

# Contractions of Representations and Algebraic Families of Harish-Chandra Modules

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## Abstract

We examine from an algebraic point of view some families of unitary group representations that arise in mathematical physics and are associated to contraction families of Lie groups. The contraction families of groups relate different real forms of a reductive group and are continuously parametrized, but the unitary representations are defined over a parameter subspace that includes both discrete and continuous parts. Both finite- and infinite-dimensional representations can occur, even within the same family. We shall study the simplest nontrivial examples, and use the concepts of algebraic families of Harish-Chandra pairs and Harish-Chandra modules, introduced in a previous paper, together with the Jantzen filtration, to construct these families of unitary representations algebraically.

## 1 Introduction

In a previous paper [BHS16] we showed how certain contraction families of Lie groups that arise in mathematical physics [Seg51, IW53] can be constructed as real points of algebraic families of complex algebraic groups. In this paper we shall examine associated families of unitary representations, and show how they can be obtained from algebraic families of Harish-Chandra modules.

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The focus of our study will be a family of groups obtained from  $SU(1, 1)$  and its Cartan decomposition

$$\mathfrak{su}(1, 1) = \mathfrak{u}(1) \oplus \mathfrak{p},$$

where  $\mathfrak{u}(1)$  is realized as the diagonal matrices in  $\mathfrak{su}(1, 1)$ , and  $\mathfrak{p}$  is the vector space of matrices in  $\mathfrak{su}(1, 1)$  with zero diagonal entries. The group  $SU(1, 1)$  may be contracted to its Cartan motion group  $U(1) \ltimes \mathfrak{p}$ . This means that a smooth family of Lie groups  $\{G_t\}_{t \in \mathbb{R}}$  may be constructed with

$$G_t \cong \begin{cases} SU(1, 1) & t \neq 0 \\ U(1) \ltimes \mathfrak{p} & t = 0. \end{cases}$$

There is also a very similar contraction of  $SU(2)$  to the same motion group. For all this see [DR85, DR83].

Every infinite-dimensional unitary irreducible representation of the motion group  $U(1) \ltimes \mathfrak{p}$  can be approximated by two different families of representations associated to these two contractions. The first is a continuous family of unitary principal series representations of  $SU(1, 1)$  and the second is a discrete family of finite-dimensional irreducible representations of  $SU(2)$ . These two families of representations are examples of the two different procedures for contraction of representations that were introduced in [IW53, Sections 2(a) and 2(b)].

The problem that we shall address in this paper is to understand these families algebraically. We shall show that the two families of representations may be obtained from one algebraic family of Harish-Chandra modules by means of a single procedure that uses Jantzen filtration techniques.

In [BHS16] we constructed an algebraic family of Harish-Chandra pairs  $(\mathfrak{g}, \mathbf{K})$  over the complex affine line<sup>1</sup> with fibers

$$(\mathfrak{g}|_z, \mathbf{K}|_z) \cong \begin{cases} (\mathfrak{su}(1, 1)_{\mathbb{C}}, U(1)_{\mathbb{C}}) & z \neq 0 \\ (\mathfrak{u}(1)_{\mathbb{C}} \ltimes \mathfrak{p}_{\mathbb{C}}, U(1)_{\mathbb{C}}) & z = 0. \end{cases}$$

It may be equipped with a real structure  $\sigma$ , and there is a corresponding

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<sup>1</sup>Actually the family extends to a family over the projective line, but the phenomena that we shall study in this paper are purely local, and so we shall work over the affine line.

family of real groups over the real line with fibers

$$\mathbf{G}^\sigma|_t \cong \begin{cases} \mathrm{SU}(1,1) & t > 0 \\ \mathrm{U}(1) \ltimes \mathbb{R}^2 & t = 0 \\ \mathrm{SU}(2) & t < 0. \end{cases}$$

We shall review these constructions in Sections 2 and 3. The family  $\mathbf{G}^\sigma$  combines the two contractions mentioned above.

Let  $\mathcal{F}$  be an admissible and generically irreducible algebraic family of Harish-Chandra modules for  $(\mathfrak{g}, K)$ , and assume that  $\mathcal{F}$  is rationally isomorphic to its  $\sigma$ -twisted dual  $\mathcal{F}^{(\sigma)}$ ; see Section 2.4. We shall obtain from  $\mathcal{F}$  a family of unitary representations of a subfamily of  $\mathbf{G}^\sigma$ .

Our method is to associate to an intertwining operator from  $\mathcal{F}$  to  $\mathcal{F}^{(\sigma)}$  (defined over the field of rational functions) Janzen-type filtrations of the fibers of  $\mathcal{F}$ . Jantzen's technique equips the subquotients with nondegenerate hermitian forms. If we isolate those subquotients on which the hermitian forms are definite, then the Harish-Chandra module structures on these fibers may be integrated to unitary group representations.

The generically irreducible, algebraic families of Harish-Chandra modules for the family  $(\mathfrak{g}, K)$  were analyzed in [BHS16]. Applying the above method to a family of “spherical principal series type” we obtain precisely the family of unitary representations described above, consisting of the family of all unitary spherical principal series representations for  $\mathrm{SU}(1,1)$  when  $t > 0$ , together with highest weight  $2m$  spherical unitary irreducible representations of  $\mathrm{SU}(2)$  when  $t = -1/(2m+1)^2$ . (In addition we obtain the unitary irreducible representation of the motion group at  $t = 0$  to which both of these families of representations converge.)

In summary, we are able to place the contraction families of unitary representations within a purely algebraic context. This is despite the fact that the “spectrum” of values  $t$  over which the contraction families are defined includes both discrete and continuous parts, together.

In fact the structure of this spectrum strongly recalls the quantum mechanics of the hydrogen atom. It will be shown elsewhere that the family of Harish-Chandra pairs studied in this paper arises as symmetries of the Schrödinger equation for the hydrogen atom, and the collection of all physical solutions of the Schrödinger equation coincides with one of the families of representations studied here. This suggests that our techniques

for obtaining unitary representations from algebraic families of Harish-Chandra modules may be useful for quantization.

Another application that we shall explore elsewhere is the Mackey bijection. Mackey showed in some specific cases how to place most of the unitary dual of a semisimple Lie group in bijection with most of the unitary dual of the corresponding Cartan motion group [Mac75]. Later on a precise bijection between the *tempered* duals was obtained—first for complex groups in [Hig08], and then for real groups in [Afg15]. The bijection for complex groups was examined from an algebraic perspective in [Hig11], and extended to admissible duals. This latter bijection fits very well with the methods of this paper, although, since unitarity is not the main issue for the Mackey bijection, we shall identify appropriate subquotients in this context using minimal K-types rather than definiteness of hermitian forms.

Finally, perhaps it is also worth mentioning that the algebraic families of Harish-Chandra pairs considered here, which place real reductive pairs together with their compact forms, provide a means to study hermitian forms on Harish-Chandra modules and c-invariant forms in the sense of [AvLTV15, Section 10] within one algebraic context. See also [Ada17] for another study of families of Harish-Chandra modules inspired by similar considerations.

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## 2 Algebraic families and real structures

In this section we fix notations and quickly recall some definitions that were given in [BHS16] (we refer the reader to that paper for more details). Throughout, by a *variety* we shall mean an irreducible, nonsingular, quasi-projective, complex algebraic variety (in all the examples and computations done in the paper, the variety will be the complex affine line). If  $X$  is a variety, then we shall denote its structure sheaf by  $\mathcal{O}_X$ .

## 2.1 Algebraic families of Harish-Chandra pairs

An *algebraic family of complex Lie algebras* over a variety  $X$  is a locally free sheaf of  $\mathcal{O}_X$ -modules that is equipped with an  $\mathcal{O}_X$ -linear Lie bracket to make it a sheaf of Lie algebras. An *algebraic family of complex algebraic groups* over  $X$  is a smooth group scheme over  $X$ . Thanks to the smoothness assumption, to any algebraic family of complex algebraic groups there is a corresponding algebraic family of complex Lie algebras.

Suppose given an algebraic family  $\mathbf{K}$  of complex algebraic groups and an algebraic family  $\mathfrak{g}$  of complex Lie algebras on which  $\mathbf{K}$  acts, along with a  $\mathbf{K}$ -equivariant embedding of families of Lie algebras  $j : \text{Lie}(\mathbf{K}) \longrightarrow \mathfrak{g}$ . This data defines an *algebraic family of Harish-Chandra pairs* if the two actions of  $\text{Lie}(\mathbf{K})$  on  $\mathfrak{g}$  coincide.

We shall deal in this paper with algebraic families of Harish-Chandra pairs  $(\mathfrak{g}, \mathbf{K})$  with a constant group scheme whose fiber is connected and reductive. That is, we shall deal here only with cases where  $\mathbf{K} = X \times K$  for some complex connected reductive group  $K$ .

## 2.2 Algebraic families of Harish-Chandra modules

Let  $(\mathfrak{g}, \mathbf{K})$  be an algebraic family of Harish-Chandra pairs over  $X$ . An *algebraic family of Harish-Chandra modules* for  $(\mathfrak{g}, \mathbf{K})$  is a flat, quasicoherent  $\mathcal{O}_X$ -module  $\mathcal{F}$  that is equipped with

- (a) an action of  $\mathbf{K}$  on  $\mathcal{F}$ , and
- (b) an action of  $\mathfrak{g}$  on  $\mathcal{F}$ ,

such that the action morphism

$$\mathfrak{g} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{F}$$

is  $\mathbf{K}$ -equivariant, and such that the differential of the  $\mathbf{K}$ -action in (a) is equal to the composition of the embedding of  $\text{Lie}(\mathbf{K})$  into  $\mathfrak{g}$  with the action of  $\mathfrak{g}$  on  $\mathcal{F}$ .

Assuming, as indicated above, that the family  $\mathbf{K}$  is constant and reductive,  $\mathcal{F}$  is said to be *admissible* if it is finitely generated as a  $\mathfrak{g}$ -module, and if  $[\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F}]^{\mathbf{K}}$  is a locally free and finitely generated sheaf of  $\mathcal{O}_X$ -modules for every family  $\mathcal{L}$  of representations of  $\mathbf{K}$  that is locally free and finitely

generated as an  $\mathcal{O}_X$ -module. In this case, there is a canonical isotypical decomposition

$$\mathcal{F} = \bigoplus_{\tau \in \widehat{\mathbb{K}}} \mathcal{F}_\tau.$$

### 2.3 Real structures

Denote by  $\overline{X}$  the complex conjugate variety of  $X$ . A *real structure* on  $X$  is a morphism of varieties  $\sigma_X : X \longrightarrow \overline{X}$  such that the composition

$$X \xrightarrow{\sigma_X} \overline{X} \xrightarrow{\overline{\sigma_X}} X$$

is the identity morphism (here  $\overline{\sigma_X}$  is equal to  $\sigma_X$  as a map of sets; it is a morphism of varieties from  $\overline{X}$  to  $X$ ). Denote by  $X^\sigma \subseteq X$  the set of points that are fixed by  $\sigma$  (recall that  $X$  and  $\overline{X}$  are equal as sets). The fixed set might be empty.

If  $\mathcal{F}$  is a sheaf of  $\mathcal{O}_X$ -modules, then its complex conjugate  $\overline{\mathcal{F}}$  is a sheaf of  $\mathcal{O}_{\overline{X}}$ -modules. Given a real structure on  $X$ , a (compatible) real structure on  $\mathcal{F}$  is a morphism

$$\sigma_{\mathcal{F}} : \mathcal{F} \longrightarrow \sigma_X^* \overline{\mathcal{F}}$$

of  $\mathcal{O}_X$ -modules for which the composition

$$\mathcal{F} \xrightarrow{\sigma_{\mathcal{F}}} \sigma_X^* \overline{\mathcal{F}} \xrightarrow{\sigma_X^*(\overline{\sigma_{\mathcal{F}}})} \sigma_X^* \overline{\sigma_X^* \overline{\mathcal{F}}} \xrightarrow{\cong} \mathcal{F}$$

is the identity morphism.

Finally, let  $(\mathfrak{g}, \mathbf{K})$  be an algebraic family of Harish-Chandra pairs over  $X$ . A *real structure on  $(\mathfrak{g}, \mathbf{K})$*  is a triplet  $\{\sigma_X, \sigma_{\mathfrak{g}}, \sigma_{\mathbf{K}}\}$  with  $\sigma_X$  a real structure on  $X$ ,  $\sigma_{\mathfrak{g}}$  a compatible real structure on the  $\mathcal{O}_X$ -module  $\mathfrak{g}$ , and  $\sigma_{\mathbf{K}}$  a real structure on  $\mathbf{K}$  for which the morphisms

$$\begin{aligned} \sigma_{\mathfrak{g}} : \mathfrak{g} &\longrightarrow \sigma_X^* \overline{\mathfrak{g}} \\ \sigma_{\mathbf{K}} : \mathbf{K} &\longrightarrow \sigma_X^* \overline{\mathbf{K}} \end{aligned}$$

are a morphism of algebraic families of Harish-Chandra pairs over  $X$ .

For further discussions about real structures see for example [Bor91, Chapter 1] or [Spr98, Chapter 11].

## 2.4 The sigma-twisted dual

If  $(\mathfrak{g}, K)$  is an ordinary Harish-Chandra pair (not a family) and if  $V$  is an admissible  $(\mathfrak{g}, K)$ -module, then its *contragredient*  $V^\dagger$  is the admissible  $(\mathfrak{g}, K)$ -module consisting of all  $K$ -finite linear functionals on  $V$ . The conjugate contragredient  $\overline{V}^\dagger$  is a  $(\overline{\mathfrak{g}}, \overline{K})$ -module, but if  $(\mathfrak{g}, K)$  is equipped with a real structure  $\sigma$ , then the conjugate contragredient becomes an admissible Harish-Chandra module  $V^{\langle\sigma\rangle}$  for  $(\mathfrak{g}, K)$ , called the  $\sigma$ -twisted dual of  $V$  (compare [AvLTV15, Section 8], where this is called the  $\sigma$ -Hermitian dual).

This construction is easily generalized to the case of families, at least when the algebraic family of groups  $K$  is constant and reductive. If  $\mathcal{F}$  is an admissible algebraic family of Harish-Chandra modules for  $(\mathfrak{g}, K)$ , with isotypical decomposition

$$\mathcal{F} = \bigoplus_{\tau \in \widehat{K}} \mathcal{F}_\tau$$

then we define its contragredient to be

$$\mathcal{F}^\dagger = \bigoplus_{\tau \in \widehat{K}} \mathcal{F}_\tau^\dagger = \bigoplus_{\tau \in \widehat{K}} \mathcal{H}\text{om}_{O_X}(\mathcal{F}_\tau, O_X),$$

which is an admissible family of Harish-Chandra modules for  $(\mathfrak{g}, K)$ .

The complex conjugate of  $\mathcal{F}^\dagger$  is an admissible algebraic family of Harish-Chandra modules for the family  $(\overline{\mathfrak{g}}, \overline{K})$  over  $\overline{X}$ .

**Definition.** Let  $(\mathfrak{g}, K)$  be an algebraic family of Harish-Chandra pairs over  $X$  for which  $K$  is a constant and reductive algebraic family of groups. Let  $\{\sigma_X, \sigma_{\mathfrak{g}}, \sigma_K\}$  be a real structure on  $(\mathfrak{g}, K)$ . If  $\mathcal{F}$  is an admissible algebraic family of Harish-Chandra modules for  $(\mathfrak{g}, K)$ , then the  $\sigma$ -twisted dual of  $\mathcal{F}$  is the sheaf

$$\mathcal{F}^{\langle\sigma\rangle} = \sigma_X^* \overline{\mathcal{F}^\dagger}$$

equipped with the  $(\mathfrak{g}, K)$ -module structure obtained by composition with the morphism of families of Harish-Chandra pairs

$$(\mathfrak{g}, K) \xrightarrow{(\sigma_{\mathfrak{g}}, \sigma_K)} (\sigma_X^* \overline{\mathfrak{g}}, \sigma_X^* \overline{K})$$

over  $X$ .

### 3 The $SU(1,1)$ family

In [BHS16] algebraic families over the complex projective line were attached to many classical symmetric pairs of algebraic groups, and real structures were constructed on these families. In this section we shall remind the reader of the construction as it applies to  $SL(2, \mathbb{C})$  and its diagonal subgroup, viewed as the fixed group of the involution

$$\Theta: \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}.$$

In [BHS16] we obtained a family over the projective line; here we shall consider only its restriction to the affine line, which is enough for our purposes.

#### 3.1 An algebraic family of groups

The family of complex algebraic groups is a subfamily of the constant family over  $\mathbb{C}$  with fiber  $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ . The fiber over  $z \in \mathbb{C}$  is

$$G|_z = \left\{ \left( \begin{bmatrix} a & b \\ zc & d \end{bmatrix}, \begin{bmatrix} a & zb \\ c & d \end{bmatrix} \right) : ad - zbc = 1 \right\}.$$

It contains as a subfamily the constant family with fiber

$$K = \left\{ \left( \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \right) : a \in \mathbb{C}^* \right\}.$$

The fibers of  $\mathfrak{g}$ , the family of complex Lie algebras corresponding to  $G$ , are

$$\mathfrak{g}|_z = \left\{ \left( \begin{bmatrix} a & b \\ zc & -a \end{bmatrix}, \begin{bmatrix} a & zb \\ c & -a \end{bmatrix} \right) : a, b, c \in \mathbb{C} \right\}.$$

The family  $(\mathfrak{g}, K)$  is an algebraic family of Harish-Chandra pairs over  $\mathbb{C}$ .

#### 3.2 Real structure

A real structure on the family  $(\mathfrak{g}, K)$  above is given by the involution

$$\sigma_X: z \mapsto \bar{z}$$

on the base space  $\mathbb{C}$ , and the real structure

$$\sigma_G: G|_z \longrightarrow \overline{G|_{\bar{z}}}, \quad \sigma_G: (S, T) \longmapsto (\Theta(T^{-1})^*, \Theta(S^{-1})^*)$$

on  $G$ , which determines real structures on  $\mathfrak{g}$  and  $K$ . The explicit formula for the real structure on  $G$  is

$$\sigma_G: \left( \begin{bmatrix} a & b \\ zc & d \end{bmatrix}, \begin{bmatrix} a & zb \\ c & d \end{bmatrix} \right) \longmapsto \left( \begin{bmatrix} \bar{d} & \bar{c} \\ \bar{zb} & \bar{a} \end{bmatrix}, \begin{bmatrix} \bar{d} & \bar{zc} \\ \bar{b} & \bar{a} \end{bmatrix} \right),$$

from which it is clear that the fixed groups of the involution over  $\mathbb{R} \subseteq \mathbb{C}^\sigma$  are the real groups

$$G|_x^\sigma = \left\{ \left( \begin{bmatrix} a & b \\ xb & \bar{a} \end{bmatrix}, \begin{bmatrix} a & xb \\ \bar{b} & \bar{a} \end{bmatrix} \right) : |a|^2 - x|b|^2 = 1 \right\}.$$

So we see that

$$G|_x^\sigma \cong \begin{cases} SU(1,1) & x > 0 \\ U(1) \times \mathbb{C} & x = 0 \\ SU(2) & x < 0, \end{cases}$$

where the action of  $U(1)$  on  $\mathbb{C}$  is scalar multiplication by the squares of elements in  $U(1)$ .

### 3.3 The Casimir section

Associated to the algebraic family of Lie algebras  $\mathfrak{g}$  there is a family of enveloping algebras. It is generated by the sections

$$\begin{aligned} H: z &\longmapsto \left( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right), \\ E: z &\longmapsto \left( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} \right), \\ F: z &\longmapsto \left( \begin{bmatrix} 0 & 0 \\ z & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right). \end{aligned}$$

The sections  $H, E, F$  are almost a standard  $\mathfrak{sl}(2)$ -triplet:

$$(3.1) \quad [H, E] = 2E, \quad [H, F] = -2F, \quad \text{and} \quad [E, F] = zH.$$

Of special concern to us is the *Casimir section*

$$C = H^2 + \frac{2}{z}EF + \frac{2}{z}FE = H^2 + 2H + \frac{4}{z}FE$$

over  $\mathbb{C} \setminus \{0\}$ , which is invariant under the adjoint action of  $G$ . (This is the same as the Casimir section considered in [BHS16], but it is expressed here in terms of different generators.)

### 3.4 Generically irreducible families of modules

In this section we recall some facts about admissible and generically irreducible families of Harish-Chandra modules for  $(\mathfrak{g}, K)$  that were noted in [BHS16].

Since we are working over the base space  $X = \mathbb{C}$ , we can and will represent sheaves of  $O_X$ -modules by their global sections, which are modules over the ring  $\mathcal{O}$  of polynomial functions on  $\mathbb{C}$ . Let  $\mathcal{K}$  be the field of rational functions on  $\mathbb{C}$ . An admissible family  $\mathcal{F}$  of Harish-Chandra modules for  $(\mathfrak{g}, K)$  is *generically irreducible* if the representation of the Lie algebra  $\mathcal{K} \otimes_{\mathcal{O}} \mathfrak{g}$  on  $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{F}$  is irreducible over the algebraic closure of  $\mathcal{K}$ . Equivalently, in the present context,  $\mathcal{F}$  is generically irreducible if the fiber  $\mathcal{F}|_z$  is an irreducible  $\mathfrak{g}|_z$ -module for all except at most countably many  $z$ .

If  $\mathcal{F}$  is admissible and generically irreducible, then the Casimir section  $C$  acts as multiplication by a regular function  $c_{\mathcal{F}}$  on  $\mathbb{C} \setminus \{0\}$ . This is a first invariant of admissible and generically irreducible families of Harish-Chandra modules. A second invariant is the set of  $K$ -types, or weights, that occur in an admissible and generically irreducible family. This set must agree with the set of weights of some irreducible  $(\mathfrak{sl}(2, \mathbb{C}), K)$ -module, where  $K$  is the diagonal subgroup in  $SL(2, \mathbb{C})$ . All the nonzero weights have multiplicity one.

These two invariants fall quite far short of determining admissible and generically irreducible families up to isomorphism, but they do determine the families up to rational isomorphism:

**Proposition.** *Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be two admissible and generically irreducible families of Harish-Chandra modules for  $(\mathfrak{g}, K)$ . If they have the same Casimir and weight invariants, then there is an isomorphism of  $\mathcal{K}$ -vector spaces*

$$\mathcal{K} \otimes_{\mathcal{O}} \mathcal{F}_1 \xrightarrow{\cong} \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F}_2$$

that intertwines the actions of  $\mathcal{K} \otimes_{\mathcal{O}} \mathfrak{g}$ .

*Proof.* It is clear from the formulas (3.1) that

$$\mathcal{K} \otimes_{\mathcal{O}} \mathfrak{g} \cong \mathfrak{sl}(2, \mathcal{K}).$$

Now from a generically irreducible admissible family of Harish-Chandra modules for  $(\mathfrak{g}, \mathbf{K})$  we obtain an Harish-Chandra module for  $(\mathfrak{sl}(2, \mathcal{K}), K_{\mathcal{K}})$  (the underlying vector space of the module is a  $\mathcal{K}$ -vector space) such that

- (i) the representations space decomposes into integer weight spaces for the action of  $K_{\mathcal{K}}$ ,
- (ii) the Casimir for  $\mathfrak{sl}(2, \mathcal{K})$  acts as multiplication by an element of  $\mathcal{K}$ , and
- (iii) the representation is irreducible over the algebraic closure of  $\mathcal{K}$ .

The classification of such modules, up to equivalence, is carried out exactly as in the standard case of complex representations of  $(\mathfrak{sl}(2, \mathbb{C}), K_{\mathbb{C}})$  (and incidentally, in the presence of (i) and (ii), irreducibility over  $\mathcal{K}$  is equivalent to irreducibility over the algebraic closure). The result follows.  $\square$

### 3.5 The sigma-twisted duals

The theorem from the previous section allows us to characterize those generically irreducible and admissible families that are rationally isomorphic to their  $\sigma$ -twisted duals:

**Proposition.** *Let  $\mathcal{F}$  be an admissible and generically irreducible family of Harish-Chandra modules for  $(\mathfrak{g}, \mathbf{K})$ . There is an isomorphism of  $\mathcal{K} \otimes_{\mathcal{O}} \mathfrak{g}$ -modules*

$$\mathcal{K} \otimes_{\mathcal{O}} \mathcal{F} \xrightarrow{\cong} \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F}^{\langle \sigma \rangle}$$

*if and only if the regular Casimir function  $c_{\mathcal{F}}$  on  $\mathbb{C} \setminus \{0\}$  is real-valued on  $\mathbb{R} \setminus \{0\}$ .*

*Proof.* This follows from the proposition in the previous section, together with the fact that the weights of  $\mathcal{F}$  and  $\mathcal{F}^{\langle \sigma \rangle}$  are equal, while

$$c_{\mathcal{F}^{\langle \sigma \rangle}} = \sigma(c_{\mathcal{F}}),$$

where  $\sigma(c)(z) = \overline{c(\bar{z})}$ .  $\square$

## 4 Families of unitary representations

In this section we shall indicate how one can use Jantzen filtration techniques to obtain families of unitary representations from admissible and generically irreducible families of Harish-Chandra modules. The families will include both discretely- and continuously-parametrized parts. In the next section we shall show that the families of unitary representations that may be obtained in this way include the contraction family that we described in the introduction.

### 4.1 The Jantzen filtration

Let  $\mathcal{F}$  and  $\mathcal{H}$  be admissible algebraic families of Harish-Chandra modules for  $(\mathfrak{g}, \mathbf{K})$  and let

$$(4.1) \quad \varphi: \mathcal{F} \longrightarrow \mathcal{H}$$

be a morphism that is generically an isomorphism. In this section we shall review the construction of canonical increasing and decreasing filtrations of the fibers of  $\mathcal{F}$  and  $\mathcal{H}$ , respectively (the Jantzen filtrations; compare [Jan79, Hum08]). We shall also recall that a choice of coordinate near any point determines isomorphisms between corresponding subquotients at that point.

We shall continue to assume that  $\mathbf{K}$  is a constant family of reductive groups with fiber  $K$ , and we shall continue to think of  $\mathcal{F}$  and  $\mathcal{H}$  as free modules over the algebra of regular functions on the line, with compatible actions of  $K$  and the Lie algebra of global sections of  $\mathfrak{g}$ . To be precise, rather than (4.1) we shall start with a map

$$(4.2) \quad \varphi: \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F} \longrightarrow \mathcal{K} \otimes_{\mathcal{O}} \mathcal{H}$$

that is a  $(\mathfrak{g}, K)$ -equivariant isomorphism of  $\mathcal{K}$ -vector spaces, where  $\mathcal{O}$  is the algebra of regular functions on the line and  $\mathcal{K}$  is the field of rational functions.

Let  $z$  be a point on the affine line, and denote by  $\mathcal{O}_z$  the localization of  $\mathcal{O}$  at  $z$ , comprised of those rational functions whose denominators are nonzero at  $z$ . In addition form the localizations

$$\mathcal{F}_z = \mathcal{O}_z \otimes_{\mathcal{O}} \mathcal{F} \quad \text{and} \quad \mathcal{H}_z = \mathcal{O}_z \otimes_{\mathcal{O}} \mathcal{H}.$$

We shall think of these as  $\mathcal{O}_z$ -submodules of  $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{F}$  and  $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{H}$ , respectively. Fix a coordinate  $p$  at  $z$ , or in other words a degree-one polynomial function that vanishes at  $z$ , and for  $n \in \mathbb{Z}$  define

$$\mathcal{F}_z^n = \{ f \in \mathcal{F}_z : \varphi(f) \in p^n \mathcal{H}_z \}$$

and

$$\mathcal{H}_z^n = \{ h \in \mathcal{H}_z : p^n h \in \varphi[\mathcal{F}_z^n] \}.$$

These constitute decreasing and increasing filtrations of  $\mathcal{F}_z$  and  $\mathcal{H}_z$ , respectively, by  $\mathcal{O}_z$ -submodules.

Consider the fibers of  $\mathcal{F}$  and  $\mathcal{H}$ ,

$$\mathcal{F}|_z = \mathbb{C} \otimes_{\mathcal{O}} \mathcal{F} = \mathbb{C} \otimes_{\mathcal{O}_z} \mathcal{F}_z \quad \text{and} \quad \mathcal{H}|_z = \mathbb{C} \otimes_{\mathcal{O}} \mathcal{H} = \mathbb{C} \otimes_{\mathcal{O}_z} \mathcal{H}_z,$$

which are complex vector spaces and  $(\mathfrak{g}|_z, K)$ -modules. Denote by

$$\mathcal{F}|_z^n \subseteq \mathcal{F}|_z$$

the image in the complex vector space  $\mathcal{F}|_z$  of the morphism

$$\mathbb{C} \otimes_{\mathcal{O}} \mathcal{F}_z^n \longrightarrow \mathbb{C} \otimes_{\mathcal{O}} \mathcal{F}_z,$$

and similarly denote by

$$\mathcal{H}|_z^n \subseteq \mathcal{H}|_z$$

the image in  $\mathcal{H}|_z$  of the morphism of complex vector spaces

$$\mathbb{C} \otimes_{\mathcal{O}} \mathcal{H}_z^n \longrightarrow \mathbb{C} \otimes_{\mathcal{O}} \mathcal{H}_z.$$

These are decreasing and increasing filtrations of  $\mathcal{F}|_z$  and  $\mathcal{H}|_z$ , respectively, by  $(\mathfrak{g}|_z, K)$ -submodules.

We shall now obtain isomorphisms between subquotients of these two filtrations. The formula

$$f \longmapsto p^{-n} \varphi(f)$$

defines an isomorphism of  $\mathcal{O}_z$ -modules from  $\mathcal{F}_z^n$  to  $\mathcal{H}_z^n$  and induces a surjective morphism of  $(\mathfrak{g}|_z, K)$ -modules

$$\varphi^n: \mathcal{F}|_z^n \longrightarrow \mathcal{H}|_z^n.$$

**Proposition.** *The above morphism  $\varphi^n$  maps the subspace  $\mathcal{F}_z^{n+1} \subseteq \mathcal{F}_z^n$  into the subspace  $\mathcal{H}_z^{n-1} \subseteq \mathcal{H}_z^n$  and induces an isomorphism of  $(\mathfrak{g}|_z, K)$ -modules*

$$\varphi^n: \mathcal{F}_z^n / \mathcal{F}_z^{n+1} \xrightarrow{\cong} \mathcal{H}_z^n / \mathcal{H}_z^{n-1}$$

*(that depends on the differential of  $p$  at  $z$ ).*

The proof is a simple calculation, but for the convenience of the reader we shall give it in a moment. The proof shows that there is an isomorphism of subquotients, as above, whether or not the morphism  $\varphi$  in (4.2) is assumed to be an isomorphism. But the hypothesis that  $\varphi$  is an isomorphism implies the following additional important fact:

**Proposition.** *If the morphism  $\varphi: \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F} \rightarrow \mathcal{K} \otimes_{\mathcal{O}} \mathcal{H}$  is an isomorphism, then the filtrations  $\mathcal{F}_z^\bullet$  and  $\mathcal{H}_z^\bullet$  are exhaustive: the intersections of the filtration spaces are zero, while the unions are  $\mathcal{F}_z$  and  $\mathcal{H}_z$ , respectively.*

*Proof.* The modules  $\mathcal{F}_z$  and  $\mathcal{H}_z$  decompose into direct sums of free and finite rank submodules according to the  $K$ -isotypical decompositions of  $\mathcal{F}$ , and  $\mathcal{H}$  and the morphism  $\varphi$  respects these decompositions. So it suffices to prove the proposition for a single summand, where it is easy.  $\square$

*Proof of the first proposition.* If  $f \in \mathcal{F}_z^n$ , and if  $f$  determines an element of the subspace  $\mathcal{F}_z^{n+1} \subseteq \mathcal{F}_z^n$ , then we can decompose  $f$  as

$$f = f_1 + pf_2,$$

where  $f_1 \in \mathcal{F}_z^{n+1}$  and  $f_2 \in \mathcal{F}_z$ . It follows from the formula

$$f_2 = p^{-1}(f - f_1)$$

that in fact  $f_2 \in \mathcal{F}_z^{n-1}$ . Consider now the formula

$$p^{-n}\varphi(f) = p \cdot p^{-(n+1)}\varphi(f_1) + p^{-(n-1)}\varphi(f_2).$$

The first term on the right lies in  $p \cdot \mathcal{H}_z$  while the second term on the right lies in  $\mathcal{H}_z^{n-1}$ . So the image of the class of  $f$  in  $\mathcal{F}_z^n$  under the morphism  $\varphi^n$  lies in  $\mathcal{H}_z^{n-1}$ , as required.

Surjectivity of the induced map on quotient spaces is clear. As for injectivity, if an element of  $\mathcal{F}_z^n / \mathcal{F}_z^{n+1}$  is represented by some  $f \in \mathcal{F}_z^n$  and maps under  $\varphi^n$  to an element of  $\mathcal{H}_z^{n-1}$ , then we may write

$$p^{-n}\varphi(f) = p^{-(n-1)}\varphi(f_1) + ph_1$$

for some  $f_1 \in \mathcal{F}_z^{n-1}$  and some  $h_1 \in \mathcal{H}_z$ . But then

$$\varphi(f - pf_1) = p^{n+1}h_1,$$

which implies that  $f - pf_1 \in \mathcal{F}_z^{n+1}$ , so that  $f$  determines the zero element of  $\mathcal{F}_z^n/\mathcal{F}_z^{n+1}$ , as required.  $\square$

## 4.2 The Jantzen filtration and real structures

Let  $(\mathfrak{g}, \mathbf{K})$  be an algebraic family of Harish-Chandra pairs over the complex affine line  $\mathbb{C}$ , as in the previous section, with  $\mathbf{K}$  a constant family of connected reductive groups with fiber  $K$ . Assume that  $(\mathfrak{g}, \mathbf{K})$  is equipped with a real structure compatible with the standard real structure  $z \mapsto \bar{z}$  on  $\mathbb{C}$ . Let  $\mathcal{F}$  be an admissible and generically irreducible algebraic family of Harish-Chandra modules for  $(\mathfrak{g}, \mathbf{K})$ , and suppose that there is an isomorphism

$$\varphi: \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F} \longrightarrow \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F}^{\langle \sigma \rangle}.$$

In this section we shall show that the Jantzen subquotients  $\mathcal{F}_x^n/\mathcal{F}_x^{n+1}$  may be equipped with canonical (up to real scalar factors) nondegenerate hermitian forms that are  $\sigma$ -invariant for the actions of the Harish-Chandra pairs  $(\mathfrak{g}|_x, K)$ .

We shall identify  $\mathcal{F}^{\langle \sigma \rangle}$  with the space of functions from  $\mathcal{F}$  to  $\mathcal{O}$  that are conjugate  $\mathcal{O}$ -linear in the sense that if  $e \in \mathcal{F}^{\langle \sigma \rangle}$ ,  $q \in \mathcal{O}$  and  $f \in \mathcal{F}$ , then

$$e(qf) = \sigma(q)e(f),$$

where  $\sigma(q)(z) = \overline{q(\bar{z})}$ , and that vanish on all but finitely many of the  $K$ -isotypical summands of  $\mathcal{F}$ . The  $\mathcal{O}$ -module structure is  $(q \cdot e)(f) = q \cdot e(f)$  and the  $\mathfrak{g}$ -module structure is  $(X \cdot e)(f) = -e(\sigma(X)f)$ . We obtain from the isomorphism  $\varphi$  above a complex-sesquilinear map

$$\langle \ , \ \rangle: (\mathcal{K} \otimes_{\mathcal{O}} \mathcal{F}) \times (\mathcal{K} \otimes_{\mathcal{O}} \mathcal{F}) \longrightarrow \mathcal{K}$$

using the formula  $\langle f_1, f_2 \rangle = \varphi(f_1)(f_2)$ . This map is in fact  $\mathcal{K}$ -sesquilinear in the sense that

$$\langle q_1 f_1, q_2 f_2 \rangle = q_1 \langle f_1, f_2 \rangle \sigma(q_2),$$

for all  $f_1, f_2 \in \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F}$  and all  $q_1, q_2 \in \mathcal{K}$ , where again  $\sigma(q)(z) = \overline{q(\bar{z})}$ . The pairing is also  $\sigma$ -invariant under the  $\mathfrak{g}$ -action in the sense that

$$\langle X \cdot f_1, f_2 \rangle + \langle f_1, \sigma(X) \cdot f_2 \rangle = 0$$

for all  $f_1, f_2 \in \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F}$  and all  $X \in \mathfrak{g}$ .

By Schur's lemma, any other isomorphism

$$\psi: \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F} \longrightarrow \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F}^{\langle \sigma \rangle}$$

is equal to  $\varphi$  times a rational function  $q \in \mathcal{K}$ . Note that the sesquilinear form associated to  $\psi$  is  $q \cdot \langle f_1, f_2 \rangle$ . Returning to  $\varphi$ , it follows from this that there is some  $q \in \mathcal{K}$  such that

$$\sigma(\langle f_2, f_1 \rangle) = q \cdot \langle f_1, f_2 \rangle$$

for all  $f_1, f_2 \in \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F}$ . This is because the left-hand side is the  $\mathcal{K}$ -sesquilinear form associated to the isomorphism

$$\psi: \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F} \longrightarrow \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F}^{\langle \sigma \rangle}$$

defined by  $\psi(f_1)(f_2) = \sigma(\langle f_2, f_1 \rangle)$ . Note that since  $\sigma$  is an involution,

$$\sigma(q) \cdot q = 1.$$

Hence the rational function  $q$  has no zeros or poles on the real axis.

Now fix  $x \in \mathbb{R}$  and choose a real coordinate  $p$  at  $x$  (meaning that  $\sigma(p) = p$ ). The Jantzen construction of the previous section produces non-degenerate, invariant, complex-sesquilinear forms

$$\langle \ , \ \rangle_{x,n}: \mathcal{F}|_x^n / \mathcal{F}|_x^{n+1} \times \mathcal{F}|_x^n / \mathcal{F}|_x^{n+1} \longrightarrow \mathbb{C}.$$

The space  $\mathcal{F}|_x^n$  is determined by elements  $f_1 \in \mathcal{F}_x$  such that

$$\text{ord}_x(\langle f_1, f_2 \rangle) \geq n$$

for every  $f_2 \in \mathcal{F}_x$ , and the scalar  $\langle f_1, f_2 \rangle_{x,n}$  is equal to the value of the function  $p^{-n} \langle f_1, f_2 \rangle$  at  $x$ .

Up to a shift in the index  $n$ , the quotient spaces  $\mathcal{F}|_x^n / \mathcal{F}|_x^{n+1}$ , are independent of  $\varphi$  and also independent of the choice of  $p$ . Moreover up to a real scalar factor the above forms are also independent of the choice of  $p$ . Finally since

$$\overline{\langle f_2, f_1 \rangle}_{x,n} = q(x) \cdot \langle f_1, f_2 \rangle_{x,n},$$

where  $|q(x)| = 1$ , after rescaling the above forms by a square root of  $q(x)$  we obtain, for all  $n$ , Hermitian forms that are independent of all choices, up to real scalar factors.

So for each  $x \in \mathbb{R}$  we have constructed a canonical (finite) set of  $(\mathfrak{g}|_x, K)$ -modules (the Jantzen quotients). Each is equipped with a canonical, up to a real scalar factor, nondegenerate Hermitian form. All the forms are  $\sigma$ -invariant for the action of  $(\mathfrak{g}|_x, K)$ .

### 4.3 Distinguished parameter values

For all but a countable set of values  $x$ , the Jantzen filtration on  $\mathcal{F}|_x$  will be trivial, so that there will be a unique Jantzen quotient at  $x$ , namely  $\mathcal{F}|_x$  itself. As a result the filtration immediately selects a distinguished, discrete collection of parameter values, where there is more than one Jantzen quotient, and a corresponding discretely parametrized set of representations.

Obviously if the fiber is an irreducible module, then the filtration must be trivial. In the particular special case that we shall analyze in the next section the distinguished values are precisely the reducibility points, but this need not be so in general.

### 4.4 Infinitesimally unitary Jantzen quotients

In this section we shall continue with the notation of the previous section; in particular we shall denote by  $x$  a fixed element of the real line. We shall consider the problem of integrating the Lie algebra representation of  $\mathfrak{g}|_x^\sigma$  on a Jantzen quotient  $\mathcal{F}|_x^n/\mathcal{F}|_x^{n+1}$  so as to obtain a unitary representation. The arguments in this section no longer involve families; they just involve the hermitian forms constructed above.

**Definition.** We shall say that a Jantzen quotient  $\mathcal{F}|_x^n/\mathcal{F}|_x^{n+1}$  is *infinitesimally unitary* if the real one-dimensional space of hermitian forms on  $\mathcal{F}|_x^n/\mathcal{F}|_x^{n+1}$  that was constructed in the previous section includes a positive-definite hermitian form (that is, an inner product).

Assume that  $\mathcal{F}|_x^n/\mathcal{F}|_x^{n+1}$  is infinitesimally unitary. Adjust the sesquilinear form  $\langle \cdot, \cdot \rangle_{x,n}$  by a (uniquely determined) complex scalar factor, if necessary, so as to make it an inner product. We can apply the following theorem of Nelson [Nel59, Theorem 5] to show that the admissible  $(\mathfrak{g}|_x^\sigma, K^\sigma)$ -module  $\mathcal{F}|_x^n/\mathcal{F}|_x^{n+1}$  integrates to a unitary representation on the Hilbert space completion of the Jantzen quotient.

**Theorem.** Let  $\mathfrak{g}$  be a real, finite-dimensional Lie algebra of skew-symmetric operators acting on a common invariant and dense domain  $\mathcal{F}$  in a Hilbert space  $H$ . Let  $X_1, \dots, X_k$  be a basis for  $\mathfrak{g}$  and form the symmetric operator

$$\Delta = X_1^2 + \dots + X_k^2$$

with domain  $\mathcal{F}$ . If  $\Delta$  is essentially self-adjoint (that is, if  $\Delta$  has a unique self-adjoint extension), then each  $X_j$  is essentially skew-adjoint, and if  $G$  is the simply connected Lie group associated to  $\mathfrak{g}$ , then there is a unique unitary representation

$$\pi: G \longrightarrow U(\mathcal{H})$$

for which the generators of the one-parameter unitary groups  $\{\pi(\exp(tX_j))\}_{t \in \mathbb{R}}$  are the skew-adjoint extensions of the operators  $X_j$ .

In order to apply Nelson's theorem we shall assume that there is an internal vector space direct sum decomposition

$$\mathfrak{g}|_x^\sigma = \mathfrak{k}^\sigma \oplus \mathfrak{p}_x^\sigma$$

in which  $\mathfrak{p}_x^\sigma$  is a real subspace of  $\mathfrak{g}|_x^\sigma$  that is invariant under the action of  $K^\sigma$ . We shall assume that both  $\mathfrak{k}^\sigma$  and  $\mathfrak{p}_x^\sigma$  carry  $K^\sigma$ -invariant inner products. This is certainly the case in our example. Indeed it is the case whenever the real group  $K^\sigma$  is compact.

Fix orthonormal bases of  $\mathfrak{k}^\sigma$  and  $\mathfrak{p}_x^\sigma$  with respect to the  $K^\sigma$ -invariant inner products, and combine them to form a basis  $\{X_j\}$  for  $\mathfrak{g}|_x^\sigma$ . The operator

$$\Delta: \mathcal{F}|_x^n / \mathcal{F}|_x^{n+1} \longrightarrow \mathcal{F}|_x^n / \mathcal{F}|_x^{n+1}$$

in Nelson's theorem is symmetric. It is also  $K^\sigma$ -invariant, and so it leaves invariant the  $K^\sigma$ -isotypical summands in  $\mathcal{F}|_x^n / \mathcal{F}|_x^{n+1}$ , which are finite-dimensional. This implies that  $\Delta$  is essentially self-adjoint, as required.

We conclude that the representation of  $\mathfrak{g}|_x^\sigma$  on each infinitesimally unitary Jantzen quotient integrates to a unitary representation of the universal cover of  $G|_x^\sigma$  on the Hilbert space completion of the Jantzen quotient.

By definition of a  $(\mathfrak{g}|_x^\sigma, K^\sigma)$ -module, we also know that the action of  $\mathfrak{k}^\sigma \subseteq \mathfrak{g}|_x^\sigma$  integrates to a unitary action of  $K^\sigma$ . So if the inclusion of  $K^\sigma$  into  $G|_x^\sigma$  induces a surjection

$$\pi_1(K^\sigma) \longrightarrow \pi_1(G|_x^\sigma),$$

then in fact the above unitary representation of the universal covering group of  $G|_x^\sigma$  factors through  $G|_x^\sigma$ . This is the case in our example (although it is not the case in all the examples in [BHS16] constructed from symmetric pairs of reductive groups).

## 5 Contraction families

In this final section we shall determine the infinitesimally unitary Jantzen quotients associated to a family of admissible and generically irreducible modules for the family of Harish-Chandra pairs  $(\mathfrak{g}, K)$  described in Section 3. We shall recover in this way the contraction family of unitary representations described in the introduction.

The infinitesimally unitary Jantzen quotients are shown in Figure 1. The quotients include both a continuous family of unitary principal series representations for  $SU(1, 1)$  and a discrete sequence of (finite-dimensional) irreducible representations for  $SU(2)$ . The figure gives some sense of the convergence phenomenon that is of interest in the theory of contractions.



Figure 1: The diagram illustrates the infinitesimally unitary Jantzen quotients for the unique family with Casimir  $-(1+x)/x$  that is generated by its 0-weight space. The real parameter  $x$  runs horizontally. At every  $x$ , the weights that appear in a unitary Jantzen quotient of  $\mathcal{F}|_x$  are shown vertically. For  $x < 0$  there is a unitary quotient for a sequence of values  $x = -(2m+1)^{-2}$  converging to zero, giving finite-dimensional unitary representations of  $SU(2)$ . At  $x = 0$  there is an irreducible unitary representation of the Cartan motion group that includes every even weight. When  $x > 0$  every even weight occurs once again, and the representations are the unitary spherical principal series for  $SU(1, 1)$  with Casimir converging to infinity as  $x$  converges to zero.

## 5.1 Bases for generically irreducible families

Let  $(\mathfrak{g}, K)$  be the algebraic family of Harish-Chandra pairs from Section 3. For simplicity, from here on we shall be calculating with those families that have nonzero  $K$ -isotypical components precisely for every even weight (recall that  $K \cong \mathbb{C}^\times$ ). These are the families that are relevant to the contraction families of unitary representations mentioned in the introduction; however the other possible types of families are listed in [BHS16] and they can be analyzed in a similar way.

Let  $\mathcal{F}$  be such a family. As a consequence of our assumption, the  $\mathcal{O}$ -module  $\mathcal{F}$  has a basis  $\{f_k\}$  indexed by the even integers  $k$ , with

$$\begin{aligned} Hf_k &= kf_k \\ Ef_k &= A_k f_{k+2} \\ Ff_{k+2} &= B_k f_k \end{aligned}$$

where  $A_k, B_k \in \mathcal{O}$ . Here  $H$ ,  $E$  and  $F$  are the sections of  $\mathfrak{g}$  defined in Section 3.3. The Casimir from Section 3.3 acts on  $f_n$  as multiplication by the polynomial

$$k^2 + 2k + \frac{4}{z} A_k B_k$$

(so this expression is independent of  $k$ ).

Let us now make a further assumption: that  $\mathcal{F}$  is generated as a family of Harish-Chandra modules by its  $0$ -isotypical component  $\mathcal{F}_0$ .<sup>2</sup> Then  $A_k$  is nowhere zero for  $k \geq 0$ . This is because the isotypical component  $\mathcal{F}_{k+2}$  is generated as a  $\mathcal{O}$ -module by the element

$$A_k A_{k-2} \cdots A_0 f_{k+2}.$$

Similarly,  $B_k$  is nowhere zero for  $k < 0$ . Hence

$$A_k = \text{constant}, \quad \text{when } k \geq 0$$

and

$$B_k = \text{constant}, \quad \text{when } k < 0,$$

where the constants might be distinct, but are all nonzero.

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<sup>2</sup>Other natural choices are possible, and are described in [BHS16, 4.9], but for brevity we shall focus on just this one example here.

In fact after adjusting the basis elements  $f_k$  by scalar factors we can, and will from now on, assume that all the above constants are 1. When that is done, the value of the Casimir section on  $f_k$  becomes

$$k^2 + 2k + \frac{4}{z}B_k, \quad \text{when } k \geq 0$$

and

$$k^2 + 2k + \frac{4}{z}A_k, \quad \text{when } k < 0.$$

So we see that the remaining  $A_k$  and  $B_k$  are completely determined by the action of the Casimir section. Hence:

**Proposition.** *Let  $\mathcal{F}$  be an admissible and generically irreducible algebraic family of Harish-Chandra modules for  $(\mathfrak{g}, \mathbf{K})$ . Assume that the weight  $k$  isotypal summands of  $\mathcal{F}$  are nonzero precisely for the even integer weights. If  $\mathcal{F}$  is generated by its weight  $k$  isotypal summand, then  $\mathcal{F}$  is determined up to isomorphism by the action of the Casimir section.  $\square$*

Conversely, a family of the type described in the proposition may be constructed for which the action of the Casimir section is by any given regular function on  $\mathbb{C} \setminus \{0\}$ , other than one of the constants  $k^2 + 2k$ , that has at most a simple pole at 0 (a family can be constructed for the excluded constants, too, but it is not generically irreducible). See [BHS16] for this and for further information on classification of admissible and generically irreducible algebraic families of modules for  $(\mathfrak{g}, \mathbf{K})$ .

*Remark.* Similar families are considered in [AvLTV15, Section 14]. The concern there is with *constant* families of Harish-Chandra pairs, but in applying Jantzen filtration techniques to our families we shall certainly be following the lead of [AvLTV15].

## 5.2 Computation of the Jantzen quotients

We shall work in this section with a family  $\mathcal{F}$  as in the previous proposition, and we shall assume in this section that there is an isomorphism

$$\varphi: \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F} \longrightarrow \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F}^{\langle \sigma \rangle}.$$

This is so precisely when the value of the Casimir section is fixed by the involution  $\sigma$ .

The  $\sigma$ -twisted dual  $\mathcal{F}^\sigma$  has the same weights as  $\mathcal{F}$ , namely a one-dimensional  $k$ -isotypic summand for every even integer  $k$ . If we choose basis elements  $e_k \in \mathcal{F}_k^{(\sigma)}$  such that  $e_k(f_k) = 1$ , then

$$\begin{aligned} He_k &= ke_k \\ Ee_k &= -\sigma(B_k)e_{k+2} \\ Fe_k &= -\sigma(A_{k-2})e_{k-2}. \end{aligned}$$

The isomorphism  $\varphi$  can be described by a sequence of formulas

$$\varphi(f_k) = \varphi_k e_k,$$

where  $\varphi_k \in \mathcal{K}$ . The rational functions  $\varphi_k$  are not independent of one another, since compatibility with the action of  $E$  and  $F$  implies that

$$\begin{aligned} A_{k-2}\varphi_k &= -\sigma(B_{k-2})\varphi_{k-2} \\ B_k\varphi_k &= -\sigma(A_k)\varphi_{k+2} \end{aligned}$$

for all even integers  $k$ . These relations imply in turn that

$$\varphi_k = (-1)^{\frac{k}{2}} \frac{B_{k-2}}{\sigma(A_{k-2})} \frac{B_{k-4}}{\sigma(A_{k-4})} \cdots \frac{B_0}{\sigma(A_0)} \varphi_0 \quad \text{if } k > 0$$

and that

$$\varphi_k = (-1)^{\frac{k}{2}} \frac{\sigma(A_k)}{B_k} \frac{\sigma(A_{k+2})}{B_{k+2}} \cdots \frac{\sigma(A_{-2})}{B_{-2}} \varphi_0 \quad \text{if } k < 0.$$

If we are working, as we may, with a basis  $\{f_n\}$  for  $\mathcal{F}$  for which  $A_k = 1$  when  $k \geq 0$  and  $B_k = 1$  when  $k < 0$ , then the formulas simplify to

$$\varphi_k = (-1)^{\frac{k}{2}} B_{k-2} B_{k-4} \cdots B_0 \varphi_0 \quad \text{if } k > 0$$

and that

$$\varphi_k = (-1)^{\frac{k}{2}} A_k A_{k+2} \cdots A_{-2} \varphi_0 \quad \text{if } k < 0.$$

Once again, each of the  $A$ - or  $B$ -polynomials appearing in these formulas is explicitly determined by the value of the Casimir section.

Let us now consider the specific example where the action of the Casimir is given by the regular function

$$c_{\mathcal{F}}(z) = -\frac{1+z}{z}$$

on  $\mathbb{C} \setminus \{0\}$  (this is choice of Casimir that will lead to the contraction families from mathematical physics that were mentioned in the introduction). We want to determine all the infinitesimally unitary Jantzen quotients.

If we work with the isomorphism  $\varphi$  for which  $\varphi_0 = 1$ , then we get

$$B_k = -\frac{1}{4} (1 + x(1+k)^2)$$

when  $k \geq 0$  and hence

$$\varphi_k(x) = \frac{1 + (k-1)^2x}{4} \cdot \frac{1 + (k-3)^2x}{4} \cdots \frac{1 + 9x}{4} \cdot \frac{1 + x}{4}.$$

The same formula holds when  $k < 0$ . So unless  $x = -(2m+1)^2$ , for some  $m$ , all the scalars  $\varphi_k(x)$  are nonzero, and hence the Jantzen filtration is trivial.

On the other hand, if  $x = -(2m+1)^2$ , then we find that

$$\text{ord}_x(\langle f_k, f_k \rangle) = \text{ord}_x(\varphi_k) = \begin{cases} 1 & |k| > 2m \\ 0 & |k| \leq 2m, \end{cases}$$

and as a result

$$\mathcal{F}|_x^n = \begin{cases} \mathcal{F}|_x & n \geq 0 \\ (\cdots \oplus \mathbb{C}_{-2m-4} \oplus \mathbb{C}_{-2m-2}) \oplus (\mathbb{C}_{2m+2} \oplus \mathbb{C}_{2m+4} \oplus \cdots) & n = 1 \\ 0 & n > 1 \end{cases}$$

So if  $x = -(2m+1)^2$ , then the Jantzen filtration is nontrivial. The set of distinguished points from Section 4.3 is therefore precisely this set of values. (These are also the  $x$  for which the fibers  $\mathcal{F}|_x$  are reducible, as we mentioned in Section 4.3.)

Let us now discuss infinitesimal unitarity. The scalars  $\varphi_k(x)$  are positive if  $x \geq 0$ , and from this it follows that the nonzero Jantzen quotient is infinitesimally unitary for these values. If  $x$  is negative but not of the form  $x = -(2m+1)^2$ , then all the  $\varphi_k(x)$  are real and nonzero, but they are not all of the same sign, so the quotient is not infinitesimally unitary. The nonzero Jantzen quotients at  $x = -(2m+1)^2$  are

$$\mathcal{F}|_x^n / \mathcal{F}|_x^{n+1} \cong \begin{cases} \mathbb{C}_{-2m} \oplus \mathbb{C}_{-2m+2} \oplus \cdots \oplus \mathbb{C}_{2m} & n = 0 \\ (\cdots \oplus \mathbb{C}_{-2m-4} \oplus \mathbb{C}_{-2m-2}) \oplus (\mathbb{C}_{2m+2} \oplus \mathbb{C}_{2m+4} \oplus \cdots) & n = 1. \end{cases}$$

The  $n = 0$  quotient is infinitesimally unitary; the other one is not.

This completes our computations of the Jantzen quotients. In summary we find that:

- (a) For  $x > 0$  the unitary representation of  $\mathbf{G}|_x^\sigma \cong \mathrm{SU}(1, 1)$  on the (completion of the) unique infinitesimally unitary Jantzen quotient is the unitary spherical principal series representation with Casimir  $-(x+1)/x$ . As  $x$  varies these representations exhaust the spherical unitary principal series except for the base of the spherical principal series.<sup>3</sup>
- (b) The unitary representation of the motion group  $\mathbf{G}|_0^\sigma$  on the unique and infinitesimally unitary Jantzen quotient is the unique spherical unitary irreducible representation  $\pi_{-1}$  on which the Casimir element  $4EF$ , which generates the center of the enveloping algebra, acts as multiplication by  $-1$ .
- (c) The remaining infinitesimally unitary Jantzen quotients only occur at  $x = -(2m+1)^{-2}$ , with  $m = 0, 1, 2, \dots$ . There is a unique one at each such  $x$ , and the unitary representation of  $\mathbf{G}|_x^\sigma \cong \mathrm{SU}(2)$  on it is the irreducible representation of highest weight  $2m$ . These representations exhaust the unitary irreducible representations of  $\mathrm{SU}(2)$  that contain the 0-weight of  $\mathrm{U}(1)$ .

Comparing the above to the contraction families we find that

- (d) The unitary representations for  $x > 0$  are those that appear in the contraction of the unitary principal series  $\mathrm{SU}(1, 1)$  to  $\pi_{-1}$  [DR85, SBBM12] (also compare [VK93, sec. 9.2.4]), while the discretely occurring unitary representations in the region  $x < 0$  are those that appear in the contraction of unitary irreducible representations of  $\mathrm{SU}(2)$  to  $\pi_{-1}$  [IW53, DR83, SBBM12].

*Remark.* Although we focused above on a family that is relevant to the theory of contractions of representations, even for the  $\mathrm{SU}(1, 1)$  case we have studied here there are other families that show quite different features,

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<sup>3</sup>In a certain sense, the base of the spherical unitary principal series is located at  $x = \infty$ . This can be made precise using the “deformation to the normal cone” family of Harish-Chandra pairs over the projective line considered in [BHS16, 2.3.1].

and may be of interest for other purposes. For instance if we analyze the family with Casimir

$$c_{\mathcal{F}}(z) = \frac{1-z}{z}$$

in the same way, then we obtain infinitesimally unitary quotients that include both discrete series and complementary series representations when  $x > 0$ . See Figure 2.

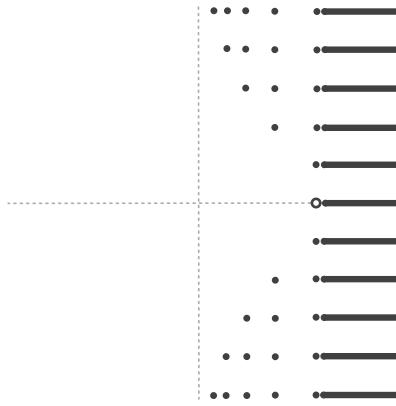


Figure 2: The infinitesimally unitary Jantzen quotients for the family with Casimir  $\frac{1-x}{x}$ . There are no infinitesimally unitary quotients in the region  $x \leq 0$ . In the region  $x > 0$  there arise all the discrete series with even weights, the trivial representation (indicated by a white dot), and all the complementary series.

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