PERIODS AND GLOBAL INVARIANTS OF AUTOMORPHIC REPRESENTATIONS

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Abstract. We consider periods of automorphic representations of adele groups defined by integrals along Gelfand subgroups. We define natural maps between local components of such periods and construct corresponding global maps using automorphic L-functions. This leads to an introduction of a global invariant of an automorphic representation arising from two such periods. We compute this invariant in some cases.

1. Introduction

1.1. **Periods and special values of L-functions.** Periods play a central role in the modern theory of automorphic functions. In particular, there are instances when periods of automorphic functions are related to L-functions. Such a relation goes back to the foundational work of E. Hecke [He], where he constructed the Hecke L-function on GL(2) as the period integral along the split torus in GL(2). This is the most basic of "period to L-function" relations. Another striking example was discovered by J.-L. Waldspurger [Wa] and connects the period along a non-split torus in GL(2) to the special value of an L-function of the appropriate base change lift. We also mention the vast generalization of the Waldspurger's result formulated as a conjecture by B. Gross and D. Prasad [GP]. Consequently, the exact form of the Gross-Prasad period relation was conjectured by A. Ichino and T. Ikeda [II]. This led to other formulas relating normalized periods and L-functions (e.g., an analog for the Whittaker functional was considered in [LM]). A general framework for period formulas in the context of Plancherel measures was recently proposed by Y. Sakellaridis and A. Venkatesh [SV].

Our main aim in this paper is to try to reformulate (at least part of) the Ichino-Ikeda approach in terms of representation theory without appealing to L-functions directly (in fact the original paper [Wa] already contains the idea we are trying to expand). By doing so we are able to treat periods which,

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as it seems to us, do not fall in the familiar framework and for which a relation to L-functions is more puzzling (e.g., see Appendix A). We first consider relations between the Whittaker (i.e., unipotent) period and the Hecke (i.e., torus) period for GL(2). One of the relations is classical and is a reformulation of the treatment given by H. Jacquet and R. Langlands [JL] to the Hecke method. However, we discover a converse relation which seems to be new (although similar local considerations appeared recently in [SV]). We then consider a non-classical example of two Whittaker periods on different unipotent subgroups. In this case our construction leads to an Euler product with a non-standard local factor which nevertheless could be regularized with the help of an appropriate L-function. This leads to an introduction of a non-trivial *global invariant* of an automorphic representation.

We note that one of the most important attributes of period to L-function formulas is the presence of the multiplicity one phenomenon (i.e., the Gelfand property of one-dimensionality of certain invariant functionals; see [Gr]). This point of view was pioneered by I. Piatetski-Shapiro [PS], and also will be essential throughout this paper.

1.2. **Action on periods.** We are interested in the following setup. Let G be an algebraic (reductive) group over a global field k (in practice a reader can assume $k = \mathbb{Q}$ for simplicity), and let $H_1, H_2 \subset G$ be two algebraic subgroups of G also defined over k (e.g., a split over k torus and an associated unipotent subgroup in $G = \mathrm{GL}(2)$). Let $G(\mathbb{A}), H_1(\mathbb{A}), H_2(\mathbb{A})$ be the corresponding adele groups, and we denote by $X_G = G(k) \setminus G(\mathbb{A}), X_{H_1} = H_1(k) \setminus H_1(\mathbb{A}), X_{H_2} = H_2(k) \setminus H_2(\mathbb{A})$ the corresponding automorphic quotient spaces. Let π be an automorphic representation of G (we will be vague at this point of what is required of π). We are interested in the period functional given by the integral $p_{H_1}(\phi) = \int_{X_{H_1}} \phi(h)dh$ over the $H_1(\mathbb{A})$-orbit $X_{H_1} \subset X_G$ of an automorphic function ϕ belonging to the space of the representation π (and similarly for the period p_{H_2} for X_{H_2}). More generally, we consider periods twisted by characters $\chi_i : H_i(k) \setminus H_i(\mathbb{A}) \to \mathbb{C}$ which are given by integrals $p_{H_i,\chi_i}(\phi) = \int_{X_{H_i}} \chi_i^{-1}(h_i)\phi(h_i)dh_i$. To define such periods one have to choose (invariant) measures on subgroups and impose certain restrictions on representation π and on spaces X_{H_i}. Assuming that all these periods are well-defined, it is natural to ask if there is a relation between functionals p_{H_1} and p_{H_2} which are defined on the same automorphic representation π. Periods p_{H_1} and p_{H_2} define functionals on π, and one possibility would be to compute their correlation (i.e., the scalar product, if it is defined of course). In fact it is possible in many cases (see [Gr]), but we found it a little bit easier to make another comparison in terms of the action of adelic groups. Namely, we can try to integrate the functional p_{H_1} with respect to the action of the adelic group $H_2(\mathbb{A})$. Assuming that such an operation is well-defined, we would obtain an $H_2(\mathbb{A})$-invariant functional $\tilde{p}_{H_2} = \int_{h\in H_2(\mathbb{A})} \pi^*(h)p_{H_1}\, dh$ on π (i.e., $\tilde{p}_{H_2}(v) = \int_{h\in H_2(\mathbb{A})}\int_{x\in X_{H_1}} v(xh)dxdh$ for any smooth vector v in the representation π). This does not identify such a functional in general, but in the case when $H_2(\mathbb{A})$ is a Gelfand subgroup of $G(\mathbb{A})$ (i.e., the space of $H_2(\mathbb{A})$-invariant functionals on π is at most one-dimensional), we should get a functional which is proportional to the period functional p_F. What we found is that the above mentioned "classical" period to L-function formulas allows one to compute the coefficient of proportionality between $\tilde{p}_{H_2}$ and p_{H_2} in some cases. Moreover, we find the "L-functions free" formulation of this relation between periods even more interesting. Such a reformulation allows us to consider cases where the relation to L-functions is somewhat more mysterious.

1.3. **The construction.** We will work only with periods satisfying the local uniqueness property (and hence also satisfying global uniqueness). Let $H \subset G$ be a subgroup of a group both defined over a global field k. For a place $\mathfrak{p}$ of k, we consider local groups $H_\mathfrak{p} \subset G_\mathfrak{p}$, (i.e., groups of points over the local field $k_\mathfrak{p}$). Let $\pi = \hat{\otimes}\pi_\mathfrak{p}$ be an irreducible representations of $G(\mathbb{A})$ and $\chi = \hat{\otimes}\chi_\mathfrak{p}$ be a character of $H(\mathbb{A})$ (more generally, one can consider an irreducible representation of $H(\mathbb{A})$ as well). We consider the complex vector space of equivariant maps, *the periods space*, $\mathcal{P}(\pi,\chi) = \mathrm{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C}_\chi)$ and its local counterparts, *the local period space*, $\mathcal{P}_\mathfrak{p}(\pi_\mathfrak{p},\chi_\mathfrak{p}) = \mathrm{Hom}_{H_\mathfrak{p}}(\pi_\mathfrak{p}, \mathbb{C}_{\chi_\mathfrak{p}})$. We call a tuple $(G_\mathfrak{p}, \pi_\mathfrak{p}, H_\mathfrak{p}, \chi_\mathfrak{p})$ local Gelfand data (or a multiplicity one tuple) if $\dim \mathcal{P}_\mathfrak{p}(\pi_\mathfrak{p},\chi_\mathfrak{p}) \le 1$. In such a case we have $\mathcal{P}(\pi,\chi) = \hat{\otimes}\mathcal{P}_\mathfrak{p}(\pi_\mathfrak{p},\chi_\mathfrak{p})$, and the global period space is also at most one-dimensional. We call the tuple (G,π,H,χ) globally Gelfand if it is locally Gelfand at every place. In fact we consider a slightly different space of maps with values in co-invariants of H (see Section 2.3). We find the language of co-invariants more appropriate when dealing with periods, and leading to more canonical constructions.

Let (G,π,H_1,χ_1) and (G,π,H_2,χ_2) be two globally Gelfand tuples and $P(\pi,\chi_i)$ corresponding *one-dimensional* complex vector spaces. Our goal is to construct a canonical map

$$I : \mathcal{P}(\pi,\chi_1) \to \mathcal{P}(\pi,\chi_2)\ . \tag{1.1}$$

between these one-dimensional vector spaces in the presence of the corresponding *automorphic* periods. We do this in two steps.

First step is purely local. It is relatively easy to construct *local* maps $I_{\mathfrak{p}} : \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \chi_{1,\mathfrak{p}}) \to \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \chi_{2,\mathfrak{p}})$ between local spaces of periods using the integration along the subgroup $H_{2,\mathfrak{p}} \subset G_{\mathfrak{p}}$. For a given vector $\xi_{\mathfrak{p}} \in \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \chi_{1,\mathfrak{p}})$, we define a vector $I_{\mathfrak{p}}(\xi_{\mathfrak{p}}) \in \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \chi_{2,\mathfrak{p}})$ by $I_{\mathfrak{p}}(\xi_{\mathfrak{p}}) := \int_{H_{2,\mathfrak{p}}} \chi_{2,\mathfrak{p}}^{-1}(h_{\mathfrak{p}})\pi_{\mathfrak{p}}^*(h_{\mathfrak{p}})(\xi_{\mathfrak{p}})dh_{\mathfrak{p}}$, where $\pi_{\mathfrak{p}}^*$ denotes the dual representation of $G_{\mathfrak{p}}$ on $V_{\pi_{\mathfrak{p}}}^*$. The integral is understood in a weak sense. This means that for any smooth vector $v_{\mathfrak{p}} \in V_{\pi_{\mathfrak{p}}}$, we have $I_{\mathfrak{p}}(\xi_{\mathfrak{p}})(v_{\mathfrak{p}}) = \int_{H_{2,\mathfrak{p}}} \chi_{2,\mathfrak{p}}^{-1}(h_{\mathfrak{p}})\xi_{\mathfrak{p}}(\pi_{\mathfrak{p}}(h_{\mathfrak{p}})v_{\mathfrak{p}})dh_{\mathfrak{p}}$. The last integral might be divergent, but in many cases could be evaluated by a standard procedure (usually involving analytic continuation; see [G1]). We stress that local maps are assumed to be defined *canonically* for *all* $\mathfrak{p}$ (i.e., the local map does not depend on parameters of local representations $\pi_{\mathfrak{p}}$, $\tau_{\mathfrak{p}}$, $\sigma_{\mathfrak{p}}$).

The next step is to "glue" local maps $I_{\mathfrak{p}}$ to a global map. This is a more subtle procedure. We construct the global map I by regularizing the tensor product $\otimes I_{\mathfrak{p}}$ of local maps with the help of appropriate weight factors. This is possible only for local maps which are coming from automorphic periods, and the weight factors are provided by the theory of automorphic L-functions. The construction of the map I (in certain cases) is the main observation of the paper. In some cases there are natural parameters (e.g., χ_1 or χ_2) for which one can notice that the corresponding Euler product is absolutely convergent in some region, and then could be analytically continued to a bigger region. In fact, in these cases the analytic continuation is based on the analytic continuation of some L-function. The relevant L-function shows up via its Euler factors appearing in the local unramified computation during the local step.

If there are no natural parameters involved, sometimes the following procedure could be employed. Let ξ be a vector in $\mathcal{P}(\pi, \chi_1)$. We write it as a product $\xi = \otimes_{\mathfrak{p}}\xi_{\mathfrak{p}}$, where $\xi_{\mathfrak{p}} \in \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \chi_{1,\mathfrak{p}})$ and for almost all $\mathfrak{p}$ we have $\xi_{\mathfrak{p}}(e_{\mathfrak{p}}^0) = 1$ for the standard vector $e_{\mathfrak{p}}^0 \in V_{\pi_{\mathfrak{p}}}$. Given a decomposable vector $v \in V_\pi$, we write it in a form $v = \otimes_{\mathfrak{p}} v_{\mathfrak{p}}$, where $v_{\mathfrak{p}} = e_{\mathfrak{p}}^0$ for almost all $\mathfrak{p}$. Now we would like to set $I(v) = \prod_{\mathfrak{p}} d_{\mathfrak{p}}$, where $d_{\mathfrak{p}} := I_{\mathfrak{p}}(\xi_{\mathfrak{p}})(v_{\mathfrak{p}})$. This product is usually not convergent. But we can use the fact that outside of a finite number of places the coefficients $d_{\mathfrak{p}}$ can be explicitly computed using an unramified computation. The unramified factors do not depend on a choice of the vector v. The result of the unramified computation allows us to use the following regularization procedure. We find an appropriate automorphic L-function (or a ratio of several L-functions) with the partial Euler product $L_S(s) = \prod_{\mathfrak{p}\notin S} L_{\mathfrak{p}}(s)$ (here S is a finite set of primes and the Euler factors for all $\mathfrak{p} \notin S$ are some rational functions of $q_{\mathfrak{p}}^{-s}$) and find some complex number s_0 such that if we replace for almost all $\mathfrak{p}$, coefficients $d_{\mathfrak{p}}$ by the normalized coefficients $d_{\mathfrak{p}}^0 := d_{\mathfrak{p}} L_{\mathfrak{p}}(s_0)$ then the product $\prod_{\mathfrak{p}} d^0_{\mathfrak{p}}$ is absolutely convergent (this condition does not depend on a specific choice of the vector v). After this we define for a large enough finite set of (ramified) primes S,

$$I(v) := L_S(s_0)^{-1} \prod_{\mathfrak{p}\notin S} d^0_{\mathfrak{p}} \prod_{\mathfrak{p}\in S} d_{\mathfrak{p}} \ .$$

Here for $Re(s) \gg 1$, $L_S(s) = \prod_{\mathfrak{p}\notin S} L_{\mathfrak{p}}(s)$ is the partial L-function. It is clear that this procedure is well defined (at least after we fix the L-function and its Euler product expansion). We note that in some examples the unramified factor $d_{\mathfrak{p}}$ *does not* coincide with an Euler factor of a Langlands L-function.

Remark. In many cases the complex number s_0 belongs to the region of the analytic continuation of $L(s)$, extension to which we will take for granted.

Having constructed the map I we can ask what is the effect of it on automorphic periods. Namely, we can try to compare the original period functional p_{H_2,χ_2} and the newly constructed functional $\tilde{p}_{H_2,\chi_2} = I(p_{H_1,\chi_1})$. This is the last (and the most interesting) step of the construction. The coefficient of proportionality (when defined) gives rise to a *global invariant* of the automorphic representation π (for χ_1 and χ_2 fixed). When $\tilde{p}_{H_2,\chi_2} = p_{H_2,\chi_2}$ this invariant is equal to 1, and we say that the collection $\{I_{\mathfrak{p}}\}$ of local maps (or the resulting global map I) is *coherent*. One can show that in many classical examples this is indeed the case, and this is equivalent to the "period to L-function" relation we mentioned above (e.g., theorems of Hecke and Waldspurger for torus periods). However, we find that sometimes the relation between $\tilde{p}_{H_2,\chi_2}$ and p_{H_2,χ_2} is more complicated and this gives rise to a non-trivial invariant. In particular for opposite unipotent subgroups of $GL(2)$, functionals $\tilde{p}_{H_2,\chi_2}$ and p_{H_2,χ_2} do *not* coincide for the Ramanujan cusp form associated with the Ramanujan tau function (see Appendix A).

Remark. We note that the alluded local map $I_{\mathfrak{p}}$ could be constructed in various ways. In particular a very general method was envisioned by H. Jaqcuet by introducing the Relative Trace Formula (see [J1] for the general framework). In fact, our construction could be interpreted as the computation of *global* Bessel distributions naturally appearing in the Relative Trace Formula. We hope to discuss this connection elsewhere.

1.3.1. *Structure of the paper.* In this paper, we will mostly discuss examples related to the classical periods considered by Hecke (and in the adelic setting by H. Jacquet and R. Langlands [JL]). We will show (see Theorem 3.2.3) how to define a procedure of integration transforming the Whittaker period (i.e., the period defining a non-trivial Fourier coefficient along the horocycle) into the Hecke period (i.e., the period against a Hecke character along the split torus of GL(2)). The proof we present is a simple reformulation of the standard argument of Hecke-Jacquet-Langlands and is based on the unfolding technique. We then prove the converse statement (i.e., we integrate the Hecke period into the Whittaker period; see Theorem 3.3.3). Here our proof is purely local (combined with the direct statement for the Whittaker to Hecke transform), and does not involve an unfolding procedure (in fact, we do not know if an appropriate unfolding exists in this case). Both of these cases are related to the classical Hecke formula for the standard L-function on GL(2).

Next we consider two unipotent periods for GL(2), that is, two Whittaker functionals: one on $N^+ = \{\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)\}$ and another on $N^- = \{\left(\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right)\}$. We consider the same question as before and discover that this example is of a completely different nature than those we discussed so far. We define the local integration procedure and show how to regularize the global map with the help of the adjoint L-function. In this case, the relation of the period map to special values of L-functions is puzzling to us. We note that local coefficients $d_{\mathfrak{p}}$ appearing in this case do *not* coincide with some familiar Euler factors from the theory of L-functions, however, $d_{\mathfrak{p}}$ coincides with the linear part of the Euler polynomial of $L(1, \pi_{\mathfrak{p}}, Ad)$ (or, as one might say, with the leading term of $L(1, \pi_{\mathfrak{p}}, Ad)$). This will be essential for the regularization of the Euler product $\prod_{\mathfrak{p}} d_{\mathfrak{p}}$. We would like to point out that this seems to be a part of a pattern and not an isolated example. In several other instances we have computed analogous local maps and found that these are connected to L-functions in a similar way (that is, coincide with linear parts of some L-functions). This should allow one to define the corresponding global maps (e.g., a map between a torus period and a *non-associated* Whittaker period). We will discuss these examples elsewhere. Note that this time the construction involves the action of the whole group GL(2) and not only the action of the Borel subgroup as in two previous examples. We will show that there is natural map between Whittaker periods on different unipotent subgroups, but in general it *does not* map the automorphic period to the automorphic period. As a result we are able to define a non-trivial invariant (a defect) of an automorphic (cuspidal) representation of GL(2). We also present a numerical computation for this invariant for the Ramanujan cusp form which indicates that the resulting invariant is not trivial (see Appendix A.1).

In Appendix B we collect information about the Kirillov model on GL(2) which we use in proofs and for computations. In Section 2 we review the basic setup and in particular discuss machinery of co-invariants which we find to be convenient in our treatment of periods of automorphic representations.

1.3.2. *Notations.* We denote the global field by k, places of k by $\mathfrak{p}$, the set of places of k by $\mathcal{P}(k)$, the corresponding ring of adeles by $\mathbb{A}$ and the group of ideles by $J_{\mathbb{A}}$. For a group $\mathcal{G}$ defined over a global field k (e.g., over $k = \mathbb{Q}$) we denote by $\mathcal{G}(k)$ the group of k-points, by $\mathcal{G}_{\mathfrak{p}} = \mathcal{G}(k_{\mathfrak{p}})$ the group of points over a local field $k_{\mathfrak{p}}$ (e.g., over $k_{\mathfrak{p}} = \mathbb{Q}_p$ or $k_\infty = \mathbb{R}$) and by $\mathcal{G}_{\mathbb{A}} = \mathcal{G}(\mathbb{A})$ the group of adelic points. For a local non-archimedian field $k_{\mathfrak{p}}$, we denote by $\mathcal{O}_{\mathfrak{p}}$ the ring of integers, by $\varpi_{\mathfrak{p}}$ a generator of the maximal ideal in $\mathcal{O}_{\mathfrak{p}}$, and by $q_{\mathfrak{p}}$ the size of the residue field. We will use the letter ψ to denote additive characters (local or global), and the letter χ to denote multiplicative characters (local or global). For a quasi-character $\chi_{\mathfrak{p}} : k_{\mathfrak{p}}^\times \to \mathbb{C}^\times$ of a local field, we have the decomposition $\chi_{\mathfrak{p}} = |\chi_{\mathfrak{p}}| \cdot \tilde{\chi}_{\mathfrak{p}}$ where $\tilde{\chi}_{\mathfrak{p}}$ is unitary. We denote by $\sigma_{\chi_{\mathfrak{p}}} = Re(\chi_{\mathfrak{p}}) \in \mathbb{R}$ the real part of $\chi_{\mathfrak{p}}$ given by the relation $|\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}})| = |\varpi_{\mathfrak{p}}|^{\sigma_{\chi_{\mathfrak{p}}}}$. Similarly, for a Hecke character $\chi : k^\times \setminus J_{\mathbb{A}} \to \mathbb{C}^\times$, there exists the unique decomposition $\chi = |\chi| \cdot \tilde{\chi}$ and $|\chi| = |\cdot|^{\sigma_\chi}$ with $\sigma_\chi = Re(\chi) \in \mathbb{R}$.

We denote by $G = \mathrm{GL}(2)$, by $Z = Z_G$ its center, by T the subgroup of diagonal matrices, by $A \subset T$ the subgroup of matrices of the form $\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)$, and by N the upper triangular matrices. We will use the following notations: $n(x) = \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)$, $\bar{a} = \left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)$, $diag(a,b) = \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix}\right)$, $z(a) = \left(\begin{smallmatrix} a & \\ & a \end{smallmatrix}\right)$, $w = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$.

For a field F (e.g., $F = k_{\mathfrak{p}}$), characters $\chi : A(F) \to \mathbb{C}^\times$ are given by $\chi(\bar{a}) = \chi(a)$, $a \in F$, and hence we can identify characters of A with those of $F^\times$. We do this for global characters as well and hence identify Hecke characters of k with those of $A(\mathbb{A})$. We use the notion of the real part for local and global characters of A as well.

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2. Measures, automorphic representations and periods

2.1. **Invariant integration.** We review invariant measures on local and adelic groups.

2.1.1. *Torsors.* By a torsor we mean a one-dimensional complex vector space. The name comes from the fact that if L is a torsor then $L \setminus \{0\}$ is a $\mathbb{C}^\times$-torsor. Torsors form a tensor category with respect to the tensor product. This category has the unit object $L_0 = \mathbb{C}$, and for every torsor L there is an inverse torsor $L^{-1} := L^*$.

2.1.2. *Moderate groups.* Let $\mathcal{A}$ be a locally compact group. We say that $\mathcal{A}$ is moderate if there exists a compact subgroup $K \subset \mathcal{A}$ with the following properties:

(i) K is totally disconnected,

(ii) The normalizer N of the group K is open in $\mathcal{A}$, and the quotient group N/K is a (smooth) Lie group.

We call a subgroup K with these properties a basic compact subgroup.

We will work only with moderate groups. In fact , as follows from Gleason-Yamabe theorem (see [T], Exercise 1.6.4, and [MZ], p. 182), any locally compact group of a finite topological dimension is moderate.

Proposition. *Any two basic subgroups K, L in a moderate group $\mathcal{A}$ are commensurable, i.e., the group $L \bigcap K$ has finite index in L and in K.*

Proof. We can assume that $\mathcal{A}$ normalizes K an L. Then the image of the group K in the Lie group $\mathcal{A}/L$ is a compact totally discontinuous subgroup in a Lie group and hence is finite. □

We define the space of test functions $\mathcal{S}(\mathcal{A})$ on a moderate group A as follows. The space $\mathcal{S}(\mathcal{A})$ consists of complex valued functions f on $\mathcal{A}$ such that

(1) f has compact support,
(2) f is left invariant with respect to some basic subgroup
(3) f is a smooth function on the smooth manifold $K \setminus \mathcal{A}$.

A function f on $\mathcal{A}$ is called smooth if in a neighborhood of any point it coincides with some test function. The algebra of smooth functions will be denoted by $C^\infty(\mathcal{A})$.

2.1.3. *Quotient spaces.* Let X be a quotient space of $\mathcal{A}$, i.e., $X = \mathcal{A}/\mathcal{B}$ for a closed moderate subgroup $\mathcal{B} \subset \mathcal{A}$ and X is endowed with the quotient topology. We call such space X a moderate space.

We denote by $C^\infty(X)$ the algebra of functions that lift to smooth functions on $\mathcal{A}$, and we denote by $\mathcal{S}(X)$ the space of test functions on X, i.e., the space of smooth functions of compact support on X.

Proposition. *Let $\alpha : \mathcal{A} \to \mathcal{A}'$ be a morphism of moderate groups, X, X' quotients spaces of $\mathcal{A}$ and $\mathcal{A}'$, and $\beta : X \to X'$ a continuous map compatible with α. Then β is smooth, i.e., $\beta^* : C^\infty(X') \to C^\infty(X)$.*

2.1.4. *Haar measure and co-invariants.* Let X be a moderate space. A Radon measure μ on X defines a functional $I_\mu : \mathcal{S}(X) \to \mathbb{C}$, i.e., $I_\mu(f) = \int_X f d\mu$ for $f \in \mathcal{S}(X)$.

For a torsor L, we can consider measures with values in L. Such a measure μ on X defines a functional $I_\mu : \mathcal{S}(X) \to L$.

Notation: We denote by $L(\mathcal{A}) = \mathcal{S}_\mathcal{A}(\mathcal{A}) := \mathcal{S}(\mathcal{A})/\langle f - a \circ f\rangle$ the space of co-invariants of the action of A acting on the left on $\mathcal{S}(\mathcal{A})$ (i.e., $a \circ f(\alpha) = f(a^{-1}\alpha)$).

Theorem. *Let $\mathcal{A}$ be a moderate group. We consider left action of $\mathcal{A}$ on itself.*

1. *The space of co-invariants $L = L(\mathcal{A})$ is a torsor. $\mathcal{A}$ acts on L trivially on the left and with some character $\Delta_{\mathcal{A}}$ (the modulus character) on the right.*
2. *The canonical morphism $I : \mathcal{S}(\mathcal{A}) \to L$ is defined by a Radon measure $\mu_{\mathcal{A}}$ with values in L.*
3. *The measure $\mu_{\mathcal{A}}$ is canonical, and it is invariant with respect to left and right actions of $\mathcal{A}$ on $\mathcal{S}(\mathcal{A})$.*

The theorem is essentially a reformulation of the Haar theorem. We call $\mu_{\mathcal{A}}$ the Haar measure of $\mathcal{A}$.

Remark. While the canonical map I is defined initially only on test functions, itcould be extended to bigger spaces, e.g., to $L^1(\mathcal{A})$. Later we will apply I also to some other classes of functions using an appropriate regularization.

We have the analogous construction for moderate quotient spaces. Let $X = \mathcal{A}/\mathcal{B}$ be a quotient space of a moderate group $\mathcal{A}$. Assume that there is a left $\mathcal{A}$-invariant measure on X. The space $L(X) = \mathcal{S}(X)_{\mathcal{A}}$ of co-invariants is then a torsor, and there exists a canonical Haar measure μ_X on X with values in $L(X)$ such that the map $I_{\mu_X} : \mathcal{S}(X) \to L(X)$ is the canonical projection.

Proposition. *We have the canonical isomorphism $L(\mathcal{A}) \simeq L(X) \otimes L(\mathcal{B})$.*

The isomorphism is given by the integration (with values in co-invariants) along fibers. In particular, for a discrete subgroup $\mathcal{B}$, we have the canonical isomorphism $L(X) \simeq L(\mathcal{A})$, and hence the canonical integration map

$$I_X : \mathcal{S}(X) \to L(\mathcal{A}) \ . \tag{2.1}$$

2.1.5. *Groups over global fields.* Let k be a global field. Let $\mathcal{G}$ be an affine algebraic group defined over k. For every place $\mathfrak{p}$ of k, we consider the group of points $\mathcal{G}_{\mathfrak{p}} = \mathcal{G}(k_{\mathfrak{p}})$ of $\mathcal{G}$ over the local field $k_{\mathfrak{p}}$. We also consider the adelic group $\mathcal{G}(\mathbb{A})$. It is defined with respect to compact open subgroups $\mathcal{G}(\mathcal{O}_{\mathfrak{p}}) \subset \mathcal{G}(k_{\mathfrak{p}})$ which are well-defined for almost all $\mathfrak{p}$.

Let $\mathcal{V} = \{V_{\mathfrak{p}}\}_{\mathfrak{p} \in \mathcal{P}(k)}$ be a collection of complex vector spaces indexed by places of k.

Definition. An adelic structure Σ on a family $\mathcal{V}$ is a choice of vectors $v_{\mathfrak{p}} \in V_{\mathfrak{p}}$ for almost all $\mathfrak{p}$ (i.e., for all except finite number of places, up to a change of vectors $v_{\mathfrak{p}}$ at finitely many places).

Definition. Let Σ be an adelic structure on a family $\mathcal{V}$. We define the restricted tensor product space V by $V = \otimes_{\Sigma} V_{\mathfrak{p}}$.

Namely, if $S \subset \mathcal{P}(k)$ is a finite set then we define $V_S = \otimes_{\mathfrak{p} \in S} V_{\mathfrak{p}}$. If $S \subset S'$ and S is sufficiently large, the adelic structure Σ defines the canonical morphism $V_S \to V_{S'}$. By the definition then, $V = \otimes_\Sigma V_{\mathfrak{p}} = \lim_{\vec{S}} V_S$.

Remark. If all spaces $V_{\mathfrak{p}}$ are torsors and vectors $v_{\mathfrak{p}}$ are non-zero for almost all $\mathfrak{p}$, then $\otimes_\Sigma V_{\mathfrak{p}}$ is also a torsor.

Example. Let $\mathcal{G}$ be an affine algebraic group defined over k. For all $\mathfrak{p}$, we have the canonical map $I_{\mathfrak{p}} : \mathcal{S}(\mathcal{G}_{\mathfrak{p}}) \to L(\mathcal{G}_{\mathfrak{p}})$.

Claim. *We have:*

(1) *The family of torsors* $L(\mathcal{G}_{\mathfrak{p}})$ *has canonical adelic structure* Σ_M*.*
(2) *There is canonical isomorphism* $L(\mathcal{G}(\mathbb{A})) \simeq \otimes_{\Sigma_M} L(\mathcal{G}_{\mathfrak{p}})$*.*

Here the canonical adelic structure Σ_M on $\{L(\mathcal{G}_{\mathfrak{p}})\}$ is obtained by taking the image $I_{\mathfrak{p}}(\chi_{K_{\mathfrak{p}}})$ of the characteristic function $\chi_{K_{\mathfrak{p}}}$ of the standard compact subgroup $K_{\mathfrak{p}} = \mathcal{G}(\mathcal{O}_{\mathfrak{p}})$ at unramified places $\mathfrak{p}$.

Remark. We note that in order to have the "usual" integral with respect to a measure with values in $\mathbb{C}$, one have to choose isomorphisms $i_{\mathfrak{p}} : \{L(\mathcal{G}_{\mathfrak{p}})\} \simeq \mathbb{C}$ for all places $\mathfrak{p}$, such that for almost all places, these satisfy $i_{\mathfrak{p}}(I_{\mathfrak{p}}(\chi_{K_{\mathfrak{p}}})) = 1 \in \mathbb{C}$. This is easily translated into the familiar normalization of the local Haar measure by the standard compact subgroup.

2.1.6. *Tamagawa structure.* There exists another remarkable adelic structure Σ_T for the family $\{L(\mathcal{G}_{\mathfrak{p}})\}$ proposed by T. Tamagawa [Ta] (see also [We]).

Let A be an algebraic group defined over k. We fix a left invariant top differential form δ on A defined over k. Such a choice gives rise to a measure $m(\delta_{\mathfrak{p}})$ on $A_{\mathfrak{p}}$, and in particular, defines the map $I_{m(\delta_{\mathfrak{p}})} : \mathcal{S}(A_{\mathfrak{p}}) \to \mathbb{C}$ given by the integration. Hence we obtain the isomorphism $i_{m(\delta_{\mathfrak{p}})} : L(A_{\mathfrak{p}}) \simeq \mathbb{C}$ of the torsor of co-invariants with the trivial torsor $\mathbb{C}$. We can now define the Tamagawa adelic structure Σ_T on the family $\{L(\mathcal{G}_{\mathfrak{p}})\}$ by choosing the vector $t_{\mathfrak{p}} = i^{-1}_{m(\delta_{\mathfrak{p}})}(1) \in L(\mathcal{G}_{\mathfrak{p}})$ for all $\mathfrak{p}$. We call the resulting restricted tensor product torsor $L^T(\mathcal{G}(\mathbb{A})) \simeq \otimes_{\Sigma_T} L(\mathcal{G}_{\mathfrak{p}})$ the Tamagawa torsor. Note that since non-zero vectors $t_{\mathfrak{p}}$ are specified for *all* places $\mathfrak{p}$, the torsor $L^T(\mathcal{G}(\mathbb{A}))$ comes with the canonical trivialization given by the "Tamagawa measure", i.e., by the vector $\mathfrak{t} = \mathfrak{t}_\delta = \otimes_{\mathfrak{p}} t_{\mathfrak{p}}$. The Tamagawa measure does not depend on the rational class of the form δ as follows from the standard product formula.

Remark. We do not claim that torsors $L^T(\mathcal{G}(\mathbb{A}))$ and $L(\mathcal{G}(\mathbb{A}))$ are isomorphic with respect to a collection of some local isomorphisms $j_{\mathfrak{p}} : L(\mathcal{G}_{\mathfrak{p}}) \to L(\mathcal{G}_{\mathfrak{p}})$ mapping the adelic structure Σ_M to Σ_T at almost all places. If this is the case, one can integrate functions in $\mathcal{S}(\mathcal{G}(\mathbb{A}))$ with respect to the Tamagawa measure $\mathfrak{t}$. Sometimes such an isomorphism exists and it is possible to integrate functions in $\mathcal{S}(\mathcal{G}(\mathbb{A})$ with respect to $\mathfrak{t}$ (e.g., for a unipotent subgroup $N \simeq k$), and this means that the Tamagawa construction provides a measure in the usual sense. However, in general, we can not integrate functions in $\mathcal{S}(\mathcal{G}(\mathbb{A}))$ with respect to $\mathfrak{t}$ since the Euler product $\prod_{\mathfrak{p}\notin S} i_{|\delta_\mathfrak{p}|}(I_\mathfrak{p}(\chi_{K_\mathfrak{p}}))$ is not absolutely convergent (e.g., for the torus $A \simeq k^\times$). This appears when two local trivializations $(L(\mathcal{G}_\mathfrak{p}), I_\mathfrak{p}(\chi_{K_\mathfrak{p}})) \simeq (\mathbb{C}, 1)$ and $(L(\mathcal{G}_\mathfrak{p}), t_\mathfrak{p}) \simeq (\mathbb{C}, 1)$ are not globally compatible, and one have to introduce a regularization procedure in order to obtain a measure out of the Tamagawa measure $\mathfrak{t}$ (i.e., another trivialization of $L^T(\mathcal{G}(\mathbb{A})))$.

2.1.7. *Characters.* We also consider integration twisted by characters.

Let (V_τ, τ) be a representation of $\mathcal{A}$ and $\chi : \mathcal{A} \to \mathbb{C}^\times$ be a character. We have the Jacquet module $J_\chi(\pi) = V_\tau / \langle v - \chi(a)\tau(a)v \rangle$, $v \in V_\tau$.

Let $X = \mathcal{A}/\mathcal{B}$ be a homogenous $\mathcal{A}$-space. We denote by $L_\chi(X) = J_\chi(\mathcal{S}(X))$ the corresponding Jacquet module. Let us assume that on X there is an invariant measure. We can describe this torsor as follows. Let $C(X, \chi)$ be the space of functions on X satisfying $f(ax) = \chi(a)f(x)$. This space is zero if $\chi|_\mathcal{B} \not\equiv 1$, and is a torsor otherwise.

Claim. *There is a canonical isomorphism* $L_\chi(X) \simeq C(X, \chi) \otimes L(X)$.

Choice of a point $x_0 \in X$ gives a trivialization $C(X, \chi) \simeq \mathbb{C}_\chi$, and hence the isomorphism $L_\chi(X) \simeq L(X) \otimes \mathbb{C}_\chi$. Hence $L_\chi(\mathcal{A})$ is a torsor on which $\mathcal{A}$ acts by χ on the left and by $\Delta_\mathcal{A}\chi^{-1}$ on the right.

The natural projection $I_\chi : \mathcal{S}(X) \to L_\chi(X)$ corresponds to the integration with some measure $\mu_{(X,\chi)}$ with values in $L_\chi(X)$.

Let $\mathcal{G}$ be an affine algebraic group defined over k. Let χ be a character $\chi = \otimes_\mathfrak{p} \chi_\mathfrak{p}$ of $\mathcal{G}(\mathbb{A})$.

Claim. *We have the isomorphism* $L_\chi(\mathcal{G}(\mathbb{A})) \simeq \otimes_{\Sigma_M} L_{\chi_\mathfrak{p}}(\mathcal{G}_\mathfrak{p})$.

Consider the automorphic space $X_\mathcal{G} = \mathcal{G}(k) \setminus \mathcal{G}(\mathbb{A})$. Let χ be a character of $\mathcal{G}(\mathbb{A})$ which is trivial on $\mathcal{G}(k)$ (i.e., $\chi : \mathcal{G}(k) \setminus \mathcal{G}(\mathbb{A}) \to \mathbb{C}$). We can trivialize the torsor $C(X_\mathcal{G}, \chi)$ by using the evaluation at the base point $x_0 = \{\mathcal{G}(k)\} \in X_\mathcal{G}$. This gives the isomorphism

$$L_\chi(X_\mathcal{G}) \simeq L(X_\mathcal{G}) \otimes \mathbb{C}_\chi \simeq L_\chi(\mathcal{G}(\mathbb{A})) \simeq \otimes_{\Sigma_M} L_{\chi_\mathfrak{p}}(\mathcal{G}_\mathfrak{p}) \ . \tag{2.2}$$

As a result, we have the corresponding integration map $I_{X_\mathcal{G},\chi} : \mathcal{S}(X_\mathcal{G}) \to L(\mathcal{G}(\mathbb{A})) \otimes \mathbb{C}_\chi$.

2.2. **Automorphic representations.** Let $\mathcal{G}$ be a reductive algebraic group defined over k. Let π be an irreducible smooth u representation of the adelic group $\mathcal{G}(\mathbb{A})$. We denote by V_π the space of smooth vectors of π and by ω_π the central character of π. We have decompositions $\pi = \otimes_{\mathfrak{p}}\pi_{\mathfrak{p}}$ and $V_\pi = \hat{\otimes}_{\mathfrak{p}} V_{\pi_{\mathfrak{p}}}$ into the restricted tensor product of local representations.

Let $X_{\mathcal{G}} = \mathcal{G}(k)\setminus\mathcal{G}(\mathbb{A})$ be the automorphic space. An automorphic structure on an (abstract) adelic representation π is an intertwining map $\nu : V_\pi \to \mathcal{F}(X_{\mathcal{G}})$ with the representation of $\mathcal{G}(\mathbb{A})$ in the space of functions on $X_{\mathcal{G}}$. We call a pair (π,ν) an automorphic representation. For a cuspidal (π,ν), the image of ν belongs to the space of rapidly decreasing smooth functions on $X_{\mathcal{G}}$.

We denote by $S(\pi)$ the set of places (including infinite places) where π is ramified (i.e., the complement to the set of unramified places $\mathfrak{p}$ where the standard $K_{\mathfrak{p}}$-fixed vector $e^0_{\mathfrak{p}} \in V_{\pi_{\mathfrak{p}}}$ is specified).

2.3. **Periods.** Let $\mathcal{H} \subset \mathcal{G}$ be an algebraic subgroup defined over k. Denote by $X_{\mathcal{H}} = \mathcal{H}(k)\setminus\mathcal{H}(\mathbb{A}) \subset X_{\mathcal{G}}$ the closed $\mathcal{H}(\mathbb{A})$-orbit. Let $\chi : \mathcal{H}(k)\setminus\mathcal{H}(\mathbb{A}) \to \mathbb{C}$ be a character. According to (2.1), we have the integration map $I_{X_{\mathcal{H}},\chi} : \mathcal{S}(X_{\mathcal{H}}) \to L_\chi(\mathcal{H}((\mathbb{A}))$. This together with the automorphic realization map ν and the restriction map $res_{X_{\mathcal{H}}} : C^\infty(X_G) \to C^\infty(X_{\mathcal{H}})$ give rise to the $\mathcal{H}(\mathbb{A})$-equivariant period map $p_{\mathcal{H},\chi} = I_{X_{\mathcal{H}},\chi} \circ res_{\mathcal{O}_{\mathcal{H}}} \circ \nu : V_\pi \to L_\chi(\mathcal{H}(\mathbb{A})) \simeq \otimes_{\Sigma_M} L_\chi(\mathcal{H}_{\mathfrak{p}})$. Formally, we need to assume that the corresponding integrals are well-defined (e.g., the orbit $X_{\mathcal{H}}$ is compact or the automorphic representation (π,ν) is cuspidal).

Definition. The space $P(V_\pi, L(\mathcal{H}(\mathbb{A}))) = \mathrm{Hom}_{H(\mathbb{A})}(V_\pi, L(\mathcal{H}(\mathbb{A})))$ is called the period space. For every place $\mathfrak{p}$, the space $P(V_{\pi_{\mathfrak{p}}}, L_\chi(\mathcal{H}_{\mathfrak{p}})) = \mathrm{Hom}_{\mathcal{H}_{\mathfrak{p}}}(V_{\pi_{\mathfrak{p}}}, L_\chi(\mathcal{H}_{\mathfrak{p}}))$ is called the local period space.

We have the factorization $P(V_\pi, L(\mathcal{H}(\mathbb{A}))) \simeq \hat{\otimes}_{\mathfrak{p}} P(V_{\pi_{\mathfrak{p}}}, L_\chi(\mathcal{H}_{\mathfrak{p}}))$.

We will assume that the local period space $P(V_{\pi_{\mathfrak{p}}}, L_\chi(\mathcal{H}_{\mathfrak{p}}))$ is at most one-dimensional. Hence any map in the period space is factorisable, and we can choose a factorization for the automorphic period $p_{\mathcal{H},\chi}$. To choose a factorization of the torsor $P(V_\pi, L_\chi(\mathcal{H}(\mathbb{A})))$ into a restricted tensor product, we need to choose for almost all places $\mathfrak{p}$, a special vector $p^0_{\mathfrak{p}} \in P(V_{\pi_{\mathfrak{p}}}, L_\chi(\mathcal{H}_{\mathfrak{p}}))$. We choose it by requiring that $p^0_{\mathfrak{p}}(e^0_{\mathfrak{p}}) = I_{\mathcal{H}_{\mathfrak{p}},\chi}(\chi_{K_{\mathcal{H}_{\mathfrak{p}}}})$ (in fact one have to check that such a normalization is possible, i.e., that there exists a non-vanishing invariant map on the standard vector $e^0_{\mathfrak{p}}$). We have then for sufficiently large finite set $S \subset \mathcal{P}(k)$, $p_{\mathcal{H},\chi} = (\otimes_{\mathfrak{p}\in S} p_{\mathfrak{p}}) \otimes (\otimes_{\mathfrak{p}\notin S} p^0_{\mathfrak{p}})$ for some choice of local ramified components $p_{\mathfrak{p}}$ for $\mathfrak{p} \in S$.

2.4. **Action on periods.** We reformulate our scheme from Section 1.3 in the language of co-invariants. Let $\mathcal{H}_1, \mathcal{H}_2 \subset \mathcal{G}$ be two algebraic subgroups as above.

In particular we will assume that all local spaces satisfy Gelfand condition of multiplicity one.

2.4.1. *Local maps.* Let $\mathfrak{p}$ be a place of k, $(\pi_\mathfrak{p}, V_\mathfrak{p})$ be an irreducible smooth representation of $\mathcal{G}_\mathfrak{p} = \mathcal{G}(k_\mathfrak{p})$. Let $\chi_{i,\mathfrak{p}} : \mathcal{H}_{i,\mathfrak{p}} \to \mathbb{C}$ be a character of $\mathcal{H}_{i,\mathfrak{p}}$. We fix a non-zero invariant differential form δ on $\mathcal{H}_1$ defined over k. Let $\delta_\mathfrak{p}$ be the corresponding invariant measure on $\mathcal{H}_{1,\mathfrak{p}}$. We use the measure $\delta_\mathfrak{p}$ to trivialize $\mathrm{ev}_{\delta_\mathfrak{p}} : L_{\chi_{1,\mathfrak{p}}}(\mathcal{H}_{1,\mathfrak{p}}) \xrightarrow{\sim} \mathbb{C}_{\chi_{1,\mathfrak{p}}}$ the corresponding co-invariants, and, correspondingly, we get the isomorphism $\mathrm{ev}^*_{\delta_\mathfrak{p}} : P(V_{\pi_\mathfrak{p}}, L_{\chi_{1,\mathfrak{p}}}(\mathcal{H}_{1,\mathfrak{p}})) \xrightarrow{\sim} \mathrm{Hom}_{A_\mathfrak{p}}(V_{\pi_\mathfrak{p}}, \mathbb{C}_{\chi_{1,\mathfrak{p}}})$ (see Section 2.1.7). We also consider the integration map $I_{\mathcal{H}_{2,\mathfrak{p}},\chi_{2,\mathfrak{p}}} : \mathcal{S}(\mathcal{H}_{2,\mathfrak{p}}) \to L_{\chi_{2,\mathfrak{p}}}(\mathcal{H}_{2,\mathfrak{p}})$.

We now construct the local map between local period spaces

$$\mathfrak{i}(\chi_{1,\mathfrak{p}}, \chi_{2,\mathfrak{p}}, \delta_\mathfrak{p}) : P(V_{\pi_\mathfrak{p}}, L_{\chi_{1,\mathfrak{p}}}(\mathcal{H}_{1,\mathfrak{p}})) \to P(V_{\pi_\mathfrak{p}}, L_{\chi_{2,\mathfrak{p}}}(\mathcal{H}_{2,\mathfrak{p}})) \tag{2.3}$$

using maps $I_{\mathcal{H}_{2,\mathfrak{p}}}$ and $\mathrm{ev}^*_{\delta_\mathfrak{p}}$. Namely, for a map $p_{(\mathcal{H}_{1,\mathfrak{p}},\chi_{1,\mathfrak{p}})} \in P(V_{\pi_\mathfrak{p}}, L_{\chi_{1,\mathfrak{p}}}(\mathcal{H}_{1,\mathfrak{p}}))$, we consider the function $f : \mathcal{H}_{2,\mathfrak{p}} \to \mathrm{Hom}_{\mathcal{H}_{1,\mathfrak{p}}}(V_{\pi_\mathfrak{p}}, \mathbb{C}_{\psi_\mathfrak{p}}) \subset V^*_{\pi_\mathfrak{p}}$ given by $f(h_2) = {\pi_\mathfrak{p}}^*(h_2)(\mathrm{ev}^*_{\delta_\mathfrak{p}}(p_{(\mathcal{H}_{1,\mathfrak{p}},\chi_{1,\mathfrak{p}})}))$ and apply to it the integration map $I_{\mathcal{H}_{2,\mathfrak{p}},\chi_{2,\mathfrak{p}}}$ (here ${\pi_\mathfrak{p}}^*$ is the dual to $\pi_\mathfrak{p}$ representation). Formally, the above function f is not compactly supported and we have to make sense of the corresponding integral. This is achieved by considering appropriate regularization procedure (e.g., by the analytic continuation method).

2.4.2. *Global maps.* In order to define the global map

$$\mathfrak{i}(\chi_1, \chi_2, \delta) : P(V_\pi, L_{\chi_1}(\mathcal{H}_1)) \to P(V_\pi, L_{\chi_2}(\mathcal{H}_2)) \tag{2.4}$$

we now want to make sense out of the Euler product $\hat{\otimes}_\mathfrak{p}\mathfrak{i}(\chi_{1,\mathfrak{p}}, \chi_{2,\mathfrak{p}}, \delta_\mathfrak{p})$. We described methods we employ to this end in Section 1.3.

3. Whittaker and Hecke periods relations

We want to illustrate how the procedure described in Section 2.4 relates Whittaker and Hecke functionals on an automorphic cuspidal representation of $G = \mathrm{GL}(2)$.

3.1. **Whittaker and Hecke periods.**

3.1.1. *Whittaker period.* We fix a nontrivial additive character $\psi : k \setminus \mathbb{A} \to \mathbb{C}^\times$ which is trivial on principal adeles. We view the character ψ as a character of $N(\mathbb{A})$. We consider the orbit $X_N = N(k) \setminus N(\mathbb{A}) \subset X_G$ and the corresponding period it induces on an automorphic (cuspidal) representation (π, ν) of $G = GL(2)$.

According to the above scheme, we view the Whittaker period $p_{(N,\psi)}$ as an element in the period space $P(V_\pi, L_\psi(N(\mathbb{A}))) = \mathrm{Hom}_{N(\mathbb{A})}(V_\pi, L_\psi(N(\mathbb{A})))$. In the factorization of the Whittaker period $p_{(N,\psi)} = \otimes_{\mathfrak{p}} p_{(N_\mathfrak{p},\psi_\mathfrak{p})}$ for almost all $\mathfrak{p}$, the local component $p_{(N_\mathfrak{p},\psi_\mathfrak{p})} \in P(V_{\pi_\mathfrak{p}}, L_{\psi_\mathfrak{p}}(N_\mathfrak{p})) = \mathrm{Hom}_{N_\mathfrak{p}}(V_{\pi_\mathfrak{p}}, L_{\psi_\mathfrak{p}}(N_\mathfrak{p}))$ is unramified (i.e., maps the standard vector $e^0_\mathfrak{p}$ to the image in co-invariants of the characteristic function $\chi_{N(\mathcal{O}_\mathfrak{p})}$ of the set $N(\mathcal{O}_\mathfrak{p})$).

Remark. In more classical terms, the Whittaker period/functional on V_π is given by the integral:

$$\mathcal{W}^\psi(v) = \int_{N(k)\backslash N(\mathbb{A})} \psi^{-1}(n)\phi_v(n)\, dn\ . \tag{3.1}$$

Here dn is the measure on $N(\mathbb{A})$ obtained from an invariant differential form. We have $\mathcal{W}^\psi \in \mathrm{Hom}_{N(\mathbb{A})}(V_\pi, \mathbb{C}_\psi)$, where $\mathbb{C}_\psi$ is the one-dimensional $N(\mathbb{A})$-module with the action given by ψ. It is well-known that $\dim \mathrm{Hom}_{N(\mathbb{A})}(V_\pi, \mathbb{C}_\psi) = 1$, the space of local functionals $\mathrm{Hom}_{N_\mathfrak{p}}(V_{\pi_\mathfrak{p}}, \mathbb{C}_{\psi_\mathfrak{p}})$ is also one-dimensional, and the global space decomposes into the restricted product of local spaces. We have the following standard decomposition of the automorphic Whittaker functional $\mathcal{W}^\psi$ into a product of local functionals. For an unramified place $\mathfrak{p} \notin S(\pi,\psi)$ (here $S(\pi,\psi)$ denotes the set of primes where π or ψ are ramified), let $\mathcal{W}^{\psi_\mathfrak{p}}_0 \in \mathrm{Hom}_{N_\mathfrak{p}}(V_{\pi_\mathfrak{p}}, \mathbb{C}_{\psi_\mathfrak{p}})$ be the local functional satisfying $\mathcal{W}^{\psi_\mathfrak{p}}_0(e^0_\mathfrak{p}) = 1$ for the standard $K_\mathfrak{p}$-fixed vector $e^0_\mathfrak{p} \in V_{\pi_\mathfrak{p}}$. We then choose local functionals $\widetilde{\mathcal{W}}^{\psi_\mathfrak{p}} \in \mathrm{Hom}_{N_\mathfrak{p}}(V_{\pi_\mathfrak{p}}, \mathbb{C}_{\psi_\mathfrak{p}})$ for ramified primes, so that $\mathcal{W}^\psi = \otimes_{\mathfrak{p}\in S(\pi,\psi)} \widetilde{\mathcal{W}}^{\psi_\mathfrak{p}} \otimes_{\mathfrak{p}\notin S(\pi,\psi)} \mathcal{W}^{\psi_\mathfrak{p}}_0$.

3.1.2. *Hecke period.* We now consider the Hecke period. Let χ be a Hecke (quasi-)character of k and let $\chi : A(\mathbb{A}) \to \mathbb{C}^\times$, $\chi\left(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right)\right) = \chi(a)$, be the corresponding (quasi-)character of $A(\mathbb{A})$ trivial on the principal subgroup $A(k)$. We consider the orbit $X_A = A(k) \setminus A(\mathbb{A}) \subset X_G$, and the corresponding (Hecke) period d_χ it induces on a cuspidal automorphic representation (π,ν). According to the above scheme we can view the Hecke period as an element in the torsor of periods $P(V_\pi, L_\chi(A(\mathbb{A}))) = \mathrm{Hom}_{A(\mathbb{A})}(V_\pi, L_\chi(A(\mathbb{A})))$. We have a factorization $p_{(A,\chi)} = \otimes_\mathfrak{p} p_{(A_\mathfrak{p},\chi_\mathfrak{p})}$ where for almost all $\mathfrak{p}$, the local component $p_{(A_\mathfrak{p},\chi_\mathfrak{p})} \in P(V_{\pi_\mathfrak{p}}, L_{\chi_\mathfrak{p}}(A_\mathfrak{p}))$ is unramified.

Remark. In more classical terms we have the following description of the functional $d_\chi = d_\chi(\pi) : V_\pi \to \mathbb{C}_\chi$ (here $\mathbb{C}_\chi$ denotes the one-dimensional $A(\mathbb{A})$-module with the action given by χ). We fix an invariant rational differential form on A and denote by μ the corresponding invariant measure. As an element in the space $\mathrm{Hom}_{A(\mathbb{A})}(V_\pi, \mathbb{C}_\chi)$ the corresponding period functional is given by the integral

$$d_\chi(v) = \int_{A(k)\backslash A(\mathbb{A})} \chi^{-1}(\bar a)\phi_v(\bar a)\ \mu\ , \tag{3.2}$$

for $v \in V_\pi$. The integral is absolutely convergent since functions in a cuspidal representation are rapidly decreasing at infinity. It is well-known that $\dim \mathrm{Hom}_{A(\mathbb{A})}(V_\pi, \mathbb{C}_\chi) = 1$, that the local space $\mathrm{Hom}_{A_\mathfrak{p}}(V_{\pi_\mathfrak{p}}, \mathbb{C}_{\chi_\mathfrak{p}})$ also satisfies the multiplicity one property, and hence $\mathrm{Hom}_{A(\mathbb{A})}(V_\pi, \mathbb{C}_\chi) = \hat\otimes_\mathfrak{p} \mathrm{Hom}_{A_\mathfrak{p}}(V_{\pi_\mathfrak{p}}, \mathbb{C}_{\chi_\mathfrak{p}})$. We again can choose a decomposition $d_\chi = \otimes_{\mathfrak{p}\in S(\pi,\chi)} \widetilde{d}_{\chi_\mathfrak{p}} \otimes_{\mathfrak{p}\notin S(\pi,\chi)} d^0_{\chi_\mathfrak{p}}$ into local components with $d^0_{\chi_\mathfrak{p}}(e^0_\mathfrak{p}) = 1$.

3.2. **Whittaker to Hecke.**

3.2.1. *Local map.* Let $\mathfrak{p}$ be a place of k, $(\pi_\mathfrak{p}, V_\mathfrak{p})$ be an irreducible smooth representation of $G_\mathfrak{p} = G(k_\mathfrak{p})$. Let $\psi_\mathfrak{p} : N_\mathfrak{p} \to \mathbb{C}$ be a nontrivial character of $N_\mathfrak{p}$ and $\chi_\mathfrak{p} : A_\mathfrak{p} \to \mathbb{C}$ be a character of $A_\mathfrak{p} \simeq k^\times_\mathfrak{p}$. We fix a (non-zero) invariant differential form δ_N on N and consider the corresponding invariant measure $dn_\mathfrak{p} = dn_\mathfrak{p}(\delta_N)$ on $N_\mathfrak{p}$. The measure $dn_\mathfrak{p}$ gives rise to the trivialization $\mathrm{ev}_{dn_\mathfrak{p}} : L_{\psi_\mathfrak{p}}(N_\mathfrak{p}) \xrightarrow{\sim} \mathbb{C}_{\psi_\mathfrak{p}}$ of the corresponding co-invariants and to the isomorphism $\mathrm{ev}^*_{dn_\mathfrak{p}} : P(V_{\pi_\mathfrak{p}}, L_{\psi_\mathfrak{p}}(N_\mathfrak{p})) \xrightarrow{\sim} \mathrm{Hom}_{N_\mathfrak{p}}(V_{\pi_\mathfrak{p}}, \mathbb{C}_{\psi_\mathfrak{p}})$. We consider the integration map $I_{A_\mathfrak{p},\chi_\mathfrak{p}} : \mathcal{S}(A_\mathfrak{p}) \to L_{\chi_\mathfrak{p}}(A_\mathfrak{p})$ (see Section 2.1.7).

Following the scheme formulated in Section 2.4, we consider the local map

$$\mathfrak{i}(\chi_\mathfrak{p}, \psi_\mathfrak{p}, dn_\mathfrak{p}) : P(V_{\pi_\mathfrak{p}}, L_{\psi_\mathfrak{p}}(N_\mathfrak{p})) \to P(V_{\pi_\mathfrak{p}}, L_{\chi_\mathfrak{p}}(A_\mathfrak{p})) \tag{3.3}$$

constructed out of maps $I_{A_\mathfrak{p},\chi_\mathfrak{p}}$ and $\mathrm{ev}^*_{dn_\mathfrak{p}}$. Namely, for a map $p_{(N_\mathfrak{p},\psi_\mathfrak{p})} \in P(V_{\pi_\mathfrak{p}}, L_{\psi_\mathfrak{p}}(N_\mathfrak{p}))$, we consider the function $f : A_\mathfrak{p} \to \mathrm{Hom}_{N_\mathfrak{p}}(V_{\pi_\mathfrak{p}}, \mathbb{C}_{\psi_\mathfrak{p}}) \subset V^*_{\pi_\mathfrak{p}}$ given by $f(\bar a) = {\pi_\mathfrak{p}}^*(\bar a)(\mathrm{ev}^*_{dn_\mathfrak{p}}(p_{(N_\mathfrak{p},\psi_\mathfrak{p})}))$ and apply to it the integration map $I_{A_\mathfrak{p},\chi_\mathfrak{p}}$ (here ${\pi_\mathfrak{p}}^*$ is the dual representation) .

Proposition.

(1) *For $Re(\chi_\mathfrak{p}) \ll 1$, the map $\mathfrak{i}(\chi_\mathfrak{p}, \psi_\mathfrak{p}, dn_\mathfrak{p})$ is well-defined (by an absolutely convergent integral* (3.6)*). It has the meromorphic continuation to the complex space of all characters (i.e., to the complex plane of characters of the form $\chi_\mathfrak{p}|\cdot|^{-s}_\mathfrak{p}$).*

(2) *For the unramified data, we obtain the Hecke-Jacquet-Langlands local L-factor. Namely,*

$$\mathfrak{i}(\chi_\mathfrak{p}, \psi_\mathfrak{p}, dn_\mathfrak{p})(\mathcal{W}^{\psi_\mathfrak{p}}_0) = L_\mathfrak{p}(\chi_\mathfrak{p}, \pi_\mathfrak{p}) \cdot d^0_{\chi_\mathfrak{p}}\ . \tag{3.4}$$

The meaning of the unramified condition above is as follows: $\pi_\mathfrak{p}$ is an unramified representation, $\psi_\mathfrak{p}$ has conductor $cond(\psi_\mathfrak{p}) = \mathcal{O}_\mathfrak{p}$, $\chi_\mathfrak{p}$ is an unramified character, and the form δ_N is normalized by $dn_\mathfrak{p}(\delta_N)(N(\mathcal{O}_\mathfrak{p}))=1$, the unramified Whittaker functional $\mathcal{W}_0^{\psi_\mathfrak{p}}\in P(V_{\pi_\mathfrak{p}},L_{\psi_\mathfrak{p}}(N_\mathfrak{p}))$ satisfies $\mathcal{W}_0^{\psi_\mathfrak{p}}(e_\mathfrak{p}^0)=l^0_{\psi_\mathfrak{p}}(N_\mathfrak{p})$ for $l^0_{\psi_\mathfrak{p}}(N_\mathfrak{p})\in L_{\psi_\mathfrak{p}}(N_\mathfrak{p})$ given by the adelic structure on $L_{\psi_\mathfrak{p}}(N_\mathfrak{p})$ described in Section 2.1.5, and correspondingly for the functional $d_0^{\chi_\mathfrak{p}}\in P(V_{\pi_\mathfrak{p}},L_{\chi_\mathfrak{p}}(A_\mathfrak{p}))$ with $d_0^{\chi_\mathfrak{p}}(e^0_\mathfrak{p})=l^0_{\chi_\mathfrak{p}}(A_\mathfrak{p})$ and $l^0_{\chi_\mathfrak{p}}(A_\mathfrak{p})\in L_{\chi_\mathfrak{p}}(A_\mathfrak{p})$.

Remark. The map $\mathfrak{i}(\chi_\mathfrak{p},\psi_\mathfrak{p},dn_\mathfrak{p})$ could be described in following terms. For a map $p_{(N_\mathfrak{p},\psi_\mathfrak{p})}\in P(V_{\pi_\mathfrak{p}},L_{\psi_\mathfrak{p}}(N_\mathfrak{p}))$ and a vector $v\in V_\mathfrak{p}$, we consider the (matrix coefficient) function $\alpha_v(\bar a_\mathfrak{p})=\mathrm{ev}_{dn_\mathfrak{p}}(p_{(N_\mathfrak{p},\psi_\mathfrak{p})}(\pi_\mathfrak{p}(\bar a_\mathfrak{p})v))\in C^\infty(A)$ and then take its image $I_{A_\mathfrak{p},\chi_\mathfrak{p}}(\alpha_v)\in L_{\chi_\mathfrak{p}}(A_\mathfrak{p})$ under the integration, i.e.,

$$p_{(A_\mathfrak{p},\chi_\mathfrak{p})}(v)=[\mathfrak{i}(\chi_\mathfrak{p},\psi_\mathfrak{p},dn_\mathfrak{p})(p_{(N_\mathfrak{p},\psi_\mathfrak{p})})](v):=I_{A_\mathfrak{p},\chi_\mathfrak{p}}(\alpha_v)\ . \tag{3.5}$$

Smooth functions $\alpha_v(\bar a_\mathfrak{p})$ with $v\in V_\mathfrak{p}$ obtained in such a way are not compactly supported on $A_\mathfrak{p}$. Hence in fact, we have to extend the integration map $I_{A_\mathfrak{p},\chi_\mathfrak{p}}:\mathcal{S}(A_\mathfrak{p})\to L_{\chi_\mathfrak{p}}(A_\mathfrak{p})$ to such functions (i.e., to the space of matrix coefficients α_v as above for an irreducible representation $\pi_\mathfrak{p}$).

We claim that the map $\mathfrak{i}(\chi_\mathfrak{p},\psi_\mathfrak{p},dn_\mathfrak{p})$ in (3.3) naturally appears, in another language, in [JL] as local zeta integrals on GL(2) of Jacquet and Langlands. Let us fix a non-zero invariant local measure $d^\times a_\mathfrak{p}$ on $A_\mathfrak{p}$. This gives rise to isomorphisms $\mathrm{ev}_{d^\times a_\mathfrak{p}}:L_\chi(A_\mathfrak{p})\xrightarrow{\sim}\mathbb{C}_{\chi_\mathfrak{p}}$ and $\mathrm{ev}^*_{d^\times a_\mathfrak{p}}:P(V_{\pi_\mathfrak{p}},L_{\chi_\mathfrak{p}}(A_\mathfrak{p}))\xrightarrow{\sim}\mathrm{Hom}_{A_\mathfrak{p}}(V_\pi,\mathbb{C}_{\chi_\mathfrak{p}})$. Using isomorphisms $\mathrm{ev}^*_{dn_\mathfrak{p}}$ and $\mathrm{ev}^*_{d^\times a_\mathfrak{p}}$, we see that the map (3.3) could be described by the following standard in GL(2) theory integral:

$$\begin{aligned}&[i(\chi_\mathfrak{p},\psi_\mathfrak{p},dn_\mathfrak{p},d^\times a_\mathfrak{p})(p_{(N_\mathfrak{p},\psi_\mathfrak{p})})](v):=\\ &\qquad[\mathrm{ev}^*_{d^\times a_\mathfrak{p}}(p_{(A_\mathfrak{p},\chi_\mathfrak{p})})](v)=\int_{A_\mathfrak{p}}\chi_\mathfrak{p}^{-1}(\bar a)l^{\psi_\mathfrak{p}}(\pi_\mathfrak{p}(\bar a)v)d^\times a_\mathfrak{p}\ ,\end{aligned}\tag{3.6}$$

for $v\in V_{\pi_\mathfrak{p}}$ and $l^{\psi_\mathfrak{p}}=\mathrm{ev}^*_{dn_\mathfrak{p}}(p_{(N_\mathfrak{p},\psi_\mathfrak{p})})\in\mathrm{Hom}_{N_\mathfrak{p}}(V_{\pi_\mathfrak{p}},\mathbb{C}_{\psi_\mathfrak{p}})$. Translated into local zeta integrals (3.6), the relation (3.4) reads

$$[i(\chi_\mathfrak{p},\psi_\mathfrak{p},dn_\mathfrak{p},d_1^\times a_\mathfrak{p})(\mathcal{W}_0^{\psi_\mathfrak{p}})](e^0_\mathfrak{p})=L_\mathfrak{p}(\chi_\mathfrak{p},\pi_\mathfrak{p})\ ,$$

for the measure $d_1^\times a_\mathfrak{p}$ on $A_\mathfrak{p}$ normalized by the condition $d_1^\times a_\mathfrak{p}(A(\mathcal{O}_\mathfrak{p}))=1$.

Note that while the integral (3.6) depends on the choice of the measure $d^\times a_\mathfrak{p}$, the map (3.3) does not. Over archimedian fields, a similar approach appeared in [Po].

3.2.2. *Global map.* Fix an automorphic cuspidal representation (π,ν), a non-trivial character $\psi:N(k)\setminus N(\mathbb{A})\to\mathbb{C}$. Choose an invariant differential form δ_N on N. We want to define a map $\mathfrak{i}(\chi,\psi,\delta_N):P(V_\pi,L_\psi(N(\mathbb{A})))\to P(V_\pi,L_\chi(A(\mathbb{A})))$ as a tensor product of local maps.

Proposition.

(1) *The tensor product* $\mathfrak{i}(\chi, \psi, \delta_N) = \otimes_{\mathfrak{p}} \mathfrak{i}(\chi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, dn_{\mathfrak{p}})$ *is absolutely convergent for* $Re(\chi) \ll 1$*, and has the meromorphic continuation to the complex space of all characters.*

(2) *The resulting map*

$$\mathfrak{i}(\chi, \psi) = \mathfrak{i}(\chi, \psi, \delta_N) : P(V_\pi, L_\psi(N(\mathbb{A}))) \to P(V_\pi, L_\chi(A(\mathbb{A})))$$

does not depend on the choice of the form δ_N.

3.2.3. *Action on automorphic periods.* We now came to the last step of our scheme where we compute the effect of the defined map on automorphic periods.

Theorem. *The global map* $\mathfrak{i}(\chi, \psi)$ *is coherent, i.e., it sends the automorphic Whittaker period* $\mathcal{W}^\psi$ *to the automorphic Hecke period* d_χ. *Namely we have*

$$\mathfrak{i}(\chi, \psi)(\mathcal{W}^\psi) = d_\chi \ .$$

3.3. **Hecke to Whittaker.**

3.3.1. *Local map.* Let $\mathfrak{p}$ be a place of k, $(\pi_{\mathfrak{p}}, V_{\mathfrak{p}})$ be an irreducible smooth representation of $G_{\mathfrak{p}} = G(k_{\mathfrak{p}})$. Let $\psi_{\mathfrak{p}} : N_{\mathfrak{p}} \to \mathbb{C}$ be a nontrivial character of $N_{\mathfrak{p}}$ and $\chi_{\mathfrak{p}} : A_{\mathfrak{p}} \to \mathbb{C}$ be a character of $A_{\mathfrak{p}}$. We fix a non-zero invariant differential form δ_A on A defined over k. Let $d^\times a_{\mathfrak{p}} = d^\times a_{\mathfrak{p}}(\delta_A)$ be the corresponding invariant measure on $A_{\mathfrak{p}}$. We use the measure $d^\times a_{\mathfrak{p}}$ to trivialize $\mathrm{ev}_{d^\times a_{\mathfrak{p}}} : L_{\chi_{\mathfrak{p}}}(A_{\mathfrak{p}}) \xrightarrow{\sim} \mathbb{C}_{\chi_{\mathfrak{p}}}$ the corresponding co-invariants, and, correspondingly, we get the isomorphism $\mathrm{ev}^*_{d^\times a_{\mathfrak{p}}} : P(V_{\pi_{\mathfrak{p}}}, L_{\chi_{\mathfrak{p}}}(A_{\mathfrak{p}})) \xrightarrow{\sim} \mathrm{Hom}_{A_{\mathfrak{p}}}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\chi_{\mathfrak{p}}})$ (see Section 2.1.7). We also consider the integration map $I_{N_{\mathfrak{p}}, \psi_{\mathfrak{p}}} : \mathcal{S}(N_{\psi_{\mathfrak{p}}}) \to L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}})$.

We now construct the local map

$$\mathfrak{i}(\psi_{\mathfrak{p}}, \chi_{\mathfrak{p}}, d^\times a_{\mathfrak{p}}) : P(V_{\pi_{\mathfrak{p}}}, L_{\chi_{\mathfrak{p}}}(A_{\mathfrak{p}})) \to P(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}})) \tag{3.7}$$

using maps $I_{N_{\mathfrak{p}}, \psi_{\mathfrak{p}}}$ and $\mathrm{ev}^*_{d^\times a_{\mathfrak{p}}}$ as in the previous case.

Proposition. *The map* $\mathfrak{i}(\psi_{\mathfrak{p}}, \chi_{\mathfrak{p}}, d^\times a_{\mathfrak{p}})$ *is well-defined for* $Re(\chi_{\mathfrak{p}}) \ll 1$ *and has the meromorphic continuation to the space of all characters.*

In other words, for a given character $\chi_{\mathfrak{p}}$, the map $\mathfrak{i}(\psi_{\mathfrak{p}}, \chi_{\mathfrak{p}}| \cdot |_{\mathfrak{p}}^{-s}, d^\times a_{\mathfrak{p}})$ is well-defined by an absolutely convergent integral for $Re(s) \gg 1$, and has the meromorphic continuation to $\mathbb{C}$ (i.e., to the complex plane of characters $\chi_{\mathfrak{p}}| \cdot |_{\mathfrak{p}}^{-s}$).

Remark. We can describe the map $\mathfrak{i}(\psi_\mathfrak{p}, \chi_\mathfrak{p}, d^\times a_\mathfrak{p})$ also as follows. For a map $p_{(A_\mathfrak{p},\chi_\mathfrak{p})} \in P(V_{\pi_\mathfrak{p}}, L_{\chi_\mathfrak{p}}(A_\mathfrak{p}))$ and a vector $v \in V_\mathfrak{p}$, we consider the (matrix coefficient) function $\beta_v(n_\mathfrak{p}) = \mathrm{ev}_{d^\times a_\mathfrak{p}}(p_{(A_\mathfrak{p},\chi_\mathfrak{p})}(\pi_\mathfrak{p}(n_\mathfrak{p})v)) \in C^\infty(N_\mathfrak{p})$ and then take its image $I_{N_\mathfrak{p},\psi_\mathfrak{p}}(\beta_v) \in L_{\psi_\mathfrak{p}}(A_\mathfrak{p})$ under the integration map, i.e.,

$$p_{(N_\mathfrak{p},\psi_\mathfrak{p})}(v) = [\mathfrak{i}(\psi_\mathfrak{p}, \chi_\mathfrak{p}, d^\times a_\mathfrak{p})(p_{(A_\mathfrak{p},\chi_\mathfrak{p})})](v) := I_{N_\mathfrak{p},\psi_\mathfrak{p}}(\beta_v) \ . \tag{3.8}$$

Smooth functions $\beta_v(n_\mathfrak{p})$ with $v \in V_\mathfrak{p}$ obtained in such a way are not compactly supported on $N_\mathfrak{p}$. Hence we have to extend the integration map $I_{N_\mathfrak{p},\psi_\mathfrak{p}} : \mathcal{S}(N_\mathfrak{p}) \to L_{\psi_\mathfrak{p}}(N_\mathfrak{p})$ to such functions (i.e., to the space of matrix coefficients β_v as above for an irreducible representation $\pi_\mathfrak{p}$). Let us fix a non-zero invariant local measure $dn_\mathfrak{p}$ on $N_\mathfrak{p}$. This gives rise to isomorphisms $\mathrm{ev}_{dn_\mathfrak{p}} : L_{\psi_\mathfrak{p}}(N_\mathfrak{p}) \xrightarrow{\sim} \mathbb{C}_{\psi_\mathfrak{p}}$ and $\mathrm{ev}^*_{dn_\mathfrak{p}} : P(V_{\pi_\mathfrak{p}}, L_{\psi_\mathfrak{p}}(N_\mathfrak{p})) \xrightarrow{\sim} \mathrm{Hom}_{N_\mathfrak{p}}(V_\pi, \mathbb{C}_{\psi_\mathfrak{p}})$. Using isomorphisms $\mathrm{ev}^*_{dn_\mathfrak{p}}$ and $\mathrm{ev}^*_{d^\times a_\mathfrak{p}}$, we see that the map (3.7) could be described by the following integral:

$$\begin{aligned}[i(\psi_\mathfrak{p}, \chi_\mathfrak{p}, d^\times a_\mathfrak{p}, dn_\mathfrak{p})(p_{(A_\mathfrak{p},\chi_\mathfrak{p})})](v) :=& \\ [\mathrm{ev}^*_{dn_\mathfrak{p}}(p_{(N_\mathfrak{p},\psi_\mathfrak{p})})](v) =& \int_{N_\mathfrak{p}} \psi_\mathfrak{p}^{-1}(n) d_{\chi_\mathfrak{p}}(\pi_\mathfrak{p}(n_\mathfrak{p})v) \ dn_\mathfrak{p} \ ,\end{aligned} \tag{3.9}$$

for $v \in V_{\pi_\mathfrak{p}}$ and $d_{\chi_\mathfrak{p}} = \mathrm{ev}^*_{d^\times a_\mathfrak{p}}(p_{(A_\mathfrak{p},\chi_\mathfrak{p})}) \in \mathrm{Hom}_{A_\mathfrak{p}}(V_{\pi_\mathfrak{p}}, \mathbb{C}_{\chi_\mathfrak{p}})$. Note that while the integral (3.9) depends on the choice of the measure $dn_\mathfrak{p}$, the map (3.7) does not. The integral (3.9) is not absolutely convergent and should be understood through the analytic continuation.

We want to point out that the integral (3.9) is not covered by the Jacquet-Langlands [JL] theory.

3.3.2. *Global map.*

Proposition. *Fix an invariant differential form δ_A on A.*

1. *The tensor product $\mathfrak{i}(\psi, \chi, \delta_A) = \otimes_\mathfrak{p} \mathfrak{i}(\psi_\mathfrak{p}, \chi_\mathfrak{p}, d^\times a_\mathfrak{p})$ is absolutely convergent for $Re(\chi) \ll 1$, and has the meromorphic continuation to the complex space of all characters.*
2. *The resulting map*

$$\mathfrak{i}(\psi, \chi) = \mathfrak{i}(\psi, \chi, \delta_A) : P(V_\pi, L_\chi(A(\mathbb{A}))) \to P(V_\pi, L_\psi(N(\mathbb{A})))$$

does not depend on the choice of the rational form δ_A.

3.3.3. *Action on automorphic periods.* As the last step we have to compute the effect on the Hecke automorphic period. We now formulate our main result in this section.

Theorem. *The global map* $\mathfrak{i}(\psi,\chi)$ *is coherent, i.e., it sends the automorphic Hecke period* d_χ *to the automorphic Whittaker period* $\mathcal{W}^\psi$. *Namely we have*

$$\mathfrak{i}(\psi,\chi)d_\chi = \mathcal{W}^\psi \ .$$

Remark. We would like to point out a subtle difference between Theorem 3.2.3 and Theorem 3.3.3. The collection of local measures $\{d^\times a_{\mathfrak{p}}(\delta_A)\}$ appearing in Proposition 3.3.2 defines the Tamagawa adelic structure on the torsor $L^T(A(\mathbb{A}))$ described in Section 2.1.6, but these measures do not define a *genuine measure* on $A(\mathbb{A})$. This differs from the situation described in Proposition 3.2.2 where the Tamagawa adelic structure on $L^T(N(\mathbb{A}))$ defines a genuine measure on $N(\mathbb{A})$. As a result, the direct map $\mathfrak{i}(\chi,\psi)$ from the Whittaker period space to the Hecke period space has the integral representation (i.e., the Hecke-Jacquet-Langlands integral (3.14)), but we do not know such integral representation for the map $\mathfrak{i}(\psi,\chi)$ in the opposite direction.

3.3.4. *The relation.* We have the following retaliation between two maps involving Whittaker and Hecke periods.

Theorem. *The following relation holds*

$$\mathfrak{i}(\psi,\chi)\circ\mathfrak{i}(\chi,\psi) = id \ ,$$

as an endomorphism of $P(V_\pi, L_\psi(N))$.

3.4. **Proofs.** The logic of the proof we present is as follows. We first prove results from the Section 3.1 by repeating arguments of Jacquet-Langlands in a slightly different language. We then prove Theorem 3.3.4 by a local computation (see Lemma 3.4.4). This then implies all the other results in Section 3.3.

3.4.1. *Proof of Proposition 3.2.1.* Both claims are standard in the Hecke-Jacquet-Langlands theory once the translation (3.6) into local zeta integrals is made.

(1) For a smooth vector $v \in V_{\pi_{\mathfrak{p}}}$, the Whittaker function $l^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}}(\bar{a})v)$ is rapidly decreasing as $||t|| \to \infty$ in the positive Weyl chamber, and has a polynomial behavior in the opposite direction. This implies the absolute convergence of the integral for the character $\chi_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^{-s}$ with $Re(s) \gg 1$.

    The meromorphic continuation is equivalent to the meromorphic continuation of the Jacquet-Langlands local zeta integrals (see [JL]).
(2) This is the standard computation in the Hecke-Jacquet-Langlands theory.

3.4.2. *Proof of Proposition 3.2.2.* Indeed this follows immediately from the analytic continuation of $L(s,\pi)$ and from Proposition 3.2.1. In fact this is a part of the Jacquet-Langlands method where the adelic integral is reduced to the absolutely convergent integral (3.14) via unfolding.

3.4.3. *Proof of Theorem 3.2.3.* As we indicated before, the following proof is the standard Hecke-Jacquet-Langlands proof. On the basis of (3.5) we want to compute

$$\left[i(\chi|\cdot|^{-s},\psi,dn,d^\times a)(\mathcal{W}^\psi)\right](v)=\prod_{\mathfrak{p}}\int_{A_\mathfrak{p}}\chi_\mathfrak{p}^{-1}(\bar a)|a|_\mathfrak{p}^s\mathcal{W}^{\psi_\mathfrak{p}}(\pi_\mathfrak{p}(\bar a)v)\ d^\times a_\mathfrak{p}\ , \tag{3.10}$$

for $Re(s)\gg 1$. Assume that $Re(s)\gg 1$. The absolute convergence of local integrals and the absolute convergence of the Euler product implies that

$$\prod_{\mathfrak{p}}\int_{A_\mathfrak{p}}\chi_\mathfrak{p}^{-1}(\bar a)|a|_\mathfrak{p}^s\mathcal{W}^{\psi_\mathfrak{p}}(\pi_\mathfrak{p}(\bar a)v)\ d^\times a_\mathfrak{p}=\int_{A(\mathbb{A})}\chi^{-1}(\bar a)|\cdot|^s\mathcal{W}^\psi(\pi(\bar a)\phi_v)\ d^\times a\ . \tag{3.11}$$

We invoke the standard unfolding technique. The rational torus acts transitively on Whittaker functionals for different characters. For an automorphic period $\mathcal{W}^\psi$ and a character $\psi_\alpha(x)=\psi(\alpha x)$ with $\alpha\in k^\times$, we have the corresponding automorphic period given by $\mathcal{W}^{\psi_\alpha}=\pi^*\left(\left(\begin{smallmatrix}\alpha&\\&1\end{smallmatrix}\right)\right)\mathcal{W}^\psi$. We have the following Fourier expansion at the identity for an automorphic function ϕ_v, $v\in V_\pi$, in a cuspidal representation π,

$$\phi_v(e)=\sum_{\alpha\in k^\times}\mathcal{W}^{\psi_\alpha}(v)=\sum_{\alpha\in k^\times}\mathcal{W}^\psi\left(\pi(\bar\alpha)v\right)\ . \tag{3.12}$$

The Fourier expansion for a cusp form ϕ_v implies that

$$\int\limits_{A(\mathbb{A})} \chi^{-1}(\bar{a})|a|^s \mathcal{W}^{\psi}(\pi(\bar{a})v)\ d^{\times}a = \tag{3.13}$$

$$\sum_{\alpha\in k^{\times}} \int\limits_{A(k)\backslash A(\mathbb{A})} \chi^{-1}(\bar{a})|a|^s \mathcal{W}^{\psi}\left(\pi(\bar{\alpha})\pi(\bar{a})v\right)\ d^{\times}a =$$

$$\int\limits_{A(k))\backslash A(\mathbb{A})} \chi^{-1}(\bar{a})|a|^s \left[\sum_{\alpha\in k^{\times}} \mathcal{W}^{\psi}\left(\pi(\bar{\alpha})\left(\pi(\bar{a})v\right)\right)\right]\ d^{\times}a =$$

$$\int\limits_{A(k)\backslash A(\mathbb{A})} \chi^{-1}(\bar{a})|a|^s \phi_{\pi(\bar{a})v}(e)\ d^{\times}a =$$

$$\int\limits_{A(k)\backslash A(\mathbb{A})} \chi^{-1}(\bar{a})|a|^s \phi_v(\bar{a})\ d^{\times}a\ . \tag{3.14}$$

This gives the integral (3.2) for the Hecke period $d_{\chi|\cdot|^{-s}}$. We have used Fubini's theorem for $Re(s) \gg 1$, to decompose the adelic integral into the integral over a quotient space of the sum over $A(k)$ since all integrals are absolutely convergent. The resulting integral defines an analytic function for all values of s and cuspidal π.

3.4.4. *Proof of Theorem 3.3.3.* Proofs of all statements leading to and of the theorem itself are purely local, granted we already know the *direct* relation between Whittaker and Hecke periods (i.e., Theorem 3.2.3). Namely, all proofs follow from the following simple computational

Lemma. *Let $\pi_{\mathfrak{p}}$ be an irreducible unitary representation of $\mathrm{GL}_2(k_{\mathfrak{p}})$, $\chi_{\mathfrak{p}}$ a quasi-character of $A_{\mathfrak{p}}$, $\psi_{\mathfrak{p}}$ a non-trivial character on $N_{\mathfrak{p}}$, and $dn_{\mathfrak{p}}$ and $d^{\times}a_{\mathfrak{p}}$ measures on $N_{\mathfrak{p}}$ and $A_{\mathfrak{p}}$ respectively corresponding to some invariant differential forms δ_N and δ_A. We have the following identity:*

$$\mathfrak{i}(\psi_{\mathfrak{p}}, \chi_{\mathfrak{p}}, d^{\times}a_{\mathfrak{p}}) \circ \mathfrak{i}(\chi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, dn_{\mathfrak{p}}) = c(dn_{\mathfrak{p}}, d^{\times}a_{\mathfrak{p}}) \cdot id\ , \tag{3.15}$$

as an endomorphism of $P(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}}))$. Here $c(dn_{\mathfrak{p}}, d^{\times}a_{\mathfrak{p}}) \in \mathbb{C}$ is the proportionality constant between the measure $d^{\times}a_{\mathfrak{p}}dn_{\mathfrak{p}}$ on $A_{\mathfrak{p}} \times N_{\mathfrak{p}} \simeq k_{\mathfrak{p}}^{\times} \times k_{\mathfrak{p}}$ and the measure $d^{\times}ydx$, i.e., $d^{\times}a_{\mathfrak{p}}dn_{\mathfrak{p}} = c(d^{\times}a_{\mathfrak{p}}, dn_{\mathfrak{p}})d^{\times}ydx$.

Remark. In this paper we consider the construction of local maps $I_{\mathfrak{p}}$ by regularizing integrals over appropriate subgroups (following the original construction of J.-L. Waldspurger [Wa]). In certain cases one can construct such maps in both directions, i.e., maps $I_{\mathfrak{p}}(H_{1,\mathfrak{p}}, H_{2,\mathfrak{p}}) : \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \sigma_{\mathfrak{p}}) \to \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \tau_{\mathfrak{p}})$ and $I_{\mathfrak{p}}(H_{2,\mathfrak{p}}, H_{1,\mathfrak{p}}) : \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \tau_{\mathfrak{p}}) \to \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \sigma_{\mathfrak{p}})$. As was pointed out to us by Y. Sakellaridis, when $\tau_{\mathfrak{p}}$ and $\sigma_{\mathfrak{p}}$ are characters (as in examples in this paper), these maps are formally adjoint in the following sense. We follow notations from Section 1.3. To an element $\xi \in \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \sigma_{\mathfrak{p}})$ we can associate a map $r_\xi : V_{\pi_{\mathfrak{p}}} \to \mathcal{F}(H_{1,\mathfrak{p}} \setminus G)$ from the space of smooth vectors in the representation $\pi_{\mathfrak{p}}$ to the space of appropriate functions on $H_{1,\mathfrak{p}} \setminus G$ given by $v \mapsto \xi(\pi_{\mathfrak{p}}(g)v)$. Similarly for an element $\eta \in \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \tau_{\mathfrak{p}})$ we have the map $q_\eta : V_{\pi_{\mathfrak{p}}} \to \mathcal{F}(H_{2,\mathfrak{p}} \backslash G)$. The integration procedure $I_{\mathfrak{p}}(H_{1,\mathfrak{p}}, H_{2,\mathfrak{p}})$ could be described then as the map $\mathcal{I}_{\mathfrak{p}}(H_{1,\mathfrak{p}}, H_{2,\mathfrak{p}}) : \mathcal{F}(H_{1,\mathfrak{p}} \backslash G) \to \mathcal{F}(H_{2,\mathfrak{p}} \backslash G)$ given by the integral $[\mathcal{I}_{\mathfrak{p}}(H_{1,\mathfrak{p}}, H_{2,\mathfrak{p}})(\phi)](g) = \int_{H_{2,\mathfrak{p}}} \phi(hg)dh$ (we are leaving aside convergence issues). We have then $\langle \mathcal{I}_{\mathfrak{p}}(H_{1,\mathfrak{p}}, H_{2,\mathfrak{p}})(\phi), \psi \rangle_{H_{2,\mathfrak{p}} \backslash G_{\mathfrak{p}}} = \int\limits_{H_{2,\mathfrak{p}} \backslash G_{\mathfrak{p}}} \left[\int\limits_{H_{2,\mathfrak{p}}} \phi(hg)dh \right] \psi(g)dg =$ $\int\limits_{G_{\mathfrak{p}}} \phi(g)\psi(g)dg = \int\limits_{H_{1,\mathfrak{p}} \backslash G_{\mathfrak{p}}} \phi(g) \left[\int\limits_{H_{1,\mathfrak{p}}} \psi(hg)dh \right] dg = \langle \phi, \mathcal{I}_{\mathfrak{p}}(H_{2,\mathfrak{p}}, H_{1,\mathfrak{p}})(\psi), \rangle_{H_{1,\mathfrak{p}} \backslash G_{\mathfrak{p}}}$.

Under certain conditions (which are satisfied for Whittaker/Hecke cases we consider in Sections 3 and 3.3) which in [SV] are called "local unfolding", it is shown by Y. Sakellaridis and A. Venkatesh [SV] that above adjoint maps are also inverse of each other. In particular, this should imply our Lemma 3.4.4 at least for tempered representations.

3.4.5. *Proof of Lemma 3.4.4.* The proof follows from a direct computation in the Kirillov model (see Appendix B). In fact, this computation is essentially identical for all representations $\pi_{\mathfrak{p}}$ since it only involves the action of the Borel subgroup of GL(2).

Let $p_{N_{\mathfrak{p}},\psi_{\mathfrak{p}}} \in P(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}}))$ and let $dn_{\mathfrak{p}} = dx$ be the standard measure on $k_{\mathfrak{p}} \simeq N_{\mathfrak{p}}$. The measure $dn_{\mathfrak{p}}$ induces the isomorphism $\mathrm{ev}^*_{dn_{\mathfrak{p}}} : P(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}})) \to \mathrm{Hom}_{N_{\mathfrak{p}}}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\psi_{\mathfrak{p}}})$. The Whittaker functional $\mathcal{W}^{\psi_{\mathfrak{p}}} = \mathrm{ev}^*_{dn_{\mathfrak{p}}}(p_{N_{\mathfrak{p}},\psi_{\mathfrak{p}}})$ gives rise to the Kirilov model realization $k^{\mathcal{W}^{\psi_{\mathfrak{p}}}} : V_{\pi_{\mathfrak{p}}} \to \mathcal{K}^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}})$ of $\pi_{\mathfrak{p}}$. Let $\delta_{\mathfrak{p}} = d^\times a_{\mathfrak{p}} = da_{\mathfrak{p}}/|a|_{\mathfrak{p}}$ be the standard local measure on $k^\times_{\mathfrak{p}} \simeq A_{\mathfrak{p}}$ and let $\mathrm{ev}^*_{\delta_{\mathfrak{p}}} : P(V_{\pi_{\mathfrak{p}}}, L_{\chi_{\mathfrak{p}}}(A_{\mathfrak{p}})) \to \mathrm{Hom}_{A_{\mathfrak{p}}}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\psi_{\mathfrak{p}}})$ be the corresponding isomorphism.

We first compute the image $d^{\#}_{\chi_{\mathfrak{p}}}$ of the Whittaker $\mathcal{W}^{\psi_{\mathfrak{p}}}$ functional under the integration with respect to $A_{\mathfrak{p}}$. We have $d^{\#}_{\chi_{\mathfrak{p}}} := (\mathrm{ev}^*_{\delta_{\mathfrak{p}}})^{-1}(\mathfrak{i}(\chi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, dn_{\mathfrak{p}})\mathcal{W}^{\psi_{\mathfrak{p}}}) = \int_{A_{\mathfrak{p}}} \chi^{-1}_{\mathfrak{p}}(\bar{a}){\pi_{\mathfrak{p}}}^*(\bar{a})\mathcal{W}^{\psi_{\mathfrak{p}}}\delta_{\mathfrak{p}}$. In the Kirillov model we have $\mathcal{W}^{\psi_{\mathfrak{p}}}(f) = f(1)$ for $f \in \mathcal{K}^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}})$, and we have

$$d^{\#}_{\chi_{\mathfrak{p}}}(f) = \int_{A_{\mathfrak{p}}} \chi^{-1}_{\mathfrak{p}}(\bar{a}) \left[\pi_{\mathfrak{p}}(\bar{a})f(x)\right]_{x=1} \delta_{\mathfrak{p}} = \int_{k^\times_{\mathfrak{p}}} \chi^{-1}_{\mathfrak{p}}(a) f(a) d^\times a_{\mathfrak{p}} \ .$$

Hence in the Kirillov model, the functional $(\mathrm{ev}^*_{\delta_\mathfrak{p}})^{-1}(\mathfrak{i}(\chi_\mathfrak{p}, \psi_\mathfrak{p}, dn_\mathfrak{p})\mathcal{W}^{\psi_\mathfrak{p}})$ is given by the kernel $d^{\#}_{\chi_\mathfrak{p}}(x) = \chi_\mathfrak{p}^{-1}(x)$ on $k_\mathfrak{p}^\times$.

We now compute the image under the second integration with respect to $N_\mathfrak{p}$: $(\mathrm{ev}^*_{dn_\mathfrak{p}})^{-1}(\mathfrak{i}(\psi_\mathfrak{p}, \chi_\mathfrak{p}, \delta_\mathfrak{p})d^{\#}_{\chi_\mathfrak{p}})$ given by

$$(\mathrm{ev}^*_{dn_\mathfrak{p}})^{-1}(\mathfrak{i}(\psi_\mathfrak{p}, \chi_\mathfrak{p}, \delta_\mathfrak{p})d^{\#}_{\chi_\mathfrak{p}})(f) = \int_{k_\mathfrak{p}} \psi_\mathfrak{p}^{-1}(x) d^{\#}_{\chi_\mathfrak{p}}(\pi_\mathfrak{p}(n(x)f)dx =$$

$$\int_{k_\mathfrak{p}} \psi_\mathfrak{p}^{-1}(x) \left[\int_{k_\mathfrak{p}^\times} \chi_\mathfrak{p}^{-1}(y)\psi_\mathfrak{p}(xy)f(y)d^\times y\right] dx \ . \tag{3.16}$$

The inner integral is absolutely convergent for $Re(\chi_\mathfrak{p}^{-1}) \gg 1$, since functions in the Kirillov model are compactly supported on $k_\mathfrak{p}$ and have a polynomial behavior at 0.

We first compute the integral (3.16) for functions which are compactly supported on $k_\mathfrak{p}^\times$. Consider $f \in \mathcal{S}(k_\mathfrak{p}^\times)$, and assume that $f(u + y) = f(y)$ for all y and $|u|_\mathfrak{p} \leq q_\mathfrak{p}^{-N}$ for some $N \geq 0$. The inner integral in this case is zero for $|x|_\mathfrak{p} \geq q_\mathfrak{p}^{N+1}$. Hence we can take the outer integral over a big enough compact set $B_{N'} = \{|x|_\mathfrak{p} \leq q_\mathfrak{p}^{N'}\}$, $N' \geq N$. Both integrals are absolutely convergent over compact sets, and we can interchange their order. We now have

$$(\mathrm{ev}^*_{dn_\mathfrak{p}})^{-1}(\mathfrak{i}(\psi_\mathfrak{p}, \chi_\mathfrak{p}, \delta_\mathfrak{p})d^{\#}_{\chi_\mathfrak{p}})(f) = \int_{k_\mathfrak{p}^\times} \chi_\mathfrak{p}^{-1}(y) \left[\int_{B_{N'}} \psi_\mathfrak{p}((y-1)x)dx\right] f(y)d^\times y \ .$$

The inner integral is zero, unless $|y - 1|_\mathfrak{p} \leq q_\mathfrak{p}^{-N'}$, and we have $f(y) = f(1)$ under such a restriction. By possibly increasing N', we also can assume that $\chi_\mathfrak{p}(y) \equiv 1$ for $|y - 1|_\mathfrak{p} \leq q_\mathfrak{p}^{-N'}$. The integration with respect to measures dx and $d^\times y$ then give 1. Hence on the space $\mathcal{S}(k_\mathfrak{p}^\times)$, we have shown that

$$(\mathrm{ev}^*_{dn_\mathfrak{p}})^{-1}(\mathfrak{i}(\psi_\mathfrak{p}, \chi_\mathfrak{p}, \delta_\mathfrak{p})d^{\#}_{\chi_\mathfrak{p}})(f) = f(1) = \mathcal{W}^{\psi_\mathfrak{p}}(f) \ .$$

This finishes the proof of the Lemma for compactly supported functions.

For induced representations, we also have to evaluate the integral (3.16) on the space $V(\chi_1, \chi_2)$ describing the Kirillov model (see Section B.1). We claim that on the space $V(\chi_1, \chi_2)$ the integral (3.16) is also given by the evaluation at the identity. Functions spanning $V(\chi_1, \chi_2)$ are supported in $\mathcal{O}_\mathfrak{p}$ and are essentially multiplicative characters near 0. Hence we consider the integral

$$\int_{k_\mathfrak{p}} \psi_\mathfrak{p}^{-1}(x) \left[\int_{\mathcal{O}_\mathfrak{p}} \chi_\mathfrak{p}^{-1}(y)\psi_\mathfrak{p}(xy)\chi'_\mathfrak{p}(y)d^\times y\right] dx \ , \tag{3.17}$$

for a fixed character $\chi'_{\mathfrak{p}}$. For $Re(\chi_{\mathfrak{p}}^{-1}) \gg 1$, the inner integral is absolutely convergent and decay polynomially in $|x|_{\mathfrak{p}}$. In fact, for $Re(\chi_{\mathfrak{p}}^{-1}\chi'_{\mathfrak{p}}) \geq 2$, the inner integral is bounded by $|x|_{\mathfrak{p}}^{-2}$ as $|x|_{\mathfrak{p}} \to \infty$, and hence the outer integral is absolutely convergent. Hence for $Re(\chi_{\mathfrak{p}}^{-1}) \gg 1$, the functional defined by the integral (3.16) extends to the space $V(\chi_1, \chi_2)$, and defines a functional on the whole space $\mathcal{W}^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}})$. On the other hand, the double integral (3.16) is clearly $(N_{\mathfrak{p}}, \psi_{\mathfrak{p}})$-equivariant. The space of such functionals is one-dimensional with a basis consisting of the Whittaker functional $\mathcal{W}^{\psi_{\mathfrak{p}}}$ we started with. Hence, for for $Re(\chi_{\mathfrak{p}}^{-1}) \gg 1$, the integral (3.16) coincides with the Whittaker functional $\mathcal{W}^{\psi_{\mathfrak{p}}}$ for induced representations as well.

We proved the statement (3.15) of the Lemma for $Re(\chi_{\mathfrak{p}}^{-1}) \gg 1$. Since the family of maps $\mathfrak{i}(\chi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, dn_{\mathfrak{p}})$ is a meromorphic family of maps by [JL], the identity (3.15) holds for all χ_p, and in fact provides the meromorphic continuation of the family of maps $\mathfrak{i}(\psi_{\mathfrak{p}}, \chi_{\mathfrak{p}}, d^{\times}a_{\mathfrak{p}})$.

Change of measures on $N_{\mathfrak{p}}$ and $A_{\mathfrak{p}}$ gives rise to the scaling factor $c(d^{\times}a_{\mathfrak{p}}, dn_{\mathfrak{p}})$.

4. Opposite Whittaker periods

We now consider two unipotent periods for GL(2), that is, two Whittaker functionals: one on $N^+ = \{\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)\}$ and another on $N^- = \{\left(\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right)\}$.

4.1. **Local map.** Let $\pi_{\mathfrak{p}}$ be an irreducible representation of G and $\psi_{\mathfrak{p}} : k_{\mathfrak{p}} \to \mathbb{C}$ be a non-trivial character. Consider the local Whittaker period space $P_{\mathfrak{p}}^-(\pi_{\mathfrak{p}}, L_{\bar{\psi}_{\mathfrak{p}}}^-) = \mathrm{Hom}_{N_{\mathfrak{p}}^-}(V_{\pi_{\mathfrak{p}}}, L_{\bar{\psi}_{\mathfrak{p}}}(N_{\mathfrak{p}}^-))$. Choose an invariant differential form δ_- on N^- and let $dn_{\mathfrak{p}}^- = dn_{\mathfrak{p}}^-(\delta^-)$ be the corresponding invariant measure on N^-. We denote by $\mathrm{ev}^*_{dn_{\mathfrak{p}}^-} : P^-(V_{\pi_{\mathfrak{p}}}, L_{\bar{\psi}_{\mathfrak{p}}}(N_{\mathfrak{p}}^-)) \to \mathrm{Hom}_{N_{\mathfrak{p}}^-}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\bar{\psi}_{\mathfrak{p}}})$ the induced isomorphism. We now construct a map

$$\mathfrak{i}({\psi_{\mathfrak{p}}}^+, {\bar{\psi}_{\mathfrak{p}}}^-, dn_{\mathfrak{p}}^-) : P^-(V_{\pi_{\mathfrak{p}}}, L_{\bar{\psi}_{\mathfrak{p}}}^-) \to P^+(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}}^+) \ , \tag{4.1}$$

given by the integration as in previous cases. Namely, for a vector $v \in V_{\pi_{\mathfrak{p}}}$ and a map $p_{\bar{\psi}_{\mathfrak{p}}}^- \in P^-(V_{\pi_{\mathfrak{p}}}, \bar{\psi}_{\mathfrak{p}})$, we consider the (matrix coefficient) function given by $\gamma_{p_{\bar{\psi}_{\mathfrak{p}}}^-, v}(n_{\mathfrak{p}}^+) = \mathrm{ev}^*_{dn_{\mathfrak{p}}^-}(p_{\bar{\psi}_{\mathfrak{p}}}^-(\pi_{\mathfrak{p}}(n_{\mathfrak{p}}^+)v)) \in C^{\infty}(N_{\mathfrak{p}}^+)$ and take its image in $L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}}^+)$ under the integration map.

Proposition.

(1) *The map*

$$\mathfrak{i}({\psi_{\mathfrak{p}}}^+, {\bar{\psi}_{\mathfrak{p}}}^-, dn_{\mathfrak{p}}^-) : P^-(V_{\pi_{\mathfrak{p}}}, L_{\bar{\psi}_{\mathfrak{p}}}^-) \to P^+(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}}^+) \ , \tag{4.2}$$

is well-defined.

(2) *For unramified $\pi_{\mathfrak{p}}$, $\psi_{\mathfrak{p}}$ and δ_-, we have*

$$\mathfrak{i}({\psi_{\mathfrak{p}}}^+, \bar{\psi_{\mathfrak{p}}}^-, dn_{\mathfrak{p}}^-)\xi_0^- = \lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}) \cdot \xi_0^+ \ , \tag{4.3}$$

where $\xi_0^\pm$ are unramified and the constant $\lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}) \in \mathbb{C}$ is given by

$$\lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}) = 1 - \mathrm{tr}(Ad(\sigma(\pi_{\mathfrak{p}}))q_{\mathfrak{p}}^{-1} \ , \tag{4.4}$$

with $\sigma(\pi_{\mathfrak{p}})$ is the Satake parameter of $\pi_{\mathfrak{p}}$ and Ad is the adjoint representation of the dual group of G.

4.2. **Global map.** Let now $\pi = \otimes_{\mathfrak{p}}\pi_{\mathfrak{p}}$ be an automorphic cuspidal representation of G and $\psi = \otimes_{\mathfrak{p}}\psi_{\mathfrak{p}}$ be a global non-trivial character. We also choose a non-zero invariant differential form δ^- on N^-. In order to construct the global map out of local maps $\mathfrak{i}({\psi_{\mathfrak{p}}}^+, \bar{\psi_{\mathfrak{p}}}^-, dn_{\mathfrak{p}}^-)$, we need to glue constants $\{\lambda_{\mathfrak{p}}\}$. The product $\prod_{\mathfrak{p}} \lambda_{\mathfrak{p}}$ is not absolutely convergent. However, due to the unramified computation (4.4) one has the natural regularization procedure. This is based on the use of the adjoint L-function of π. For an unramified $\pi_{\mathfrak{p}}$, let $L(s, \pi_{\mathfrak{p}}, Ad)$ be the local adjoint L-function and for a finite set S of primes, including primes where π is ramified, let $L_S(s, \pi, Ad)$ be the partial adjoint L-function of π.

Proposition. *The Euler product*

$$\prod_{\mathfrak{p}\ \text{unramified}} L(1, \pi_{\mathfrak{p}}, Ad) \cdot \mathfrak{i}({\psi_{\mathfrak{p}}}^+, \bar{\psi_{\mathfrak{p}}}^-, dn_{\mathfrak{p}}^-) \tag{4.5}$$

is absolutely convergent.

On the basis of this proposition, we consider for a large enough set S, the following absolutely convergent Euler product

$$\mathfrak{i}(\psi^+, \bar{\psi}^-, \delta^-) = \tag{4.6}$$
$$L_S(1, \pi, Ad)^{-1} \prod_{\mathfrak{p}\in S} \mathfrak{i}({\psi_{\mathfrak{p}}}^+, \bar{\psi_{\mathfrak{p}}}^-, dn_{\mathfrak{p}}^-) \quad \prod_{\mathfrak{p}\notin S} L(1, \pi_{\mathfrak{p}}, Ad) \cdot \mathfrak{i}({\psi_{\mathfrak{p}}}^+, \bar{\psi_{\mathfrak{p}}}^-, dn_{\mathfrak{p}}^-) \ .$$

The resulting map clearly does not depend on a set S if it is large enough.

Theorem. *The resulting map*

$$\mathfrak{i}(\psi^+, \bar{\psi}^-) = \mathfrak{i}(\psi^+, \bar{\psi}^-, \delta^-) : P^-(V_\pi, L_{\bar{\psi}}^-) \to P^+(V_\pi, L_{\psi}^+) \ . \tag{4.7}$$

is well-defined and does not depend on the choice of the form δ^-.

4.3. **Action on automorphic periods. The invariant.** We now consider the action of the map $\mathfrak{i}(\psi^+, \bar{\psi}^-)$ on automorphic periods. According to the Theorem 4.2, there exists a constant $\lambda(\pi, \psi) \in \mathbb{C}$ such that

$$\mathfrak{i}(\psi^+, \bar{\psi}^-)(\mathcal{W}^{\psi^-}) = \lambda(\pi, \psi) \cdot \mathcal{W}^{\psi^+} \ . \tag{4.8}$$

We call it the *period invariant* associated to the map $\mathfrak{i}(\psi^+, \bar{\psi}^-)$. The constant $\lambda(\pi, \psi)$ gives rise to a *global invariant* of π (and ψ) depending only on its automorphic realization (or even only on the isomorphism class of π if it appears with the multiplicity one in the automorphic space, as is the case for $GL(2)$).

The constant $\lambda(\pi, \psi)$ measures to what extent the integration map $\mathfrak{i}(\psi^+, \bar{\psi}^-)$ fails to be *coherent* (e.g., $\lambda(\pi, \psi) = 1$ if it is coherent, as was the case for maps considered in Section 3). In Appendix A we make a numerical evaluation of this invariant for the Ramanujan holomorphic cusp form Δ of weight 12 and level 1. In particular we will see that $\lambda(\Delta, e^{2\pi i x}) \neq 1$, i.e., in that case the corresponding map is not *coherent*.

4.3.1. *Product formula.* We claim that the invariant $\lambda(\pi, \psi) \in \mathbb{C}$ could be computed via an absolutely convergent Euler product (i.e., it has local to global representation). To write the product formula for $\lambda(\pi, \psi)$ we use the element $w = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right) \in G$. We know that w maps the automorphic period $\mathcal{W}^{\psi^-}$ to the automorphic period $\mathcal{W}^{\psi^+}$, and we also know how it acts on local period spaces.

Fix a local representation $\pi_{\mathfrak{p}}$ and a local character $\psi_{\mathfrak{p}}$. We have the isomorphism of co-invariants $c(w) : L_{\psi_{\mathfrak{p}}}(N^+_{\mathfrak{p}}) \simeq L_{\bar{\psi}_{\mathfrak{p}}}(N^-_{\mathfrak{p}})$ arising from the conjugation map $wN^+w^{-1} \to N^-$. We have the natural map:

$$m({\psi_{\mathfrak{p}}}^+, {\bar{\psi}_{\mathfrak{p}}}^-) : P^-(V_{\pi_{\mathfrak{p}}}, L^-_{\bar{\psi}_{\mathfrak{p}}}) \to P^+(V_{\pi_{\mathfrak{p}}}, L^+_{\psi_{\mathfrak{p}}}) \ , \tag{4.9}$$

given by the action of the element w, i.e., $\left[m({\psi_{\mathfrak{p}}}^+, {\bar{\psi}_{\mathfrak{p}}}^-)\xi\right](v) = c(w)(\xi(v))$ for any $v \in V_{\pi_{\mathfrak{p}}}$ and $\xi \in P^-(V_{\pi_{\mathfrak{p}}}, L^-_{\bar{\psi}_{\mathfrak{p}}})$. Hence there exists a constant $\lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}) \in \mathbb{C}$ such that

$$\mathfrak{i}({\psi_{\mathfrak{p}}}^+, {\bar{\psi}_{\mathfrak{p}}}^-, dn^-_{\mathfrak{p}})\xi = \lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}) \cdot m({\psi_{\mathfrak{p}}}^+, {\bar{\psi}_{\mathfrak{p}}}^-)\xi \ , \tag{4.10}$$

for any $\xi \in P^-(V_{\pi_{\mathfrak{p}}}, L^-_{\bar{\psi}_{\mathfrak{p}}})$. It is easy to see that for unramified $\mathfrak{p}$, these coefficients coincide with those defined in (4.3).

Theorem. *For a sufficiently large set S, the following relation holds*

$$\lambda(\pi, \psi) = L_S(1, \pi, Ad)^{-1} \prod_{\mathfrak{p} \in S} \lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}) \prod_{\mathfrak{p} \notin S} \lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}) \cdot L(1, \pi_{\mathfrak{p}}, Ad) \ , \tag{4.11}$$

where $L_S(1, \pi, Ad)$ is the (analytically continued) partial adjoint L-function.

Remark. Instead of using the element w to obtain the product formula for $\lambda(\pi,\psi)$, it is possible to use the construction from Section 3.1 to obtain another factoring of $\lambda(\pi,\psi)$ into local factors. These two representations lead to different local factors with the difference canceling out globally due to appropriate functional equation for the Hecke L-function of π.

4.4. **Proofs.**

4.4.1. *The integral.* Functions $\gamma_{p^-_{\bar\psi_\mathfrak{p}},v}$ defined in Section 4.1 are not compactly supported on $N^+_\mathfrak{p}$ and corresponding integrals should be understood in the regularized sense as follows. Denote by $\delta_+ = w^*\delta_-$ the form on N_+ and let $dn^+_\mathfrak{p}$ be the corresponding invariant measure on $N^+_\mathfrak{p}$. We denote by $\mathrm{ev}^*_{dn^+_\mathfrak{p}}$: $P^+(V_{\pi_\mathfrak{p}}, L^+_{\psi_\mathfrak{p}}) \to \mathrm{Hom}_{N^+_\mathfrak{p}}(V_{\pi_\mathfrak{p}}, \mathbb{C}_{\psi_\mathfrak{p}})$ the induced isomorphism. Hence we obtain the map

$$i({\psi_\mathfrak{p}}^+, \bar{\psi_\mathfrak{p}}^-, dn^-_\mathfrak{p}, dn^+_\mathfrak{p}) : \mathrm{Hom}_{N^-_\mathfrak{p}}(V_{\pi_\mathfrak{p}}, \mathbb{C}_{\bar\psi_\mathfrak{p}}) \to \mathrm{Hom}_{N^+_\mathfrak{p}}(V_{\pi_\mathfrak{p}}, \mathbb{C}_{\psi_\mathfrak{p}}) \ , \tag{4.12}$$

given by $i({\psi_\mathfrak{p}}^+, \bar{\psi_\mathfrak{p}}^-, dn^-_\mathfrak{p}, dn^+_\mathfrak{p}) = \mathrm{ev}^*_{dn^+_\mathfrak{p}}(\mathfrak{i}({\psi_\mathfrak{p}}^+, \bar{\psi_\mathfrak{p}}^-, dn^-_\mathfrak{p}))$ which has the following integral representation

$$\left[i({\psi_\mathfrak{p}}^+, \bar{\psi_\mathfrak{p}}^-, dn^-_\mathfrak{p}, dn^+_\mathfrak{p})p^-_{\bar\psi_\mathfrak{p}}\right](v) = \int_{N^+_\mathfrak{p}} \bar\psi_\mathfrak{p}(n^+_\mathfrak{p})\gamma_{p^-_{\bar\psi_\mathfrak{p}},v}(n^+_\mathfrak{p})dn^+_\mathfrak{p} \ . \tag{4.13}$$

The integral (4.13) does not converge absolutely, and should be understood in the following regularized sense. For a non-archimedian field $k_\mathfrak{p}$, we will understand under the integral (4.13) the limit

$$\lim_{l\to\infty}\int_{B_l} \bar\psi_\mathfrak{p}(n^+_\mathfrak{p})\gamma_{p^-_{\bar\psi_\mathfrak{p}},v}(n^+_\mathfrak{p})dn^+_\mathfrak{p} \ , \tag{4.14}$$

where $B_l = \{n(x),\ |x|_\mathfrak{p} \le q^l_\mathfrak{p}\} \subset N^+_\mathfrak{p}$. We will show that for any given smooth vector $v \in V_{\pi_\mathfrak{p}}$, the integral (4.14) stabilizes as $l \to \infty$.

For $k_\mathfrak{p} = \mathbb{R}$, we will use the asymptotic expansion of the Bessel function of π_∞ in order to regularize integral (4.13) by means of analytic continuation (this procedure could be interpreted as the analytic continuation of the integral (4.13) in the space of parameters of representations of $\mathrm{GL}_2(\mathbb{R})$).

4.4.2. *Proof of Proposition 4.1.* The proof of the proposition is based on the same idea as the proof of Lemma 3.15, i.e., we compute the local map in terms of the Bessel function.

Lemma. *The following relation holds*

$$\mathfrak{i}({\psi_\mathfrak{p}}^+, \bar{\psi_\mathfrak{p}}^-, dn^-_\mathfrak{p})\xi = j_{\pi_\mathfrak{p},\psi_\mathfrak{p}}(1)\cdot m({\psi_\mathfrak{p}}^+, \bar{\psi_\mathfrak{p}}^-)\xi \ , \tag{4.15}$$

for any $\xi \in P^-(V_{\pi_\mathfrak{p}}, L^-_{\bar{\psi}_\mathfrak{p}})$*. Here* $j_{\pi_\mathfrak{p},\psi_\mathfrak{p}}$ *is the Bessel function of the representation* $\pi_\mathfrak{p}$ *(see Appendix B) and* $m({\psi_\mathfrak{p}}^+, {\bar{\psi}_\mathfrak{p}}^-)$ *is given by the action of* w *as in* (4.9).

The lemma clearly implies that the local map is well-defined. From the lemma we see that $\lambda_\mathfrak{p}(\pi_\mathfrak{p}, \psi_\mathfrak{p}) = j_{\pi_\mathfrak{p},\psi_\mathfrak{p}}(1)$.

Proof. We fix an additive character $\psi_\mathfrak{p}$ and choose a non-zero $(N^+_\mathfrak{p}, \psi_\mathfrak{p})$-Whittaker functional $\mathcal{W}^{\psi_\mathfrak{p}}_+$ on $\pi_\mathfrak{p}$. Let $\mathcal{K}^{\psi_\mathfrak{p}}(\pi_\mathfrak{p})$ be the corresponding $(N^+_\mathfrak{p}, \psi_\mathfrak{p})$-Kirillov model of $\pi_\mathfrak{p}$. In the Kirillov model, the original Whittaker functional is then given by the delta function δ_1 at $1 \in k^\times_\mathfrak{p}$. Let $j_{\pi_\mathfrak{p},\psi_\mathfrak{p}}$ be the $\psi_\mathfrak{p}$-Bessel function of $\pi_\mathfrak{p}$. We have

$$\left[m({\bar{\psi}_\mathfrak{p}}^-, {\psi_\mathfrak{p}}^+)\delta_1\right](v) = \pi^*_\mathfrak{p}(w)\delta_1(v) = \delta_1(\pi_\mathfrak{p}(w)v) = \langle \delta_{a=1}, \int_{k^\times_\mathfrak{p}} j_{\pi_\mathfrak{p},\psi_\mathfrak{p}}(at)v(t)d^\times t\rangle \ . \tag{4.16}$$

Hence the functional $\delta_- = m({\bar{\psi}_\mathfrak{p}}^-, {\psi_\mathfrak{p}}^+)\delta_1$ is given by the kernel $j_{\pi_\mathfrak{p},\psi_\mathfrak{p}}$ in the $(N^+_\mathfrak{p}, \psi_\mathfrak{p})$-Kirillov model. We now compute

$$\left[\mathfrak{i}({\psi_\mathfrak{p}}^+, {\bar{\psi}_\mathfrak{p}}^-)\delta_-\right](v) = \int_{N^+_\mathfrak{p}} \bar{\psi}_\mathfrak{p}(x)\delta_-(\pi_\mathfrak{p}(n(x))v)dx = \int_{k_\mathfrak{p}} \bar{\psi}_\mathfrak{p}(x)\left[\int_{k^\times_\mathfrak{p}} \psi_\mathfrak{p}(xt)j_{\pi_\mathfrak{p}}(t)v(t)d^\times t\right]dx \ . \tag{4.17}$$

For $v \in \mathcal{S}(k^\times_\mathfrak{p})$, this immediately implies that $\mathfrak{i}({\psi_\mathfrak{p}}^+, {\bar{\psi}_\mathfrak{p}}^-)\delta_- = j_{\pi_\mathfrak{p}}(1)\delta_1$ as in the proof of Lemma 3.4.4. For a non-archimedian field, this finishes the proof for the space of compactly supported functions.

For induced representations over a non-archimedian field, we note that the inner integral in (4.17) is absolutely convergent. This follows from the bound (B.3) $|j_{\pi_\mathfrak{p}}(x)| \leq C_{\pi_\mathfrak{p}}|x|^{-1/4}_\mathfrak{p}$ (we assume that the central character is trivial), and the fact that functions in $V(\chi_1, \chi_2) \subset \mathcal{K}^{\psi_\mathfrak{p}}(\pi_\mathfrak{p}(\chi_1, \chi_2))$ satisfy the bound $|f(x)| \ll |x|^{1/2}_\mathfrak{p} \log|x|_\mathfrak{p}$ (both bounds hold for small enough $|x|_\mathfrak{p}$). Hence we can interchange the order of integration in (4.17) if we understand under the outer integral the limit $\lim_{N\to\infty}\int_{|x|_\mathfrak{p}\leq p^N} \dots$. For every $N \geq 1$, we consider the absolutely convergent double integral

$$\int_{|x|_\mathfrak{p}\leq q^N_\mathfrak{p}} \int_{k^\times_\mathfrak{p}} \bar{\psi}_\mathfrak{p}(x)\psi_\mathfrak{p}(xt)j_{\pi_\mathfrak{p}}(t)v(t)d^\times t dx \ .$$

Integrating now over x first, we see that for any given smooth function v, the integral stabilizes as $N \to \infty$. Hence the functional $\mathfrak{i}({\psi_\mathfrak{p}}^+, {\bar\psi_\mathfrak{p}}^-)\delta_-$ extends to the space $\mathcal{K}^{\psi_\mathfrak{p}}(\pi_\mathfrak{p})$ for induced representations as well. The uniqueness of the Whittaker functional implies again that the resulting functional is δ_1, and hence we proved that for any unitary infinite-dimensional representation $\pi_\mathfrak{p}$ of G over a non-archimedian field, the following relation holds:

$$\mathfrak{i}({\psi_\mathfrak{p}}^+, {\bar\psi_\mathfrak{p}}^-) \circ m({\bar\psi_\mathfrak{p}}^-, {\psi_\mathfrak{p}}^+)\delta_1 = \mathfrak{i}({\psi_\mathfrak{p}}^+, {\bar\psi_\mathfrak{p}}^-)\delta_- = j_{\pi_\mathfrak{p}}(1)\delta_1 \ . \tag{4.18}$$

We now prove the same statement over reals. As before we have $\delta_- = \pi(w)\delta_1 = j_{\pi_\infty,\psi_\infty}$, and we consider the integral

$$\begin{aligned}\left[\mathfrak{i}(\psi^+_\infty, \bar\psi^-_\infty)\delta_-\right](v) &= \int_{N^+_\infty} \bar\psi_\infty(x)\delta_-(\pi_\infty(n(x))v)dx = \\ &\int_\mathbb{R} \bar\psi_\infty(x)\left[\int_{\mathbb{R}^\times} \psi_\infty(xt)j_{\pi_\infty}(t)v(t)d^\times t\right]dx \ .\end{aligned} \tag{4.19}$$

For Schwartz functions $v \in \mathcal{S}(\mathbb{R}^\times)$, the integral is absolutely convergent and rapidly decaying in $|x| \to \infty$. Hence we can split the outer integral into a compact part $|x| \le N$ and the rest: $|x| > N$. The non-compact part tends to 0 as $N \to \infty$, and in the compact part we can change the order of integration. As a result, we arrive at $\left[\mathfrak{i}(\psi^+_\infty, \bar\psi^-_\infty)\delta_-\right](v) = j_{\pi_\infty,\psi_\infty}(1)\delta_1(v)$ as in the non-archimedian case. We need to show that $i(\psi^+_\infty, \bar\psi^-_\infty)\delta_-$ extends to a functional on $\mathcal{K}^{\psi_\infty}(\pi_\infty)$. The inner integral in (4.19) is absolutely convergent for all $v \in \mathcal{K}^{\psi_\infty}(\pi_\infty)$ as follows from asymptotic of Whittaker functions of smooth vectors and from asymptotic of Bessel functions (e.g., asymptotic (B.5) for the J-Bessel function). Using these asymptotic we see that the inner integral also has polynomial asymptotic expansion of the type $\sum_{i=0}^M a_i|x|^{\lambda-i} + O(|x|^{Re(\lambda)-M-1})$ as $|x| \to \infty$ where $\lambda \in \mathbb{C}$ is the parameter of the representation π_∞. Such an integral could be regularized by the analytic continuation method (see [G1]). Hence we extended the functional to the whole space $\mathcal{K}^{\psi_\infty}(\pi_\infty)$, and from the uniqueness of Whittaker functional it follows that $\mathfrak{i}(\psi^+_\infty, \bar\psi^-_\infty)\delta_- = j_{\pi_\infty,\psi_\infty}(1)\delta_1$. □

Remark. One can use the asymptotic expansion for the Bessel function obtained in [JY], Proposition 2.3, to give a proof for the Lemma for a non-archimedian field arguing as in the case of reals.

We now prove (4.3). This is a simple computation following [S], [BM1]. The Bessel function of an induced representation $\pi_\mathfrak{p}(\chi_1, \chi_2)$ is given by

$$j_{\pi_\mathfrak{p}}(1) = \lim_{N\to\infty} \int_{|x|_\mathfrak{p} \le p^N} \chi_1^{-1}\chi_2(x)\psi_\mathfrak{p}(x - x^{-1})|x|_\mathfrak{p}^{-1}dx \ . \tag{4.20}$$

The integral stabilizes as $N \to \infty$, and for unramified $\psi_{\mathfrak{p}}$, χ_i, in fact, stabilizes at $N = 1$. We obtain

$$j_{\pi_{\mathfrak{p}},\psi_{\mathfrak{p}}}(1) = 1 - q_{\mathfrak{p}}^{-1} - \chi_1^{-1}\chi_2(\varpi)q_{\mathfrak{p}}^{-1} - \chi_1\chi_2^{-1}(\varpi)q_{\mathfrak{p}}^{-1} \ . \tag{4.21}$$

4.4.3. *Proof of Proposition 4.2.* Let $\pi_{\mathfrak{p}} \simeq \pi_{\mathfrak{p}}(\chi_1, \chi_2)$ be an unramified representation. Denote by $\alpha_{\mathfrak{p}}^2 = \chi_1^{-1}\chi_2(\varpi)$. We have

$$L(1, \pi_{\mathfrak{p}}, Ad) = 1/(1 - q_{\mathfrak{p}}^{-1})(1 - \alpha_{\mathfrak{p}}^2 q_{\mathfrak{p}}^{-1})(1 - \alpha_{\mathfrak{p}}^{-2} q_{\mathfrak{p}}^{-1}) \ .$$

Using (4.4) we write $\lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}) = 1 - q_{\mathfrak{p}}^{-1} - \alpha_{\mathfrak{p}}^2 q_{\mathfrak{p}}^{-1} - \alpha_{\mathfrak{p}}^{-2} q_{\mathfrak{p}}^{-1} = L(1, \pi_{\mathfrak{p}}, Ad)^{-1} - r_{\mathfrak{p}}(\alpha_{\mathfrak{p}}, q_{\mathfrak{p}})$, where $r_{\mathfrak{p}}(\alpha_{\mathfrak{p}}, q_{\mathfrak{p}}) = q_{\mathfrak{p}}^{-2} + \alpha_{\mathfrak{p}}^2 q_{\mathfrak{p}}^{-2} + \alpha_{\mathfrak{p}}^{-2} q_{\mathfrak{p}}^{-2} - q_{\mathfrak{p}}^{-3}$. From this we deduce that

$$\begin{aligned} \lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})L(1, \pi_{\mathfrak{p}}, Ad) &= (L(1, \pi_{\mathfrak{p}}, Ad)^{-1} - r_{\mathfrak{p}}(\alpha_{\mathfrak{p}}, q_{\mathfrak{p}}))L(1, \pi_{\mathfrak{p}}, Ad) = \\ &1 - r_{\mathfrak{p}}(\alpha_{\mathfrak{p}}, q_{\mathfrak{p}})L(1, \pi_{\mathfrak{p}}, Ad) = 1 - Q_{\mathfrak{p}}(\alpha_{\mathfrak{p}}, q_{\mathfrak{p}}) \ . \end{aligned} \tag{4.22}$$

Here the term $Q_{\mathfrak{p}}(\alpha_{\mathfrak{p}}, q_{\mathfrak{p}}) = q_{\mathfrak{p}}^{-2} \frac{(1+\alpha_{\mathfrak{p}}^2+\alpha_{\mathfrak{p}}^{-2}-q_{\mathfrak{p}}^{-1})}{(1-q_{\mathfrak{p}}^{-1})(1-\alpha_{\mathfrak{p}}^2 q_{\mathfrak{p}}^{-1})(1-\alpha_{\mathfrak{p}}^{-2} q_{\mathfrak{p}}^{-1})}$ is expected to be bounded by $q_{\mathfrak{p}}^{-2+\varepsilon}$ according to the Ramanujan-Peterson conjecture, and hence this leads to an absolutely convergent Euler product. Namely, according to any non-trivial bound towards Ramanujan, there exists $\sigma > 0$ such that $|\alpha_{\mathfrak{p}}| \leq q_{\mathfrak{p}}^{\frac{1}{2}-\sigma}$. Hence $|Q_{\mathfrak{p}}(\alpha_{\mathfrak{p}}, q_{\mathfrak{p}})| \ll q_{\mathfrak{p}}^{-1-\sigma'}$ for some $\sigma' > 0$. This implies that $|\lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})L(1, \pi_{\mathfrak{p}}, Ad)| \leq 1 - q_{\mathfrak{p}}^{-1-\sigma'}$ and hence the Euler product $\prod_{\mathfrak{p}} \lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}})L(1, \pi_{\mathfrak{p}}, Ad)$ is absolutely convergent.

Remark. The procedure of regularization described in Proposition 4.2 is similar to the one used to normalize the Tamagawa measure on the non-split torus (for the split torus the corresponding L-function has a pole) and is widely used in the theory of automorphic functions (e.g., see [Wa]). We do not know why the adjoint L-function shows up in our example.

Proposition 4.2 could be formulated without mentioning L-functions explicitly, but using instead the language of maps between periods satisfying uniqueness property. The Rankin-Selberg method allows one to relate the adjoint L-function to the integration map from the diagonal period on $X_G \times X_G$ defining the invariant Hermitian form on $\pi \otimes \pi^\vee$ to the Hermitian form $\mathcal{W}^\psi \otimes \overline{\mathcal{W}}^\psi$ coming from the Whittaker functional. Hence the statement in Proposition 4.2 could be interpreted as the statement about ratio for certain maps between appropriate periods. The advantage of such a reformulation is that one does not need to know local components of π in order to construct the regularization (4.6) (i.e., one can think of the local factor $L(1, \pi_{\mathfrak{p}}, Ad)$ as a map between one-dimensional spaces of certain local period spaces). Moreover, we have more examples of regularization of period maps similar to appearing in Proposition 4.2. We hope to return to this subject elsewhere.

Remark. As was pointed out by Y. Sakellaridis, the difference between examples in Sections 3 and the example form this section could be seen in the language of [SV] as follows. Following the general setup from Section 1.3, we note that a choice of an invariant Hermitian form on $\pi_{\mathfrak{p}}$ (and on the relevant Gelfand data) gives rise to norm on the corresponding local period spaces $\mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \sigma_{\mathfrak{p}})$ and $\mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \tau_{\mathfrak{p}})$ (at least for tempered representations). Once the local map $I_{\mathfrak{p}} : \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \sigma_{\mathfrak{p}}) \to \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \tau_{\mathfrak{p}})$ is constructed, one can ask if it is unitary with respect to these norms. It is easy to see that for the Hecke and converse to Hecke cases, the map is unitary and for the case of opposite Whittaker periods it is not unitary. Hence in the latter case the map $I_{\mathfrak{p}}$ is locally non-trivial and this is the origin of the defect.

Appendix A. Computation for the Ramanujan cusp form

A.1. **Numerical evaluation.** Let $\Delta(z) = \sum_{n\geq 1} \tau(n)q^n$ be the classical cusp form, studied by Ramanujan in [Ra], with $\tau(n)$ the Ramanujan tau function. The holomorphic cusp form Δ has weight 12 and level 1. In the adelic language, Δ corresponds to a cuspidal automorphic representation $\pi_\Delta = \otimes_{p\leq\infty}\pi_p$ of $GL(2)$ over $\mathbb{Q}$ with trivial central character. The corresponding local components are unramified for all finite primes p, and π_∞ is isomorphic to the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ with the lowest weight vector of weight 12 (in [BM1] such a representation is denoted by π_6 surprisingly; see Appendix B.2 below). The Satake parameters $(\alpha_p, \alpha_p^{-1})$ of a local representation π_p at $p < \infty$ are given by $\alpha_p + \alpha_p^{-1} = \tau(p)p^{-\frac{11}{2}}$ (since the Ramanujan conjecture is known for Δ, we have $|\alpha_p + \alpha_p^{-1}| \leq 2$). Below we attempt to calculate the constant $\lambda(\pi_\Delta, \psi)$ for the additive character $\psi(x) = e^{2\pi i x}$ of $\mathbb{Q} \setminus \mathbb{A}_{\mathbb{Q}}$. We have

$$\lambda_p(\pi_p, \psi_p) = 1 - p^{-1} - \alpha_p^2 p^{-1} - \alpha_p^{-2}p^{-1} = 1 - (1 + \alpha_p^2 + \alpha_p^{-2})p^{-1} = \\ 1 - ((\alpha_p + \alpha_p^{-1})^2 - 1)p^{-1} = 1 - (\tau^2(p)p^{-11} - 1)p^{-1} \ .$$

We also have

$$L(1, \pi_p, Ad) = 1/(1 - p^{-1})(1 - \alpha_p^2 p^{-1})(1 - \alpha_p^{-2}p^{-1}) = \\ (1 - p^{-1} - \alpha_p^2 p^{-1} - \alpha_p^{-2}p^{-1} + \alpha_p^2 p^{-2} + \alpha_p^{-2}p^{-2} + p^{-2} - p^{-3})^{-1} = \\ (1 - (\tau^2(p)p^{-11} - 1)p^{-1} + (\tau^2(p)p^{-11} - 1)p^{-2} - p^{-3})^{-1} \ .$$

We compute now numerically the product

$$\tilde{\lambda}^f_N(\pi_\Delta,\psi)=\prod_{p\leq N}\lambda(\pi_p,\psi_p)L(1,\pi_p,Ad) \tag{A.1}$$

for $N\geq 1$, and $\lambda^f_N(\pi_\Delta,\psi)=\tilde{\lambda}^f_N(\pi_\Delta,\psi)/L(1,\pi,Ad)$ (we have $L(1,\pi,Ad)=0.63179294573\ldots$ according to the computation kindly provided by M. Rubinstein). We computed the numerical approximation $\lambda^f_{100}(\pi_\Delta,\psi)=1.49154\ldots$ for the first hundred primes, and the infinite counterpart $\lambda^\infty(\pi_6,\psi_\infty)$ given by the value of the classical Bessel function $j_{\pi_d,\psi_\infty}(1)=2\pi J_{11}(4\pi)=1.8305\ldots$ (see (B.4)).

Hence we obtain the following numerical approximation for $\lambda(\Delta,e^{2\pi ix})=4.32145\ldots$. In particular, one can see that it is different from 1 (since it is easy to estimate the absolutely convergent remainder).

A.2. **An infinite product.** While we do not understand the nature of the constant $\lambda(\pi,\psi)$, we would like to point out the following curious observation. According to the local unramified computation (see (4.4)), the local constant $\lambda_\mathfrak{p}(\pi_\mathfrak{p},\psi_\mathfrak{p})$ and the Euler polynomial $L(1,\pi_\mathfrak{p},Ad)$ have the same linear part. This allowed us to prove Theorem 4.2 by approximating $\lambda_\mathfrak{p}(\pi_\mathfrak{p},\psi_\mathfrak{p})$ with $L(1,\pi_\mathfrak{p},Ad)^{-1}$. We can iterate this process. Following the classical argument of T. Estermann [E] (see also [K1], [K2]), one would expect that the Euler product obtained from the product $\prod_\mathfrak{p}\lambda_\mathfrak{p}$ by substituting q^{-s} in place of q^{-1} in (4.4) will have the natural boundary at $Re(s)=0$.

We consider the polynomial $l(a,x)=1-x-ax-a^{-1}x$ in x (so that $\lambda_\mathfrak{p}(\pi_\mathfrak{p}(\alpha_\mathfrak{p},\alpha_\mathfrak{p}^{-1}))=l(\alpha_\mathfrak{p}^2,q_\mathfrak{p}^{-1})$). Let us introduce a family of polynomials

$$p_l(a,x)=\prod_{i=-l}^{l}(1-a^ix)\ ,\ l\geq 0 \tag{A.2}$$

(e.g., $p_0(a,x)=1-x$, $p_l(a,x)=(1-a^lx)(1-a^{-l}x)p_{l-1}(a,x)$). In particular, we have $L(s,\pi_\mathfrak{p}(\alpha_\mathfrak{p},\alpha_\mathfrak{p}^{-1}),Sym^{2l})=p_l(\alpha_\mathfrak{p}^2,q_\mathfrak{p}^{-s})^{-1}$ for the symmetric power L-function. From an easy inductive claim, it follows that there are integer coefficients $m_{kl}\in\mathbb{Z}$, $k,\ l\in\mathbb{Z}_+$ such that

$$l(a,x)=\prod_{k=1}^{\infty}\left[\prod_{l=0}^{k-1}p_l(a,x^k)^{m_{kl}}\right] \tag{A.3}$$

as a formal identity.

We now introduce the constant $\lambda^f(\pi,\psi)=\prod_{\mathfrak{p}<\infty}\lambda_\mathfrak{p}(\pi_\mathfrak{p},\psi_\mathfrak{p})$. Assuming coefficients m_{kl} do not grow too fast (although they do grow exponentially), the formula (A.3) suggests the following representation (at least in the unramified case):

$$\lambda^f(\pi,\psi) = \prod_{k=1}^{\infty}\left[\prod_{l=0}^{k-1} L(k,\pi,Sym^{2l})^{-m_{kl}}\right] . \tag{A.4}$$

We note that all L-functions appearing in such a product are in the region of absolute convergence (assuming the Ramanujan conjecture), except the first term which is $L(1,\pi,Ad)^{-1}$.

The sequence m_{kl} could be interpreted as a virtual representation of $\mathrm{SL}(2,\mathbb{C})\times\mathbb{G}_m$. Even some basic properties of the sequence m_{kl} are not clear to us. In particular, we do not know if these coefficients are non-negative (i.e., is it true that $m_{kl}\geq 0$; this would mean that the corresponding virtual representation is a genuine representation). Also we would like to have an estimate for the growth rate of m_{kl} in order to justify convergence of the infinite product (A.4).

Here we list the first few coefficients m_{kl} (kindly computed by S. Miller):

$k\setminus l$	0	1	2	3	4	5	6	7	8	9	10
1		1									
2		1									
3		1	1								
4		2	1	1							
5	1	3	3	2	1						
6	1	7	6	5	2	1					
7	5	13	15	12	7	3	1				
8	9	31	33	31	18	10	3	1			
9	25	67	84	74	52	29	12	4	1		
10	55	163	198	192	137	85	39	16	4	1	
11	144	383	500	483	375	240	127	55	19	5	1

Appendix B. Kirillov model

Here we collect various facts about the Kirillov model for representations of GL(2) (for more detail, see [JL], [B], [Ba], [BS], [BM1], [BM2], [S]).

B.1. **Non-archimedian Kirillov model.** Let $\pi_{\mathfrak{p}}$ be an irreducible infinite dimensional unitary representation of GL(2) over a local field $k_{\mathfrak{p}}$. We fix a non-trivial character $\psi_{\mathfrak{p}}$ of $N_{\mathfrak{p}}$, and choose a non-zero Whittaker functional $\mathcal{W}^{\psi_{\mathfrak{p}}}$ on $\pi_{\mathfrak{p}}$. Such a functional gives rise to the Kirillov model for $\pi_{\mathfrak{p}}$. Let $S^+(k_{\mathfrak{p}}^\times)$ be the space of smooth (locally constant for $\mathfrak{p}<\infty$) functions on $k_{\mathfrak{p}}^\times$ of rapid decay at infinity (relative to the completion $k_{\mathfrak{p}}^\times\subset k_{\mathfrak{p}}$ at 0). Consider the map $k^{\mathcal{W}^{\psi_{\mathfrak{p}}}}:V_{\pi_{\mathfrak{p}}}\to S^+(k_{\mathfrak{p}}^\times)$ given by $\left(k^{\mathcal{W}^{\psi_{\mathfrak{p}}}}(v)\right)(a)=\mathcal{W}^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}}(\bar{a}))$, $a\in k_{\mathfrak{p}}^\times$, for any vector $v\in V_{\pi_{\mathfrak{p}}}$ in the space of smooth vectors in $\pi_{\mathfrak{p}}$. The image $\mathcal{K}^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}})$ of this map is called the (smooth) Kirillov model of $\pi_{\mathfrak{p}}$.

We now describe the structure of $\mathcal{K}^{\psi_\mathfrak{p}}(\pi_\mathfrak{p})$ where $k_\mathfrak{p}$ is a non-archimedian local field. Let $\mathcal{S}(k_\mathfrak{p}^\times)$ be the space of Schwartz functions on $k_\mathfrak{p}^\times$ (i.e., locally constant functions of compact support on $k_\mathfrak{p}^\times$). For a supercuspidal representation $\pi_\mathfrak{p}$, we have $\mathcal{K}^{\psi_\mathfrak{p}}(\pi_\mathfrak{p}) = \mathcal{S}(k_\mathfrak{p}^\times)$. For induced representations $\pi_\mathfrak{p}(\chi_1,\chi_2)$, the space $\mathcal{K}^{\psi_\mathfrak{p}}(\pi_\mathfrak{p})$ is spent over $\mathbb{C}$ by $\mathcal{S}(k_\mathfrak{p}^\times)$ and a finite-dimensional space $V(\chi_1,\chi_2)$ with $\dim V(\chi_1,\chi_2) = 1$ or 2. One can take as a basis of $V(\chi_1,\chi_2)$ functions on $k_\mathfrak{p}^\times$ with the support in $\mathcal{O}_\mathfrak{p} \cap k_\mathfrak{p}^\times$. More precisely, for an irreducible $\pi_\mathfrak{p}(\chi_1,\chi_2)$, $V(\chi_1,\chi_2) = \mathbb{C}\text{-span}(f_1,f_2)$ with $f_i = \chi_i(t)|t|_\mathfrak{p}^{\frac{1}{2}}\chi_{\mathcal{O}_\mathfrak{p}}$ and $\chi_{\mathcal{O}_\mathfrak{p}}$ which is the characteristic function of $\mathcal{O}_\mathfrak{p}$, for $\pi_\mathfrak{p}(\chi_1,\chi_2)$ with $\chi_1=\chi_2$, $V(\chi_1,\chi_2) = \mathbb{C}\text{-span}(f_1)$ with $f_1 = \chi_1(t)|t|_\mathfrak{p}^{\frac{1}{2}}\chi_{\mathcal{O}_\mathfrak{p}}$, and for for $\pi_\mathfrak{p}(\chi_1,\chi_2)$ with $\chi_1 = \chi_2|\cdot|_\mathfrak{p}$, $V(\chi_1,\chi_2) = \mathbb{C}$-span$(f_1,f_2)$ with $f_1 = \chi_1(t)|t|_\mathfrak{p}^{\frac{1}{2}}\chi_{\mathcal{O}_\mathfrak{p}}$ and $f_2 = \chi_2(t)|t|_\mathfrak{p}^{\frac{1}{2}}\log|t|_\mathfrak{p}\chi_{\mathcal{O}_\mathfrak{p}}$

The action of $\mathrm{GL}(2,k_\mathfrak{p})$ on $\mathcal{K}^{\psi_\mathfrak{p}}(\pi_\mathfrak{p})$ can be described as follows. The action of the Borel subgroup does not depend on the representation (however, the space of smooth vectors does!), but only on its central character, and is given by:

$$\begin{aligned} &\pi_\mathfrak{p}(n(x))f(a) = \psi_\mathfrak{p}(x)f(a),\\ &\pi_\mathfrak{p}(\bar{t})f(a) = f(ta), \qquad\qquad \text{(B.1)}\\ &\pi_\mathfrak{p}(z(t))f(a) = \omega_{\pi_\mathfrak{p}}(t)f(a),\end{aligned}$$

where $a \in k^\times$ and $f \in \mathcal{K}^{\psi_\mathfrak{p}}(\pi_\mathfrak{p})$. Hence the Whittaker functional $\mathcal{W}^{\psi_\mathfrak{p}}$ we started with is given by the evaluation at $a=1$ (i.e., is given by the delta function $\delta_1(f) = f(1)$).

The action of w defines the action of G (via the Bruhat decomposition), and it is known that $\pi_\mathfrak{p}(w)$ is given by the integral transform

$$\pi_\mathfrak{p}(w)f(a) = \int_{k_\mathfrak{p}^\times} \omega_{\psi_\mathfrak{p}}^{-1}(t)j_{\pi_\mathfrak{p}}(ta)f(t)d^\times t\ , \qquad \text{(B.2)}$$

where the kernel $j_{\pi_\mathfrak{p}} = j_{\pi_\mathfrak{p},\psi_\mathfrak{p}}$ is called the Bessel function of the representation $\pi_\mathfrak{p}$. The function $j_{\pi_\mathfrak{p}}$ is a smooth function (i.e., a locally constant for non-archimedian $k_\mathfrak{p}$ and smooth for $k_\mathfrak{p}$ archimedian). We will need a non-trivial bound on $j_{\pi_\mathfrak{p}}$ near 0. We have

$$|j_{\pi_\mathfrak{p}}(x)| \le C_{\pi_\mathfrak{p}}|\omega_{\pi_\mathfrak{p}}(x)|^{-1/2}|x|_\mathfrak{p}^{-1/4} \qquad \text{(B.3)}$$

for $|x|_\mathfrak{p} \le 1$. This is proved in [Ba], Corollary 4.2 (see also [JY] for the essential computation of the germ of the corresponding orbital integral).

B.2. **Kirillov model for** $\mathrm{GL}_2(\mathbb{R})$**.** We recall here the structure of the Kirillov model for unitary representations of $\mathrm{GL}_2(\mathbb{R})$. The results we quote are discussed at length (and proved) in [BM2] from where we borrow notations as well. We will cover only representations with the *trivial* center character.

Let $\eta \in \{0,1\}$ and $s \in \mathbb{C}$, $Re(s) \geq 0$. Let $\Pi_{\eta,s}$ be the (induced) representation of G in the space of smooth functions $f : G \to \mathbb{C}$ satisfying $f(n(x)\bar{a}z(b)h) = sign^{\eta}(a)|a|^{1/2+s}f(h)$. For $s \neq d - \frac{1}{2}$ where d is a positive integer, the representation $\Pi_{\eta,s}$ is irreducible and we denote it by $\pi_{\eta,s}$. In that case, it is a unitarazable representation for $Re(s) = 0$ (the principal series representations) and for real s satisfying $0 < s < 1/2$ (the complimentary series representations). For $s = d - \frac{1}{2}$ with $d \in \mathbb{N}$, the representation $\Pi_{\eta,s}$ has the unique irreducible subspace which we denote by π_d (suppressing η since $\pi_{1,d-1/2} \simeq \pi_{0,d-1/2}$). We note that the quotient space $W_d = V_{\Pi_{1,d-1/2}}/V_{\pi_{1,d-1/2}}$ is finite-dimensional of dimension $2d-1$ (e.g., $W_1 \simeq \mathbb{C}$). The lowest weight vector in the representation π_d has the weight $2d$. (Note that in notations of [B], Theorem 2.5.3, we have $k = 2d$.)

Let $\psi_\infty(x) = e^{2\pi i x}$. The Bessel function j_{π_d,ψ_∞} for representation of discrete series π_d, $d \in \mathbb{N}$, is given by

$$j_{\pi_d,\psi_\infty}(x) = (-1)^d 2\pi |x|^{\frac{1}{2}} J_{2d-1}(4\pi|x|^{\frac{1}{2}}) \tag{B.4}$$

for $x > 0$ and $j_{\pi_d,\psi_\infty}(x) = 0$ for $x < 0$. Here J_n is the classical J-Bessel function (see [Ma]).

For a principal series representation $\pi_{0,ir}$, $ir \in i\mathbb{R}$, we have

$$j_{\pi_{0,ir},\psi_\infty}(x) = \pi|x|^{\frac{1}{2}} \sin(\pi i r)^{-1}(J_{2ir}(4\pi|x|^{\frac{1}{2}}) - J_{-2ir}(4\pi|x|^{\frac{1}{2}}))$$

for $x > 0$, and $j_{\pi_{0,ir},\psi_\infty}(x) = \pi|x|^{\frac{1}{2}} \sin(\pi i r)^{-1}(I_{2ir}(4\pi|x|^{\frac{1}{2}}) - I_{-2ir}(4\pi|x|^{\frac{1}{2}}))$ for $x < 0$. Analogous formulas are known for representations of complimentary series (e.g., see [BM2]).

We note that the classical Bessel function has a well-developed theory of asymptotic behavior (see [Ma]). In particular, the classical J-Bessel function satisfies for $0 < z \ll 1$,

$$J_\alpha(z) = \sum_{m=0}^{N} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{z}{2}\right)^{2m+\alpha} + O(|z|^{2N+\alpha+1}) \ , \tag{B.5}$$

and $J_\alpha(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)$ for $|z| \to \infty$ and $Im(z)$ bounded. As a result, we have a similar asymptotic expansion at 0 and ∞ for Bessel functions of all representations.

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