearly, $p(\partial_i)=p(x_i)$ and $[\partial_i,\partial_j]=0$. Any element $\mathfrak{D}\in W(n,m)$ is of the form $\mathfrak{D}=\Sigma f_i\partial_i$, where $f_i=\mathfrak{D}(x_i)\in\mathfrak{F}$. We will denote $W(n,m)$ by $\mathfrak{L}$. In $\mathfrak{L}$, define a filtration of the form $\mathfrak{L}=\mathfrak{L}_{-1}\supset\mathfrak{L}_0 \supset\mathfrak{L}_1\supset\cdots$, putting

$$\mathfrak{L}_r=\{\mathfrak{D}\in W(n,m)\,|\,\mathfrak{D}(\mathfrak{F})\subset(x)^{r+1}\}=\Sigma(x)^{i+1}\partial_i.$$

This filtration defines a topology on $\mathfrak{L}$, where $\mathfrak{L}_r$ are neighborhoods of zero.

Denote by $L=\oplus L_r$ the associated graded Lie superalgebra, where $L_r=\mathfrak{L}_r/\mathfrak{L}_{r+1}$. Let us identify L_0 with $gl(n,m)$ via $E_{ij}\leftrightarrow x_j\partial_i$.

5. *Tensor Fields.* Let ρ be a representation of the Lie superalgebra $L_0=gl(n,m)$ in a finite-dimensional superspace V. Define a $W(n,m)$-module $T(\rho)$ putting $T(V)=\mathfrak{F}\otimes_k V$. The superspace $T(V)$ evidently inherits the topology of $\mathfrak{F}$. Let us assign to a vector field $\mathfrak{D}$ the operator $L_{\mathfrak{D}}: T(V)\to T(V)$ such that

$$L_{\mathfrak{D}}(fv)=\mathfrak{D}(f)v+(-1)^{p(\mathfrak{D})p(f)}\Sigma\mathfrak{D}^{ij}\rho(E_{ij})(v)$$

where $f\in\mathfrak{F}$, $v\in V$ and $\mathfrak{D}^{ij}=(-1)^{p(x_i)(p(f)+1)}\partial_i f_j$. The operator $L_{\mathfrak{D}}$ is called the *Lie derivative* (with respect to $\mathfrak{D}$). We will usually write simply $\mathfrak{D}$ instead of $L_{\mathfrak{D}}$.

Elements $t\in T(V)$ will be called *tensor fields of type V*. The superspace V is embedded in $T(V)$ as $1\otimes V$. Elements $t\in V$ will be called *tensor fields with constant coefficients*. The latter are defined by

$$\partial_i t=0,\qquad i=1,2,\ldots,n+m.$$

For any tensor field $t=\Sigma f_i v_i$ we put

$$t(0)=\Sigma f_i(0)v_i\in V.$$

Example. The superspace $\mathfrak{L}$ considered as an $\mathfrak{L}$-module is the space of tensor fields of type *id*, where *id* is the standard (identity) representation of $gl(n,m)$. We have $L_{\mathfrak{D}}(\mathfrak{D}')=[\mathfrak{D},\mathfrak{D}']$.

The operator $L_{\mathfrak{D}}$ is a derivation of the $\mathcal{F}$-module $T(V)$, i.e., the analog of the Leibnitz rule holds

$$L_{\mathfrak{D}}(ft) = \mathfrak{D}(f)t + (-1)^{p(f)p(\mathfrak{D})} f L_{\mathfrak{D}}(t), \quad \text{where } f \in \mathcal{F}, \mathfrak{D} \in \mathfrak{L}, t \in T(V). \tag{*}$$

6. We will also use the following equivalent definition of tensor fields. Denote by $U(\mathfrak{L})$ and $U(\mathfrak{L}_0)$ the universal enveloping algebras of $\mathfrak{L}$ and $\mathfrak{L}_0$, respectively. Let us extend the representation ρ of $L_0 = \mathfrak{L}_0/\mathfrak{L}_1$ to a representation of $\mathfrak{L}_0$, then to a representation of $U(\mathfrak{L}_0)$.

Let us define a morphism $\varphi: T(V) \to \mathrm{Hom}_{U(\mathfrak{L}_0)}(U(\mathfrak{L}), V)$, via $\varphi(t)(X) = (-1)^{p(t)p(X)} X(t)(0)$. The Poincaré–Birkhoff–Witt theorem implies that $U(\mathfrak{L}) = k[\partial_1, \ldots, \partial_{n+m}] \otimes U(\mathfrak{L}_0)$. Since $\mathrm{Hom}(k[\partial], k] \cong \mathcal{F}$ we have that φ is an isomorphism so we may put

$$T(V) \overset{\text{def}}{=} \mathrm{Hom}_{U(\mathfrak{L}_0)}(U(\mathfrak{L}), V).$$

7. The space of tensor fields can be defined axiomatically. Let T be a finitely generated $\mathcal{F}$-module with a consistent $\mathfrak{L}$-action, i.e., the Leibnitz rule is satisfied. Suppose that

$$\mathfrak{L}_1 T \subset (x)T. \tag{*}$$

Then, on the space $V = T/(x)T$ a representation ρ of L_0 arises. Let us assign to $t \in T$ a homomorphism $t^h: U(\mathfrak{L}) \to V$, via

$$t^h(u) = (-1)^{p(t)p(u)} ut \; mod(x)T.$$

Thus we obtain a morphism $\varphi: T \to \mathrm{Hom}_{U(\mathfrak{L}_0)}(U(\mathfrak{L}), V)$. It is easy to verify that φ preserves $\mathfrak{L}$- and $\mathcal{F}$-module structures. In addition, φ defines isomorphisms

$$T/(x)T \cong V \cong T(V)/(x)T(V).$$

Since $T(V)$ is a free $\mathcal{F}$-module, the Nakayama lemma implies that φ is an isomorphism.

8. Differential forms. Let us introduce variables $\hat{x}_i$, so that $p(\hat{x}_i) = p(x_i) + 1$, and call elements of $\Omega = \mathcal{F}[\hat{x}]$ *differential forms*. (Informally speaking $\hat{x}_i$ is simply dx_i.) In Ω we introduce a grading with respect to the degree of $\hat{x}_i$. Each Ω^i is a finitely generated $\mathcal{F}$-module with basis $\hat{x}^\kappa = \hat{x}_1^{\kappa_1}, \ldots, \hat{x}_{n+m}^{\kappa_{n+m}}$, where κ runs over multi-indices $(\kappa_1, \ldots, \kappa_{n+m})$ such that $\kappa_i \in \mathbf{Z}_+$ and $\kappa_i = 0, 1$ for $i \leqslant n$.

Let us define an odd derivation $d: \Omega^i \to \Omega^{i+1}$ putting $d = \Sigma \hat{x}_i \partial_i$.

For any vector field $\mathfrak{D} = \Sigma f_i \partial_i$ let us define a derivation $i_{\mathfrak{D}}: \Omega^i \to \Omega^{i-1}$ by putting $i_{\mathfrak{D}} = \Sigma(-1)^{p(\mathfrak{D})} f_i(\partial/\partial\hat{x}_i)$ (we call it the *inner product* on $\mathfrak{D}$). Define also the *Lie derivative* $L_{\mathfrak{D}}$ with respect to $\mathfrak{D}$, putting $L_{\mathfrak{D}} = [d, i_{\mathfrak{D}}]$.

Lemma

a) $p(d) = \bar{1}$, $\deg d = 1$;

b) $p(i_{\mathfrak{D}}) = p(\mathfrak{D}) + \bar{1}$, $\deg i_{\mathfrak{D}} = -1$;

c) $p(L_{\mathfrak{D}}) = p(\mathfrak{D})$, $\deg L_{\mathfrak{D}} = 0$;

d) $d^2 = \frac{1}{2}[d,d] = 0$, $[i_{\mathfrak{D}'}, i_{\mathfrak{D}'}] = 0$,

$$i_{f\mathfrak{D}} = (-1)^{p(f)} f i_{\mathfrak{D}}, \qquad [d, L_{\mathfrak{D}}] = 0, \qquad [L_{\mathfrak{D}}, i_{\mathfrak{D}'}] = (-1)^{p(\mathfrak{D})} i_{[\mathfrak{D},\mathfrak{D}']},$$
$$[L_{\mathfrak{D}}, L_{\mathfrak{D}'}] = L_{[\mathfrak{D},\mathfrak{D}']}.$$

All these statements are easy to prove. The $\mathfrak{L}$-action on Ω satisfies (*) of Section 1.7 so that Ω is a superspace of tensor fields.

9. Integrable forms. Consider a superalgebra $\mathfrak{J}$ consisting of operators generated by operators of multiplication by f for $f \in \mathfrak{F}$, and by $i_{\mathfrak{D}}$ for $\mathfrak{D} \in \mathfrak{L}$. It is clear that $\mathfrak{J} = \mathfrak{F}[\hat{\partial}_1, \ldots, \hat{\partial}_{n+m}]$, where $\hat{\partial}_j = i_{\partial_j}$. Introduce a grading in $\mathfrak{J}$, putting $\deg \hat{\partial}_j = -1$, $\deg x_i = 0$. Putting $\hat{x}_j = \partial/\partial(\hat{\partial}_j)$, define on $\mathfrak{J}$ an Ω-module structure. Denote by Σ a graded $\mathfrak{J}$-module with one generator Δ such that $\deg \Delta = n - m$ and $p(\Delta) = (n-m) \bmod 2$. Elements $\sigma \in \Sigma_{n-m-r}$ we will call *integrable forms of degree r*, since Σ is the formal analog of smooth forms that can be integrated, see [1].

On Σ we will consider the following structures.

a) The structure of a graded Ω-module defined by the formula

$$\hat{x}_j(P\Delta) = \hat{x}_j(P) \cdot \Delta, \qquad \text{where} \quad P \in \mathfrak{J}.$$

b) The structure of a graded $\mathfrak{J}$-module. In particular, multiplication by $i_{\mathfrak{D}}$ for $\mathfrak{D} \in \mathfrak{L}$ is defined.

c) The odd derivation d is defined by the formula

$$d(P\Delta) = (\Sigma \hat{x}_i \partial_i P)\Delta = \left(\Sigma \frac{\partial^2 P}{\partial(\hat{\partial}_i)\partial x_i}\right)\Delta = [d, P]\Delta.$$

d) The Lie derivative $L_{\mathfrak{D}}$ for $\mathfrak{D} \in \mathfrak{L}$ is defined by the formula $L_{\mathfrak{D}} = [d, i_{\mathfrak{D}}]$.

Evidently

$$X(\omega\sigma) = X(\omega)\sigma + (-1)^{p(X)p(\omega)}\omega X(\sigma),$$

where $\omega \in \Omega$, $\sigma \in \Sigma$, and X is one of d, $i_{\mathfrak{D}}$, or $L_{\mathfrak{D}}$.

It is easy to verify that the $\mathfrak{L}$-action on Σ satisfies (*) of Section 1.7 so that Σ is a space of tensor fields. The corresponding space $\Sigma/(x)\Sigma = V_\Sigma$ is isomorphic to $k[\hat{\partial}]\Delta$.

10. Special cases

a) $m = 0$. We have that $\Omega^i = 0$ for $i > n$ and $\Sigma_i = 0$ for $i < 0$. In addition, the mapping $\Delta \mapsto \hat{x}_1 \dots \hat{x}_n$ defines an isomorphism of Ω^i with Σ_i preserving all structures.

b) $n = 0$. In this case there is an even $\mathfrak{L}$-module morphism $\int : \Sigma_{-m} \to k$ called the ***Berezin integral***. It is defined by the formula [9]

$$\int \xi_1 \dots \xi_m \Delta = 1, \quad \text{and} \quad \int \xi_1^{\nu_1} \dots \xi_m^{\nu_m} \Delta = 0 \quad \text{if } \Pi \nu_i = 0.$$

We will also denote by $\int$ the composition $\int : \Sigma_{-m} \to k \hookrightarrow \Omega^0$ of the Berezin integral and the natural embedding.

c) $m = 1$. We generalize Ω^i and Σ_j to the spaces Φ^λ containing Ω^i and Σ_j, where $\lambda \in k$. Let $x = (u_1, \dots, u_n, \xi)$. Consider a k-graded Ω-module $\Phi = \oplus \Phi^\lambda$ (we assume that $\deg \hat{x}_i = 1 \in k$) generated by $\hat{\xi}^\lambda$, where $\deg \hat{\xi}^\lambda = \lambda$ and $p(\hat{\xi}^\lambda) = \bar{0}$, with relations $\hat{\xi} \cdot \hat{\xi}^\lambda = \hat{\xi}^{\lambda+1}$. Define ∂_i and $\hat{\partial}_i = \overline{\partial \hat{x}_i}$ via $\partial_i(\hat{\xi}^\lambda) = 0$, $\partial \hat{u}_i(\hat{\xi}^\lambda) = 0$ and $\partial \hat{\xi}(\hat{\xi}^\lambda) = \lambda \hat{\xi}^{\lambda-1}$.

Now on Φ, derivations d, $i_{\mathfrak{D}}$ and $L_{\mathfrak{D}}$ consistent with the derivations d, $i_{\mathfrak{D}}$ and $L_{\mathfrak{D}}$ in Ω are evidently defined.

Evidently, $\Phi = \oplus \Phi^\lambda$ is a commutative superalgebra.

Clearly, Φ is a superspace on tensor fields and for $\Phi^{\mathbf{Z}} = \oplus_{r \in \mathbf{Z}} \Phi^r$ we have a decomposition

$$0 \to \Omega \xrightarrow{\alpha} \Phi^{\mathbf{Z}} \xrightarrow{\beta} \Sigma \to 0, \qquad (*)$$

where

$$\alpha(\omega) = \omega \hat{\xi}^0, \qquad \beta(\hat{u}_1 \dots \hat{u}_n \hat{\xi}^{-1}) = \Delta.$$

Evidently, the homomorphisms α and β are consistent with the Ω-module structure and the operators d, $i_{\mathfrak{D}}$ and $L_{\mathfrak{D}}$. The explicit form of the $\mathfrak{F}$-basis in Ω, Σ and Φ easily implies the exactness of $(*)$.

2. Results

1. An analog of the Poincaré lemma

Theorem 1.

1. *The sequence*

$$\Omega^0 \xrightarrow{d} \Omega^i \xrightarrow{d} \cdots$$

is exact. $\operatorname{Ker} d \cap \Omega^0 = k$ *(constants).*

2. *The sequence*

$$\cdots \xrightarrow{d} \Sigma_{-m-1} \xrightarrow{d} \Sigma_{-m} \xrightarrow{d} \cdots \xrightarrow{d} \Sigma_{n-m} \xrightarrow{d} 0$$

is exact everywhere except Σ_{-m}. *The space* $\operatorname{Ker} d / \operatorname{Im} d$ *is generated by* $\xi_1 \dots \xi_m \hat{\partial}_1 \dots \hat{\partial}_n \Delta$.

3. *For $m = 1$ the sequence*

$$\cdots \to \Phi^{\lambda-1} \to \Phi^{\lambda} \to \Phi^{\lambda+1} \to \cdots$$

is exact for any $\lambda \neq 0$. For $\lambda = 0$ we have $\operatorname{Ker} d/\operatorname{Im} d = k$ *(generated by ξ^0).*

4. *For $m = 0$ the sequence*

$$\cdots \xrightarrow{d} \Sigma_{-1-n} \xrightarrow{d} \Sigma_{-n} \xrightarrow{d} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots$$

is exact.

2. *Description of invariant operators.* Let V_1, V_2 be irreducible finite-dimensional $gl(n,m)$-modules and $c: T(V_1) \to T(V_2)$ a nonzero $\mathfrak{L}$-invariant operator.

Theorem 2. *Replacing, if necessary, V_1 by $\Pi(V_1)$ and V_2 by $\Pi(V_2)$, we arrive at one of the following possibilities.*

a) *There is an isomorphism $c': V_1 \to V_2$ that defines the isomorphism $c: T(V_1) \to T(V_2)$.*

b) *$T(V_1)$ and $T(V_2)$ are isomorphic to neighboring terms in one of sequences of Theorem 1 and c is a multiple of d or $\int$.*

3. *Description of irreducible $\mathfrak{L}$-modules.* We will consider two categories of $\mathfrak{L}$-modules—discrete and topological. An $\mathfrak{L}$-module I such that for any $i \in I$ the superspace $U(\mathfrak{L}_0)i$ is finite-dimensional and $\mathfrak{L}_1^{r(i)}\, i = 0$ for large $r(i) \in \mathbf{N}$, we will call *discrete.*

To any discrete $\mathfrak{L}$-module I there corresponds a *topological* $\mathfrak{L}$-module $I^* = \operatorname{Hom}_k(I,k)$ with a basis of neighborhoods of zero consisting of annihilators of finite dimensional subsuperspaces in I. The module I can be recovered from I^* since $I = \operatorname{Hom}_k^c(I^*,k)$ where Hom_k^c stands for a superspace of continuous homomorphisms.

These two categories of discrete and topological $\mathfrak{L}$-modules are analogs of the category of finite-dimensional gl-modules, where gl is a finite-dimensional simple Lie algebra. In particular, for $m = 0$ these categories coincide with that of finite-dimensional $\mathfrak{L}$-modules.

The following theorem describes irreducible $\mathfrak{L}$-modules in the category of topological $\mathfrak{L}$-modules and automatically yields a description of irreducible discrete $\mathfrak{L}$-modules.

Theorem 3. *If V is an irreducible $gl(n,m)$-module, then $T(V)$ contains the unique irreducible $\mathfrak{L}$-submodule* $\operatorname{irr} T(V)$, *where* $\operatorname{irr} T(V) = T(V)$ *if $T(V)$ does not occur in any of the sequences of Theorem* 1, *and*

$$\operatorname{irr}\Omega^i = \operatorname{Ker} d \cap \Omega^i, \qquad \operatorname{irr}\Sigma_i = \operatorname{Im} d \cap \Sigma_i,$$

$$\operatorname{irr}\Phi^{\lambda} = \operatorname{Ker} d \cap \Phi^{\lambda}.$$

The modules $\operatorname{irr} T(V)$ *and* $\Pi(\operatorname{irr} T(V))$ *are mutually inequivalent and exhaust all irreducible topological $\mathfrak{L}$-modules.*

4. *Characters of finite-dimensional $\mathfrak{L}$-modules.* Let $\mathfrak{L} = W(0,m)$. Let W be the Weil group, φ the first fundamental weight, ρ half the sum of the positive roots (of the Lie algebra L_0), and $\{\varphi_i\}$ the weights of the L_0-module L_{-1}. Let $\mathfrak{D} = \sum_{w \in W} \operatorname{sgn} w e^{w\rho}$ and $N = \Pi(1 + \epsilon e^{\varphi_i})$.

Theorem 4. *Let L_χ be an irreducible L_0-module with highest weight χ and even highest weight vector. Then*

1) $\cosh T(L_\chi) = N \operatorname{ch} L_\chi$

2) $\cosh d\Omega^r = (1/\mathfrak{D}) \sum_{w \in W} \operatorname{sgn} w \cdot w\left[e^{r\varphi+\rho} \prod_{\varphi_i \neq (1,0,\ldots,0)} (1 + \epsilon e^{\varphi_i})\right]$

3) $\cosh d\Sigma_{-m-r} = -(\epsilon)^m (1/\mathfrak{D}) \sum_{w \in W} \operatorname{sgn} w$

$$\times w\left[e^{-r\varphi+\rho} \prod_{\varphi_i \neq (1,0,\ldots,0)} (1 + \epsilon e^{\varphi_i})\right].$$

3. Proofs

1. *Proof of Poincaré lemma.* Let K be one of complexes

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \text{ or } \cdots \xrightarrow{d} \Sigma_{n-m-1} \xrightarrow{d} \Sigma_{n-m}.$$

Put $L_j = L_{x_j \partial_j}$. The formula $L_j = [d, i_{x_j \partial_j}]$ implies that L_j act trivially on the cohomology $H(K)$ of K.

It is easy to verify that $x^\kappa \hat{x}^\nu$ and $x^\kappa \hat{\partial}^\nu \Delta$ are eigenvectors with respect to L_j. Hence, $K = K_j' \oplus K_j''$ where $K_j' = \operatorname{Ker} L_j$, $K_j'' = \operatorname{Im} L_j$. Since L_j is invertible on K_j'', it is also invertible on $H(K_j'')$, implying $H(K_j'') = 0$. Thus, $H(K) = H(K_j')$.

Since all L_j commute, we have $H(K) = H(\cap K_j')$. Let us describe $\cap K_j'$.

a) The case of differential forms. The operator L_j multiplies $x^\kappa \hat{x}^\nu$ by $\lambda_j = \kappa_j + \nu_j$. The condition $\lambda_j = 0$ means that $\kappa_j = \nu_j = 0$. Hence

$$\cap K_j' = k \cdot 1 \subset \Omega^0.$$

b) The case of integrable forms. The operator L_j multiplies $x^\kappa \hat{\partial}^\nu \Delta$ by $\lambda_j = \kappa_j - \nu_j + (-1)^{p(x_j)}$. If $p(x_j) = 0$, then $\nu_j \leqslant 1$ and $\lambda_j = 0$ implies $\kappa_j = 0$, $\nu_j = 1$. If $p(x_j) = 1$, then $\kappa_j \leqslant 1$ and $\lambda_j = 0$ implies $\kappa_j = 1$, $\nu_j = 0$. Thus $\cap K_j' = k \cdot \xi_1 \ldots \xi_m \hat{\partial}_1 \ldots \partial_n \Delta \subset \Sigma_{-m}$.

This proves 1 and 2 of Theorem 1. Statement 3 is proved in the same way. Statement 4 follows from 1 and 2.

2. *Discrete and topological $\mathfrak{L}$-modules.* It is more convenient to prove Theorems 2 and 3 in terms of discrete modules.

The following statements are evident:

a) $\operatorname{Hom}_{\mathfrak{L}}(I_1, I_2) \cong \operatorname{Hom}_{\mathfrak{L}}^c(I_2^*, I_1^*)$.

b) Closed $\mathfrak{L}$-submodules in I^* are in one-to-one correspondence with $\mathfrak{L}$-submodules in I (we assign to $I' \subset I$ the module $(I/I')^* = \operatorname{Ann} I' \subset I^*$).

Put $I(V) = U(\mathfrak{L}) \otimes_{U(\mathfrak{L}_0)} V$ for any finite-dimensional L_0 module V, where as in Section 1.6, we extend the action of L_0 to a $U(\mathfrak{L}_0)$-action. Clearly $I(V)$ is a discrete $\mathfrak{L}$-module.

Lemma. $I(V)^* \cong T(V^*)$.

Proof

$$T(V^*) = \operatorname{Hom}_{U(\mathfrak{L}_0)}(U(\mathfrak{L}), V^*)$$
$$\cong \operatorname{Hom}_k(U(\mathfrak{L}) \otimes_{U(\mathfrak{L}_0)} V, k) = \operatorname{Hom}_k(I(V), k).$$

Thus, instead of studying $T(V)$ we may study $I(V)$, which sometimes is somewhat easier to describe.

3. Peculiar vectors. Let I be a discrete $\mathfrak{L}$-module. The subsuperspace $I^{\mathfrak{L}_1} = \{z \in I \mid \mathfrak{L}_1 z = c\}$ is $\mathfrak{L}_0$-invariant, so it is naturally endowed with an L_0-module structure. Vectors $z \in I^{\mathfrak{L}_1}$ are called *peculiar*; see [9].

Lemma. $\operatorname{Hom}_{\mathfrak{L}}(I(V), I) = \operatorname{Hom}_{L_0}(V, I^{\mathfrak{L}_1})$.

Proof

$$\operatorname{Hom}_{\mathfrak{L}}(I(V), I) = \operatorname{Hom}_{\mathfrak{L}_0}(U(\mathfrak{L}) \otimes_{U(\mathfrak{L}_0)} V, I)$$
$$= \operatorname{Hom}_{\mathfrak{L}_0}(V, I) = \operatorname{Hom}_{L_0}(V, I^{\mathfrak{L}_1}).$$

This lemma reduces the problem of describing $\mathfrak{L}$-invariant operators $c: I(V_1) \to I(V_2)$ to that of describing L_0-homomorphisms $c_0: V_1 \to I(V_2)^{\mathfrak{L}_1}$. If V_1 is irreducible, then a homomorphism c_0 is completely defined by the image $c_0(v)$ of the highest-weight vector $v \in V_1$. The vector $c_0(v)$ is the highest-weight peculiar vector. Hence a description of the various homomorphisms c is reduced to the description of the highest-weight peculiar vectors. This description will be carried out in the following Section 3.5.

Remark. The vector $c_0(v)$ need not correspond to any homomorphism $c: I(V_1) \to I(V_2)$ with irreducible L_0-module V_1. This happens when the L_0-module $U(L_0)c_0(v)$ is reducible.

4. Description of peculiar vectors in $I(V)$. The decomposition $U(\mathfrak{L}) \cong k[\partial_1, \ldots, \partial_{n+m}] \otimes U(\mathfrak{L}_0)$ implies $I(V) \cong k[\partial] \otimes V$. We will describe peculiar vectors in a somewhat more general situation.

Lemma. *Let I be an $\mathfrak{L}$-module, V a subsuperspace in I satisfying*

a) $\mathfrak{L}_0 V \subset V$, $\mathfrak{L}_1 V = 0$,

b) $I = k[\partial] \otimes V$.

Let $z \in I$ be a highest-weight peculiar vector and λ its weight. Then there is a highest-weight vector $v \in V$ of weight μ such that one of the following *holds*:

a) $z = v$, $\lambda = \mu$;

b) *For some* r *we have*

$$z = \partial_r v_r + \sum_{i>r} \partial_i v_i + v,$$

where $v_i \in V$, $v_i \neq 0$, $i \geqslant r$ *and* v *is the highest-weight vector of weight* λ. *We have either* $1 \leqslant r \leqslant n + i$ *or* $r = n + m$ *and the following may occur (all of them satisfying* $\mu_i = \lambda_i + \delta_{ir}$):

α) *If* $r \leqslant n$, *then* $\lambda = (0, \ldots, 0, -1, \ldots, -1; 1, \ldots, 1; \bar{0})$, *where* 0 *occurs* $r - 1$ *times*;

β) *If* $m > 1$, $r = n + 1$, *then* $\lambda = (0, \ldots, 0; p, 1, \ldots, 1; \bar{0})$, *where* $p \in \mathbf{Z}$, $p \geqslant 1$.

γ) *If* $m > 1$, $r = n + m$, *then* $\lambda = (0, \ldots, 0; p, 0, \ldots, 0; \bar{0})$, *where* $p \in \mathbf{Z}$, $p \geqslant 1$.

δ) *If* $m = 1$, $r = n + 1$, *then* $\lambda = (0, \ldots, 0; s; \bar{0})$, *where* $s \in k$.

c) *If* $n = 0$, $m > 1$ *then* $\lambda = (0, \ldots, 0; \bar{0})$ *while* $z = \partial_1 \ldots \partial_m v + \tilde{v}$.

Let us deduce Theorem 2 from this lemma. Let $c: T(V_1) \to T(V_2)$ be a nonzero invariant operator. The homomorphism $c^*: I(V_2^*) \to I(V_1^*)$ corresponds to this operator, hence there is a highest-weight peculiar vector $c^*(v^*) \in I(V_1^*)$. Let λ be the weight of $c^*(v^*)$. Consider the highest-weight vector $v \in V_1^*$ described in the lemma. Since the L_0-module V_1^* is irreducible, the highest-weight theorem implies that μ is the highest weight of V_1^* and v is a multiple of a highest-weight vector $v^* \in V_1^*$. By multiplying by a scalar, we may assume that $v = v^*$. This condition uniquely defines c; therefore $\dim \mathrm{Hom}_{\mathfrak{L}}^{c}(T(V_1), T(V_2)) = (1, 0)$.

In fact the weights λ and μ tell which of (a), (b), or (c) takes place. If z and z' are vectors corresponding to different homomorphisms c and c', then $z - z'$ depends only on v_i and v, their weights being different from μ, which is only possible for $z - z' = 0$.

It is clear that the case $\lambda = \mu$, $z = v$ corresponds to an isomorphism $V_1 \cong V_2$. Let us describe which cases of the lemma correspond to d and $\int$. Denote by $\lambda(T)$ the highest weight of the L_0-module V^* for $T = T(V)$. Then

$$\lambda(\Omega^r) = (0, \ldots, 0; 0, \ldots, 0, -r), \qquad \text{if} \quad m > 0;$$

$$\lambda(\Sigma_{m-n-r}) = (0, \ldots, 0, -1, \ldots, -1; 1, \ldots, 1) \quad \text{if } 0 \leqslant r \leqslant n \quad (0 \text{ is } r \text{ times});$$

$$\lambda(\Sigma_{m-n-r}) = (0, \ldots, 0; 1 + r - n, 1, \ldots, 1) \qquad \text{if} \quad r \geqslant n;$$

$$\lambda(\Phi^s) = (0, \ldots, 0; -s).$$

The proof uses the explicit description of the $\mathcal{F}$-basis in Ω, Σ and Φ, and is quite straightforward.

Thus, the following possibilities of the lemma correspond to d and $\int$:

$d : \Omega^r \to \Omega^{r+1}$—cases (b$\lambda$) and (b$\delta$) for $s \in \mathbf{Z}, s < 0$;

$d : \Sigma_r \to \Sigma_{r+1}$—cases (b$\alpha$), (b$\beta$), and (b$\delta$) for $s \in \mathbf{Z}, s > 0$;

$d : \Phi^s \to \Phi^{s+1}$—case (b$\delta$) for $s \notin \mathbf{Z}$;

$\int : \Sigma_{-m} \to \Omega^0$—case (c).

Note that only one peculiar vector, that of the case (bδ) for $s = 0$, does not correspond to any invariant operator acting on tensor fields with *irreducible* fibre.

Indeed, since the highest weight of $U(L_0)z$ is 0, irreducibility implies its triviality due to the highest-weight theorem. But

$$E_{n+1,1}(z) = E_{n+1,1}(\partial_{n+1}v) = \partial_1 v - \partial_{n+1}E_{n+1,1}v \neq 0.$$

5. Proof of Lemma 4. In Section 3.6 we will prove Lemmas 5.1 and 5.2, which describe peculiar vectors for Lie superalgebras $W(n, m)$ for small n and m. In this section we will deduce the general case from this description; it is based on the following idea (compare with [11], [12]).

Let $R \subset [1, 2, \ldots, n + m]$ be a subset of indices. Consider a Lie subalgebra

$$\mathfrak{L}^R = \{\Sigma f_j(\bar{x})\partial_j \mid j \in R, \bar{x} = \{x_i\}_{i \in R}\}$$

and subsuperspace $V^R = k[\partial_j]_{j \notin R} \otimes V \subset I$. Then the $\mathfrak{L}^R$-modules I and V^R satisfy conditions a) and b) of Lemma 4, and z is a highest-weight peculiar vector with respect to $\mathfrak{L}^R$. Using the description of peculiar vectors for $\mathfrak{L}^R$, we will obtain restrictions for the weight and the form of z. Applying this trick for various R we will describe z completely.

Let us prove Lemma 4. Let $z = \Sigma \partial^\nu v_\nu$, so that all $v_\nu \neq 0$. The *depth* of a vector z is $d(z) = \max|\nu|$.

Case 1. $d(z) = 0$, i.e., $z \in V$. This corresponds to (a).

Case 2. $d(z) = 1$, i.e., $z = \Sigma\partial_i v_i + v$, where v_i and v are $\mathfrak{h}$-invariant and $v_i \neq 0$ for some i. Since z is a highest-weight vector, $E_{ij}z = 0$ for $i < j$. The coefficients of ∂_j and 1 being zero, we have

$$v_i = (-1)^{p(E_{ij})}\rho(E_{ij})v_j, \qquad \rho(E_{ij})\tilde{v} = 0. \tag{$*$}$$

In particular, $\tilde{v}$ is a highest-weight vector of weight λ.

Let r be the smallest subscript such that $v_r \neq 0$. From ($*$) it follows that $v = v_r$ is the highest-weight vector. But, for $j > r$ we have $E_{rj}v_j = \pm v_r$ so that $v_j \neq 0$.

Lemma 1. *A peculiar vector z of depth 1 has the following weight:*

1. $W(1,0)$. *If* $z = \partial v + \tilde{v}$, *then* $\lambda = -1$.
2. $W(2,0)$. *If* $z = \partial_1 v_1 + \partial_2 v_2 + \tilde{v}$, *then* $\lambda = (-1, -1)$.
   If $z = \partial_2 v_2 + \tilde{v}$, *then* $\lambda = (0, -1)$.
3. $W(1,1)$. *If* $z = \partial_1 v_1 + \partial_2 v_2 + \tilde{v}$ *then* $\lambda = (-1, 1)$.
   If $z = \partial_2 v_2 + \tilde{v}$, *then* $\lambda = (0, s)$ *where* $s \in k$.
4. $W(0,2)$. *If* $z = \partial_1 v_1 + \partial_2 v_2 + \tilde{v}$, *then* $\lambda = (p, 1)$, *where* $p \in \mathbf{Z}$, $p \geqslant 1$.
   If $z = \partial_2 v_2 + \tilde{v}$, *then* $\lambda = (0, -p)$, $p \in \mathbf{Z}$, $p \geqslant 1$.

Using this lemma we will consider the various cases of Lemma 4.

$(b\alpha), r \leqslant n$. Applying Lemma 5.1.1 to $\mathfrak{L}^r$, we have that $\lambda_r = -1$. If $i < r$, then applying 5.1.2 to $\mathfrak{L}^{ir}$ we have that $\lambda_i = 0$. If $i > r$, then 5.1.2 and 5.1.3 imply that $\lambda_i = -1$ for $i \leqslant n$ and $\lambda_j = 1$ for $j \geqslant n$. That implies case $(b\alpha)$ of Lemma 4.

$(b\beta, \gamma, \delta), r > n$. From 5.1.3 it follows that $\lambda_i = 0$ for $i \leqslant n$. Further, it follows from 5.1.4 that if $n < i < r$ then $\lambda_i = 0$, $\lambda_r < 0$ and if $r < i$ then $\lambda_i = 1$, $\lambda_2 > 0$. In particular, this implies that $n + 1 < r < n + m$ is impossible. The case $(b\beta)$ holds for $m > 1$, $i = n + 1$, the case $(b\gamma)$ holds for $m > 1$, $r = n + m$, and $(b\delta)$ holds for $m = 1$.

Case 3. $d(z) > 1$.

Lemma 2. *Let* $z \neq 0$ *be a highest-weight peculiar vector.*

1. *If* $\mathfrak{L}$ *is either* $W(1,0)$, $W(2,0)$, *or* $W(1,1)$, *or* $W(1,2)$, *then* $z = 0$.
2. *For* $\mathfrak{L} = W(0,2)$ *we have* $z = \partial_1\partial_2 v + \tilde{v}$, $\lambda = (0,0)$.

Let us show that if $d(z) > 1$ then $n = 0$. Let $n > 1$ and $z = \Sigma\partial^\nu v_\nu$, where $v_\nu \neq 0$ and $|\nu| > 1$. Choose indices i, j so that $i < j$ and $\nu_i + \nu_j > 1$. Applying Lemma 5.2.1 to $\mathfrak{L}^R$ where $R = \{1, i, j\}$, we obtain a contradiction.

Now let $n = 0$ and $z = \Sigma\partial^\nu v_\nu$, where $v_\nu \neq 0$ and $|\nu| > 1$. Since all ∂_i are odd we see that $\nu_i \leqslant 1$. Let $\nu_i = \nu_j = 1$. From Lemma 5.2.2 we obtain that $\lambda_i = \lambda_j = 0$. Now if r differs from i and j, then applying Lemma 5.1.4 to $\mathfrak{L}^{ir}$ we see that $\nu_r = 1$ since $\nu_i = 1$, $\nu_r = 0$ implies $\nu_i \neq 0$. From Lemma 5.2.2 we see that $\lambda_r = 0$. Thus $\nu = (1, \ldots, 1)$, $\lambda = (0, \ldots, 0)$.

Suppose that in the decomposition $z = \Sigma\partial^\nu v_\nu$ there is a multi-index ν such that $|\nu| = 1$. Let $\nu_i = 1$, $\nu_j = 0$. Applying Lemma 5.1.4 to $\mathfrak{L}^{ij}$ we obtain a contradiction to $\lambda_i = \lambda_j = 0$.

Finally $z = \partial_1 \ldots \partial_m v + \tilde{v}$ and we see that v and $\tilde{v}$ are highest-weight vectors. Hence we obtain case (c) of Lemma 4.

6. *Proof of Lemmas 5.1 and 5.2*

a) Put $h_i = x_i\partial_i$.

5.1.1. $(u^2\delta)(\partial v + \tilde{v}) = -2hv$ since $(u^2\delta)(v) = u^2\partial(\tilde{v}) = 0$. Hence $\mu = 0$ and $\lambda = -1$.

5.1.2. Let $z_1 = \partial_1 v_1 + \partial_2 v_2 + \tilde{v}$. Applying 5.1.1 to $\mathcal{L}^1$ and $\mathcal{L}^2$ we have $\lambda = (-1, -1)$. If $z = \partial_2 v$, we similarly obtain $\lambda_2 = -1$. Now consider a vector $z' = E_{21} z = -\partial_1 v + \partial_2 E_{21} v$ and let us apply 5.1.1 to $\mathcal{L}^1$. We have $\mu_1 = 0$. Hence $\lambda_1 = 0$, i.e., $\lambda = (0, -1)$.

5.1.3. Let $z = \partial_1 v_1 + \partial_2 v_2 + \tilde{v}$. From 5.1.1 it follows that $\lambda_1 = -1$. We have $(u\xi\partial_2)z = -h_2 v_1 - E_{12} v_2$. From the formula (*) of Section 5 we have $E_{12} v_2 = -v_1$. Thus $v_1 = h_2 v_1$, i.e., $\mu_2 = 1$ and $\lambda_2 = 1$, i.e., $\lambda = (-1, 1)$.

The case $z = \partial_2 v + \tilde{v}$ is dealt with in the same way as 5.1.2.

5.1.4. Let $z = \partial_1 v_1 + \partial_2 v_2 + \tilde{v}$. As in 5.1.3, we have that $\lambda_2 = 1$.

From the highest-weight theorem we have that $\lambda_1 \in \mathbf{Z}$ and $\lambda_1 \geqslant 1$.

If $z = \partial_2 v + \tilde{v}$, then $(\xi\eta\partial_1)z = -h_1 v$; hence $\mu_1 = 0$.

The highest-weight theorem implies $\mu_2 \in \mathbf{Z}$, $\mu_2 \leqslant 0$. Hence $\lambda = (0, -l)$, where $-l = \mu_2 - 1 \leqslant -1$.

5.2.1. a) $\mathcal{L} = W(1,0)$ see [18]. Put $t = u^2\partial$, $s = u^3\partial$. An induction easily shows that

$$[t, \partial^r] = -\partial^{r-1}(2rh - r(r-1)),$$

$$[s, \partial^r] = -3\partial^{r-1}t + \partial^{r-2}[3r(r-1)h - r(r-1)(r-2)].$$

If $d(z) = r > 1$ and $z = \partial^r v + \dots$, then

$$tz = -\partial^{r-1}[2rh - r(r-1)]v + \dots = [-2r\mu - r(r-1)]\partial^{r-1}v + \dots$$

$$sz = [3r(r-1)\mu - r(r-1)(r-2)]\partial^{r-2} + \dots$$

Hence $2\mu = r - 1$, $3\mu = r - 2$, or $3(r-1) = 2(r-2)$, i.e., $r = 1$.

Contradiction

b) Let $\mathcal{L}$ be any of the Lie superalgebras under consideration, and $z \in I$ a peculiar vector such that $z = \Sigma\partial^r v_r$, where $v_r \neq 0$.

Then all r_i are either 0 or 1. For $i \leqslant n$ the statement follows from the case $W(1,0)$, and for $i > n$ from the fact that $\partial_i^2 = 0$.

c) Let $\mathcal{L} = W(2,0)$ and $z = \partial_1\partial_2 v + \dots$, where $v \neq 0$. Then $z' = E_{21}z = -\partial_1^2 v + \dots$ is a peculiar vector contradicting b).

The case $W(1,1)$ is treated in the same way.

For case $W(1,2)$ it may happen that $z = \partial_2\partial_3 v + \dots$, but then $E_{21}E_{31}z = \pm\partial_1^2 v + \dots$ is a peculiar vector that contradicts b).

5.2.2. Let $z = \partial_1\partial_2 v + \dots$. Then $(\xi\eta\partial_2)z = [\partial_2(\eta\partial_1) + \partial_2(\xi\partial_1) - \partial_2]v + \tilde{v} = \partial_2 E_{21} v + (\mu_1 - 1)\partial_1 v + \tilde{v}$. Hence $\mu_1 = 1$ and $E_{21} v = 0$. In the same way we prove that $\mu_2 = 1$ and $E_{12} v = 0$. Thus v is a highest-weight vector and $\lambda = (0, 0)$.

These formulas imply that $z' = z - \partial_1\partial_2 v$ is a highest-weight peculiar vector, $\lambda = (0,0)$ and $d(z') \leqslant 1$. It follows from 5.1.3 that $d(z') = 0$, i.e., $z' \in V$.

7. Let us prove Theorem 3. Let V be an irreducible L_0-module, $T \subset T(V)$ a proper closed submodule.

a) Let us prove that there are an irreducible L_0-module V' and a nonzero invariant operator $c: T(V) \to T(V')$ such that $T \subset \operatorname{Ker} c$. In fact, as is shown in 3.2.b, $T(V)/T = I^*$ for a discrete module I and $\operatorname{Hom}^c_{\mathfrak{L}}(T(V)/T, T(V')) = \operatorname{Hom}_{\mathfrak{L}}(I(V'^*), I) = \operatorname{Hom}_{L_0}(V'^*, I^{\mathfrak{L}_1})$. By the definition of a discrete module, we have $I^{\mathfrak{L}_1} \neq 0$ and $I^{\mathfrak{L}_1}$ contains a finite-dimensional irreducible L_0-submodule V''. Choose V' so that $V'^* = V''$. Then $\operatorname{Hom}^c_{\mathfrak{L}}(T(V)/T, T(V')) \neq 0$, i.e., there is an invariant operator c: $T(V) \to T(V')$ such that $c(T) = 0$.

b) Since c is not an isomorphism, Theorem 2 implies that $T(V) = K^r$, $T(V') = K^{r+1}$ are neighboring terms in one of the sequences of Theorem 1 and $c: K^r \to K^{r+1}$ is the corresponding homomorphism. Let us write down the preceding term of the sequence: $K^{r-1} \to K^r \to K^{r+1}$ (if $K^r = \Omega^0$ this is impossible, but now $T \subset \operatorname{Ker} d = K \cdot 1$, hence $T = K = \operatorname{irr} \Omega^0$). Put $T' = c^{-1}(T)$. By the same reasoning for T' we see that either $T' = cK^{r-1}$, i.e., $T = \operatorname{Ker} c$, or $cT' = 0$, and then $T \cap \operatorname{Im} c = 0$. Since $T \subset \operatorname{Ker} c$ the Poincaré lemma implies that $T = \operatorname{Im} c = \operatorname{Ker} c$ for $K^r \neq \Sigma_{-m}$.

If $K^r = \Sigma_{-m}$, then either $T = \operatorname{Im} d$, $T = \operatorname{Ker} d$, or T is a one-dimensional complement to $\operatorname{Im} d$ in $\operatorname{Ker} d$. It is clear then that $T = k \cdot H$ where $H = \xi_1 \ldots \xi_m \hat{\partial}_1 \ldots \tilde{\partial}_n \Delta$. But it is easy to verify that this space is not $\mathfrak{L}$-invariant (e.g., $\partial_{n+1} H$ is not proportional to H); hence this is impossible.

We have obtained a complete description of all closed submodules in every $T(V)$, which implies Theorem 3.

8. *Proof of Theorem 4.* Formula 1 is evident, since $\operatorname{ch} \Omega^0 = N$. To prove formula 2 let us use a sequence $d\Omega^r \xrightarrow{i} \Omega^{r+1} \xrightarrow{d} \Omega^{r+2} \ldots$, where $p(i) = \bar{0}$ (recall that $p(d) = \bar{1}$). The exactness of this sequence implies that

$$\begin{aligned} \operatorname{ch} d\Omega^r &= \sum_{l \geqslant 0} (-\epsilon)^l \operatorname{ch} \Omega^{r+l} \\ &= (N/\mathfrak{D}) \sum_{w \in W} \operatorname{sgn} w \cdot w \left[\sum_{l \geqslant 0} (-\epsilon)^l e^{\rho + r\varphi_1 + l\varphi_1} \right] \\ &= (n/\mathfrak{D}) \sum_{w \in W} \operatorname{sgn} w \cdot w \left(\frac{e^{\rho + r\varphi_1}}{1 + \epsilon e^{\varphi_1}} \right). \end{aligned}$$

Inserting N under the summation sign, we obtain formula 2. Formula 3 is a generalization of formula 2 to negative r and is proved in the same way using

$$\ldots \xrightarrow{d} \Sigma_{-r-m-1} \xrightarrow{d} \Sigma_{-r-m} \xrightarrow{d} d\Sigma_{-r-m}.$$

Appendix. Invariant operators in smooth and formal tensor fields

Let $\mathcal{U} = (U, \mathcal{O}_{\mathcal{U}})$ be a connected superdomain (see [1,9]) of dimension (n,m) with coordinates $x = (u,\xi)$. Let $\mathcal{F}(\mathcal{U}) = C^\infty(\mathcal{U})$ be the commutative superalgebra of smooth functions on $\mathcal{U}$ and $\mathrm{Vect}(\mathcal{U})$ be the Lie superalgebra of vector fields on $\mathcal{U}$. To any representation ρ of $gl(n,m)$ in a finite-dimensional superspace V, we assign a representation of $\mathrm{Vect}(\mathcal{U})$ in the superspace $T(\mathcal{U}; V) = \mathcal{F}(\mathcal{U}) \otimes V$ of tensor fields. $\mathrm{Vect}(\mathcal{U})$-action on $T(\mathcal{U}; V)$ is defined by the same formula as in the formal case (see Section 1.4).

On $\mathcal{F}(\mathcal{U}), \mathrm{Vect}(\mathcal{U}), T(\mathcal{U}; V)$ we define a topology of uniform convergence on all compacta in all derivatives. An *invariant operator* is a continuous homomorphism of $\mathrm{Vect}(\mathcal{U})$-modules $c: T(\mathcal{U}; V_1) \to T(\mathcal{U}; V_2)$. An operator c is *differential* if it is *local*, i.e., it does not enlarge the support: $\mathrm{supp}\, c(t) \subset \mathrm{supp}\, t$ where $t \in T(\mathcal{U}; V_1)$. Denote $\mathrm{Hom}^d_{\mathrm{Vect}(\mathcal{U})}$ the superspace of invariant differential operators.

Proposition. $\mathrm{Hom}^d_{\mathrm{Vect}(\mathcal{U})}(T(\mathcal{U}; V_1), T(\mathcal{U}; V_2)) = \mathrm{Hom}^c_{W(n,m)}(T(V_1), T(V_2))$. *Elements of this space are differential operators with constant coefficients.*

Proof. Let $c \in \mathrm{Hom}^d_{\mathrm{Vect}(\mathcal{U})}$ and $p \in V$. Define $\varphi: T(\mathcal{U}; V_1) \to V_2$ via $\varphi(t) = c(t)(p)$. Evidently φ is continuous. Since c is local we have that $\varphi(t) = 0$ if $p \notin \mathrm{supp}\, t$. From standard advanced calculus we have that $\varphi(t) = \Sigma a_\kappa \partial^\kappa t(p)$, where κ runs over a finite set of multi-indices and $a_\kappa \in \mathrm{Hom}_k(V_1, V_2)$. From the continuity of c it follows that the $|\kappa|$ are bounded for every point in a small neighborhood of the point p, i.e., c is a differential operator in a neighborhood of p. Since $[c, \partial_i] = 0$ for every i, we see that c is an operator with constant coefficients. Since this is true for every $p \in \mathcal{U}$ we see that c is a global differential operator with constant coefficients. Similarly we prove that $\mathrm{Hom}^c_{W(n,m)}$ consists of differential operators with constant coefficients.

The condition that $c = \Sigma a_\kappa \partial^\kappa$, where $a_\kappa \in \mathrm{Hom}_k(V_1, V_2)$ belongs to $\mathrm{Hom}^d_{\mathrm{Vect}(\mathcal{U})}$ or $\mathrm{Hom}^c_{W(n,m)}$, means that $[c, \mathcal{D}] = 0$ for $\mathcal{D} \in \mathrm{Vect}(\mathcal{U})$ or $\mathcal{D} \in W(n,m)$. For each $p \in \mathcal{U}$ let us consider a Lie superalgebra morphism $\alpha_p: \mathrm{Vect}(\mathcal{U}) \to W(n,m)$ (the decomposition into the Taylor series at the point p). Since the image of α_p is dense, then $[c, \mathrm{Vect}(\mathcal{U})] = 0$ implies that $[c, W(n,m)] = 0$. Conversely, let $[c, W(n,m)] = 0$. Then $[c, \mathcal{D}]$, where $\mathcal{D} \in \mathrm{Vect}(\mathcal{U})$, has a zero Taylor series expansion in each point $p \in \mathcal{U}$, implying $[c, \mathcal{D}] = 0$. Hence $[c, \mathrm{Vect}(\mathcal{U})] = 0$. The proposition is proved.

Theorem 5. *Let V_1 and V_2 be irreducible $gl(n,m)$-modules, $n > 0$ and $c: T(\mathcal{U}; V_1) \to T(\mathcal{U}; V_2)$ a nonlocal invariant operator. Then $T(\mathcal{U}; V_1) = \Sigma_{n-m}(\mathcal{U})$ and $T(\mathcal{U}; V_2) = \Omega^0(\mathcal{U})$, and c is a multiple of the integral.*

We will omit the proof of this theorem.

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