# DELIGNE-LUSZTIG DUALITY AND WONDERFUL COMPACTIFICATION

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Abstract. We use geometry of the wonderful compactification to obtain a new proof of the relation between Deligne-Lusztig (or Alvis-Curtis) duality for $p$-adic groups to homological duality. This provides a new way to introduce an involution on the set of irreducible representations of the group, which has been earlier defined by A. Zelevinsky for $G = GL(n)$ by A.-M. Aubert in general. As a byproduct we describe the Serre functor for representations of a $p$-adic group.

*To Sasha Beilinson, with admiration and best wishes for his birthday.*

## 1. Introduction

The goal of this note is to present a new conceptual proof of the relation between homological and Deligne-Lusztig dualities and apply it to define an involutive auto-equivalence of the category of admissible representations, which induces an involution on the set of irreducible representations.

Let $G$ be a semisimple $p$-adic group. For every smooth $G$-module $M$ one can form a complex

$$0 \to M \to \bigoplus_P i_P^G r_P^G(M) \to \cdots \to i_B^G r_B^G(M) \to 0,$$

where $i_P^G$, $r_P^G$ denote, respectively, the parabolic induction and Jacquet functors, and summation in the $i$-th term runs over conjugacy classes of parabolic subgroups of corank $i$. The differential is the sum of the natural maps coming from adjunction between induction and Jacquet functors and transitivity of parabolic induction, taken with appropriate signs. We call this complex the Deligne-Lusztig complex associated to $M$, and denote it by $DL(M)$. Analogous complexes for representations of a finite Chevalley group have been considered in [11], the corresponding automorphisms of the Grothendieck group has been studied earlier in [1], [10] for finite Chevalley groups and in [15] for $p$-adic groups.

Starting from a complex of smooth representations $M^\bullet$ one gets a bicomplex $DL(M^\bullet)$, thus we get a functor $DL : D^b(Sm) \to D^b(Sm)$.

The main result of this note is as follows.

**Theorem 1.1.** *For a complex $M$ with admissible cohomology we have a canonical quasi-isomorphism*

$$DL(\check{M}) \cong RHom(M, H),$$

*where $\check{M}$ denotes the contragradient representation and $H \cong C_c^\infty(G)$ is the regular bimodule for the Hecke algebra.*

Theorem 1.1 is proved in section 2 by using an explicit resolution for the regular bimodule over $H$ coming from geometry of the wonderful compactification $\overline{G}$ of $G$. The idea of this proof is as follows.

Recall that $\overline{G}$ admits a stratification where the strata are indexed by conjugacy classes of parabolic subgroups in $G$. A sheaf on a stratified space admits a standard resolution with terms indexed by the strata. We apply this to the sheaf of smooth sections (see 2.1 for the definiton) of the sheaf $j_*(\underline{k}_G)$, where $j : G \to \overline{G}$ is the embedding and $\underline{k}_G$ is the constant sheaf on $G$. Taking global sections, we get a resolution for the regular bimodule over $H$.

Next, specialization to the normal bundle for functions on a $p$-adic manifold [7] allows us to describe the terms of this resolution via functions on the normal bundles to the strata. A normal bundle to a stratum $\Sigma$ contains an open $G \times G$ orbit $O_\Sigma$, and functions on $O_\Sigma$ form an injective $G$-module, dual to the one appearing in the study of second adjointness [7]. Taking $Hom$ from a finitely generated module $M$ into the space of functions on $O_\Sigma$ yields a term of the complex $DL(\check{M})$. On the other hand, a term of the resolution for $H$ described in the previous paragraph receives a surjective (but not injective) map from the space of functions on $O_\Sigma$. It turns out that this map induces an isomorphism on $Ext$'s from an admissible module. This yields Theorem 1.1.

A less direct geometric proof of Theorem 1.1, based on Borel-Serre compactification of the Bruhat-Tits building of $G$ and a localization type theorem relating equivariant sheaves on the building to $G$-modules has appeared in the thesis of the second author [6].

In section 3 we present a generalization and algebraic consequences of Theorem 1.1. This section (except for §3.4 which describes the Serre functor for admissible modules over a $p$-adic group) follows the strategy devised by the first author several decades ago, see announcement in [5].

The main application recorded in section 3 is Corollary 3.3 which says that for an irreducible module $M$ the complex $DL(M)$ has cohomology in only one degree $d$, where $d$ is dimension of the component of Bernstein center containing $M$. This implies that in fact $H^d(DL(M))$ is an irreducible representation, thus we get an automorphism on the set of representations. This result has already appeared in [2]. In fact, the automorphism on the set of irreducible representations can be shown to be an involution, which for $G = GL(n)$ has been defined and studied in [17], thus it is often called the Aubert, or the Zelevinsky involution. While the proof in [2] is short, Aubert-Zelevinsky involution has many interesting known or expected properties related to Langlands conjectures,[1] Koszul duality (cf. [16]) etc., which are

[1]Namely, it is expected to interchange the Arthur and the Deligne-Langlands parameters, see [14] for details.

far from being completely understood. The central idea of the present note is a link between this involution and geometry of the wonderful compactification, which is, on the other hand, related, as we plan to argue elsewhere, to the local trace formula; see also [12], [13] for other apparently related constructions. We hope this link will help clarify some of the outstanding questions about the involution.

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## 2. Proof of Theorem 1.1

2.1. **Notation.** We work over an algebraically closed characteristic zero field $k$.

Let $F$ be a local non-Archemidian field and $G$ be the group of $F$ points of a reductive algebraic group of $F$. We let $\overline{G}$ denote its wonderful, or De Concini – Procesi (partial) compactification. The definition of wonderful compactification is standard in the case of an adjoint semi-simple group only. For a general semi-simple group $\overline{G}$ is defined to be the normalization of $\overline{G}_{ad}$ in $G$; for a general reductive group we let $\overline{G} = (\overline{G}' \times C)/(G' \cap C)$, where $C$ is the center of $G$, $G'$ is the derived group and $G_{ad} = G/C$. Thus $\overline{G}$ is proper if $G$ is semi-simple.

For a totally topological space $X$ we write $C_c(X)$ for the space of compactly supported functions on $X$, $C^\infty(X)$ for the space of locally constant function and $C_c^\infty = C_c(X) \cap C^\infty(X)$; all functions are assumed to be $k$-valued.

We have the *Hecke algebra* $H$ which is the space of locally constant compactly supported measures on $G$ equipped with the convolution algebra structure; a choice of a Haar measure yields an isomorphism $H \cong C_c^\infty(G)$. For an open compact subgroup $K \subset G$ we have the unital subalgebra $H(G,K) \subset H$ of $K$-biinvariant measures.

Recall that for a module $M$ over a totally disconnected group a vector $v \in M$ is called smooth if its stabilizer is open. If $M$ is a module over an open subgroup in $G$ we let $M^{sm}$ denote the submodule of smooth vectors. If $M = M^{sm}$ then $M$ is called smooth.

We let $Sm(G)$ denote the category of smooth $G$-modules. For $M \in Sm(G)$ the contragradient module $(M^*)^{sm}$ is denoted by $\check{M}$ or $M^{\check{}}$. We let $Adm(G) \subset Sm(G)$ be the subcategory of admissible representations.

A $G$-equivariant sheaf $\mathcal{F}$ on a $G$-space $X$ will be called smooth if for every point $x \in X$ and every local section $s$ of $\mathcal{F}$ defined on a neighborhood of $x$ there exists an open subgroup $K$ in $G$ and a $K$-invariant neighborhood $U$ of $x$, such that $s|_U$ is (well defined and) $K$-invariant. For a $G$-equivariant sheaf $\mathcal{F}$ we let $\mathcal{F}^{sm}$ denote the maximal smooth subsheaf. In other words, $s \in \Gamma(U, \mathcal{F})$ belongs $\mathcal{F}^{sm}$ if $U$ admits an open covering $U_i$ such that $U_i$ is invariant under an open subgroup $K_i$ and $s|_{U_i}$ is $K_i$-invariant.

It is easy to see that $\Gamma(X, \mathcal{F})$ is a smooth $G$-module provided that $\mathcal{F}$ is a smooth $G$-equivariant sheaf on a compact $G$-space $X$.

The constant sheaf with stalk $M$ over a topological space $X$ will be denoted by $\underline{M}_X$.

2.2. **The specialization complex.** We start by introducing an auxiliary complex of $G \times G$ modules.

For a subset $S$ in the set $I$ of simple roots let $X_S = (G/U_S \times G/U_S^-)/L_S$, where $P_S = L_S U_S$ is a parabolic subgroup in the conjugacy class determined by $S$ and $P_S^- = L_S U_S^-$ is an opposite parabolic. In [7] we have defined[2] the cospecialization map $c_S : C_c^\infty(X_S) \to C_c^\infty(G)$ for all $S$. Similar considerations yield a map $c_{S,S'} : C_c^\infty(X_S) \to C_c^\infty(X_{S'})$ for $S \subset S'$, such that $c_{S',S''} c_{S,S'} = c_{S,S''}$ and $c_{S,I} = c_S$ which we also call the cospecialization map.

Taking the sum of these maps with appropriate signs we get a complex of $G \times G$ modules:

$$0 \to C_c^\infty(X_I) \to \cdots \to \bigoplus_{|S|=1} C_c^\infty(X_S) \to C_c^\infty(G) \to 0,$$

which we call the cospecialization complex. Passing to the contragradient $G \times G$-modules we get a complex which we call the specialization complex $\mathfrak{C}$.

**Lemma 2.1.** *a) The terms of the specialization complex $\mathfrak{C}$ are injective as left $G$-modules.*

*b) For a finitely generated $G$-module $M$ the complex $Hom(M, \mathfrak{C})$ is canonically isomorphic to the complex $DL(\check{M})$.*

**Proof.** It is clear that

$$C_c^\infty(X) = i_{P \times P^-}(C_c^\infty(L)) \cong i_P^G(C_c^\infty(G/U^-)).$$

Since passing to the contragradient module commutes with (unitary) parabolic induction, we see that the terms of $\mathfrak{C}$ are of the form

$$i_{P \times P^-}^{G \times G}((C_c^\infty(L)^*)^{sm}) = i_P^G(C(G/U^-)^{sm}),$$

where in the last expression the superscript denotes taking smooth vectors with respect to the $G \times L$ action, and compactness of $G/P^-$ is used.

[2] In [7] we work under the running assumption that the group $G$ is split. This assumption is not used in proving any of the facts referenced in the present paper.

The space $C(G/U^-)^{sm}$ considered as an $L$-module via the right $L$ action is clearly a union of cofree modules over the Hecke algebras $H(L,K)$, where $K$ runs over open compact subgroups in $L$ (here by a cofree module we mean a module isomorphic to a sum of copies of the module $H(L,K)^* \cong C(L)^K$), hence $C(G/U^-)^{sm}$ is an injective object in $Sm(L)$. The standard Frobenius adjointness shows that parabolic induction sends injective modules to injective ones, this proves (a).

For $M \in Sm(G)$ we have:

$$Hom_G(M, i_P^G(C(G/U^-)^{sm})) = Hom_L(r_P^G(M), C(G/U^-)^{sm}).$$

If $M$ is finitely generated then $r_P^G(M)$ is also finitely generated, and for a finitely generated smooth $L$-module $N$ we have

$$Hom_L(N, C(G/U^-)^{sm}) \cong i_{P^-}^G(Hom(N, C(L)^{sm})) = i_{P^-}^G(\check{N}).$$

Recalling the isomorphism $r_P^G(M)\check{} \cong r_{P^-}^G(\check{M})$ which follows directly from the second adjointness, we get statement (b). □

We do not describe cohomology of $\mathfrak{C}$; instead, in the next subsection we define a certain quotient of $\mathfrak{C}$ which will be shown to be a resolution for $C_c^\infty(G)$.

2.3. **The resolution.** For a subset $S$ in the set $I$ of simple roots let $\overline{G}_S$ be the corresponding closed stratum in $\overline{G}$ and let $i_S : \overline{G}_S \to \overline{G}$, $j : G \to \overline{G}$ be the embeddings.

We let $\mathcal{F}_S = i_S^*(j_*(\underline{k}_G)^{sm})$.

We set $M_S := \Gamma(\mathcal{F}_S)$. In particular, $M_\emptyset$ is the space of sections of the constant sheaf on $G$, i.e. the space $C^\infty(G)$ of locally constant functions on $G$.

We now proceed to construct a resolution of the regular bimodule for $H$. This will be derived from a standard resolution for a sheaf on a stratified space.

Let $X$ be a topological space and $Z_1, \ldots, Z_n$ be closed subspaces. For a subset $S \subset [1,n]$ let $X_S$ denote the intersection of $Z_i$, $i \in S$ (in particular, $X_\emptyset = X$) and $i_S : X_S \to X$ be the embedding; let $j : U \to X$ be the embedding of the complement to $\cup_i Z_i$. Then for a sheaf $\mathcal{F}$ on $X$ we can form a complex of sheaves on $X$:

$$0 \to (i_\emptyset)_* i_\emptyset^*(\mathcal{F}) \to \cdots \to \bigoplus_{|S|=i} (i_S)_* i_S^*(\mathcal{F}) \to \cdots \to (i_{[1,n]})_* i_{[1,n]}^* \mathcal{F} \to 0, \tag{2.1}$$

where the differential is the sum of restriction maps with appropriate signs.

**Lemma 2.2.** *The complex* (2.1) *is exact in positive degree and its cohomology in degree zero is* $j_! j^*(\mathcal{F})$.

**Proof.** A complex of sheaves is exact if an only if the complex of stalks at every point is exact. Applying this criterion to the complex

$$0 \to j_! j^*(\mathcal{F}) \to (i_\emptyset)_* i_\emptyset^*(\mathcal{F}) \to \cdots \to \bigoplus_{|S|=i} (i_S)_* i_S^*(\mathcal{F}) \to \cdots \to (i_{[1,n]})_* i_{[1,n]}^* \mathcal{F} \to 0$$

we get the claim. □

**Corollary 2.3.** *We have a complex of $G \times G$ modules*

$$0 \to M_\emptyset \to \bigoplus_{|S|=1} M_S \to \cdots \to M_I \to 0, \tag{2.2}$$

*whose cohomology in concentrated in degree zero.*

*If $G$ is semisimple, then the 0-th cohomology of this complex is isomorphic to $C_c^\infty(G)$.*

**Proof.** We apply Lemma 2.2 to the sheaf $j_*(\underline{k})^{sm}$ on $\overline{G}$. Since $\overline{G}$ is totally disconnected, the sheaves have no higher cohomology, so the resolution for a sheaf yields a resolution for its global sections. It is clear that if $\overline{G}$ is compact, then $C_c^\infty(G) = \Gamma_c(\underline{k}_G) = \Gamma(j_!(\underline{k}_G))$. □

Our next goal is a more explicit description of the complex (2.2).

If $G$ is adjoint, then $\overline{G}$ is (the space of $F$-points of) a nonsingular projective algebraic variety. Let $\bar{X}_S$ denote the normal bundle to $\overline{G}_S$ in $\overline{G}$. For a general $G$ we let $\bar{X}_S$ be the quasi-normal cone in the sense of [7, §11.1]. Let $\mathfrak{i}_S$ be the zero section embedding of $\overline{G}_S$ into $\bar{X}_S$.

Recall that $\bar{X}_S$ contains an open dense $G \times G$-orbit $X_S$ isomorphic to $(G/U_S \times G/U_S^-)/L$. The projection $X_S \to G/P \times G/P^-$ extends to a map $\bar{X}_S \to G/P \times G/P^-$; let $L^+$ denote a fiber of that map. Thus $L^+$ is partial compactification of $L$.

For future reference we recall the following description of $L_+$. Let $C$ be the center of $L$ and $C_+$ be its partial compactification defined as $Spec(k[\Lambda_C^+])$ where $\Lambda_C$ is the character lattice of $C$ and $\Lambda_C^+ \subset \Lambda_C$ is the semigroup of dominant weights. Then we have

$$L_+ = (\overline{L} \times C_+)/C, \tag{2.3}$$

where $\overline{L}$ is the (partial) wonderful compactification of $L$.

Let $\mathfrak{j}_S : X_S \to \bar{X}_S$ denote the embedding.

**Proposition 2.4.** *a) We have canonical $G \times G$-equivariant isomorphisms:*

$$\mathcal{F}_S \cong \mathfrak{i}_S^*(\mathfrak{j}_{S*}(\underline{k}_{X_S})^{sm}); \tag{2.4}$$

$$M_S = i_{P_S \times P_S^-}^{G \times G}((C(L)^{sm}/C_0(L)^{sm}), \tag{2.5}$$

*where $C_0(L)$ is the space of functions on $L$ whose support is closed in $L^+$.*

*b) The surjections $i_{P_S \times P_S^-}^{G \times G}(C(L)^{sm}) = C(X_S)^{sm} \to M_S$ from (2.5) provide a morphism of complexes of $G \times G$-modules from the specialization complex $\mathfrak{C}$ to the complex (2.2).*

**Proof.** We first show that isomorphism (2.5) follows from (2.4). We have an exact sequence of sheaves on $X_S$:

$$0 \to \mathfrak{j}_{S!}(\underline{k}_{X_S})^{sm} \to \mathfrak{j}_{S*}(\underline{k}_{X_S})^{sm} \to \mathfrak{i}_S^*(\mathfrak{j}_{S*}(\underline{k}_{X_S})^{sm}) \to 0.$$

It is clear that

$$\Gamma((\mathfrak{j}_{S*}\underline{k}_{X_S})^{sm}) \supset \Gamma(\mathfrak{j}_{S*}\underline{k}_{X_S})^{sm} = C(X_S)^{sm} = i_{P_S\times P_S^-}^{G\times G}(C(L)^{sm}),$$

$$\Gamma(\mathfrak{j}_{S!}(\underline{k}_{X_S}))^{sm} = i_{P_S\times P_S^-}^{G\times G}(C_0(L)^{sm}).$$

It is also easy to see that the functor $\mathcal{F} \mapsto \Gamma(\mathcal{F})^{sm}$ is exact on the category of smooth $G$-equivariant sheaves on a totally disconnected $G$-space, this yields (2.5).

We proceed to check (2.4). We deduce it from the next general Lemma, to state it we need some notation.

Let $X$ be a smooth algebraic variety over $F$ and $D \subset X$ be a divisor with normal crossings. Let $D_1, \dots, D_n$ be the components of $D$; for $I \subset [1,n]$ let $D_I = \bigcap_{s\in I} D_s$, let $i_I : D_I \hookrightarrow X$ be the embedding and let $D_I^o \subset D_I$ be the complement to the union of $D_J$, $J \supsetneq I$. Set $U = D_\emptyset^o$ and let $j : U \to X$ be the embedding.

Let $N_I = N_X(D_I)$ be the normal bundle to $D_I$ in $X$ and $\mathfrak{i}_I : D_I \to N_I$ be the zero section embedding and let $\mathfrak{j}_I : U_I \hookrightarrow N_I$ be the embedding, where $U_I$ is the complement to the union of $N_{D_i}(D_I)$, $i \in [1,n]$.

**Lemma 2.5.** *Suppose that an algebraic group $G$ acts on $X$ preserving $D_i$, so that the action on $D_I^o$ is transitive for each $I$. Assume*[3] *also that for every $I$ and for any (equivalently some) point $x \in D_I^o$ the action of the stabilizer $Stab_G(x)$ on the fiber at $x$ of the normal bundle to $D_I$ in $X$ decomposes as a sum of linearly independent characters.*[4]

*Then for each $I$ we have a canonical $G$-equivariant isomorphism of sheaves on $D_I$:*

$$i_I^* j_*(\underline{k}_U)^{sm} \cong \mathfrak{i}^* \mathfrak{j}_{I*}(\underline{k}_{U_I})^{sm}.$$

**Proof.** Choose a neighborhood $V$ of $D_I$ in $X$ and a map $\phi$ from $V$ to $N_I$ which is admissible in the sense of [7, Definition 3.2]. It is clear that $\phi$ induces an isomorphism $\iota_\phi : i_I^* j_*(\underline{k}_U) \to \mathfrak{i}_I^* \mathfrak{j}_*(\underline{k}_{U_I})$. It is also easy to see that $\iota_\phi$ restricts to an isomorphism $\iota_\phi^{sm} : i_S^* j_*(\underline{k})^{sm} \to \mathfrak{i}_S^* \mathfrak{j}_{I*}(\underline{k})^{sm}$.

We claim that $\iota_\phi^{sm}$ (though possibly not $\iota_\phi$) is independent of the choice of $\phi$. We deduce this from [7, Lemma 3.4].

Let $\phi'$ be another such map. Notice that the bundle $N_I$ splits as a sum of line bundles $N_{D_J}(D_I)$, where $J \subset I$ and $|J| = |I| - 1$. Thus the torus $(F^*)^d$, $d = |I|$ acts on $N_I$. Fixing a uniformizer $\varpi \in F$ we get an embedding

[3]This assumption may in fact be redundant.

[4]It is easy to see that the running assumptions imply it decomposes as a sum of one-dimensional representations.

$\mathbb{Z}^d \hookrightarrow (F^*)^d$, $\lambda \mapsto \varpi^\lambda$. According to *loc. cit.*, for any locally constant function $f$ with compact support on $U_I$ there exists $\lambda_0 \in \mathbb{Z}^d$, such that

$$\phi_*(\varpi_*^\lambda(f)) = \phi'_*(\varpi_*^\lambda(f))$$

for $\lambda \geq \lambda_0$, where "$\geq$" refers to the coordinatewise partial order on $\mathbb{Z}^d$, i.e. $\lambda = (\lambda_1, \ldots, \lambda_d) \geq \mu = (\mu_1, \ldots, \mu_d)$ if $\lambda_i \geq \mu_i$ for all $i = 1, .., d$.

It is clear that for $f$ as above and a Taylor series in $d$ variables $\alpha = \sum a_\lambda t^\lambda$ the function $\alpha(f) := \sum a_\lambda \varpi_*^\lambda(f)$ is well defined, the above implies that

$$\phi_*(\alpha(f)) = \phi'_*(\alpha(f))$$

for $\alpha \in t^{\lambda_0} k[[t]]$.

Let now $K$ be an open compact subgroup in $G$ and $C$ a compact open $K$-invariant subset in $D_I^o$. The second assumption of Lemma implies that $G$ acts transitively on $U_I$. It is easy to deduce that there exist a finite collection of functions $f_i$ with compact support on the preimage of $C$ in $U_I$, such that the sections of the form $\mathfrak{i}_I^*(\alpha(f_i))$, $\alpha \in k[[t]]$ span the space of $K$-invariant sections of $\mathfrak{i}_I^*(\mathfrak{j}_I)_*(\underline{k}_{U_I})|_C$. Furthermore, this remains true if we only allow $\alpha \in t^{\lambda_0} k[[t]]$ for a fixed $\lambda_0 \in \mathbb{Z}^d$. Now we see that $\phi_*(\sigma) = \phi'_*(\sigma)$ for any $K$-invariant section of $\mathfrak{i}_I^*(\mathfrak{j}_I)_*(\underline{k}_{U_I})|_C$, thus we have checked independence of $\iota_\phi^{sm}|_{D_I^o}$ on $\phi$.

We finish the proof by checking that the isomorphism on stalks at every point in $x \in D_I$ induced by $\iota_\phi^{sm}$ is independent of $\phi$. Let $x$ be such a point. The statement is equivalent to saying that an automorphism of a neighborhood $V$ of $D_I$ in $X$ which is identity on $D_I$, preserves $D_J \cap V$ for all $J$ and has normal component of differential at points of $D_I$ equal identity acts by identity on the stalk of $i_I^* j_*(\underline{k}_U)$ at $x$. In view of the previous paragraph we already know this is true for points $x \in D_I^o$. Let now $x \in D_I$ be arbitrary, we have $x \in D_J^o$ for some $J$. Applying the argument of the previous paragraph to $D_J$ and observing that an automorphism of a neighborhood of $D_I$ as above restricts to an automorphism of a neighborhood of $D_J$ $(J \supset I)$ satisfying similar conditions, we get independence of $\iota_\phi^{sm}$ on $\phi$.

Independence of $\phi$ also shows that the isomorphism is compatible with the natural $G$-equivariant structures on the two sheaves. This proves the Lemma. $\square$

We return to the proof of Proposition 2.4. Isomorphism (2.4) in the adjoint case follows directly from Lemma 2.5. In the general case it is constructed in a similar way, relying on a generalization of Lemma 2.5 to the setting of [7, Appendix], the proof of this generalization is parallel to the proof of Lemma 2.5. This proves part (a) of the Proposition.

Compatibility with the differential claimed in part (b) follows by comparing definition of $c_{S,S'}$ from [7] with the proof of the Lemma. $\square$

**Proposition 2.6.** *For an admissible $L$-module $M$ we have*

$$Ext_L^i(M, C_0^\infty(L)) = 0$$

*for all $i$.*

**Proof.** Let $g \in C$ be an element which has a positive pairing with elements of $C_+$. We claim that for a nonzero constant $c$ the element $(c - g)$ acts on $C_0^\infty(L)$ by an invertible operator, this yields the Proposition since for an admissible module $M$ Schur Lemma shows that $\prod(c_i - g)$ kills $M$ for some nonzero constants $c_1, \ldots, c_n$.

The operator inverse to $c-g$ is obtained by expanding $(1-c\cdot g^{-1})^{-1}$ in a series $1 + cg^{-1} + c^2g^{-2} + \ldots$. In order to check that this expression for the inverse operator is well defined it suffices to check that for a subset $X$ in $L$ which is closed in $L_+$ and for a point $x \in L$ there exist only finitely many natural numbers $n$ for which $g^{-n}(X) \ni x$. This is clear from (2.3) which implies that the sequence $g^n(x)$ has a limit in $\partial L = L_+ \setminus L$. □

2.4. **The main result.** We are now ready to prove:

**Theorem 2.7.** *Assume that $G$ is semisimple.*

*For a complex $M$ with admissible cohomology we have a canonical isomorphism in the derived category:*

$$RHom(M, C_c^\infty(G)) \cong DL(\check{M}).$$

**Proof.** Compare Corollary 2.3 and Proposition 2.4(b) with Proposition 2.6. □

## 3. A generalization and algebraic consequences

3.1. **Cohen-Macaulay property.** Let $\mathsf{A}$ be an indecomposable summand in the category $Sm(G)$. According to [3], the center $Z$ of $\mathsf{A}$ (i.e. the endomorphism ring of the identity functor $Id_{\mathsf{A}}$) is the ring of functions on an algebraic torus invariant under an action of a finite group; in particular, $Z$ is a *Cohen-Macaulay* commutative algebra.

By [3] the category $\mathsf{A}$ is equivalent to the category of $H_{\mathsf{A}}$-modules for a certain $Z$-algebra $H_{\mathsf{A}}$, which is finitely generated as a $Z$-module. The algebra $H_{\mathsf{A}}$ is not uniquely defined, but it is defined uniquely up to a (canonical) Morita equivalence. Notice that the homological duality functor $M \mapsto RHom_R(M, R)$ defined on the derived category of $R$-modules commutes with Morita equivalences; it is clear that for $R = H_{\mathsf{A}}$ we recover the above homological duality.

**Proposition 3.1.** *$H_{\mathsf{A}}$ is a Cohen-Macaulay $Z$-module.*

**Proof.** It suffices to show that a projective generator for $\mathsf{A}$ is a Cohen-Macaulay $Z$-module. In view of second adjointness [5], [7], a parabolic induction functor sends projective objects to projective ones. Recall that to the indecomposable summand $\mathsf{A}$ there corresponds a Levi subgroup $L$ in $G$ and a cuspidal representation $\rho$ of $L$, which is defined uniquely up to twisting with an unramified character of $L$. Let $L^0 \subset L$ be the kernel of undramified characters, thus $L^0$ is the subgroup generated by all compact subgroups. Let $P = L \cdot U$ be a parabolic. Then the module $\Pi_\rho = i_P^G(ind_{L^0}^L(\rho|_{L^0}))$ is a projective generator for $\mathsf{A}$. The action of $Z$ on $\Pi_\rho$ extends to an action of the ring $\tilde{Z} = k[L/L^0]$, the ring of regular functions on the algebraic torus. Moreover, $\Pi_\rho$ is easily seen to be a free module over $\tilde{Z}$, hence it is a Cohen-Macaulay module over $Z$. □

Following [4], by a (possibly noncommutative) *Cohen-Macaulay* ring of dimension $d$ we will understand a ring $A$ for which there exists a homomorphism from a commutative ring $C$ to the center of $A$, such that $C$ is a Cohen-Macaulay ring of dimension $d$ and $A$ is a finitely generated Cohen-Macaulay module over $C$ of full dimension. Standard properties of Cohen-Macauliness show that when $A$ is commutative this agrees with the standard definition.

The following statement is standard for commutative rings, the noncommutative generalization is proved in a similar way.

**Lemma 3.2.** *Let $A$ be a Cohen-Macaulay (possibly non-commutative) ring of dimension $d$ and $M$ a finite dimensional module. Then $Ext^i_A(M, H) = 0$ for $i < d$.*

**Proof.** We use induction in $d$. Let $C \to A$ be as above. We can find a non-zero divisor $f \in C$ which annihilates $M$. Then $\bar{A} = A/fA$ is a Cohen-Macaulay ring of dimension $d - 1$ and

$$Ext^i_A(M, A) = Ext^i_{\bar{A}}(M, RHom_A(\bar{A}, A)) = Ext^{i-1}_{\bar{A}}(M, \bar{A}),$$

where we used that $Ext^i_A(\bar{A}, A) = \bar{A}$ for $i = 1$ and this Ext vanishes for $i \neq 1$. □

## 3.2. **Aubert-Zelevinsky involution.**

**Theorem 3.3.** *Assume that the group $G$ is semisimple.*

*a) Let $M$ be an admissible module belonging to a component of Bernstein center of dimension $d$. Then $DL(M)$ has cohomology in degree $d$ only.*

*b) The functor $M \mapsto H^d(DL(M))$ is an autoequivalence of the summand in $Adm(G)$ corresponding to a component in the Bernstein center of dimension $d$.*

**Proof.** a) The complex $DL(M)$ is concentrated in degrees from 0 to $d$. On the other hand, Proposition 3.1 and Lemma 3.2 show that $Ext^i_H(\check{M}^K, H_{\mathsf{A}}) = 0$ for $i \leq d$.

b) The functor $DL$ is an autoequivalence of the derived category $D^b(Sm(G))_{adm}$ (complexes with admissible cohomology). Part (a) shows that on a summand of dimension $d$ this autoequivalence shifted by $d$ preserves the abelian heart.

**Corollary 3.4.** *Given an irreducible module $M$ belonging to a component of dimension $d$ there exists an irreducible module $M'$ such that: $[DL(M)] = (-1)^d[M']$.*

3.3. **Homological duality and Deligne-Lusztig duality for non-admissible modules.** Notice that both the homological duality and $DL$ are defined on the derived category of finitely generated smooth modules, so it is natural to ask if they can be related as functors on that larger category.

To state the answer we introduce the *Grothendieck-Serre* duality $\mathcal{S}$ on $Sm$.

If $A$ is noncommutative ring over a commutative ring $R$, then for an $A$-module $M$ and a $R$-module $N$ the space $Hom_R(M,N)$ carries a right $A$-module structure. Passing to derived functors we get a functor $RHom$ : $D^b(A-mod)^{op} \times D^+(R-mod) \to D^+(A^{op}-mod)$.

We now consider the category $Sm_R$ of smooth $G$-modules over a base commutative ring $R$ which is finitely generated over $k$.

Since we have a standard isomorphsim $H^{op} \cong H$, we get a functor $RHom$ : $D^b(Sm_R)^{op} \times D^+(R-mod) \to D^+(Sm_R)$.

**Proposition 3.5.** *Let $D^b(Sm_R)_{adm}$ be the full subcategory in $D^b(Sm_R)$ consisting of complexes whose cohomology is $R$-admissible. Then for $M \in D^b(Sm_R)_{adm}$ and a $N \in D^b(R-mod)$ we have a natural isomorphism*

$$RHom_{D^b(Sm_R)}(M, H \otimes N) \cong DL(RHom_R(M,N)).$$

**Proof.** The proof is similar to the proof of Theorem 1.1. Let $\mathfrak{C}_N$ denote the specialization complex tensored by $N$ over $k$, this is a resolution for $H \otimes N$. It is clear that

$$RHom_{Sm_R}(M, \mathfrak{C}_N) \cong DL(RHom_R(M,N)).$$

Thus it remains to generalize Proposition 2.6 by checking that for a parabolic $P = LU$ we have: $Ext^i_{Sm_R(L)}(r^G_P(M), C_0^\infty(L) \otimes N) = 0$ for all $i$. Let $g \in C$ be as in the proof of Proposition 2.6. Since $r^G_P(M)$ is an $R$-admissible finitely generated $L$-module, we can find a monic polynomial $P \in R[t]$, such that $P(g)$ annihilates $r^G_P(M)$. It remains to see that $P$ acts invertibly on $C_0^\infty(L) \otimes N$. If $P = t^n + r_{n-1}t^{n-1} + \cdots + r_0$, then the inverse operator is given by the Taylor expansion of $(1+x)^{-1}$, $x = r_{n-1}g^{-1} + \cdots + r_0 g^{-n}$. $\square$

We now define the *Grothendieck-Serre duality* functor $\mathcal{S} : D^b(Sm)^{op} \to D^b(Sm)$, $\mathcal{S} : M \mapsto RHom_Z(M, D)$, where $D$ stands for the Grothendieck-Serre dualizing complex.

Let $Sm_{fg}(G) \subset Sm(G)$ be the full subcategory of finitely generated modules.

**Theorem 3.6.** *Assume that the group $G$ is semisimple.*

*For $M \in D^b(Sm_{fg})$ we have a canonical isomorphism*

$$DL \circ \mathcal{S}(M) \cong RHom(M, C_c^\infty(G)).$$

**Proof.** For every $M \in Sm_{fg}(G)$ and an open compact subgroup $K \subset G$ the space $M^K$ is a finitely generated module over the center. Thus we have a functor $i : Sm_{fg}(G) \to Sm_Z(G)$ which sends an object $M$ in $Sm$ to the same

$G$-module $M$ equipped with the natural $Z$-action; it lands in $Z$-admissible modules. We use the same notation for its extension to derived categories, $i : D^b(Sm_{fg}) \to D^b(Sm_Z(G))_{adm}$.

The Theorem now follows from Proposition 3.5 and the isomorphism

$$i^!(H \otimes D) \cong H, \tag{3.1}$$

where $i^!$ is the right adjoint to $i$. To establish (3.1) we use the standard isomorphism $\delta^!(\mathcal{F} \boxtimes D) \cong \mathcal{F}$ where $\delta$ is the diagonal embedding for $Spec(Z)$ and $\mathcal{F} \in D^b(Coh(Spec(Z))$. The isomorphism (3.1) follows from this in view of the isomorphism

$$Forg \circ i^! i \cong \delta^! \delta_* \circ Forg,$$

where $Forg : Sm \cong H - mod \to Z - mod$ is the forgetful functor, and the fact that the latter isomorphism is compatible with the action of $H$ on the functor $Forg$.

3.4. **Serre functor.** Recall the notion of a *Serre functor* on a $k$-linear triangulated category with finite dimensional (graded) Hom spaces [9, Definition 3.1].

Recall that for an algebra $A$ over a base field $k$ the functor $M \mapsto RHom_{A^{op}}(M^*, A^{op})$ is an endo-functor of the derived category of $A$-modules known as the *Nakayama functor*.

Suppose that $A$ has finite homological dimension, and also is finite as a module over its center which is of finite type over $k$. Let $D^b_{fd}(A - mod)$ be the full subcategory in the bounded derived category consisting of modules with finite dimensional cohomology. It is well known[5] that in this case the Nakayama functor restricted to $D^b_{fd}(A - mod)$ is in fact a Serre functor for that category. Thus we arrive at the following

**Corollary 3.7.** *Assume that the group $G$ is semi-simple. The functor $DL$ is a Serre functor on the category $D^b(Sm(G))_{adm}$.*

*Remark* 3.8. A similar argument shows that the functor $DL$ on $D^b(Sm(G))$ is a relative Serre functor over the Bernstein center, see e.g. [8, Definition 2.5].

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