

PERIODS AND GLOBAL INVARIANTS OF AUTOMORPHIC REPRESENTATIONS

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ABSTRACT. We consider periods of automorphic representations of adèle groups defined by integrals along Gelfand subgroups. We define natural maps between local components of such periods and construct corresponding global maps using automorphic L -functions. This leads to an introduction of a global invariant of an automorphic representation arising from two such periods. We compute this invariant in some cases.

1. INTRODUCTION

1.1. Periods and relations between them. Periods play a central role in the modern theory of automorphic functions. In particular, there are many instances where periods of automorphic functions are related to L -functions. In this paper, we propose to study relations between different periods defined on the same automorphic representation. We find that in this new setup L -functions also appear naturally.

Our main aim in this paper is to highlight a relation between different periods which, we believe underlies “period to L -function” relation (in fact, the celebrated paper of J-L. Walsdpurger [Wa] that was our starting point, already contains the idea we are trying to expand). By doing so, we are able make a very natural reformulation of of some classical results and to include in our scheme periods which, apparently, do not fall in the familiar framework (see Section 4).

We explore a general scheme that allows to compare different periods and consider few (classical and not so classical) examples. We first consider relations between the Whittaker (i.e., unipotent) period and the Hecke (i.e., torus) period for $GL(2)$. One of the relations (see Theorem 3.2.3) is classical and is a mere reformulation of the treatment given by H. Jacquet and R. Langlands [JL] to the Hecke method. However, we discover a converse relation (see Theorem 3.3.3) which seems to be new (although similar local considerations

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appeared recently in [SV]). We then consider a non-classical example of two Whittaker periods on different unipotent subgroups of $\mathrm{GL}(2)$. In this case, our construction leads to an Euler product with a non-standard local factor which nevertheless could be regularized with the help of an appropriate L -function (see Section 4). This leads to an introduction of a non-trivial *global invariant* of an automorphic representation of $\mathrm{GL}(2)$.

Our motivation comes from the desire to understand the fundamental “period to L -function” relation in the theory of automorphic functions. Such a relation goes back to the foundational work of E. Hecke [He], where he constructed the Hecke L -function on $\mathrm{GL}(2)$ as the period integral along the split torus in $\mathrm{GL}(2)$. This is the most basic of “period to L -function” relations. Another striking example was discovered by J.-L. Waldspurger [Wa] and connects the period along a non-split torus in $\mathrm{GL}(2)$ to the special value of an L -function of the appropriate base change lift (i.e., a period on another group). Our starting point was an attempt to reformulate Waldspurger’s theorem in representation theoretic terms which do not require the notion of L -function.

We mention a vast generalization of the Waldspurger’s result formulated as a conjecture by B. Gross and D. Prasad [GP]. Consequently, the exact form of the Gross-Prasad period relation was conjectured by A. Ichino and T. Ikeda [II]. This led to other formulas relating normalized periods and L -functions (e.g., an analog for the Whittaker functional was considered in [LM1]). A general framework for period formulas in the context of Plancherel measures was recently proposed by Y. Sakellaridis and A. Venkatesh [SV]. We also note recent work [LM2] where similar to ours ideas discussed for much more sophisticated higher rank examples (under the name of “model transition”). However, our interest is slightly different from [LM2] as we are interested in understanding the nature of the global invariant associated to two periods.

We note that one of the most important attributes of period to L -function formulas is the presence of the multiplicity one phenomenon (i.e., the Gelfand property of one-dimensionality of certain invariant functionals; see [Gr]). This point of view was pioneered by I. Piatetski-Shapiro [PS], and also will be essential throughout this paper.

1.2. Action on periods. We are interested in the following setup. Let G be an algebraic (reductive) group over a global field k (in practice a reader can assume $k = \mathbb{Q}$ for simplicity), and let $H_1, H_2 \subset G$ be two algebraic subgroups of G also defined over k (e.g., a split over k torus and an associated unipotent subgroup in $G = \mathrm{GL}(2)$). Let $G(\mathbb{A}), H_1(\mathbb{A}), H_2(\mathbb{A})$ be the corresponding adèle groups, and we denote by $X_G = G(k) \backslash G(\mathbb{A}), X_{H_1} = H_1(k) \backslash H_1(\mathbb{A}), X_{H_2} = H_2(k) \backslash H_2(\mathbb{A})$ the corresponding automorphic quotient

spaces. Let π be an automorphic representation of G (we will be vague at this point of what is required of π). We are interested in the period functional given by the integral $p_{H_1}(\phi) = \int_{X_{H_1}} \phi(h)dh$ over the $H_1(\mathbb{A})$ -orbit $X_{H_1} \subset X_G$ of an automorphic function ϕ belonging to the space of the representation π (and similarly for the period p_{H_2} for X_{H_2}). More generally, we consider periods twisted by characters $\chi_i : H_i(k) \setminus H_i(\mathbb{A}) \rightarrow \mathbb{C}$ which are given by integrals $p_{H_i, \chi_i}(\phi) = \int_{X_{H_i}} \chi_i^{-1}(h_i)\phi(h_i)dh_i$. To define such periods, one has to choose (invariant) measures on subgroups and impose certain restrictions on representation π and on spaces X_{H_i} .

Assuming that all these periods are well-defined, it is natural to ask if there is a relation between functionals p_{H_1} and p_{H_2} which are defined on the same space of smooth vectors for the automorphic representation π . Periods p_{H_1} and p_{H_2} define functionals on π , and one possibility would be to compute their correlation (i.e., the scalar product, properly understood of course). In fact it is possible in many cases (see [Gr]), but we found it a little bit easier to make another comparison in terms of the action of adelic groups. Namely, we can try to integrate the functional p_{H_1} with respect to the action of the adelic group $H_2(\mathbb{A})$. Assuming that such an operation is well-defined, we would obtain an $H_2(\mathbb{A})$ -invariant functional $\tilde{p}_{H_2} = \int_{h \in H_2(\mathbb{A})} \pi^*(h)p_{H_1} dh$ on π (i.e., $\tilde{p}_{H_2}(v) = \int_{h \in H_2(\mathbb{A})} \int_{x \in X_{H_1}} v(xh)dx dh$ for any smooth vector v in the representation π). This does not identify such a functional in general, but in the case when $H_2(\mathbb{A})$ is a Gelfand subgroup of $G(\mathbb{A})$ (i.e., the space of $H_2(\mathbb{A})$ -invariant functionals on π is at most one-dimensional), we should get a functional which is proportional to the period functional p_{H_2} . What we found is that the above mentioned ‘‘classical’’ period to L -function formulas allows one to compute the coefficient of proportionality between \tilde{p}_{H_2} and p_{H_2} in some cases. Moreover, we find the ‘‘ L -functions free’’ formulation of this relation between periods even more interesting. Such a reformulation allows us to consider cases where the relation to L -functions is not a standard one.

1.3. The construction. We will work only with periods satisfying the local uniqueness property (and hence also satisfying global uniqueness). Let $H \subset G$ be a subgroup of a group both defined over a global field k . For a place \mathfrak{p} of k , we consider local groups $H_{\mathfrak{p}} \subset G_{\mathfrak{p}}$, (i.e., groups of points over the local field $k_{\mathfrak{p}}$). Let $\pi = \hat{\otimes}_{\mathfrak{p}} \pi_{\mathfrak{p}}$ be an irreducible representation of $G(\mathbb{A})$ and $\chi = \hat{\otimes}_{\mathfrak{p}} \chi_{\mathfrak{p}}$ be a character of $H(\mathbb{A})$ (more generally, one can consider an irreducible representation of $H(\mathbb{A})$ as well). We consider the complex vector space of equivariant maps, *the periods space*, $\mathcal{P}(\pi, \chi) = \text{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C}_{\chi})$ and its local counterparts, *the local period space*, $\mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \chi_{\mathfrak{p}}) = \text{Hom}_{H_{\mathfrak{p}}}(\pi_{\mathfrak{p}}, \mathbb{C}_{\chi_{\mathfrak{p}}})$. We call a tuple $(G_{\mathfrak{p}}, \pi_{\mathfrak{p}}, H_{\mathfrak{p}}, \chi_{\mathfrak{p}})$

a local Gelfand data (or a multiplicity one tuple) if $\dim \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \chi_{\mathfrak{p}}) \leq 1$. In such a case we have $\mathcal{P}(\pi, \chi) = \hat{\otimes} \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \chi_{\mathfrak{p}})$, and the global period space is also at most one-dimensional. We call the tuple (G, π, H, χ) globally Gelfand if it is locally Gelfand at every place. We note that in practice we consider a slightly different space of H -maps from π with values in co-invariants of H (see Section 2.3). We find the language of co-invariants more appropriate when dealing with periods, and it leads to more canonical constructions.

Let (G, π, H_1, χ_1) and (G, π, H_2, χ_2) be two globally Gelfand tuples and $P(\pi, \chi_i)$ corresponding *one-dimensional* complex vector spaces. Our goal is to construct a *canonical* map

$$(1.1) \quad I : \mathcal{P}(\pi, \chi_1) \rightarrow \mathcal{P}(\pi, \chi_2)$$

between these one-dimensional vector spaces in the presence of the corresponding *automorphic* periods. We do this in two steps.

1.4. Local step. The first step is purely local. It is relatively easy to construct *local* maps $I_{\mathfrak{p}} : \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \chi_{1,\mathfrak{p}}) \rightarrow \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \chi_{2,\mathfrak{p}})$ between local spaces of periods using the integration along the subgroup $H_{2,\mathfrak{p}} \subset G_{\mathfrak{p}}$. For a given vector $\xi_{\mathfrak{p}} \in \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \chi_{1,\mathfrak{p}})$, we define a vector $I_{\mathfrak{p}}(\xi_{\mathfrak{p}}) \in \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \chi_{2,\mathfrak{p}})$ by $I_{\mathfrak{p}}(\xi_{\mathfrak{p}}) := \int_{H_{2,\mathfrak{p}}} \chi_{2,\mathfrak{p}}^{-1}(h_{\mathfrak{p}}) \pi_{\mathfrak{p}}^*(h_{\mathfrak{p}})(\xi_{\mathfrak{p}}) dh_{\mathfrak{p}}$, where $\pi_{\mathfrak{p}}^*$ denotes the dual representation of $G_{\mathfrak{p}}$ on $V_{\pi_{\mathfrak{p}}}^*$. The integral is understood in a weak sense. This means that for any smooth vector $v_{\mathfrak{p}} \in V_{\pi_{\mathfrak{p}}}$, we have $I_{\mathfrak{p}}(\xi_{\mathfrak{p}})(v_{\mathfrak{p}}) = \int_{H_{2,\mathfrak{p}}} \chi_{2,\mathfrak{p}}^{-1}(h_{\mathfrak{p}}) \xi_{\mathfrak{p}}(\pi_{\mathfrak{p}}(h_{\mathfrak{p}})v_{\mathfrak{p}}) dh_{\mathfrak{p}}$ (in a case the intersection $H_{1,\mathfrak{p}} \cap H_{2,\mathfrak{p}}$ in non-trivial one have to take integration over the appropriate quotient). The last integral might be divergent, but in many cases could be evaluated by a standard procedure (see [G1] for a general strategy based on analytic continuation and [LM3] for a compact exhaustion ubiquitous in a non-archimedean setup). We stress that local maps are assumed to be defined *canonically* for *all* \mathfrak{p} . This of course might constitute a non-trivial local problem, but it is solved in many cases.

Remark. We note that the alluded local map $I_{\mathfrak{p}}$ could be constructed (or normalized) in various ways. In particular, a very general method was envisioned by H. Jacquet by introducing the Relative Trace Formula (see [J1] for the general framework). We hope to discuss this connection elsewhere.

1.5. Global step. The next step is to “glue” local maps $I_{\mathfrak{p}}$ to form a global map I . This is a more subtle procedure. We construct the global map I by regularizing the tensor product $\otimes I_{\mathfrak{p}}$ of local maps with the help of appropriate weight factors. This is possible only for local maps which are coming from automorphic periods, and the weight factors are provided by the theory of

automorphic L -functions. The construction of the map I (in certain cases) is the main observation of the paper.

1.5.1. *Regularization by L -functions.* We propose the following procedure, having its origin in [Wa], for the regularization of the map $\otimes I_{\mathfrak{p}}$. Let ξ be a vector in $\mathcal{P}(\pi, \chi_1)$. We write it as a product $\xi = \otimes_{\mathfrak{p}} \xi_{\mathfrak{p}}$, where $\xi_{\mathfrak{p}} \in \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \chi_{1,\mathfrak{p}})$ and for almost all \mathfrak{p} we have $\xi_{\mathfrak{p}}(e_{\mathfrak{p}}^0) = 1$ for the standard vector $e_{\mathfrak{p}}^0 \in V_{\pi_{\mathfrak{p}}}$ (see Section 2.4.1 for a slightly more precise formulation in terms of co-invariants). Given a decomposable vector $v \in V_{\pi}$, we write it in a form $v = \otimes_{\mathfrak{p}} v_{\mathfrak{p}}$, where $v_{\mathfrak{p}} = e_{\mathfrak{p}}^0$ for almost all \mathfrak{p} . Now we would like to set $I(v) = \prod_{\mathfrak{p}} d_{\mathfrak{p}}(v_{\mathfrak{p}})$, where $d_{\mathfrak{p}}(v_{\mathfrak{p}}) := I_{\mathfrak{p}}(\xi_{\mathfrak{p}})(v_{\mathfrak{p}})$.

This product is usually not convergent. To regularize such an infinite product we can use the fact that outside of a finite number of places maps $d_{\mathfrak{p}}$ can be explicitly computed using an unramified computation. Unramified factors do not depend on a choice of the vector v . In many cases maps $d_{\mathfrak{p}}$ are connected to local L -factors of automorphic representations. In such cases this allows us to use the following regularization procedures.

The regularization procedure is based on a renormalization of local unramified maps $d_{\mathfrak{p}}$ by scalar factors which we call L -weights. These L -weights are constructed from Langlands L -factors.

First, assume we can find an " L -weight" consisting of an L -function, i.e., an appropriate (partial) automorphic L -function given by $L_S(s) = \prod_{\mathfrak{p} \notin S} L_{\mathfrak{p}}(s)$ for $\text{Re}(s) \gg 1$, and a complex number s_0 such that $d_{\mathfrak{p}}(e_{\mathfrak{p}}^0) = L_{\mathfrak{p}}(s_0)$ for $\mathfrak{p} \notin S$. Here S is a finite set of primes and for a fixed $\pi_{\mathfrak{p}}$, the Euler factors for all $\mathfrak{p} \notin S$ are rational functions of $q_{\mathfrak{p}}^{-s}$ given by a local Langlands L -function. Note that an existence of the factor $L_{\mathfrak{p}}(s_0)$ as above is a condition on the unramified representation $\pi_{\mathfrak{p}}$ and does not depend on the vector $v_{\mathfrak{p}}$. We define the normalized maps $d_{\mathfrak{p}}^0(v_{\mathfrak{p}}) := L_{\mathfrak{p}}(s_0)^{-1} d_{\mathfrak{p}}(v_{\mathfrak{p}})$. In particular, for an unramified \mathfrak{p} , we have $d_{\mathfrak{p}}^0(e_{\mathfrak{p}}^0) = 1$. Here we assume that $L_{\mathfrak{p}}(s_0) \neq 0$ for all but a finite number of primes \mathfrak{p} . We define now for a large enough finite set of primes S ,

$$(1.2) \quad I(v) := L_S(s_0) \prod_{\mathfrak{p} \in S} d_{\mathfrak{p}}(v_{\mathfrak{p}}) .$$

Here the set S depends on the vector $v = \otimes_{\mathfrak{p}} v_{\mathfrak{p}}$. In many cases, the complex number s_0 belongs to the region of the analytic continuation of $L(s)$, an extension to which we will take for granted.

Remark. More generally, one have to consider L -weights given by a ratio of various Langlands L -functions evaluated at possibly different points instead of just one L -function as above. We defer to Section 4.3.2 a discussion of why one might expect to see such weights in a regularization procedure.

The regularization (1.2) is in fact the standard way to define infinite Euler products in many instances arising in the theory of automorphic functions. Our new point is that it could be extended to cover more cases which are less standard for the theory of automorphic functions (although the basic argument is well-known in other situations in number theory).

Namely, we propose a slight elaboration of the above method. In Section 4 we will provide an example where a local L -function (or a ratio of L -functions) such that $d_{\mathfrak{p}}(e_{\mathfrak{p}}^0) = L_{\mathfrak{p}}(s_0)$ for $\mathfrak{p} \notin S$ does not exist. On the other hand, we can find a partial L -function (or an L -weight) and some complex number s_0 such that if we replace for almost all \mathfrak{p} , maps $d_{\mathfrak{p}}$ by $d_{\mathfrak{p}}^0 := L_{\mathfrak{p}}(s_0)^{-1}d_{\mathfrak{p}}$, then the product $\prod_{\mathfrak{p}} d_{\mathfrak{p}}^0(v_{\mathfrak{p}})$ is absolutely convergent (this condition does not depend on a specific choice of the vector $v = \otimes_{\mathfrak{p}} v_{\mathfrak{p}}$).

Hence we propose the following modified regularization procedure. For a large enough finite set of primes S (depending on a vector v), we define

$$(1.3) \quad I(v) := L_S(s_0) \prod_{\mathfrak{p} \notin S} d_{\mathfrak{p}}^0(v_{\mathfrak{p}}) \prod_{\mathfrak{p} \in S} d_{\mathfrak{p}}(v_{\mathfrak{p}}) .$$

Note that (1.3) is a generalization of (1.2) since in the former case we have $d_{\mathfrak{p}}^0(v_{\mathfrak{p}}) = 1$ for $\mathfrak{p} \notin S$, and hence can omit the middle term in the product.

In principle, there might be different regularizing factors L_S giving for the same periods a different regularized map. We expect however that this is not the case and in fact each period map could be regularized by essentially unique combination of L -functions (if a regularization exists at all which we do not claim in general).

The basic idea behind regularization (1.3) is well known since the work of T. Estermann [E] (see also [K1], [K2]) and usually does not produce interesting Euler products. Our point here is that notwithstanding this there is a *natural* construction in the theory of automorphic functions which leads to such Euler products.

1.6. Period invariant. Having constructed the map I , we can ask what is its effect on automorphic periods. Namely, we can try to compare the original period functional p_{H_2, χ_2} and the newly constructed functional $\tilde{p}_{H_2, \chi_2} = I(p_{H_1, \chi_1})$. This is the last (and the most interesting) step of the construction. The coefficient of proportionality (when defined) gives rise to a *global invariant* of the automorphic representation π (for χ_1 and χ_2 fixed). When $\tilde{p}_{H_2, \chi_2} = p_{H_2, \chi_2}$, this invariant is equal to 1, and we say that the collection $\{I_{\mathfrak{p}}\}$ of local maps (or the resulting global map I) is *coherent*, that is, the *local to global principle* is satisfied for these automorphic periods. One can show that in many

classical examples this is indeed the case, and this is equivalent to the “period to L -function” relation we mentioned above (e.g., theorems of Hecke and Waldspurger on torus periods). However, we find that sometimes the relation between \tilde{p}_{H_2, χ_2} and p_{H_2, χ_2} is more complicated and this gives rise to a non-trivial invariant. In particular, in Section 4 we provide an example of two opposite unipotent subgroups of $GL(2)$ and Whittaker functionals \tilde{p}_{H_2, χ_2} and p_{H_2, χ_2} which do *not* coincide for the cusp form associated with the Ramanujan tau function. As a result we obtain some new, as far as we know, numerical invariant canonically associated to the Ramanujan cusp form (see Appendix A).

1.7. Structure of the paper. In this paper, we start with a discussion of examples related to the classical periods considered by Hecke (and in the adelic setting by H. Jacquet and R. Langlands [JL]). We will show (see Theorem 3.2.3) how to define a procedure of integration transforming the Whittaker period (i.e., the period defining a non-trivial Fourier coefficient along the horocycle) into the Hecke period (i.e., the period against a Hecke character along the split torus of $GL(2)$). The proof we present is a simple reformulation of the standard argument of Hecke-Jacquet-Langlands and is based on the unfolding technique.

Next we prove the converse statement (i.e., we integrate the Hecke period into the Whittaker period; see Theorem 3.3.3). This case is not covered by the Hecke-Jacquet-Langlands theory and the result is new. Here our proof is purely local (combined with the direct statement for the Whittaker to Hecke transform), and does not involve an unfolding procedure. In fact, we do not know if some kind of an unfolding exists in this case.

In Section 4 we consider two unipotent periods for $GL(2)$, that is, two Whittaker functionals: one for $N^+ = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \right\}$ and another for $N^- = \left\{ \begin{pmatrix} 1 & & \\ & 1 & x \\ & & 1 \end{pmatrix} \right\}$. We consider the same question as before and discover that this example is of a completely different nature than those we discussed so far. We define the local integration procedure and show how to regularize the global map with the help of the adjoint L -function $L(1, \pi_{\mathfrak{p}}, Ad)$. In this case, the relation of the period map to special values of L -functions is more puzzling as compared to the Hecke case. We note that local coefficients $d_{\mathfrak{p}}$ appearing in this case do *not* coincide with some familiar Euler factors from the theory of L -functions; however, coefficients $d_{\mathfrak{p}}$ coincide with the *linear part* of the Euler polynomial of $L(1, \pi_{\mathfrak{p}}, Ad)$ (or, as one might say, with the leading term of $L(1, \pi_{\mathfrak{p}}, Ad)$). This will be essential for the regularization of the Euler product $\prod_{\mathfrak{p}} d_{\mathfrak{p}}$. We will show that there is natural map between Whittaker periods on different unipotent subgroups, but in general it *does not* map the automorphic period to the

automorphic period. As a result we are able to define a non-trivial invariant (a "defect" or an "obstruction") of an automorphic (cuspidal) representation of $GL(2)$. We also present a numerical computation for this invariant for the Ramanujan cusp form which indicates that the resulting invariant is not trivial (see Appendix A.1).

We would like to point out that the above phenomenon seems to be part of a pattern and not an isolated example. In several other instances, we have computed analogous local maps and found that these are connected to L -functions in a similar way (that is, coincide with linear parts of some L -functions). This should allow one to define the corresponding global maps (e.g., a map between a torus period and a *non-associated* Whittaker period). We will discuss these examples elsewhere.

We discuss in Section 4.3.3 a geometric reason why the particular L -factor $L(1, \pi_{\mathfrak{p}}, Ad)$ might be relevant for the regularization procedure for two Whittaker functionals. In Section 4.3.2 we discuss a reason why one might expect Langlands L -functions be relevant to the regularization procedure we propose.

In Section 2, we review the basic setup and in particular discuss machinery of co-invariants which we find to be convenient in our treatment of periods of automorphic representations.

In Appendix B, we collect information about the Kirillov model for $GL(2)$ which we use in proofs and for computations.

1.7.1. *Notations.* We denote the global field by k , places of k by \mathfrak{p} , the set of places of k by $\mathcal{P}(k)$, the corresponding ring of adèles by \mathbb{A} , and the group of ideles by $J_{\mathbb{A}}$. For a group \mathcal{G} defined over a global field k (e.g., over $k = \mathbb{Q}$), we denote by $\mathcal{G}(k)$ the group of k -points, by $\mathcal{G}_{\mathfrak{p}} = \mathcal{G}(k_{\mathfrak{p}})$ the group of points over a local field $k_{\mathfrak{p}}$ (e.g., over $k_{\mathfrak{p}} = \mathbb{Q}_p$ or $k_{\infty} = \mathbb{R}$) and by $\mathcal{G}_{\mathbb{A}} = \mathcal{G}(\mathbb{A})$ the group of adelic points. For a local non-archimedean field $k_{\mathfrak{p}}$, we denote by $\mathcal{O}_{\mathfrak{p}}$ the ring of integers, by $\varpi_{\mathfrak{p}}$ a generator of the maximal ideal in $\mathcal{O}_{\mathfrak{p}}$, and by $q_{\mathfrak{p}}$ the size of the residue field. We will use the letter ψ to denote additive characters (local or global), and the letter χ to denote multiplicative characters (local or global). For a quasi-character $\chi_{\mathfrak{p}} : k_{\mathfrak{p}}^{\times} \rightarrow \mathbb{C}^{\times}$ of a local field, we have the decomposition $\chi_{\mathfrak{p}} = |\chi_{\mathfrak{p}}| \cdot \tilde{\chi}_{\mathfrak{p}}$ where $\tilde{\chi}_{\mathfrak{p}}$ is unitary. We denote by $\sigma_{\chi_{\mathfrak{p}}} = Re(\chi_{\mathfrak{p}}) \in \mathbb{R}$ the real part of $\chi_{\mathfrak{p}}$ given by the relation $|\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}})| = |\varpi_{\mathfrak{p}}|^{\sigma_{\chi_{\mathfrak{p}}}}$. Similarly, for a Hecke character $\chi : k^{\times} \backslash J_{\mathbb{A}} \rightarrow \mathbb{C}^{\times}$, there exists the unique decomposition $\chi = |\chi| \cdot \tilde{\chi}$ and $|\chi| = |\cdot|^{\sigma_{\chi}}$ with $\sigma_{\chi} = Re(\chi) \in \mathbb{R}$.

We denote by $G = GL(2)$, by $Z = Z_G$ its center, by T the subgroup of diagonal matrices, by $A \subset T$ the subgroup of matrices of the form $\begin{pmatrix} a & \\ & 1 \end{pmatrix}$, and by $N = N^+$ the upper triangular matrices. We will use the following notations: $n(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$, $\bar{a} = \begin{pmatrix} a & \\ & 1 \end{pmatrix}$, $diag(a, b) = \begin{pmatrix} a & \\ & b \end{pmatrix}$, $z(a) = \begin{pmatrix} a & \\ & a \end{pmatrix}$, $w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$.

For a field F (e.g., $F = k_p$), characters $\chi : A(F) \rightarrow \mathbb{C}^\times$ are given by $\chi(\bar{a}) = \chi(a)$, $a \in F$, and hence we can identify characters of A with those of F^\times . We do this for global characters as well and hence identify Hecke characters of k with those of $A(\mathbb{A})$. We use the notion of the real part for local and global characters of A as well.

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2. MEASURES, AUTOMORPHIC REPRESENTATIONS AND PERIODS

2.1. **Invariant integration.** We review invariant measures on local and adelic groups. We make an emphasis on the language of co-invariants.

2.1.1. *Torsors.* By a torsor we mean a one-dimensional complex vector space. The name comes from the fact that if L is a torsor then $L \setminus \{0\}$ is a \mathbb{C}^\times -torsor. Torsors form a tensor category with respect to the tensor product. This category has the unit object $L_0 = \mathbb{C}$, and for every torsor L there is an inverse torsor $L^{-1} := L^*$.

2.1.2. *Moderate groups.* Let \mathcal{A} be a locally compact group. We say that \mathcal{A} is moderate if there exists a compact subgroup $K \subset \mathcal{A}$ with the following properties:

- (i) K is totally disconnected,
- (ii) The normalizer N of the group K is open in \mathcal{A} , and the quotient group N/K is a (smooth) Lie group.

We call a subgroup K with these properties a basic compact subgroup.

We will work only with moderate groups. In fact, as follows from Gleason-Yamabe's theorem (see [T], Exercise 1.6.4, [MZ], p. 182), any locally compact group of a finite topological dimension is moderate.

Proposition. *Any two basic compact subgroups K, L in a moderate group \mathcal{A} are commensurable, i.e., the group $L \cap K$ has finite index in L and in K .*

Proof. We can assume that \mathcal{A} normalizes K and L . Then the image of the group K in the Lie group \mathcal{A}/L is a compact totally discontinuous subgroup in a Lie group and hence is finite. \square

We define the space of test functions $\mathcal{S}(\mathcal{A})$ on a moderate group \mathcal{A} as follows. The space $\mathcal{S}(\mathcal{A})$ consists of complex valued functions f on \mathcal{A} such that

- (1) f has compact support;
- (2) f is left invariant with respect to some basic compact subgroup;

(3) f is a smooth function on the smooth manifold $K \setminus \mathcal{A}$.

A function f on \mathcal{A} is called smooth if in a neighborhood of any point it coincides with some test function. The algebra of smooth functions will be denoted by $C^\infty(\mathcal{A})$.

2.1.3. *Quotient spaces.* Let X be a quotient space of \mathcal{A} , i.e., $X = \mathcal{A}/\mathcal{B}$ for a closed moderate subgroup $\mathcal{B} \subset \mathcal{A}$ and X is endowed with the quotient topology. We call such a space X a moderate space.

We denote by $C^\infty(X)$ the algebra of functions that lift to smooth functions on \mathcal{A} , and we denote by $\mathcal{S}(X)$ the space of test functions on X , i.e., the space of smooth functions of compact support on X .

Proposition. *Let $\alpha : \mathcal{A} \rightarrow \mathcal{A}'$ be a morphism of moderate groups, X, X' quotients spaces of \mathcal{A} and \mathcal{A}' , and $\beta : X \rightarrow X'$ a continuous map compatible with α . Then β is smooth, i.e., $\beta^* : C^\infty(X') \rightarrow C^\infty(X)$.*

2.1.4. *Haar measure and co-invariants.* Let X be a moderate space. A Radon measure μ on X defines a functional $I_\mu : \mathcal{S}(X) \rightarrow \mathbb{C}$, i.e., $I_\mu(f) = \int_X f d\mu$ for $f \in \mathcal{S}(X)$.

For a torsor L , we can consider measures with values in L . Such a measure μ on X defines a functional $I_\mu : \mathcal{S}(X) \rightarrow L$.

Notation: We denote by $L(\mathcal{A}) = \mathcal{S}_\mathcal{A}(\mathcal{A}) := \mathcal{S}(\mathcal{A})/\langle f - a \circ f \rangle$ the space of co-invariants of the action of \mathcal{A} acting on the left on $\mathcal{S}(\mathcal{A})$ (i.e., $a \circ f(\alpha) = f(a^{-1}\alpha)$).

Theorem. *Let \mathcal{A} be a moderate group. We consider the left action of \mathcal{A} on itself.*

- (1) *The space of co-invariants $L = L(\mathcal{A})$ is a torsor. \mathcal{A} acts on L trivially on the left and with some character $\Delta_\mathcal{A}$ (the modulus character) on the right.*
- (2) *The canonical morphism $I : \mathcal{S}(\mathcal{A}) \rightarrow L$ is defined by a Radon measure $\mu_\mathcal{A}$ with values in L .*
- (3) *The morphism I and the measure $\mu_\mathcal{A}$ are canonical, and it is invariant with respect to left and right actions of \mathcal{A} on $\mathcal{S}(\mathcal{A})$.*

The theorem is essentially a reformulation of the Haar theorem. We call $\mu_\mathcal{A}$ the Haar measure of \mathcal{A} .

Remark. While the canonical map I is defined initially only on the space of test functions, it could be extended to bigger spaces. Later we will apply I also to some other classes of functions using an appropriate regularization.

We have the analogous construction for moderate quotient spaces. Let $X = \mathcal{A}/\mathcal{B}$ be a quotient space of a moderate group \mathcal{A} . Assume that there is a left \mathcal{A} -invariant measure on X . The space $L(X) = \mathcal{S}(X)_{\mathcal{A}}$ of co-invariants is then a torsor, and there exists a canonical Haar measure μ_X on X with values in $L(X)$ such that the map $I_{\mu_X} : \mathcal{S}(X) \rightarrow L(X)$ is the canonical projection.

Proposition. *We have the canonical isomorphism $L(\mathcal{A}) \simeq L(X) \otimes L(\mathcal{B})$.*

The isomorphism is given by the integration (with values in co-invariants) along fibers. In particular, for a discrete subgroup \mathcal{B} , we have the canonical isomorphism $L(X) \simeq L(\mathcal{A})$, and hence the canonical integration map

$$(2.1) \quad I_X : \mathcal{S}(X) \rightarrow L(\mathcal{A}) .$$

2.1.5. *Groups over global fields.* Let k be a global field. Let \mathcal{G} be an affine algebraic group defined over k . For every place \mathfrak{p} of k , we consider the group of points $\mathcal{G}_{\mathfrak{p}} = \mathcal{G}(k_{\mathfrak{p}})$ of \mathcal{G} over the local field $k_{\mathfrak{p}}$. We also consider the adelic group $\mathcal{G}(\mathbb{A})$. It is defined with respect to compact open subgroups $\mathcal{G}(\mathcal{O}_{\mathfrak{p}}) \subset \mathcal{G}(k_{\mathfrak{p}})$ which are well-defined for almost all \mathfrak{p} .

Let $\mathcal{V} = \{V_{\mathfrak{p}}\}_{\mathfrak{p} \in \mathcal{P}(k)}$ be a collection of complex vector spaces indexed by places of k .

Definition. An adelic structure Σ on a family \mathcal{V} is a choice of vectors $v_{\mathfrak{p}} \in V_{\mathfrak{p}}$ for almost all \mathfrak{p} (i.e., for all except finite number of places, up to a change of vectors $v_{\mathfrak{p}}$ at finitely many places).

Definition. Let Σ be an adelic structure on a family \mathcal{V} . We define the restricted tensor product space V by $V = \otimes_{\Sigma} V_{\mathfrak{p}}$.

Namely, if $S \subset \mathcal{P}(k)$ is a finite set then we define $V_S = \otimes_{\mathfrak{p} \in S} V_{\mathfrak{p}}$. If $S \subset S'$ and S is sufficiently large, the adelic structure Σ defines the canonical morphism $V_S \rightarrow V_{S'}$. Then, by the definition, $V = \otimes_{\Sigma} V_{\mathfrak{p}} = \varinjlim_S V_S$.

Remark. If all spaces $V_{\mathfrak{p}}$ are torsors and vectors $v_{\mathfrak{p}}$ are non-zero for almost all \mathfrak{p} , then $\otimes_{\Sigma} V_{\mathfrak{p}}$ is also a torsor.

Example. Let \mathcal{G} be an affine algebraic group defined over k . For all \mathfrak{p} , we have the canonical map $I_{\mathfrak{p}} : \mathcal{S}(\mathcal{G}_{\mathfrak{p}}) \rightarrow L(\mathcal{G}_{\mathfrak{p}})$.

Claim. *We have:*

- (1) *The family of torsors $L(\mathcal{G}_{\mathfrak{p}})$ has canonical adelic structure Σ_M .*
- (2) *There is canonical isomorphism $L(\mathcal{G}(\mathbb{A})) \simeq \otimes_{\Sigma_M} L(\mathcal{G}_{\mathfrak{p}})$.*

Here the canonical adelic structure Σ_M on $\{L(\mathcal{G}_{\mathfrak{p}})\}$ is obtained by taking the image $I_{\mathfrak{p}}(\chi_{K_{\mathfrak{p}}})$ of the characteristic function $\chi_{K_{\mathfrak{p}}}$ of the standard compact subgroup $K_{\mathfrak{p}} = \mathcal{G}(\mathcal{O}_{\mathfrak{p}})$ at unramified places \mathfrak{p} .

Remark. We note that in order to have the “usual” integral with respect to a measure with values in \mathbb{C} , one has to choose isomorphisms $i_{\mathfrak{p}} : \{L(\mathcal{G}_{\mathfrak{p}})\} \simeq \mathbb{C}$ for all places \mathfrak{p} , such that for almost all places, these satisfy $i_{\mathfrak{p}}(I_{\mathfrak{p}}(\chi_{K_{\mathfrak{p}}})) = 1 \in \mathbb{C}$. This is easily translated into the familiar normalization of the local Haar measure by the standard compact subgroup.

2.1.6. *Tamagawa structure.* There exists another remarkable adelic structure Σ_T for the family $\{L(\mathcal{G}_{\mathfrak{p}})\}$ proposed by T. Tamagawa [Ta] (see also [We]).

Let A be an algebraic group defined over k . We fix a left invariant top differential form δ on A defined over k . Such a choice gives rise to a measure $m(\delta_{\mathfrak{p}})$ on $A_{\mathfrak{p}}$, and in particular, defines the map $I_{m(\delta_{\mathfrak{p}})} : \mathcal{S}(A_{\mathfrak{p}}) \rightarrow \mathbb{C}$ given by the integration. Hence we obtain the isomorphism $i_{m(\delta_{\mathfrak{p}})} : L(A_{\mathfrak{p}}) \simeq \mathbb{C}$ of the torsor of co-invariants with the trivial torsor \mathbb{C} . We can now define the Tamagawa adelic structure Σ_T on the family $\{L(\mathcal{G}_{\mathfrak{p}})\}$ by choosing the vector $t_{\mathfrak{p}} = i_{m(\delta_{\mathfrak{p}})}^{-1}(1) \in L(\mathcal{G}_{\mathfrak{p}})$ for all \mathfrak{p} . We call the resulting restricted tensor product torsor $L^T(\mathcal{G}(\mathbb{A})) \simeq \otimes_{\Sigma_T} L(\mathcal{G}_{\mathfrak{p}})$ the Tamagawa torsor. Note that since non-zero vectors $t_{\mathfrak{p}}$ are specified for *all* places \mathfrak{p} , the torsor $L^T(\mathcal{G}(\mathbb{A}))$ comes with the canonical trivialization given by the “Tamagawa measure”, i.e., by the vector $\mathfrak{t} = \mathfrak{t}_{\delta} = \otimes_{\mathfrak{p}} t_{\mathfrak{p}}$. The Tamagawa structure \mathfrak{t}_{δ} does not depend on the rational class of the form δ as follows from the standard product formula.

Remark. We do not claim that torsors $L^T(\mathcal{G}(\mathbb{A}))$ and $L(\mathcal{G}(\mathbb{A}))$ are isomorphic with respect to a collection of some local isomorphisms $j_{\mathfrak{p}} : L(\mathcal{G}_{\mathfrak{p}}) \rightarrow L(\mathcal{G}_{\mathfrak{p}})$ mapping the adelic structure Σ_M to Σ_T at almost all places. If this is the case, one can integrate functions in $\mathcal{S}(\mathcal{G}(\mathbb{A}))$ with respect to the Tamagawa structure \mathfrak{t} (i.e., this gives the usual Tamagawa measure). Sometimes such an isomorphism exists and it is possible to integrate functions in $\mathcal{S}(\mathcal{G}(\mathbb{A}))$ with respect to \mathfrak{t} (e.g., for a unipotent subgroup $N \simeq k^+$), and this means that the Tamagawa construction provides a measure in the usual sense. However, in general, we cannot integrate functions in $\mathcal{S}(\mathcal{G}(\mathbb{A}))$ with respect to \mathfrak{t} since the Euler product $\prod_{\mathfrak{p} \notin S} i_{|\delta_{\mathfrak{p}}|}(I_{\mathfrak{p}}(\chi_{K_{\mathfrak{p}}}))$ is not absolutely convergent (e.g., for the torus $A \simeq k^{\times}$). This appears when two local trivializations $(L(\mathcal{G}_{\mathfrak{p}}), I_{\mathfrak{p}}(\chi_{K_{\mathfrak{p}}})) \simeq (\mathbb{C}, 1)$ and $(L(\mathcal{G}_{\mathfrak{p}}), t_{\mathfrak{p}}) \simeq (\mathbb{C}, 1)$ are not globally compatible, and one has to introduce a regularization procedure in order to obtain a measure out of the Tamagawa structure \mathfrak{t} (i.e., to construct another trivialization of $L^T(\mathcal{G}(\mathbb{A}))$).

2.1.7. *Characters.* We also consider integration twisted by characters.

Let (V_{τ}, τ) be a representation of \mathcal{A} , and $\chi : \mathcal{A} \rightarrow \mathbb{C}^{\times}$ be a character. We have the Jacquet module $J_{\chi}(\pi) = V_{\tau} / \langle v - \chi^{-1}(a)\tau(a)v \rangle$, $v \in V_{\tau}$.

Let $X = \mathcal{A}/\mathcal{B}$ be a homogenous \mathcal{A} -space. We denote by $L_{\chi}(X) = J_{\chi}(\mathcal{S}(X))$ the corresponding Jacquet module. Let us assume that on X there is an

invariant measure. We can describe this torsor as follows. Let $C(X, \chi)$ be the space of functions on X satisfying $f(ax) = \chi(a)f(x)$. This space is zero if $\chi|_{\mathcal{B}} \neq 1$, and is a torsor otherwise.

Claim. *There is a canonical isomorphism $L_\chi(X) \simeq C(X, \chi) \otimes L(X)$.*

Choice of a point $x_0 \in X$ gives a trivialization $C(X, \chi) \simeq \mathbb{C}_\chi$, and hence the isomorphism $L_\chi(X) \simeq L(X) \otimes \mathbb{C}_\chi$. Hence $L_\chi(\mathcal{A})$ is a torsor on which \mathcal{A} acts by χ on the left and by $\Delta_{\mathcal{A}}\chi^{-1}$ on the right.

The natural projection $I_\chi : \mathcal{S}(X) \rightarrow L_\chi(X)$ corresponds to the integration with some measure $\mu_{(X, \chi)}$ with values in $L_\chi(X)$.

Let \mathcal{G} be an affine algebraic group defined over k . Let χ be a character $\chi = \otimes_{\mathfrak{p}} \chi_{\mathfrak{p}}$ of $\mathcal{G}(\mathbb{A})$.

Claim. *We have the isomorphism $L_\chi(\mathcal{G}(\mathbb{A})) \simeq \otimes_{\Sigma_M} L_{\chi_{\mathfrak{p}}}(G_{\mathfrak{p}})$.*

Consider the automorphic space $X_{\mathcal{G}} = \mathcal{G}(k) \backslash \mathcal{G}(\mathbb{A})$. Let χ be a character of $\mathcal{G}(\mathbb{A})$ which is trivial on $\mathcal{G}(k)$ (i.e., $\chi : \mathcal{G}(k) \backslash \mathcal{G}(\mathbb{A}) \rightarrow \mathbb{C}$). We can trivialize the torsor $C(X_{\mathcal{G}}, \chi)$ by using the evaluation at the base point $x_0 = \{\mathcal{G}(k)\} \in X_{\mathcal{G}}$. This gives the isomorphism

$$(2.2) \quad L_\chi(X_{\mathcal{G}}) \simeq L(X_{\mathcal{G}}) \otimes \mathbb{C}_\chi \simeq L_\chi(\mathcal{G}(\mathbb{A})) \simeq \otimes_{\Sigma_M} L_{\chi_{\mathfrak{p}}}(G_{\mathfrak{p}}) .$$

As a result, we have the corresponding integration map $I_{X_{\mathcal{G}}, \chi} : \mathcal{S}(X_{\mathcal{G}}) \rightarrow L(\mathcal{G}(\mathbb{A})) \otimes \mathbb{C}_\chi$.

2.2. Automorphic representations. Let \mathcal{G} be a reductive algebraic group defined over k . Let π be an irreducible smooth representation of the adelic group $\mathcal{G}(\mathbb{A})$. We denote by V_π the space of smooth vectors of π and by ω_π the central character of π . We have decompositions $\pi = \otimes_{\mathfrak{p}} \pi_{\mathfrak{p}}$ and $V_\pi = \hat{\otimes}_{\mathfrak{p}} V_{\pi_{\mathfrak{p}}}$ into the restricted tensor product of local representations.

Let $X_{\mathcal{G}} = \mathcal{G}(k) \backslash \mathcal{G}(\mathbb{A})$ be the automorphic space. An automorphic structure on an (abstract) adelic representation π is an intertwining map $\nu : V_\pi \rightarrow \mathcal{F}(X_{\mathcal{G}})$ with the representation of $\mathcal{G}(\mathbb{A})$ in the space of functions on $X_{\mathcal{G}}$. We call a pair (π, ν) an automorphic representation. For a cuspidal (π, ν) , the image of ν belongs to the space of rapidly decreasing smooth functions on $X_{\mathcal{G}}$. For a vector $v \in V_\pi$, we will denote by $\phi_v = \nu(v) \in \mathcal{G}(\mathbb{A})$ the corresponding automorphic function.

We denote by $S(\pi)$ the set of places (including infinite places) where π is ramified (i.e., the complement to the set of unramified places \mathfrak{p} where the standard $K_{\mathfrak{p}}$ -fixed vector $e_{\mathfrak{p}}^0 \in V_{\pi_{\mathfrak{p}}}$ is specified).

2.3. Periods. Let $\mathcal{H} \subset \mathcal{G}$ be an algebraic subgroup defined over k . Denote by $X_{\mathcal{H}} = \mathcal{H}(k) \backslash \mathcal{H}(\mathbb{A}) \subset X_G$ the closed $\mathcal{H}(\mathbb{A})$ -orbit. Let $\chi : \mathcal{H}(k) \backslash \mathcal{H}(\mathbb{A}) \rightarrow \mathbb{C}$ be a character of $\mathcal{H}(\mathbb{A})$ which is trivial on $\mathcal{H}(k)$. According to (2.1), we have the integration map $I_{X_{\mathcal{H}}, \chi} : \mathcal{S}(X_{\mathcal{H}}) \rightarrow L_{\chi}(\mathcal{H}(\mathbb{A}))$. This together with the automorphic realization map ν and the restriction map $res_{X_{\mathcal{H}}} : C^{\infty}(X_G) \rightarrow C^{\infty}(X_{\mathcal{H}})$ give rise to the $\mathcal{H}(\mathbb{A})$ -equivariant period map $p_{\mathcal{H}, \chi} = I_{X_{\mathcal{H}}, \chi} \circ res_{\mathcal{O}_{\mathcal{H}}} \circ \nu : V_{\pi} \rightarrow L_{\chi}(\mathcal{H}(\mathbb{A})) \simeq \otimes_{\Sigma_M} L_{\chi}(\mathcal{H}_{\mathfrak{p}})$, where $\mathcal{H}_{\mathfrak{p}} = \mathcal{H}(k_{\mathfrak{p}})$. Formally, we need to assume that the corresponding integrals are well-defined (e.g., the orbit $X_{\mathcal{H}}$ is compact or the automorphic representation (π, ν) is cuspidal).

Definition.

- (1) The space $P(V_{\pi}, L_{\chi}(\mathcal{H}(\mathbb{A}))) := \text{Hom}_{H(\mathbb{A})}(V_{\pi}, L_{\chi}(\mathcal{H}(\mathbb{A})))$ is called the period space.
- (2) For every place \mathfrak{p} , the space $P(V_{\pi_{\mathfrak{p}}}, L_{\chi}(\mathcal{H}_{\mathfrak{p}})) := \text{Hom}_{\mathcal{H}_{\mathfrak{p}}}(V_{\pi_{\mathfrak{p}}}, L_{\chi}(\mathcal{H}_{\mathfrak{p}}))$ is called the local period space.

With the decomposition $\chi = \otimes \chi_{\mathfrak{p}}$, we have the factorization

$$P(V_{\pi}, L_{\chi}(\mathcal{H}(\mathbb{A}))) \simeq \hat{\otimes}_{\mathfrak{p}} P(V_{\pi_{\mathfrak{p}}}, L_{\chi_{\mathfrak{p}}}(\mathcal{H}_{\mathfrak{p}})) .$$

We will assume that the local period space $P(V_{\pi_{\mathfrak{p}}}, L_{\chi_{\mathfrak{p}}}(\mathcal{H}_{\mathfrak{p}}))$ is at most one-dimensional. Hence any map in the period space is factorizable, and we can choose a factorization for the automorphic period $p_{\mathcal{H}, \chi}$. To choose a factorization of the torsor $P(V_{\pi}, L_{\chi}(\mathcal{H}(\mathbb{A})))$ into a restricted tensor product, we need to choose for almost all places \mathfrak{p} , a special vector $p_{\mathfrak{p}}^0 \in P(V_{\pi_{\mathfrak{p}}}, L_{\chi_{\mathfrak{p}}}(\mathcal{H}_{\mathfrak{p}}))$. We choose it by requiring that $p_{\mathfrak{p}}^0(e_{\mathfrak{p}}^0) = I_{\mathcal{H}_{\mathfrak{p}}, \chi_{\mathfrak{p}}}(\chi_{K_{\mathcal{H}_{\mathfrak{p}}}})$ (in fact, one has to check that such a normalization is possible, i.e., that there exists a non-vanishing invariant map on the standard vector $e_{\mathfrak{p}}^0$). Then for a sufficiently large finite set $S \subset \mathcal{P}(k)$, we have $p_{\mathcal{H}, \chi} = (\otimes_{\mathfrak{p} \in S} p_{\mathfrak{p}}) \otimes (\otimes_{\mathfrak{p} \notin S} p_{\mathfrak{p}}^0)$ for some choice of local ramified components $p_{\mathfrak{p}}$ for $\mathfrak{p} \in S$.

2.4. Action on periods. We reformulate our scheme from Section 1.3 in the language of co-invariants. Let $\mathcal{H}_1, \mathcal{H}_2 \subset \mathcal{G}$ be two algebraic subgroups as above. In particular, we will assume that all local spaces satisfy the Gelfand condition of multiplicity one.

2.4.1. Local maps. Let \mathfrak{p} be a place of k , $(\pi_{\mathfrak{p}}, V_{\mathfrak{p}})$ be an irreducible smooth representation of $\mathcal{G}_{\mathfrak{p}} = \mathcal{G}(k_{\mathfrak{p}})$. Let $\chi_{i, \mathfrak{p}} : \mathcal{H}_{i, \mathfrak{p}} \rightarrow \mathbb{C}$ be a character of $\mathcal{H}_{i, \mathfrak{p}}$. We fix a non-zero invariant differential form δ on \mathcal{H}_1 defined over k . Let $\delta_{\mathfrak{p}}$ be the corresponding invariant measure on $\mathcal{H}_{1, \mathfrak{p}}$. We use the measure $\delta_{\mathfrak{p}}$ to trivialize by the map $ev_{\delta_{\mathfrak{p}}} : L_{\chi_{1, \mathfrak{p}}}(\mathcal{H}_{1, \mathfrak{p}}) \xrightarrow{\sim} \mathbb{C}_{\chi_{1, \mathfrak{p}}}$ the corresponding space of co-invariants,

and, correspondingly, we get the isomorphism (see Section 2.1.7)

$$\mathrm{ev}_{\delta_{\mathfrak{p}}}^* : P(V_{\pi_{\mathfrak{p}}}, L_{\chi_{1,\mathfrak{p}}}(\mathcal{H}_{1,\mathfrak{p}})) \xrightarrow{\sim} \mathrm{Hom}_{A_{\mathfrak{p}}}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\chi_{1,\mathfrak{p}}}) .$$

We also consider the integration map $I_{\mathcal{H}_{2,\mathfrak{p}}, \chi_{2,\mathfrak{p}}} : \mathcal{S}(\mathcal{H}_{2,\mathfrak{p}}) \rightarrow L_{\chi_{2,\mathfrak{p}}}(\mathcal{H}_{2,\mathfrak{p}})$.

We now construct the local map between local period spaces

$$(2.3) \quad \mathbf{i}(\chi_{1,\mathfrak{p}}, \chi_{2,\mathfrak{p}}, \delta_{\mathfrak{p}}) : P(V_{\pi_{\mathfrak{p}}}, L_{\chi_{1,\mathfrak{p}}}(\mathcal{H}_{1,\mathfrak{p}})) \rightarrow P(V_{\pi_{\mathfrak{p}}}, L_{\chi_{2,\mathfrak{p}}}(\mathcal{H}_{2,\mathfrak{p}}))$$

using the co-invariant map $I_{\mathcal{H}_{2,\mathfrak{p}}}$ and the evaluation map $\mathrm{ev}_{\delta_{\mathfrak{p}}}^*$. Namely, for a given local period $p_{(\mathcal{H}_{1,\mathfrak{p}}, \chi_{1,\mathfrak{p}})} \in P(V_{\pi_{\mathfrak{p}}}, L_{\chi_{1,\mathfrak{p}}}(\mathcal{H}_{1,\mathfrak{p}}))$, we consider the function $f : \mathcal{H}_{2,\mathfrak{p}} \rightarrow \mathrm{Hom}_{\mathcal{H}_{1,\mathfrak{p}}}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\psi_{\mathfrak{p}}}) \subset V_{\pi_{\mathfrak{p}}}^*$ given by $f(h_2) = \pi_{\mathfrak{p}}^*(h_2)(\mathrm{ev}_{\delta_{\mathfrak{p}}}^*(p_{(\mathcal{H}_{1,\mathfrak{p}}, \chi_{1,\mathfrak{p}})}))$ (here $\pi_{\mathfrak{p}}^*$ is the dual to $\pi_{\mathfrak{p}}$ representation). We now consider the matrix coefficient function $f_v(h_2) = \pi_{\mathfrak{p}}^*(h_2)(\mathrm{ev}_{\delta_{\mathfrak{p}}}^*(p_{(\mathcal{H}_{1,\mathfrak{p}}, \chi_{1,\mathfrak{p}})}))(v)$, $v \in V$, and apply to the function $f_v(h_2)$ on $\mathcal{H}_{2,\mathfrak{p}}$ the integration map $I_{\mathcal{H}_{2,\mathfrak{p}}, \chi_{2,\mathfrak{p}}}(f_v) \in L_{\chi_{2,\mathfrak{p}}}(\mathcal{H}_{2,\mathfrak{p}})$. Formally, the function f_v is not compactly supported and we have to make sense of the corresponding integral. This is achieved by considering appropriate regularization procedure (e.g., by the analytic continuation method).

2.4.2. *Global maps.* In order to define the global map

$$(2.4) \quad \mathbf{i}(\chi_1, \chi_2, \delta) : P(V_{\pi}, L_{\chi_1}(\mathcal{H}_1)) \rightarrow P(V_{\pi}, L_{\chi_2}(\mathcal{H}_2)) ,$$

we make sense out of the Euler product $\hat{\otimes}_{\mathfrak{p}} \mathbf{i}(\chi_{1,\mathfrak{p}}, \chi_{2,\mathfrak{p}}, \delta_{\mathfrak{p}})$ using methods we described in Section 1.3.

3. WHITTAKER AND HECKE PERIODS RELATIONS

We want to illustrate how the procedure described in Section 2.4 relates Whittaker and Hecke functionals on an automorphic cuspidal representation of $G = \mathrm{GL}(2)$.

3.1. Whittaker and Hecke periods.

3.1.1. *Whittaker period.* We fix a nontrivial additive character $\psi : \mathbb{A} \rightarrow \mathbb{C}^{\times}$ which is trivial on principal adèles. We view the character ψ as a character of the unipotent group $N(\mathbb{A})$. We consider the $N(\mathbb{A})$ -orbit $X_N = N(k) \backslash N(\mathbb{A}) \subset X_G$ and the corresponding period it induces on an automorphic representation (π, ν) of $G = \mathrm{GL}(2)$.

According to the above scheme, we view the Whittaker period $p_{(N, \psi)}$ as an element in the period space $P(V_{\pi}, L_{\psi}(N(\mathbb{A}))) = \mathrm{Hom}_{N(\mathbb{A})}(V_{\pi}, L_{\psi}(N(\mathbb{A})))$. In the factorization of the Whittaker period $p_{(N, \psi)} = \hat{\otimes}_{\mathfrak{p}} p_{(N_{\mathfrak{p}}, \psi_{\mathfrak{p}})}$ for almost all \mathfrak{p} , the local component $p_{(N_{\mathfrak{p}}, \psi_{\mathfrak{p}})} \in P(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}})) = \mathrm{Hom}_{N_{\mathfrak{p}}}(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}}))$ is unramified (i.e., maps the standard vector $e_{\mathfrak{p}}^0$ to the image in co-invariants of the characteristic function $\chi_{N(\mathcal{O}_{\mathfrak{p}})}$ of the set $N(\mathcal{O}_{\mathfrak{p}})$).

Remark. In more classical terms, the Whittaker period/functional on V_π is given by the integral

$$(3.1) \quad \mathcal{W}^\psi(v) = \int_{N(k) \backslash N(\mathbb{A})} \psi^{-1}(n) \phi_v(n) \, dn .$$

Here dn is the measure on $N(\mathbb{A})$ obtained from an invariant differential form and ϕ_v is the automorphic function corresponding to the vector $v \in V_\pi$. We have $\mathcal{W}^\psi \in \text{Hom}_{N(\mathbb{A})}(V_\pi, \mathbb{C}_\psi)$, where \mathbb{C}_ψ is the one-dimensional $N(\mathbb{A})$ -module with the action given by ψ . It is well-known that $\dim \text{Hom}_{N(\mathbb{A})}(V_\pi, \mathbb{C}_\psi) = 1$, the space of local functionals $\text{Hom}_{N_{\mathfrak{p}}}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\psi_{\mathfrak{p}}})$ is also one-dimensional, and the global space decomposes into the restricted product of local spaces. We have the following standard decomposition of the automorphic Whittaker functional \mathcal{W}^ψ into a product of local functionals. For an unramified place $\mathfrak{p} \notin S(\pi, \psi)$ (here $S(\pi, \psi)$ denotes the set of primes where π or ψ are ramified), let $\mathcal{W}_0^{\psi_{\mathfrak{p}}} \in \text{Hom}_{N_{\mathfrak{p}}}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\psi_{\mathfrak{p}}})$ be the local functional satisfying $\mathcal{W}_0^{\psi_{\mathfrak{p}}}(e_{\mathfrak{p}}^0) = 1$ for the standard $K_{\mathfrak{p}}$ -fixed vector $e_{\mathfrak{p}}^0 \in V_{\pi_{\mathfrak{p}}}$. We then choose local functionals $\widetilde{\mathcal{W}}^{\psi_{\mathfrak{p}}} \in \text{Hom}_{N_{\mathfrak{p}}}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\psi_{\mathfrak{p}}})$ for ramified primes, so that $\mathcal{W}^\psi = \otimes_{\mathfrak{p} \in S(\pi, \psi)} \widetilde{\mathcal{W}}^{\psi_{\mathfrak{p}}} \otimes_{\mathfrak{p} \notin S(\pi, \psi)} \mathcal{W}_0^{\psi_{\mathfrak{p}}}$.

3.1.2. Hecke period. We now consider the Hecke period. Let χ be a Hecke (quasi-)character of k and let $\chi : A(\mathbb{A}) \rightarrow \mathbb{C}^\times$, $\chi \left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} \right) = \chi(a)$, be the corresponding (quasi-)character of $A(\mathbb{A})$ trivial on the principal subgroup $A(k)$. We consider the orbit $X_A = A(k) \backslash A(\mathbb{A}) \subset X_G$, and the corresponding (Hecke) period d_χ it induces on a cuspidal automorphic representation (π, ν) . According to the above scheme, we can view the Hecke period as an element in the torsor of periods $P(V_\pi, L_\chi(A(\mathbb{A}))) = \text{Hom}_{A(\mathbb{A})}(V_\pi, L_\chi(A(\mathbb{A})))$. We have a factorization $p_{(A, \chi)} = \otimes_{\mathfrak{p}} p_{(A_{\mathfrak{p}}, \chi_{\mathfrak{p}})}$ where for almost all \mathfrak{p} , the local component $p_{(A_{\mathfrak{p}}, \chi_{\mathfrak{p}})} \in P(V_{\pi_{\mathfrak{p}}}, L_{\chi_{\mathfrak{p}}}(A_{\mathfrak{p}}))$ is unramified (i.e., maps the standard vector $e_{\mathfrak{p}}^0$ to the image in co-invariants of the characteristic function $\chi_{A(\mathcal{O}_{\mathfrak{p}})}$ of the set $A(\mathcal{O}_{\mathfrak{p}})$).

Remark. In more classical terms, we have the following description of the functional $d_\chi = d_\chi(\pi) : V_\pi \rightarrow \mathbb{C}_\chi$ (here \mathbb{C}_χ denotes the one-dimensional $A(\mathbb{A})$ -module with the action given by χ). We fix an invariant rational differential form on A and denote by μ the corresponding invariant measure. As an element in the space $\text{Hom}_{A(\mathbb{A})}(V_\pi, \mathbb{C}_\chi)$, the corresponding period functional is given by the integral

$$(3.2) \quad d_\chi(v) = \int_{A(k) \backslash A(\mathbb{A})} \chi^{-1}(\bar{a}) \phi_v(\bar{a}) \, \mu ,$$

for $v \in V_\pi$ and ϕ_v the corresponding automorphic function. The integral is absolutely convergent since functions in a cuspidal representation are rapidly decreasing at infinity. It is well-known that $\dim \text{Hom}_{A(\mathbb{A})}(V_\pi, \mathbb{C}_\chi) = 1$, that the local space $\text{Hom}_{A_{\mathfrak{p}}}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\chi_{\mathfrak{p}}})$ also satisfies the multiplicity one property, and hence $\text{Hom}_{A(\mathbb{A})}(V_\pi, \mathbb{C}_\chi) = \hat{\otimes}_{\mathfrak{p}} \text{Hom}_{A_{\mathfrak{p}}}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\chi_{\mathfrak{p}}})$. We again can choose a decomposition $d_\chi = \otimes_{\mathfrak{p} \in S(\pi, \chi)} \tilde{d}_{\chi_{\mathfrak{p}}} \otimes_{\mathfrak{p} \notin S(\pi, \chi)} d_{\chi_{\mathfrak{p}}}^0$ into local components with $d_{\chi_{\mathfrak{p}}}^0(e_{\mathfrak{p}}^0) = 1$. In what follows we recall Hecke-Jacquet-Langlands recipe how to choose local components consistently at all places.

3.2. Action from Whittaker to Hecke. Let us describe a morphism from the Whittaker torsor to the Hecke torsor.

3.2.1. Local map. Let \mathfrak{p} be a place of k and $(\pi_{\mathfrak{p}}, V_{\mathfrak{p}})$ be an irreducible smooth representation of $G_{\mathfrak{p}} = G(k_{\mathfrak{p}})$. Let $\psi_{\mathfrak{p}} : N_{\mathfrak{p}} \rightarrow \mathbb{C}$ be a nontrivial character of $N_{\mathfrak{p}}$ and $\chi_{\mathfrak{p}} : A_{\mathfrak{p}} \rightarrow \mathbb{C}$ be a character of $A_{\mathfrak{p}} \simeq k_{\mathfrak{p}}^\times$. We fix a (non-zero) invariant differential form δ_N on N and consider the corresponding invariant measure $dn_{\mathfrak{p}} = dn_{\mathfrak{p}}(\delta_N)$ on $N_{\mathfrak{p}}$. The measure $dn_{\mathfrak{p}}$ gives rise to the trivialization $\text{ev}_{dn_{\mathfrak{p}}} : L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}}) \xrightarrow{\sim} \mathbb{C}_{\psi_{\mathfrak{p}}}$ of the corresponding co-invariants and to the isomorphism $\text{ev}_{dn_{\mathfrak{p}}}^* : P(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}})) \xrightarrow{\sim} \text{Hom}_{N_{\mathfrak{p}}}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\psi_{\mathfrak{p}}})$. We consider the integration map $I_{A_{\mathfrak{p}}, \chi_{\mathfrak{p}}} : \mathcal{S}(A_{\mathfrak{p}}) \rightarrow L_{\chi_{\mathfrak{p}}}(A_{\mathfrak{p}})$ (see Section 2.1.7).

Following the scheme formulated in Section 2.4, we consider the local map

$$(3.3) \quad \mathbf{i}(\chi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, dn_{\mathfrak{p}}) : P(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}})) \rightarrow P(V_{\pi_{\mathfrak{p}}}, L_{\chi_{\mathfrak{p}}}(A_{\mathfrak{p}}))$$

constructed out of the co-invariant map $I_{A_{\mathfrak{p}}, \chi_{\mathfrak{p}}}$ and out of the evaluation map $\text{ev}_{dn_{\mathfrak{p}}}^*$. The map $\mathbf{i}(\chi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, dn_{\mathfrak{p}})$ could be described in the following terms. For a map $p_{(N_{\mathfrak{p}}, \psi_{\mathfrak{p}})} \in P(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}}))$ and a vector $v \in V_{\mathfrak{p}}$, we consider the (matrix coefficient) function $\alpha_v(\bar{a}_{\mathfrak{p}}) = \text{ev}_{dn_{\mathfrak{p}}}(p_{(N_{\mathfrak{p}}, \psi_{\mathfrak{p}})}(\pi_{\mathfrak{p}}(\bar{a}_{\mathfrak{p}})v)) \in C^\infty(A)$ and then take its image $I_{A_{\mathfrak{p}}, \chi_{\mathfrak{p}}}(\alpha_v) \in L_{\chi_{\mathfrak{p}}}(A_{\mathfrak{p}})$ under the integration, i.e.,

$$(3.4) \quad p_{(A_{\mathfrak{p}}, \chi_{\mathfrak{p}})}(v) = [\mathbf{i}(\chi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, dn_{\mathfrak{p}})(p_{(N_{\mathfrak{p}}, \psi_{\mathfrak{p}})})](v) := I_{A_{\mathfrak{p}}, \chi_{\mathfrak{p}}}(\alpha_v) .$$

Smooth functions $\alpha_v(\bar{a}_{\mathfrak{p}})$ with $v \in V_{\mathfrak{p}}$ obtained in such a way are not compactly supported on $A_{\mathfrak{p}}$. Hence, in fact, we have to extend the integration map $I_{A_{\mathfrak{p}}, \chi_{\mathfrak{p}}} : \mathcal{S}(A_{\mathfrak{p}}) \rightarrow L_{\chi_{\mathfrak{p}}}(A_{\mathfrak{p}})$ to such functions (i.e., to the space of matrix coefficients α_v as above for an irreducible representation $\pi_{\mathfrak{p}}$).

Proposition.

- (1) For $\text{Re}(\chi_{\mathfrak{p}}) \ll 1$, the map $\mathbf{i}(\chi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, dn_{\mathfrak{p}})$ is well-defined (by an absolutely convergent integral (3.6)). It has the meromorphic continuation to the complex space of all characters (i.e., to the complex plane of characters of the form $\chi_{\mathfrak{p}} | \cdot |_{\mathfrak{p}}^{-s}$).

(2) *For the unramified data, we obtain the Hecke-Jacquet-Langlands local L -factor. Namely,*

$$(3.5) \quad \mathbf{i}(\chi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, dn_{\mathfrak{p}})(\mathcal{W}_0^{\psi_{\mathfrak{p}}}) = L_{\mathfrak{p}}(\chi_{\mathfrak{p}}, \pi_{\mathfrak{p}}) \cdot d_{\chi_{\mathfrak{p}}}^0 .$$

The meaning of the unramified condition above is as follows (see Section 3.1): $\pi_{\mathfrak{p}}$ is an unramified representation, $\psi_{\mathfrak{p}}$ has conductor $\text{cond}(\psi_{\mathfrak{p}}) = \mathcal{O}_{\mathfrak{p}}$, $\chi_{\mathfrak{p}}$ is an unramified character, and the form δ_N comes from a rational invariant form on N which is unramified at \mathfrak{p} (and as a result satisfies $dn_{\mathfrak{p}}(\delta_N)(N(\mathcal{O}_{\mathfrak{p}})) = 1$), the unramified Whittaker functional $\mathcal{W}_0^{\psi_{\mathfrak{p}}} \in P(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}}))$ satisfies $\mathcal{W}_0^{\psi_{\mathfrak{p}}}(e_{\mathfrak{p}}^0) = l_{\psi_{\mathfrak{p}}}^0(N_{\mathfrak{p}})$ for $l_{\psi_{\mathfrak{p}}}^0(N_{\mathfrak{p}}) \in L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}})$ given by the adelic structure on $L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}})$ described in Section 2.1.5, and correspondingly for the functional $d_0^{\chi_{\mathfrak{p}}} \in P(V_{\pi_{\mathfrak{p}}}, L_{\chi_{\mathfrak{p}}}(A_{\mathfrak{p}}))$ with $d_0^{\chi_{\mathfrak{p}}}(e_{\mathfrak{p}}^0) = l_{\chi_{\mathfrak{p}}}^0(A_{\mathfrak{p}})$ and $l_{\chi_{\mathfrak{p}}}^0(A_{\mathfrak{p}}) \in L_{\chi_{\mathfrak{p}}}(A_{\mathfrak{p}})$.

Remark. We claim that the map $\mathbf{i}(\chi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, dn_{\mathfrak{p}})$ in (3.3) naturally appears, in another language, in [JL] as local zeta integrals on $\text{GL}(2)$ of Jacquet and Langlands. Let us fix a non-zero invariant local measure $d^{\times}a_{\mathfrak{p}}$ on $A_{\mathfrak{p}}$. This gives rise to isomorphisms $\text{ev}_{d^{\times}a_{\mathfrak{p}}} : L_{\chi}(A_{\mathfrak{p}}) \xrightarrow{\sim} \mathbb{C}_{\chi_{\mathfrak{p}}}$ and $\text{ev}_{d^{\times}a_{\mathfrak{p}}}^* : P(V_{\pi_{\mathfrak{p}}}, L_{\chi_{\mathfrak{p}}}(A_{\mathfrak{p}})) \xrightarrow{\sim} \text{Hom}_{A_{\mathfrak{p}}}(V_{\pi}, \mathbb{C}_{\chi_{\mathfrak{p}}})$. Using isomorphisms $\text{ev}_{dn_{\mathfrak{p}}}^*$ and $\text{ev}_{d^{\times}a_{\mathfrak{p}}}^*$, we see that the map (3.3) could be described by the following standard in $\text{GL}(2)$ theory integral:

$$(3.6) \quad [i(\chi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, dn_{\mathfrak{p}}, d^{\times}a_{\mathfrak{p}})(p_{(N_{\mathfrak{p}}, \psi_{\mathfrak{p}})})](v) := \\ [\text{ev}_{d^{\times}a_{\mathfrak{p}}}^*(p_{(A_{\mathfrak{p}}, \chi_{\mathfrak{p}})})](v) = \int_{A_{\mathfrak{p}}} \chi_{\mathfrak{p}}^{-1}(\bar{a}) l^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}}(\bar{a})v) d^{\times}a_{\mathfrak{p}} ,$$

for $v \in V_{\pi_{\mathfrak{p}}}$ and $l^{\psi_{\mathfrak{p}}} = \text{ev}_{dn_{\mathfrak{p}}}^*(p_{(N_{\mathfrak{p}}, \psi_{\mathfrak{p}})}) \in \text{Hom}_{N_{\mathfrak{p}}}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\psi_{\mathfrak{p}}})$. Translated into local zeta integrals (3.6), the relation (3.5) reads

$$[i(\chi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, dn_{\mathfrak{p}}, d_1^{\times}a_{\mathfrak{p}})(\mathcal{W}_0^{\psi_{\mathfrak{p}}})](e_{\mathfrak{p}}^0) = L_{\mathfrak{p}}(\chi_{\mathfrak{p}}, \pi_{\mathfrak{p}}) ,$$

for the measure $d_1^{\times}a_{\mathfrak{p}}$ on $A_{\mathfrak{p}}$ normalized by the condition $d_1^{\times}a_{\mathfrak{p}}(A(\mathcal{O}_{\mathfrak{p}})) = 1$.

Note that while the integral (3.6) depends on the choice of the measure $d^{\times}a_{\mathfrak{p}}$, the map (3.3) does not. Over archimedean fields, a similar approach appeared in [Po].

3.2.2. Global map. Fix an automorphic cuspidal representation (π, ν) and a non-trivial character $\psi : N(k) \setminus N(\mathbb{A}) \rightarrow \mathbb{C}$. Choose an invariant differential form δ_N on N . We want to define a map $\mathbf{i}(\chi, \psi, \delta_N) : P(V_{\pi}, L_{\psi}(N(\mathbb{A}))) \rightarrow P(V_{\pi}, L_{\chi}(A(\mathbb{A})))$ as a tensor product of local maps.

Proposition. *Fix an invariant rational differential form δ_N on N .*

- (1) The tensor product $\mathbf{i}(\chi, \psi, \delta_N) = \otimes_{\mathfrak{p}} \mathbf{i}(\chi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, dn_{\mathfrak{p}})$ is absolutely convergent for $\operatorname{Re}(\chi) \ll 1$, and has the meromorphic continuation to the complex space of all characters.
- (2) The resulting map

$$\mathbf{i}(\chi, \psi) = \mathbf{i}(\chi, \psi, \delta_N) : P(V_{\pi}, L_{\psi}(N(\mathbb{A}))) \rightarrow P(V_{\pi}, L_{\chi}(A(\mathbb{A})))$$

does not depend on the choice of the form δ_N .

3.2.3. *Action on automorphic periods.* We now come to the last step of our scheme where we compute the effect of the defined map on automorphic periods.

Theorem. *The global map $\mathbf{i}(\chi, \psi)$ is coherent, i.e., it sends the automorphic Whittaker period \mathcal{W}^{ψ} to the automorphic Hecke period d_{χ} . Namely, we have*

$$\mathbf{i}(\chi, \psi)(\mathcal{W}^{\psi}) = d_{\chi} .$$

Remarks. 1. The set of Hecke quasi-characters $\chi : A(k) \backslash A(\mathbb{A}) \rightarrow \mathbb{C}^{\times}$ has the natural structure of a complex manifold (with infinitely many connected components). For a given quasi-character χ we denote by $C_{\chi} := \{\chi_s = \chi \cdot |^s, s \in \mathbb{C}\} \simeq \mathbb{C}$ the corresponding leaf. The set $N(k) \backslash \widehat{N(\mathbb{A})}$ is discrete and we denote by pt_{ψ} the point corresponding to a character ψ . For given character $\psi : N(k) \backslash N(\mathbb{A}) \rightarrow \mathbb{C}$ and a quasi-character $\chi : A(k) \backslash A(\mathbb{A}) \rightarrow \mathbb{C}^{\times}$, consider the trivial line bundle $W_{\psi, \chi}(\pi)$ over $pt_{\psi} \times C_{\chi} \simeq C_{\chi}$, with the constant fiber $P_{\pi, \psi} = P(V_{\pi}, L_{\psi}(N(\mathbb{A})))$, and the line bundle $H_{\chi}(\pi)$ over C_{χ} with the fiber $P(V_{\pi}, L_{\chi_s}(A(\mathbb{A})))$. Both line bundles have natural holomorphic (for a cuspidal representation π) sections $S_{\psi}^{aut} = \mathcal{W}^{\psi}$ and $S_{\chi_s}^{aut} = d_{\chi}$ coming from automorphic periods (the section for the Whittaker bundle is a constant non-zero section providing a trivialization, but the automorphic Hecke section of $H_{\chi}(\pi)$ has zeroes).

One can naturally view the map $\mathbf{i}(\chi, \psi)$ as a holomorphic map between two bundles which is trivial on the base and which maps the section S_{ψ}^{aut} to the section $S_{\chi_s}^{aut}$.

2. Our treatment of the map $\mathbf{i}(\chi, \psi)$ is based on the possibility to include it in an analytic family of maps $\mathbf{i}(\chi_s, \psi)$ and use analytic continuation. One can use the single character χ and follow the recipe formulated in the second part of Section 1.3 which is applicable when there are no deformations. Namely, by looking at the local unramified computation (3.5), we see that the partial Euler product $\otimes_{\mathfrak{p} \notin S} L_{\mathfrak{p}}(\chi_{\mathfrak{p}}, \pi_{\mathfrak{p}})^{-1} \mathbf{i}(\chi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, dn_{\mathfrak{p}})$ is absolutely convergent (since all but finite number of terms are trivial). Assuming that $L_S(\chi \cdot |^s, \pi)$ have no pole at $s = 0$ (e.g., π is cuspidal), the formula (1.3) defines the map $\mathbf{i}(\chi, \psi)(v) = L_S(\chi, \pi) \prod_{\mathfrak{p} \notin S} d_{\mathfrak{p}}^0(v_{\mathfrak{p}}) \prod_{\mathfrak{p} \in S} d_{\mathfrak{p}}(v_{\mathfrak{p}})$. Note that this map might be

zero. Hence we can define the map $\mathbf{i}(\chi, \psi)$ without considering the whole family $\mathbf{i}(\chi_s, \psi)$ (of course this is a purely cosmetic change since we use the equivalent property of $L_S(\chi|\cdot|^s, \pi)$). What we can not see that way is the fact that the resulting map is coherent (i.e., Theorem 3.2.3), and in fact we will see in Section 3.3 that for other periods this does not always holds. To show that $\mathbf{i}(\chi_s, \psi)$ is coherent, we use the Hecke-Jacquet-Langlands unfolding which does not have an analog for other periods (e.g., for the map from Hecke period to Whittaker period considered in the next section).

3.3. Action from Hecke to Whittaker. Here we treat the opposite direction which inspite of similarities is not classical.

3.3.1. Local map. Let \mathfrak{p} be a place of k and $(\pi_{\mathfrak{p}}, V_{\mathfrak{p}})$ be an irreducible smooth representation of $G_{\mathfrak{p}} = G(k_{\mathfrak{p}})$. Let $\psi_{\mathfrak{p}} : N_{\mathfrak{p}} \rightarrow \mathbb{C}$ be a nontrivial character of $N_{\mathfrak{p}}$ and $\chi_{\mathfrak{p}} : A_{\mathfrak{p}} \rightarrow \mathbb{C}$ be a character of $A_{\mathfrak{p}}$. We fix a non-zero invariant differential form δ_A on A defined over k . Let $d^{\times}a_{\mathfrak{p}} = d^{\times}a_{\mathfrak{p}}(\delta_A)$ be the corresponding invariant measure on $A_{\mathfrak{p}}$. We use the measure $d^{\times}a_{\mathfrak{p}}$ to trivialize $\text{ev}_{d^{\times}a_{\mathfrak{p}}} : L_{\chi_{\mathfrak{p}}}(A_{\mathfrak{p}}) \xrightarrow{\sim} \mathbb{C}_{\chi_{\mathfrak{p}}}$ the corresponding co-invariants, and, correspondingly, we get the isomorphism $\text{ev}_{d^{\times}a_{\mathfrak{p}}}^* : P(V_{\pi_{\mathfrak{p}}}, L_{\chi_{\mathfrak{p}}}(A_{\mathfrak{p}})) \xrightarrow{\sim} \text{Hom}_{A_{\mathfrak{p}}}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\chi_{\mathfrak{p}}})$ (see Section 2.1.7). We also consider the integration map $I_{N_{\mathfrak{p}}, \psi_{\mathfrak{p}}} : \mathcal{S}(N_{\psi_{\mathfrak{p}}}) \rightarrow L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}})$.

We now construct the local map

$$(3.7) \quad \mathbf{i}(\psi_{\mathfrak{p}}, \chi_{\mathfrak{p}}, d^{\times}a_{\mathfrak{p}}) : P(V_{\pi_{\mathfrak{p}}}, L_{\chi_{\mathfrak{p}}}(A_{\mathfrak{p}})) \rightarrow P(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}}))$$

using maps $I_{N_{\mathfrak{p}}, \psi_{\mathfrak{p}}}$ and $\text{ev}_{d^{\times}a_{\mathfrak{p}}}^*$ as in the previous case. Namely, for a map $p_{(A_{\mathfrak{p}}, \chi_{\mathfrak{p}})} \in P(V_{\pi_{\mathfrak{p}}}, L_{\chi_{\mathfrak{p}}}(A_{\mathfrak{p}}))$ and a vector $v \in V_{\mathfrak{p}}$, we consider the (matrix coefficient) function $\beta_v(n_{\mathfrak{p}}) = \text{ev}_{d^{\times}a_{\mathfrak{p}}}(p_{(A_{\mathfrak{p}}, \chi_{\mathfrak{p}})}(\pi_{\mathfrak{p}}(n_{\mathfrak{p}})v)) \in C^{\infty}(N_{\mathfrak{p}})$ and then take its image $I_{N_{\mathfrak{p}}, \psi_{\mathfrak{p}}}(\beta_v) \in L_{\psi_{\mathfrak{p}}}(A_{\mathfrak{p}})$ under the integration map, i.e.,

$$(3.8) \quad p_{(N_{\mathfrak{p}}, \psi_{\mathfrak{p}})}(v) = [\mathbf{i}(\psi_{\mathfrak{p}}, \chi_{\mathfrak{p}}, d^{\times}a_{\mathfrak{p}})(p_{(A_{\mathfrak{p}}, \chi_{\mathfrak{p}})})](v) := I_{N_{\mathfrak{p}}, \psi_{\mathfrak{p}}}(\beta_v).$$

Smooth functions $\beta_v(n_{\mathfrak{p}})$ with $v \in V_{\mathfrak{p}}$ obtained in such a way are not compactly supported on $N_{\mathfrak{p}}$. Hence we have to extend the integration map $I_{N_{\mathfrak{p}}, \psi_{\mathfrak{p}}} : \mathcal{S}(N_{\mathfrak{p}}) \rightarrow L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}})$ to such functions (i.e., to the space of matrix coefficients β_v as above for an irreducible representation $\pi_{\mathfrak{p}}$). Indeed it is easy to see that such an extension exists and is unique as explained in Section 3.4.1.

Proposition. *For the unramified data, we have*

$$(3.9) \quad \mathbf{i}(\psi_{\mathfrak{p}}, \chi_{\mathfrak{p}}, d^{\times}a_{\mathfrak{p}})(d_{\chi_{\mathfrak{p}}}^0) = L_{\mathfrak{p}}(\chi_{\mathfrak{p}}, \pi_{\mathfrak{p}})^{-1} \cdot \mathcal{W}_0^{\psi_{\mathfrak{p}}}$$

with the Hecke-Jacquet-Langlands local L -factor $L_{\mathfrak{p}}(\chi_{\mathfrak{p}}, \pi_{\mathfrak{p}})$.

For the meaning of the unramified data $d_{\chi_{\mathfrak{p}}}^0$ and $\mathcal{W}_0^{\psi_{\mathfrak{p}}}$, see Section 3.1.

Remark. Let us fix a non-zero invariant local measure $dn_{\mathfrak{p}}$ on $N_{\mathfrak{p}}$. This gives rise to isomorphisms $\text{ev}_{dn_{\mathfrak{p}}} : L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}}) \xrightarrow{\sim} \mathbb{C}_{\psi_{\mathfrak{p}}}$ and $\text{ev}_{dn_{\mathfrak{p}}}^* : P(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}})) \xrightarrow{\sim} \text{Hom}_{N_{\mathfrak{p}}}(V_{\pi}, \mathbb{C}_{\psi_{\mathfrak{p}}})$.

Using isomorphisms $\text{ev}_{dn_{\mathfrak{p}}}^*$ and $\text{ev}_{d^{\times}a_{\mathfrak{p}}}^*$, we see that the map (3.7) could be described by the following integral:

$$(3.10) \quad [i(\psi_{\mathfrak{p}}, \chi_{\mathfrak{p}}, d^{\times}a_{\mathfrak{p}}, dn_{\mathfrak{p}})(p_{(A_{\mathfrak{p}}, \chi_{\mathfrak{p}})})](v) := \\ [e\text{v}_{dn_{\mathfrak{p}}}^*(p_{(N_{\mathfrak{p}}, \psi_{\mathfrak{p}})})](v) = \int_{N_{\mathfrak{p}}} \psi_{\mathfrak{p}}^{-1}(n) d_{\chi_{\mathfrak{p}}}(\pi_{\mathfrak{p}}(n_{\mathfrak{p}})v) dn_{\mathfrak{p}},$$

for $v \in V_{\pi_{\mathfrak{p}}}$ and $d_{\chi_{\mathfrak{p}}} = \text{ev}_{d^{\times}a_{\mathfrak{p}}}^*(p_{(A_{\mathfrak{p}}, \chi_{\mathfrak{p}})}) \in \text{Hom}_{A_{\mathfrak{p}}}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\chi_{\mathfrak{p}}})$. Note that while the integral (3.10) depends on the choice of the measure $dn_{\mathfrak{p}}$, the map (3.7) does not. The integral (3.10) is not absolutely convergent and should be understood in a regularized sense.

We want to point out that the integral (3.10) is not covered by the Jacquet-Langlands [JL] theory, unlike the integral (3.6) for the opposite map.

3.3.2. Global map.

Proposition. *Fix an invariant rational differential form δ_A on A .*

- (1) *The tensor product $\mathbf{i}(\psi, \chi, \delta_A) = \otimes_{\mathfrak{p}} \mathbf{i}(\psi_{\mathfrak{p}}, \chi_{\mathfrak{p}}, d^{\times}a_{\mathfrak{p}})$ is absolutely convergent for $\text{Re}(\chi) \ll 1$, and has the meromorphic continuation to the complex space of all characters.*
- (2) *The resulting map*

$$\mathbf{i}(\psi, \chi) = \mathbf{i}(\psi, \chi, \delta_A) : P(V_{\pi}, L_{\chi}(A(\mathbb{A}))) \rightarrow P(V_{\pi}, L_{\psi}(N(\mathbb{A})))$$

does not depend on the choice of the rational form δ_A .

3.3.3. *Action on automorphic periods.* As the final step we have to compute the effect of the map $\mathbf{i}(\psi, \chi)$ on the Hecke automorphic period. To this end, we now formulate our main result in this section.

Theorem. *Let χ be such that the partial L -function satisfies $L_S(\pi, \chi) \neq 0$ for some finite set S of primes. The global map $\mathbf{i}(\psi, \chi)$ is coherent, i.e., it sends the automorphic Hecke period d_{χ} to the automorphic Whittaker period \mathcal{W}^{ψ} . Namely, we have*

$$\mathbf{i}(\psi, \chi)d_{\chi} = \mathcal{W}^{\psi}.$$

Remarks. 1. We would like to point out a subtle (but crucial) difference between Theorem 3.2.3 and Theorem 3.3.3. The collection of local measures $\{d^{\times}a_{\mathfrak{p}}(\delta_A)\}$ appearing in Proposition 3.3.2 defines the Tamagawa adelic structure on the torsor $L^T(A(\mathbb{A}))$ described in Section 2.1.6, but these local measures do not define a *genuine measure* on $A(\mathbb{A})$. This differs from the situation described in Proposition 3.2.2 where the Tamagawa adelic structure on $L^T(N(\mathbb{A}))$

defines a genuine measure on $N(\mathbb{A})$. As a result, the direct map $\mathbf{i}(\chi, \psi)$ from the Whittaker period space to the Hecke period space has the integral representation (i.e., the Hecke-Jacquet-Langlands integral (3.15) given by an integral over a closed (non-compact) cycle $A(k) \setminus A(\mathbb{A})$ in the automorphic space X), but we do not know of such an integral representation for the map $\mathbf{i}(\psi, \chi)$ in the opposite direction.

2. Similarly to Remark 3.2.3 we can interpret Theorem 3.3.3 as a fiberwise meromorphic map from the bundle $H_\chi(\pi)$ to the constant bundle $W_{\psi, \chi}(\pi)$. This time however the corresponding map has poles.

3.3.4. *The relation.* We have the following relation between two maps we defined for Whittaker and Hecke periods.

Theorem. *The following relation holds*

$$\mathbf{i}(\psi, \chi) \circ \mathbf{i}(\chi, \psi) = id ,$$

as an endomorphism of $P(V_\pi, L_\psi(N))$.

3.4. **Proofs.** The logic of the proof we present is as follows. We first prove results from the Section 3.1 by repeating arguments of Jacquet-Langlands in a slightly different language. We then prove Theorem 3.3.4 by a local computation (see Lemma 3.4.5). This then implies all the other results in Section 3.3.

3.4.1. *Regularization.* Many integrals appearing in our considerations require a regularization. There are various techniques to achieve regularization, e.g., via analytic continuation (see [G1]). In what follows we often use the following more elementary standard construction.

Let F be a local non-archimedean field and A the additive group of F . Fix a non-trivial additive character $\psi : A \rightarrow \mathbb{C}^\times$.

Let $V = V(A)$ denote the space of locally constant functions on A . It contains the subspace $\mathcal{S}(A)$ of functions with compact support.

Let $V_0 \subset V$ be the space of functions $f \in V$ such that the image of f in the space $V/\mathcal{S}(A)$ is F^\times -finite.

Consider the Whittaker functional W on the space $\mathcal{S}(A)$ given by the integral $W(u) = \int_A \psi(a) \cdot u(a) da$.

Claim. *There exists the unique extension of the Whittaker functional W to the space V_0 that satisfies the identity $W(t_a(u)) = \psi(a)W(u)$, where $t_a(u)(x) = u(x - a)$.*

Uniqueness is easy since F acts trivially on $V_0/\mathcal{S}(A)$. To prove existence one can extend the functional W by setting $W(u) = \lim_{\{B\}} \int_B \psi(a) \cdot u(a) da$ where the limit is taken over a family $\{B\}$ of the standard compact balls exhausting A . It is easy to see that the limit exists for functions in V_0 .

3.4.2. *Proof of Proposition 3.2.1.* Both claims are standard in the Hecke-Jacquet-Langlands theory once the translation (3.6) into local zeta integrals is made.

- (1) For a smooth vector $v \in V_{\pi_{\mathfrak{p}}}$, the Whittaker function $l^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}}(\bar{a})v)$ is rapidly decreasing as $\|t\| \rightarrow \infty$ in the positive Weyl chamber, and has a polynomial behavior in the opposite direction. This implies the absolute convergence of the integral for the character $\chi_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^{-s}$ with $Re(s) \gg 1$.

The meromorphic continuation is equivalent to the meromorphic continuation of the Jacquet-Langlands local zeta integrals (see [JL]).

- (2) This is the standard computation in the Hecke-Jacquet-Langlands theory.

3.4.3. *Proof of Proposition 3.2.2.* Indeed this follows immediately from the analytic continuation of $L(s, \pi)$ and from Proposition 3.2.1. In fact, this is a part of the Jacquet-Langlands method where the adelic integral is reduced to the absolutely convergent integral (3.15) via unfolding.

3.4.4. *Proof of Theorem 3.2.3.* As we indicated before, the following proof is the standard Hecke-Jacquet-Langlands proof. On the basis of (3.4), we want to compute

$$(3.11) \quad [i(\chi|\cdot|^{-s}, \psi, dn, d^{\times}a)(\mathcal{W}^{\psi})](v) = \prod_{\mathfrak{p}} \int_{A_{\mathfrak{p}}} \chi_{\mathfrak{p}}^{-1}(\bar{a})|a|_{\mathfrak{p}}^s \mathcal{W}^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}}(\bar{a})v) d^{\times}a_{\mathfrak{p}},$$

for $Re(s) \gg 1$. The absolute convergence of local integrals and the absolute convergence of the Euler product imply that

$$(3.12) \quad \prod_{\mathfrak{p}} \int_{A_{\mathfrak{p}}} \chi_{\mathfrak{p}}^{-1}(\bar{a})|a|_{\mathfrak{p}}^s \mathcal{W}^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}}(\bar{a})v) d^{\times}a_{\mathfrak{p}} = \int_{A(\mathbb{A})} \chi^{-1}(\bar{a})|a|^s \mathcal{W}^{\psi}(\pi(\bar{a})\phi_v) d^{\times}a.$$

We invoke the standard unfolding technique. The rational torus acts transitively on Whittaker functionals for different characters. For an automorphic period \mathcal{W}^{ψ} and a character $\psi_{\alpha}(x) = \psi(\alpha x)$ with $\alpha \in k^{\times}$, we have the corresponding automorphic period given by $\mathcal{W}^{\psi_{\alpha}} = \pi^*((\begin{smallmatrix} \alpha & \\ & 1 \end{smallmatrix}))\mathcal{W}^{\psi}$. We have the following Fourier expansion at the identity for an automorphic function ϕ_v ,

$v \in V_\pi$, in a cuspidal representation π ,

$$(3.13) \quad \phi_v(e) = \sum_{\alpha \in k^\times} \mathcal{W}^{\psi_\alpha}(v) = \sum_{\alpha \in k^\times} \mathcal{W}^\psi(\pi(\bar{\alpha})v) .$$

The Fourier expansion for a cusp form ϕ_v implies that

$$(3.14) \quad \int_{A(\mathbb{A})} \chi^{-1}(\bar{a})|a|^s \mathcal{W}^\psi(\pi(\bar{a})v) d^\times a =$$

$$\sum_{\alpha \in k^\times} \int_{A(k) \backslash A(\mathbb{A})} \chi^{-1}(\bar{a})|a|^s \mathcal{W}^\psi(\pi(\bar{\alpha})\pi(\bar{a})v) d^\times a =$$

$$\int_{A(k) \backslash A(\mathbb{A})} \chi^{-1}(\bar{a})|a|^s \left[\sum_{\alpha \in k^\times} \mathcal{W}^\psi(\pi(\bar{\alpha})(\pi(\bar{a})v)) \right] d^\times a =$$

$$\int_{A(k) \backslash A(\mathbb{A})} \chi^{-1}(\bar{a})|a|^s \phi_{\pi(\bar{a})v}(e) d^\times a =$$

$$(3.15) \quad \int_{A(k) \backslash A(\mathbb{A})} \chi^{-1}(\bar{a})|a|^s \phi_v(\bar{a}) d^\times a .$$

This gives the integral (3.2) for the Hecke period $d_{\chi|\cdot|^{-s}}$. We have used Fubini's theorem for $Re(s) \gg 1$, to decompose the adelic integral into the integral over a quotient space of the sum over $A(k)$ since all integrals are absolutely convergent. The resulting integral defines an analytic function for all values of s and for a cuspidal π .

3.4.5. *Proof of Theorem 3.3.3.* Proofs of all statements leading to and of the theorem itself are purely local, granted we already know the *direct* relation between Whittaker and Hecke periods (i.e., Theorem 3.2.3). Namely, all proofs follow from the following local lemma

Lemma. *Let $\pi_{\mathfrak{p}}$ be an irreducible unitary representation of $GL_2(k_{\mathfrak{p}})$, $\chi_{\mathfrak{p}}$ a quasi-character of $A_{\mathfrak{p}}$, $\psi_{\mathfrak{p}}$ a non-trivial character on $N_{\mathfrak{p}}$, and $dn_{\mathfrak{p}}$ and $d^\times a_{\mathfrak{p}}$ measures on $N_{\mathfrak{p}}$ and $A_{\mathfrak{p}}$, respectively, corresponding to some invariant differential forms δ_N and δ_A . We have the following identity:*

$$(3.16) \quad \mathbf{i}(\psi_{\mathfrak{p}}, \chi_{\mathfrak{p}}, d^\times a_{\mathfrak{p}}) \circ \mathbf{i}(\chi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, dn_{\mathfrak{p}}) = c(dn_{\mathfrak{p}}, d^\times a_{\mathfrak{p}}) \cdot id ,$$

as an endomorphism of $P(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}}))$. Here $c(dn_{\mathfrak{p}}, d^\times a_{\mathfrak{p}}) \in \mathbb{C}$ is the proportionality constant between the measure $d^\times a_{\mathfrak{p}} dn_{\mathfrak{p}}$ on $A_{\mathfrak{p}} \times N_{\mathfrak{p}} \simeq k_{\mathfrak{p}}^\times \times k_{\mathfrak{p}}$ and the measure $d^\times y dx$, i.e., $d^\times a_{\mathfrak{p}} dn_{\mathfrak{p}} = c(d^\times a_{\mathfrak{p}}, dn_{\mathfrak{p}}) d^\times y dx$.

Remark. In this paper, we consider the construction of local maps $I_{\mathfrak{p}}$ by regularizing integrals over appropriate subgroups (following the original construction of J.-L. Waldspurger [Wa]). In certain cases one can construct such maps in both directions, i.e., maps $I_{\mathfrak{p}}(H_{1,\mathfrak{p}}, H_{2,\mathfrak{p}}) : \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \sigma_{\mathfrak{p}}) \rightarrow \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \tau_{\mathfrak{p}})$ and $I_{\mathfrak{p}}(H_{2,\mathfrak{p}}, H_{1,\mathfrak{p}}) : \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \tau_{\mathfrak{p}}) \rightarrow \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \sigma_{\mathfrak{p}})$. As was pointed out to us by Y. Sakellaridis, when $\tau_{\mathfrak{p}}$ and $\sigma_{\mathfrak{p}}$ are characters (as in examples in this paper), these maps are formally adjoint in the following sense. We follow notations from Section 1.3. To an element $\xi \in \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \sigma_{\mathfrak{p}})$, we can associate a map $r_{\xi} : V_{\pi_{\mathfrak{p}}} \rightarrow \mathcal{F}(H_{1,\mathfrak{p}} \backslash G)$ from the space of smooth vectors in the representation $\pi_{\mathfrak{p}}$ to the space of appropriate functions on $H_{1,\mathfrak{p}} \backslash G$ given by $v \mapsto \xi(\pi_{\mathfrak{p}}(g)v)$. Similarly for an element $\eta \in \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \tau_{\mathfrak{p}})$, we have the map $q_{\eta} : V_{\pi_{\mathfrak{p}}} \rightarrow \mathcal{F}(H_{2,\mathfrak{p}} \backslash G)$. The integration procedure $I_{\mathfrak{p}}(H_{1,\mathfrak{p}}, H_{2,\mathfrak{p}})$ could be described then as the map $\mathcal{I}_{\mathfrak{p}}(H_{1,\mathfrak{p}}, H_{2,\mathfrak{p}}) : \mathcal{F}(H_{1,\mathfrak{p}} \backslash G) \rightarrow \mathcal{F}(H_{2,\mathfrak{p}} \backslash G)$ given by the integral $[\mathcal{I}_{\mathfrak{p}}(H_{1,\mathfrak{p}}, H_{2,\mathfrak{p}})(\phi)](g) = \int_{H_{2,\mathfrak{p}}} \phi(hg)dh$ (we are leaving aside convergence issues).

We have then $\langle \mathcal{I}_{\mathfrak{p}}(H_{1,\mathfrak{p}}, H_{2,\mathfrak{p}})(\phi), \psi \rangle_{H_{2,\mathfrak{p}} \backslash G_{\mathfrak{p}}} = \int_{H_{2,\mathfrak{p}} \backslash G_{\mathfrak{p}}} \left(\int_{H_{2,\mathfrak{p}}} \phi(hg)dh \right) \psi(g)dg =$

$$\int_{G_{\mathfrak{p}}} \phi(g)\psi(g)dg = \int_{H_{1,\mathfrak{p}} \backslash G_{\mathfrak{p}}} \phi(g) \left(\int_{H_{1,\mathfrak{p}}} \psi(hg)dh \right) dg = \langle \phi, \mathcal{I}_{\mathfrak{p}}(H_{2,\mathfrak{p}}, H_{1,\mathfrak{p}})(\psi), \rangle_{H_{1,\mathfrak{p}} \backslash G_{\mathfrak{p}}}.$$

Under certain conditions (which are satisfied for Whittaker/Hecke cases, we consider in Sections 3 and 3.3) which in [SV] are called “local unfolding”, it is shown by Y. Sakellaridis and A. Venkatesh [SV] that above adjoint maps are also inverse of each other. In particular, this should imply our Lemma 3.4.5 at least for tempered representations.

3.4.6. Proof of Lemma 3.4.5. The proof follows from a direct computation in the Kirillov model (see Appendix B). In fact, this computation is essentially identical for all representations $\pi_{\mathfrak{p}}$ since it only involves the action of the Borel subgroup of $\mathrm{GL}(2)$.

Non-archimedean fields. Let $p_{N_{\mathfrak{p}}, \psi_{\mathfrak{p}}} \in P(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}}))$ and let $dn_{\mathfrak{p}} = dx$ be the standard measure on $N_{\mathfrak{p}} \simeq k_{\mathfrak{p}}$. The measure $dn_{\mathfrak{p}}$ induces the isomorphism $\mathrm{ev}_{dn_{\mathfrak{p}}}^* : P(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}}(N_{\mathfrak{p}})) \rightarrow \mathrm{Hom}_{N_{\mathfrak{p}}}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\psi_{\mathfrak{p}}})$. The Whittaker functional $\mathcal{W}^{\psi_{\mathfrak{p}}} = \mathrm{ev}_{dn_{\mathfrak{p}}}^*(p_{N_{\mathfrak{p}}, \psi_{\mathfrak{p}}})$ gives rise to the Kirillov model realization $k^{\mathcal{W}^{\psi_{\mathfrak{p}}}} : V_{\pi_{\mathfrak{p}}} \rightarrow \mathcal{K}^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}})$ of $\pi_{\mathfrak{p}}$. Let $\delta_{\mathfrak{p}} = d^{\times}a_{\mathfrak{p}} = da_{\mathfrak{p}}/|a|_{\mathfrak{p}}$ be the standard local measure on $A_{\mathfrak{p}} \simeq k_{\mathfrak{p}}^{\times}$ and let $\mathrm{ev}_{\delta_{\mathfrak{p}}}^* : P(V_{\pi_{\mathfrak{p}}}, L_{\chi_{\mathfrak{p}}}(A_{\mathfrak{p}})) \rightarrow \mathrm{Hom}_{A_{\mathfrak{p}}}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\psi_{\mathfrak{p}}})$ be the corresponding isomorphism.

We first compute the image $d_{\chi_{\mathfrak{p}}}^{\#}$ of the Whittaker $\mathcal{W}^{\psi_{\mathfrak{p}}}$ functional under the integration with respect to $A_{\mathfrak{p}}$. We have $d_{\chi_{\mathfrak{p}}}^{\#} := (\mathrm{ev}_{\delta_{\mathfrak{p}}}^*)^{-1}(\mathfrak{i}(\chi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, dn_{\mathfrak{p}})\mathcal{W}^{\psi_{\mathfrak{p}}}) = \int_{A_{\mathfrak{p}}} \chi_{\mathfrak{p}}^{-1}(\bar{a})\pi_{\mathfrak{p}}^*(\bar{a})\mathcal{W}^{\psi_{\mathfrak{p}}}\delta_{\mathfrak{p}}$. In the Kirillov model of $\pi_{\mathfrak{p}}$, we have $\mathcal{W}^{\psi_{\mathfrak{p}}}(f) = f(1)$

for $f \in \mathcal{K}^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}})$, and we have

$$d_{\chi_{\mathfrak{p}}}^{\#}(f) = \int_{A_{\mathfrak{p}}} \chi_{\mathfrak{p}}^{-1}(\bar{a}) [\pi_{\mathfrak{p}}(\bar{a})f(x)]_{x=1} \delta_{\mathfrak{p}} = \int_{k_{\mathfrak{p}}^{\times}} \chi_{\mathfrak{p}}^{-1}(a) f(a) d^{\times} a_{\mathfrak{p}} .$$

Hence in the Kirillov model, the functional $(\text{ev}_{\delta_{\mathfrak{p}}}^*)^{-1}(\mathbf{i}(\chi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, dn_{\mathfrak{p}})\mathcal{W}^{\psi_{\mathfrak{p}}})$ is given by the kernel $d_{\chi_{\mathfrak{p}}}^{\#}(x) = \chi_{\mathfrak{p}}^{-1}(x)$ on $k_{\mathfrak{p}}^{\times}$.

We now compute the image under the second integration with respect to $N_{\mathfrak{p}}$: $(\text{ev}_{dn_{\mathfrak{p}}}^*)^{-1}(\mathbf{i}(\psi_{\mathfrak{p}}, \chi_{\mathfrak{p}}, \delta_{\mathfrak{p}})d_{\chi_{\mathfrak{p}}}^{\#})$ given by

$$(3.17) \quad \begin{aligned} (\text{ev}_{dn_{\mathfrak{p}}}^*)^{-1}(\mathbf{i}(\psi_{\mathfrak{p}}, \chi_{\mathfrak{p}}, \delta_{\mathfrak{p}})d_{\chi_{\mathfrak{p}}}^{\#})(f) &= \int_{k_{\mathfrak{p}}} \psi_{\mathfrak{p}}^{-1}(x) d_{\chi_{\mathfrak{p}}}^{\#}(\pi_{\mathfrak{p}}(n(x)f)) dx = \\ & \int_{k_{\mathfrak{p}}} \psi_{\mathfrak{p}}^{-1}(x) \left[\int_{k_{\mathfrak{p}}^{\times}} \chi_{\mathfrak{p}}^{-1}(y) \psi_{\mathfrak{p}}(xy) f(y) d^{\times} y \right] dx . \end{aligned}$$

The inner integral is absolutely convergent for $\text{Re}(\chi_{\mathfrak{p}}^{-1}) \gg 1$, since functions in the Kirillov model are compactly supported on $k_{\mathfrak{p}}$ and have a polynomial behavior at 0.

We first compute the integral (3.17) for functions which are compactly supported on $k_{\mathfrak{p}}^{\times}$. Consider $f \in \mathcal{S}(k_{\mathfrak{p}}^{\times})$ and assume that $f(u+y) = f(y)$ for all y and $|u|_{\mathfrak{p}} \leq q_{\mathfrak{p}}^{-N}$ for some $N \geq 0$. The inner integral in this case is zero for $|x|_{\mathfrak{p}} \geq q_{\mathfrak{p}}^{N+1}$. Hence we can take the outer integral over a big enough compact set $B_{N'} = \{|x|_{\mathfrak{p}} \leq q_{\mathfrak{p}}^{N'}\}$, $N' \geq N$. Both integrals are absolutely convergent over compact sets, and we can interchange their order. We now have

$$(\text{ev}_{dn_{\mathfrak{p}}}^*)^{-1}(\mathbf{i}(\psi_{\mathfrak{p}}, \chi_{\mathfrak{p}}, \delta_{\mathfrak{p}})d_{\chi_{\mathfrak{p}}}^{\#})(f) = \int_{k_{\mathfrak{p}}^{\times}} \chi_{\mathfrak{p}}^{-1}(y) \left[\int_{B_{N'}} \psi_{\mathfrak{p}}((y-1)x) dx \right] f(y) d^{\times} y .$$

The inner integral is zero, unless $|y-1|_{\mathfrak{p}} \leq q_{\mathfrak{p}}^{-N'}$, and we have $f(y) = f(1)$ under such a restriction. By possibly increasing N' , we also can assume that $\chi_{\mathfrak{p}}(y) \equiv 1$ for $|y-1|_{\mathfrak{p}} \leq q_{\mathfrak{p}}^{-N'}$. The integration with respect to measures dx and $d^{\times}y$ then give 1. Hence on the space $\mathcal{S}(k_{\mathfrak{p}}^{\times})$, we have shown that

$$(\text{ev}_{dn_{\mathfrak{p}}}^*)^{-1}(\mathbf{i}(\psi_{\mathfrak{p}}, \chi_{\mathfrak{p}}, \delta_{\mathfrak{p}})d_{\chi_{\mathfrak{p}}}^{\#})(f) = f(1) = \mathcal{W}^{\psi_{\mathfrak{p}}}(f) .$$

This finishes the proof of the Lemma for compactly supported functions (and finishes the proof for supercuspidal representations).

For induced representations, we also have to evaluate the integral (3.17) on the space $V(\chi_1, \chi_2)$ describing the Kirillov model (see Section B.1). We claim that on the space $V(\chi_1, \chi_2)$ the integral (3.17) is also given by the evaluation at the identity. Modulo compactly supported functions, functions spanning

$V(\chi_1, \chi_2)$ are supported in $\mathcal{O}_{\mathfrak{p}}$ and are essentially multiplicative characters near 0. Hence we consider the integral

$$(3.18) \quad \int_{k_{\mathfrak{p}}} \psi_{\mathfrak{p}}^{-1}(x) \left[\int_{\mathcal{O}_{\mathfrak{p}}} \chi_{\mathfrak{p}}^{-1}(y) \psi_{\mathfrak{p}}(xy) \chi'_{\mathfrak{p}}(y) d^{\times} y \right] dx ,$$

for a fixed character $\chi'_{\mathfrak{p}}$. For $Re(\chi_{\mathfrak{p}}^{-1}) \gg 1$, the inner integral is absolutely convergent and decays polynomially in $|x|_{\mathfrak{p}}$. In fact, for $Re(\chi_{\mathfrak{p}}^{-1} \chi'_{\mathfrak{p}}) \geq 2$, the inner integral is bounded by $|x|_{\mathfrak{p}}^{-2}$ as $|x|_{\mathfrak{p}} \rightarrow \infty$, and hence the outer integral is absolutely convergent. Hence for $Re(\chi_{\mathfrak{p}}^{-1}) \gg 1$, the functional defined by the integral (3.17) extends to the space $V(\chi_1, \chi_2)$, and defines a functional on the whole space $\mathcal{W}^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}})$. On the other hand, the double integral (3.17) is clearly $(N_{\mathfrak{p}}, \psi_{\mathfrak{p}})$ -equivariant. The space of such functionals is one-dimensional with a basis consisting of the Whittaker functional $\mathcal{W}^{\psi_{\mathfrak{p}}}$ we started with. Hence, for $Re(\chi_{\mathfrak{p}}^{-1}) \gg 1$, the integral (3.17) coincides with the Whittaker functional $\mathcal{W}^{\psi_{\mathfrak{p}}}$ for induced representations as well.

We proved the statement (3.16) of the Lemma for $Re(\chi_{\mathfrak{p}}^{-1}) \gg 1$. Since the family of maps $\mathfrak{i}(\chi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, dn_{\mathfrak{p}})$ is a meromorphic family of maps by [JL], the identity (3.16) holds for all $\chi_{\mathfrak{p}}$, and in fact provides the meromorphic continuation of the family of maps $\mathfrak{i}(\psi_{\mathfrak{p}}, \chi_{\mathfrak{p}}, d^{\times} a_{\mathfrak{p}})$.

Change of measures on $N_{\mathfrak{p}}$ and $A_{\mathfrak{p}}$ gives rise to the scaling factor $c(d^{\times} a_{\mathfrak{p}}, dn_{\mathfrak{p}})$.

Archimedean fields. We assume for simplicity that $k_{\mathfrak{p}} = \mathbb{R}$. The integral analogous to (3.17) now takes the form

$$(3.19) \quad \int_{\mathbb{R}} e^{-ix} \left[\int_{\mathbb{R}_+} |y|^{-s} e^{ixy} f(y) d^{\times} y \right] dx ,$$

where f is a smooth vector in the Kirillov space of a unitary irreducible representation of $\mathrm{PGL}_2(\mathbb{R})$. In particular, f is smooth, rapidly decreasing as $y \rightarrow \infty$, has a polynomial asymptotic at 0 and $f \in L^2(\mathbb{R}_+, d^{\times} y)$. Hence, for $Re(s) \ll 0$, the inner integral is absolutely convergent and defines a bounded polynomially decreasing function as $|x| \rightarrow \infty$. This implies that the outer integral is also absolutely convergent for $Res(s) \ll 0$. Hence (3.19) is given by the repeated Fourier transform and we obtain $\mathcal{F}\mathcal{F}(f | \cdot |^{-s-1})(1) = f(1)$ for $Re(s) \ll 0$, i.e., the integral (3.19) gives the Whittaker functional in the Kirillov model. By analytic continuation we obtain this relation for all s .

4. OPPOSITE WHITTAKER PERIODS

We denote by $N^+ = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \right\}$ and $N^- = \left\{ \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \right\}$ two standard unipotent subgroups of $\mathrm{GL}(2)$. Our aim is to compare two period data coming from

Whittaker functionals associated with (N^+, ψ) and (N^-, ψ') , where $\psi, \psi' : \mathbb{Q} \setminus \mathbb{A} \rightarrow \mathbb{C}$ are two non-trivial additive characters.

4.1. Local map. Let $\pi_{\mathfrak{p}}$ be an irreducible representation of G , $\psi_{\mathfrak{p}}, \psi'_{\mathfrak{p}} : k_{\mathfrak{p}} \rightarrow \mathbb{C}$ be two non-trivial additive characters. We denote by $\psi_{\mathfrak{p}}^+ : N_{\mathfrak{p}}^+ \rightarrow \mathbb{C}$ the character given by $\psi_{\mathfrak{p}}^+(n^+(x)) = \psi_{\mathfrak{p}}(x)$ and by $\psi_{\mathfrak{p}}^- : N_{\mathfrak{p}}^- \rightarrow \mathbb{C}$ the character given by $\psi_{\mathfrak{p}}^-(n^-(x)) = \psi'_{\mathfrak{p}}(x)$. Consider the local Whittaker period space $P_{\mathfrak{p}}^-(\pi_{\mathfrak{p}}, L_{\psi_{\mathfrak{p}}^-}) = \text{Hom}_{N_{\mathfrak{p}}^-}(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}^-}(N_{\mathfrak{p}}^-))$. Choose an invariant differential form δ_- on N^- and let $dn_{\mathfrak{p}}^- = dn_{\mathfrak{p}}^-(\delta_-)$ be the corresponding invariant measure on N^- . We denote by $\text{ev}_{dn_{\mathfrak{p}}^-}^* : P^-(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}^-}(N_{\mathfrak{p}}^-)) \rightarrow \text{Hom}_{N_{\mathfrak{p}}^-}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\psi_{\mathfrak{p}}^-})$ the induced isomorphism. We now construct a map

$$(4.1) \quad \mathbf{i}(\psi_{\mathfrak{p}}^+, \psi_{\mathfrak{p}}^-, dn_{\mathfrak{p}}^-) : P^-(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}^-}) \rightarrow P^+(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}^+}) ,$$

given by the integration. Namely, for a vector $v \in V_{\pi_{\mathfrak{p}}}$ and a map $p_{\psi_{\mathfrak{p}}^-}^- \in P^-(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}^-})$, we consider the following (matrix coefficient) function given by $\gamma_{p_{\psi_{\mathfrak{p}}^-}^-, v}(n_{\mathfrak{p}}^+) = \text{ev}_{dn_{\mathfrak{p}}^-}^*(p_{\psi_{\mathfrak{p}}^-}^-(\pi_{\mathfrak{p}}(n_{\mathfrak{p}}^+)v)) \in C^\infty(N_{\mathfrak{p}}^+)$ and take its image in $L_{\psi_{\mathfrak{p}}^+}(N_{\mathfrak{p}}^+)$ under the integration/co-invariant map.

Proposition.

(1) *The map*

$$(4.2) \quad \mathbf{i}(\psi_{\mathfrak{p}}^+, \psi_{\mathfrak{p}}^-, dn_{\mathfrak{p}}^-) : P^-(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}^-}) \rightarrow P^+(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}^+}) ,$$

is well-defined.

(2) *For unramified $\pi_{\mathfrak{p}}$, $\psi_{\mathfrak{p}}, \psi_{\mathfrak{p}}'$ and δ_- , we have*

$$(4.3) \quad \mathbf{i}(\psi_{\mathfrak{p}}^+, \psi_{\mathfrak{p}}^-, dn_{\mathfrak{p}}^-)\xi_0^- = \lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, \psi_{\mathfrak{p}}') \cdot \xi_0^+ ,$$

where ξ_0^\pm are unramified and the constant $\lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, \psi_{\mathfrak{p}}') \in \mathbb{C}$ is given by

$$(4.4) \quad \lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, \psi_{\mathfrak{p}}') = 1 - \text{tr}(Ad(\sigma(\pi_{\mathfrak{p}}))q_{\mathfrak{p}}^{-1}) ,$$

where $\sigma(\pi_{\mathfrak{p}}) \in \text{SL}_2(\mathbb{C})$ denotes the Satake parameter of $\pi_{\mathfrak{p}}$ and Ad is the adjoint representation of the dual group of G .

4.2. Global map. Let now $\pi = \otimes_{\mathfrak{p}} \pi_{\mathfrak{p}}$ be an automorphic cuspidal representation of G , $\psi = \otimes_{\mathfrak{p}} \psi_{\mathfrak{p}}$ and $\psi' = \otimes_{\mathfrak{p}} \psi_{\mathfrak{p}}'$ are two global non-trivial characters. We also choose a non-zero invariant differential form δ^- on N^- . In order to construct the global map out of local maps $\mathbf{i}(\psi_{\mathfrak{p}}^+, \psi_{\mathfrak{p}}^-, dn_{\mathfrak{p}}^-)$, we need to glue constants $\{\lambda_{\mathfrak{p}}\}$. The product $\prod_{\mathfrak{p}} \lambda_{\mathfrak{p}}$ is not absolutely convergent. However, due to the unramified computation (4.4) one has the natural regularization

procedure. This is based on the use of the adjoint L -function of π . For an unramified $\pi_{\mathfrak{p}}$, let $L(s, \pi_{\mathfrak{p}}, Ad)$ be the local adjoint L -function and for a finite set S of primes, including primes where π is ramified, let $L_S(s, \pi, Ad)$ be the partial adjoint L -function of π .

Proposition. *The Euler product*

$$(4.5) \quad \prod_{\mathfrak{p} \text{ unramified}} [L(1, \pi_{\mathfrak{p}}, Ad) \cdot \mathbf{i}(\psi_{\mathfrak{p}}^+, \psi_{\mathfrak{p}}^-, dn_{\mathfrak{p}}^-)]$$

is absolutely convergent.

On the basis of this proposition, we consider for a large enough set S , the following absolutely convergent Euler product

$$(4.6) \quad \mathbf{i}(\psi^+, \psi^-, \delta^-) = L_S(1, \pi, Ad)^{-1} \prod_{\mathfrak{p} \in S} \mathbf{i}(\psi_{\mathfrak{p}}^+, \psi_{\mathfrak{p}}^-, dn_{\mathfrak{p}}^-) \prod_{\mathfrak{p} \notin S} [L(1, \pi_{\mathfrak{p}}, Ad) \cdot \mathbf{i}(\psi_{\mathfrak{p}}^+, \psi_{\mathfrak{p}}^-, dn_{\mathfrak{p}}^-)] .$$

The resulting map clearly does not depend on a set S if it is large enough. We also use the well-known fact that $L_S(1, \pi, Ad) \neq 0$.

Theorem. *The resulting map*

$$(4.7) \quad \mathbf{i}(\psi^+, \psi^-) = \mathbf{i}(\psi^+, \psi^-, \delta^-) : P^-(V_{\pi}, L_{\psi^-}^-) \rightarrow P^+(V_{\pi}, L_{\psi^+}^+)$$

is well-defined and does not depend on the choice of the form δ^- .

4.3. Action on automorphic periods. The invariant. We now consider the action of the map $\mathbf{i}(\psi^+, \psi^-)$ on automorphic periods. According to the Theorem 4.2, there exists a constant $\lambda(\pi, \psi, \psi') \in \mathbb{C}$ such that

$$(4.8) \quad \mathbf{i}(\psi^+, \psi^-)(\mathcal{W}^{\psi^-}) = \lambda(\pi, \psi, \psi') \cdot \mathcal{W}^{\psi^+} .$$

We call it the *period invariant* associated to the map $\mathbf{i}(\psi^+, \psi^-)$. The constant $\lambda(\pi, \psi, \psi')$ gives rise to a *global invariant* of π (and of characters ψ, ψ') depending only on its automorphic realization (or even only on the isomorphism class of π if it appears with multiplicity one in the automorphic space, as is the case for $GL(2)$).

The constant $\lambda(\pi, \psi)$ measures to what extent the integration map $\mathbf{i}(\psi^+, \psi^-)$ fails to be *coherent* (e.g., $\lambda(\pi, \psi, \psi') = 1$ if it is coherent, i.e., maps the automorphic period to the automorphic period, as was the case for maps considered in Section 3). In Appendix A, we make a numerical evaluation of this invariant for the Ramanujan holomorphic cusp form Δ of weight 12 and level 1. In particular, we will see that $\lambda(\Delta, e^{2\pi ix}) \neq 1$, i.e., in that case the corresponding map is not *coherent*. This leads to a new numerical invariant of the Ramanujan function.

4.3.1. *Product formula.* We claim that the invariant $\lambda(\pi, \psi, \psi') \in \mathbb{C}$ could be computed via an absolutely convergent Euler product (i.e., it has local to global representation). Let $a \in k^\times$ be the scalar such that $\psi(x) = \psi'(ax)$. To write the product formula for $\lambda(\pi, \psi, \psi')$, we use the element $w_a = \begin{pmatrix} a \\ & 1 \end{pmatrix} \in G$. We know that w_a maps the automorphic period \mathcal{W}^{ψ^-} to the automorphic period \mathcal{W}^{ψ^+} (i.e., $\pi^*(w_a)\mathcal{W}^{\psi^-} = \mathcal{W}^{\psi^+}$), and we also know how w_a acts on local period spaces.

Fix a local representation $\pi_{\mathfrak{p}}$ and local character $\psi_{\mathfrak{p}}, \psi'_{\mathfrak{p}}$. We have the isomorphism of co-invariants $c(w) : L_{\psi_{\mathfrak{p}}^+}(N_{\mathfrak{p}}^+) \simeq L_{\psi_{\mathfrak{p}}^-}(N_{\mathfrak{p}}^-)$ arising from the conjugation map $wN^+w^{-1} \rightarrow N^-$. We have the natural map:

$$(4.9) \quad m(\psi_{\mathfrak{p}}^+, \psi_{\mathfrak{p}}^-) : P^-(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}^-}^-) \rightarrow P^+(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}^+}^+) ,$$

given by action of the element w , i.e., $[m(\psi_{\mathfrak{p}}^+, \psi_{\mathfrak{p}}^-)\xi](v) = c(w_a)(\xi(v))$ for any $v \in V_{\pi_{\mathfrak{p}}}$ and $\xi \in P^-(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}^-}^-)$. Hence there exists a constant $\lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}) \in \mathbb{C}$ such that

$$(4.10) \quad \mathfrak{i}(\psi_{\mathfrak{p}}^+, \psi_{\mathfrak{p}}^-, dn_{\mathfrak{p}}^-)\xi = \lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, \psi'_{\mathfrak{p}}) \cdot m(\psi_{\mathfrak{p}}^+, \psi_{\mathfrak{p}}^-)\xi ,$$

for any $\xi \in P^-(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}^-}^-)$. It is easy to see that for unramified \mathfrak{p} , these coefficients coincide with those defined in (4.3).

Theorem. *For a sufficiently large set S , the following relation holds*

$$(4.11) \quad \lambda(\pi, \psi, \psi') = L_S(1, \pi, Ad)^{-1} \prod_{\mathfrak{p} \in S} \lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, \psi'_{\mathfrak{p}}) \prod_{\mathfrak{p} \notin S} [\lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, \psi'_{\mathfrak{p}}) \cdot L(1, \pi_{\mathfrak{p}}, Ad)] ,$$

where $L_S(1, \pi, Ad)$ is the (analytically continued) partial adjoint L -function.

Remarks. 1. Instead of using the element w_a to obtain the product formula for $\lambda(\pi, \psi, \psi')$, it is possible to compare spaces $P^\pm(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}^\pm}^\pm)$ of Whittaker periods with the space of Hecke period (for some character) following the construction from Section 3.1. This gives another factoring of $\lambda(\pi, \psi, \psi')$ into local factors. Such a comparison will differ locally from the one we considered above (i.e., coming from the action of w_a), and will lead to different *local* factors. However, *globally* this will give the same result due to the functional equation for the corresponding Hecke L -function of π .

2. The procedure of regularization described in Proposition 4.2 is similar to the one used to normalize the Tamagawa measure on the non-split torus (for the split torus the corresponding L -function has a pole) and is widely used in the theory of automorphic functions (e.g., see [Wa]). That the adjoint L -function shows up in our example is surprising since it is typically appear

as a comparison between automorphic invariant Hermitian norm and the one coming from the Whittaker model which is absent in our example. We discuss in Section 4.3.3 a possible geometric explanation for the appearance of the adjoint L -function.

We have more examples of regularization of period maps similar to those appearing in Proposition 4.2. We hope to return to this subject elsewhere.

3. Proposition 4.2 could be formulated without mentioning L -functions explicitly, but using instead the language of maps between periods satisfying uniqueness property. The Rankin-Selberg method allows one to relate the adjoint L -function to the integration map from the diagonal period on $X_G \times X_G$ defining the invariant Hermitian form on $\pi \otimes \pi^\vee$ to the Hermitian form $\mathcal{W}^\psi \otimes \overline{\mathcal{W}}^\psi$ coming from the Whittaker functional. Hence the statement in Proposition 4.2 could be interpreted as the statement about ratio for certain maps between appropriate periods. The advantage of such a reformulation is that one does not need to know local components of π in order to construct the regularization (4.6) (i.e., one can think of the local factor $L(1, \pi_{\mathfrak{p}}, Ad)$ as a map between one-dimensional spaces of certain local period spaces).

4. As was pointed out by Y. Sakellaridis, the difference between examples in Section 3 and the example from this section could be seen in the language of [SV] as follows. Following the general setup from Section 1.3, we note that a choice of an invariant Hermitian form on $\pi_{\mathfrak{p}}$ (and on the relevant Gelfand data) gives rise to norm on the corresponding local period spaces $\mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \sigma_{\mathfrak{p}})$ and $\mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \tau_{\mathfrak{p}})$ (at least for tempered representations). Once the local map $I_{\mathfrak{p}} : \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \sigma_{\mathfrak{p}}) \rightarrow \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \tau_{\mathfrak{p}})$ is constructed, one can ask if it is unitary with respect to these norms. It is easy to see that for the Hecke and converse to Hecke cases, the map is unitary and for the case of opposite Whittaker periods it is not unitary.

5. We can apply the construction of the global invariant (4.8) presented in this section to an Eisenstein series representation $E(s)$ instead of a cuspidal representation π . An argument similar to the proof presented above shows that the function $\lambda(E(s), \psi, \psi')$ is well-defined for s in the strip $|Re(s)| < \frac{1}{2}$ (in particular, $\lambda(E(s), \psi, \psi')$ is well-defined for the unitary Eisenstein series, i.e., for $s \in i\mathbb{R}$). Hence we obtain the function $\lambda(E(s), \psi, \psi')$ depending on the parameter $s \in \mathbb{C}$. We note however that, following the classical argument of T. Estermann [E] (see also [K1], [K2]), one would expect that the Euler product (4.11) $\prod_{\mathfrak{p}} \lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, \psi'_{\mathfrak{p}})$ with $\pi_{\mathfrak{p}} = \pi_s$ for all \mathfrak{p} , and unramified factors $\lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, \psi'_{\mathfrak{p}})$ as in (4.4), has the natural boundary at $Re(s) = \pm \frac{1}{2}$.

4.3.2. *Functions on the dual group.* Let us explain why it is natural to expect that the regularizing L -weights should be given in terms of Langlands

L -functions. We stress again that we do not expect that for any period data as in Section 1.4 some regularizing L -weights exist.

Instead of dealing with the local period data as in Section 1.4, we will work in this section with the equivalent setup consisting of a group $\mathfrak{G} = G_{\mathfrak{p}} \times H_{1,\mathfrak{p}} \times G_{\mathfrak{p}} \times H_{2,\mathfrak{p}}$, its subgroup $\mathfrak{H} = H_{1,\mathfrak{p}} \times H_{2,\mathfrak{p}} \subset \mathfrak{G}$ under the natural embedding and the representation $\Pi_{\mathfrak{p}} = \pi_{\mathfrak{p}}^* \otimes \chi_{1,\mathfrak{p}} \otimes \pi_{\mathfrak{p}} \otimes \chi_{2,\mathfrak{p}}^*$ of \mathfrak{G} . We will assume for simplicity that \mathfrak{G} is a split reductive group (we note that some of our examples do not fit this restriction, but one can easily modify this setup). We have then $\mathcal{P}_{\mathfrak{p}}(\Pi_{\mathfrak{p}}, \mathbb{C}) \simeq \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \chi_{1,\mathfrak{p}})^* \otimes \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \chi_{2,\mathfrak{p}})$. The local procedure of Section 1.4 should provide us with the canonical element $\mathcal{I}_{\mathfrak{p}} \in \mathcal{P}_{\mathfrak{p}}(\Pi_{\mathfrak{p}}, \mathbb{C})$ corresponding to the canonical map $I_{\mathfrak{p}} : \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \chi_{1,\mathfrak{p}}) \rightarrow \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \chi_{2,\mathfrak{p}})$ and resulting local unramified factors $D_{\mathfrak{p}}(\Pi_{\mathfrak{p}})$ as in 1.5.1. Our scheme calls for a regularization of the Euler product $\prod_{\mathfrak{p} \notin S} D_{\mathfrak{p}}(\Pi_{\mathfrak{p}})$ by the Langlands L -functions. The main issue here is how to identify the correct local L -function. Note that we do not assume that there is a natural parameter $s \in \mathbb{C}$ present in the problem and this complicates the matter. Sometimes there is such a parameter (e.g., the parameter s of the character $|\cdot|^s$ for a split tori) and in that case one can avoid the discussion below of the extended dual group.

We view local Langlands L -functions as functions on the extended dual group $\tilde{\mathfrak{G}} = \check{\mathfrak{G}} \times \mathbb{R}_+^{\times}$, where $\check{\mathfrak{G}}$ is the dual group of \mathfrak{G} . For a finite dimensional complex representation $r : \check{\mathfrak{G}} \rightarrow \text{End}_{\mathbb{C}}(V_r)$ and $s \in \mathbb{C}$, we consider a function on $\tilde{\mathfrak{G}}$ given by $L(s, \check{g} \times t, r) = [\det(1 - r(\check{g}))t^{-s}]^{-1}$. For an unramified representation $\Pi_{\mathfrak{p}}$ of $\mathfrak{G}_{\mathfrak{p}}$, the local L -factor is then given by $L_{\mathfrak{p}}(s, \Pi_{\mathfrak{p}}, r) = L(s, \sigma(\Pi_{\mathfrak{p}}) \times q_{\mathfrak{p}}^{-s}, r)$, where $\sigma(\Pi_{\mathfrak{p}}) \in \check{\mathfrak{G}}$ is the Satake parameter of $\Pi_{\mathfrak{p}}$ and $q_{\mathfrak{p}}$ is the size of the residue field at \mathfrak{p} .

We denote by $\mathcal{C}(\tilde{\mathfrak{G}})$ the space of complex valued regular central functions on $\tilde{\mathfrak{G}}$. Functions $L(s, \check{g} \times t, r)$ form a basis of $\mathcal{C}(\tilde{\mathfrak{G}})$.

Let us assume that local factors $D_{\mathfrak{p}}(\Pi_{\mathfrak{p}})$ are defined for a rich family of unramified representations $\Pi_{\mathfrak{p}}$ (e.g., for all unramified tempered representations). Under such an assumption, for each \mathfrak{p} , we can view $D_{\mathfrak{p}}(\Pi_{\mathfrak{p}})$ as a function defined on an open set $\Sigma_{\mathfrak{p}} \subset \tilde{\mathfrak{G}}$ of the dual group and resulting functions are central. Hence we obtain a collection of central functions $\{D_{\mathfrak{p}}\}_{\mathfrak{p} \notin S}$ each defined on the corresponding sheet of the set $\Sigma = \bigcup_{\mathfrak{p} \notin S} [\Sigma_{\mathfrak{p}} \times q_{\mathfrak{p}}] \subset \tilde{\mathfrak{G}}$. Any regular central function on $\tilde{\mathfrak{G}}$ is uniquely determined by its values on the set Σ .

In view of J. Bernstein rationality theorem (see [Be], [Ban]), it is natural to expect that function $D_{\mathfrak{p}}(\Pi_{\mathfrak{p}})$ is a rational function in $q_{\mathfrak{p}}^{-s}$, and such a function is determined by its values for an infinite set of unramified \mathfrak{p} and a fixed $s \neq 0$.

For a given local factors $D_{\mathfrak{p}}(\Pi_{\mathfrak{p}})$ coming from an unramified computation in the local step 1.4, it is easy to check if the factorization into local Langlands

L -functions exists since a rational central function on $\tilde{\mathfrak{G}}$ is uniquely determined by its values on *infinite* number of sets $\Sigma_{\mathfrak{p}} \times q_{\mathfrak{p}} \subset \tilde{\mathfrak{G}}$ with \mathfrak{p} unramified. The use of Langlands local factors is natural since functions $L(s, \check{g} \times t, r)$ form a basis for central functions.

The above discussion supports our expectation that Langlands L -functions could provide regularization in our global step, at least in some cases. This does not however explicitly describe the relevant L -factor. In Section 4.3.3 we will discuss an example where we can identify the regularizing factor by a geometric construction.

4.3.3. Geometry of periods and the regularizing factor. We present a geometric observation of how one can see what L -factor appears naturally in the regularization for two Whittaker periods discussed above. We hope to discuss more examples elsewhere.

In the local step 1.4 we construct a canonical map $I_{\mathfrak{p}} : \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \chi_{1,\mathfrak{p}}) \rightarrow \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \chi_{2,\mathfrak{p}})$. In particular, we obtain the collection of unramified maps $d_{\mathfrak{p}} \in \mathcal{P}_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \chi_{2,\mathfrak{p}})$ defined for unramified primes \mathfrak{p} . The local regularizing factor then comes from this unramified computation. We note that in fact it is natural to ask how this factor varies as we change the period data. Namely, pairs of period data (G, π, H_1, χ_1) and (G, π, H_2, χ_2) naturally form a “moduli” space (or a stack).

For the Hecke case from Section 3, we have a pair of a tori and a unipotent subgroup in the same Borel subgroup (and their corresponding characters). It is easy to see that this moduli space consists of a point and the regularizing factor will not change as we vary the tori in an *unramified* fashion (and change accordingly the character), i.e., conjugate the tori by an element in $N_{\mathfrak{p}} \cap K_{\mathfrak{p}}$. On the other hand, in Section 4 we consider two Whittaker functionals for unipotent subgroups which are not in the same Borel (in the example we choose them to be conjugated by the Weyl element w_a). In this case the moduli space is non-trivial. If we fix one such a functional, the second one is given up to a conjugation by an element $g \in N_{\mathfrak{p}} \setminus G_{\mathfrak{p}}$. To consider an unramified data, we should conjugate by an element $g \in (N_{\mathfrak{p}} \cap K_{\mathfrak{p}}) \setminus K_{\mathfrak{p}}$. This correspond to the compact part of the moduli space (i.e., its points over $\mathbb{Z}_{\mathfrak{p}}$). It turns out that unlike in the Hecke case, here the result of an unramified computation depends on the conjugating element g (i.e., it depends on the relative position of two unramified Whittaker functionals). We can average the resulting factor $d_{\mathfrak{p}}$ over the compact part of the moduli space, i.e., over $(N_{\mathfrak{p}} \cap K_{\mathfrak{p}}) \setminus K_{\mathfrak{p}}$. As in Lemma 4.15, $d_{\mathfrak{p}}$ is given by the Bessel function at the point $g \in G_{\mathfrak{p}}$ depending on two Whittaker functionals. The Bessel function at g represents essentially the (regularized) value of the pairing $\langle \pi_{\mathfrak{p}}(g)\mathcal{W}^{\psi}, \mathcal{W}^{\psi} \rangle$. Hence, the averaged over

$g \in (N_{\mathfrak{p}} \cap K_{\mathfrak{p}}) \setminus K_{\mathfrak{p}}$ value is given by the absolute value squared of a Whittaker functional evaluated at the standard unramified vector. This gives the factor $L(1, \pi_{\mathfrak{p}}, Ad)$ according to the well known computation (see, e.g. [J2]) which is exactly the regularizing L -factor from Theorem 4.3.1 and might be the reason we see the adjoint L -function in (4.11).

4.4. Proofs. Proofs below are based on repeated use of regularization of Whittaker functional as formulated in Section 3.4.1.

4.4.1. *Local maps.* Functions $\gamma_{p_{\bar{\psi}_{\mathfrak{p}}}, v}$ defined in Section 4.1 are not compactly supported on $N_{\mathfrak{p}}^+$ and corresponding integrals should be understood in the regularized sense as in Section 3.4.1 .

Denote by $\delta_+ = w_a^* \delta_-$ the form on N_+ and let $dn_{\mathfrak{p}}^+$ be the corresponding invariant measure on $N_{\mathfrak{p}}^+$. We denote by $\text{ev}_{dn_{\mathfrak{p}}^+}^* : P^+(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}}^+) \rightarrow \text{Hom}_{N_{\mathfrak{p}}^+}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\psi_{\mathfrak{p}}})$ the induced isomorphism. Hence we obtain the map

$$(4.12) \quad i(\psi_{\mathfrak{p}}^+, \psi_{\mathfrak{p}}^-, dn_{\mathfrak{p}}^-, dn_{\mathfrak{p}}^+) : \text{Hom}_{N_{\mathfrak{p}}^-}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\psi_{\mathfrak{p}}}') \rightarrow \text{Hom}_{N_{\mathfrak{p}}^+}(V_{\pi_{\mathfrak{p}}}, \mathbb{C}_{\psi_{\mathfrak{p}}}) ,$$

given by $i(\psi_{\mathfrak{p}}^+, \bar{\psi}_{\mathfrak{p}}^-, dn_{\mathfrak{p}}^-, dn_{\mathfrak{p}}^+) = \text{ev}_{dn_{\mathfrak{p}}^+}^*(i(\psi_{\mathfrak{p}}^+, \psi_{\mathfrak{p}}^-, dn_{\mathfrak{p}}^-))$ which has the following integral representation

$$(4.13) \quad \left[i(\psi_{\mathfrak{p}}^+, \bar{\psi}_{\mathfrak{p}}^-, dn_{\mathfrak{p}}^-, dn_{\mathfrak{p}}^+) p_{\bar{\psi}_{\mathfrak{p}}}^- \right] (v) = \int_{N_{\mathfrak{p}}^+} \bar{\psi}_{\mathfrak{p}}(n_{\mathfrak{p}}^+) \gamma_{p_{\bar{\psi}_{\mathfrak{p}}}, v}(n_{\mathfrak{p}}^+) dn_{\mathfrak{p}}^+ .$$

The integral (4.13) does not converge absolutely, and should be regularized. For a non-archimedean field $k_{\mathfrak{p}}$, we will understand under the integral (4.13) the limit

$$(4.14) \quad \lim_{l \rightarrow \infty} \int_{B_l} \bar{\psi}_{\mathfrak{p}}(n_{\mathfrak{p}}^+) \gamma_{p_{\bar{\psi}_{\mathfrak{p}}}, v}(n_{\mathfrak{p}}^+) dn_{\mathfrak{p}}^+ ,$$

where $B_l = \{n(x), |x|_{\mathfrak{p}} \leq q_{\mathfrak{p}}^l\} \subset N_{\mathfrak{p}}^+$. We will show that for any given smooth vector $v \in V_{\pi_{\mathfrak{p}}}$, the integral (4.14) stabilizes as $l \rightarrow \infty$.

For $k_{\mathfrak{p}} = \mathbb{R}$, we will use the asymptotic expansion of the Bessel function of π_{∞} in order to regularize integral (4.13) by means of analytic continuation (this procedure could be interpreted as the analytic continuation of the integral (4.13) in the space of parameters of representations of $\text{GL}_2(\mathbb{R})$).

4.4.2. *Proof of Proposition 4.1.* The proof of the proposition is based on the same idea as the proof of Lemma 3.16, i.e., we compute the local map in terms of the Bessel function.

Lemma. *The following relation holds*

$$(4.15) \quad \mathbf{i}(\psi_{\mathfrak{p}}^+, \psi_{\mathfrak{p}}^-, dn_{\mathfrak{p}}^-)\xi = j_{\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}}(a) \cdot m(\psi_{\mathfrak{p}}^+, \psi_{\mathfrak{p}}^-)\xi ,$$

for any $\xi \in P^-(V_{\pi_{\mathfrak{p}}}, L_{\psi_{\mathfrak{p}}^-}^-)$. Here $j_{\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}}$ is the Bessel function of the representation $\pi_{\mathfrak{p}}$ (see Appendix B) and $m(\psi_{\mathfrak{p}}^+, \psi_{\mathfrak{p}}^-)$ is given by the action of w as in (4.9).

The lemma clearly implies that the local map is well-defined. From the lemma we see that $\lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}, \psi_{\mathfrak{p}}') = j_{\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}}(a)$.

Proof of the lemma.

4.4.3. *Non-archimedean field.* We fix an additive character $\psi_{\mathfrak{p}}$ and choose a non-zero $(N_{\mathfrak{p}}^+, \psi_{\mathfrak{p}})$ -Whittaker functional $\mathcal{W}_+^{\psi_{\mathfrak{p}}}$ on $\pi_{\mathfrak{p}}$. Let $\mathcal{K}^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}})$ be the corresponding $(N_{\mathfrak{p}}^+, \psi_{\mathfrak{p}})$ -Kirillov model of $\pi_{\mathfrak{p}}$. In the Kirillov model, the original Whittaker functional is then given by the delta function δ_1 at $1 \in k_{\mathfrak{p}}^{\times}$. Let $j_{\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}}$ be the $\psi_{\mathfrak{p}}$ -Bessel function of $\pi_{\mathfrak{p}}$. We have

$$(4.16) \quad \begin{aligned} [m(\psi_{\mathfrak{p}}^+, \psi_{\mathfrak{p}}^-)\delta_1](v) &:= \pi_{\mathfrak{p}}^*(w_a)\delta_1(v) = \delta_1(\pi_{\mathfrak{p}}(w_a)v) \\ &= \langle \delta_{\alpha=a}, \int_{k_{\mathfrak{p}}^{\times}} j_{\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}}(\alpha t)v(t)d^{\times}t \rangle . \end{aligned}$$

Hence the functional $\delta_- = m(\psi_{\mathfrak{p}}^+, \psi_{\mathfrak{p}}^-)\delta_1$ is given by the kernel $j_{\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}}(at)$ in the $(N_{\mathfrak{p}}^+, \psi_{\mathfrak{p}})$ -Kirillov model. We now compute

$$(4.17) \quad \begin{aligned} [\mathbf{i}(\psi_{\mathfrak{p}}^+, \psi_{\mathfrak{p}}^-)\delta_-](v) &= \int_{N_{\mathfrak{p}}^+} \bar{\psi}_{\mathfrak{p}}(x)\delta_-(\pi_{\mathfrak{p}}(n_+(x))v)dx \\ &= \int_{k_{\mathfrak{p}}} \bar{\psi}_{\mathfrak{p}}(x) \left[\int_{k_{\mathfrak{p}}^{\times}} \psi_{\mathfrak{p}}(xt)j_{\pi_{\mathfrak{p}}}(at)v(t)d^{\times}t \right] dx . \end{aligned}$$

For $v \in \mathcal{S}(k_{\mathfrak{p}}^{\times})$, this immediately implies that $\mathbf{i}(\psi_{\mathfrak{p}}^+, \psi_{\mathfrak{p}}^-)\delta_- = j_{\pi_{\mathfrak{p}}}(a)\delta_1$ as in the proof of Lemma 3.4.5. For a non-archimedean field, this finishes the proof for the space of compactly supported functions.

For induced representations over a non-archimedean field, we note that the inner integral in (4.17) is absolutely convergent. This follows from the bound (B.3) $|j_{\pi_{\mathfrak{p}}}(t)| \leq C_{\pi_{\mathfrak{p}}}|t|_{\mathfrak{p}}^{-1/4}$ (we assume that the central character is trivial), and the fact that functions in $V(\chi_1, \chi_2) \subset \mathcal{K}^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}}(\chi_1, \chi_2))$ satisfy the bound $|f(t)| \ll |t|_{\mathfrak{p}}^{1/2} \log |t|_{\mathfrak{p}}$ (both bounds hold for small enough $|t|_{\mathfrak{p}}$). Hence we can interchange the order of integration in (4.17) if we understand under the outer

integral the limit $\lim_{N \rightarrow \infty} \int_{|x|_p \leq p^N} \dots$. For every $N \geq 1$, we consider the absolutely convergent double integral

$$\int_{|x|_p \leq q_p^N} \int_{k_p^\times} \bar{\psi}_p(x) \psi_p(xt) j_{\pi_p}(at) v(t) d^\times t dx .$$

Integrating now over x first, we see that for any given smooth function v , the integral stabilizes as $N \rightarrow \infty$. Hence the functional $\mathbf{i}(\psi_p^+, \psi_p^-) \delta_-$ extends to the space $\mathcal{K}^{\psi_p}(\pi_p)$ for induced representations as well. The uniqueness of the Whittaker functional implies again that the resulting functional is δ_1 , and hence we proved that for any unitary infinite-dimensional representation π_p of G over a non-archimedean field, the following relation holds:

$$(4.18) \quad \mathbf{i}(\psi_p^+, \psi_p^-) \circ m(\bar{\psi}_p^-, \psi_p^+) \delta_1 = \mathbf{i}(\psi_p^+, \psi_p^-) \delta_- = j_{\pi_p}(a) \delta_1 .$$

4.4.4. *Archimedean field.* We now prove the same statement over reals. As before we have $\delta_- = \pi(w_a) \delta_1 = j_{\pi_\infty, \psi_\infty}(a \cdot)$, and we consider the integral

$$(4.19) \quad [\mathbf{i}(\psi_\infty^+, \psi_\infty^-) \delta_-](v) = \int_{N_\infty^\pm} \bar{\psi}_\infty(x) \delta_-(\pi_\infty(n_+(x))v) dx = \\ \int_{\mathbb{R}} \bar{\psi}_\infty(x) \left[\int_{\mathbb{R}^\times} \psi_\infty(xt) j_{\pi_\infty}(at) v(t) d^\times t \right] dx .$$

For Schwartz functions $v \in \mathcal{S}(\mathbb{R}^\times)$, the integral is absolutely convergent and rapidly decaying in $|t| \rightarrow \infty$. Hence we can split the outer integral into a compact part $|t| \leq N$ and the rest: $|t| > N$. The non-compact part tends to 0 as $N \rightarrow \infty$, and in the compact part we can change the order of integration. As a result, we arrive at $[\mathbf{i}(\psi_\infty^+, \psi_\infty^-) \delta_-](v) = j_{\pi_\infty, \psi_\infty}(a) \delta_1(v)$ as in the non-archimedean case. We need to show that $\mathbf{i}(\psi_\infty^+, \psi_\infty^-) \delta_-$ extends to a functional on $\mathcal{K}^{\psi_\infty}(\pi_\infty)$. The inner integral in (4.19) is absolutely convergent for all $v \in \mathcal{K}^{\psi_\infty}(\pi_\infty)$ as follows from asymptotic of Whittaker functions of smooth vectors and from asymptotic of Bessel functions (e.g., asymptotic (B.5) for the J -Bessel function). Using these asymptotic, we see that the inner integral also has polynomial asymptotic expansion of the type $\sum_{i=0}^M a_i |t|^{\lambda-i} + O(|t|^{Re(\lambda)-M-1})$ as $|t| \rightarrow \infty$ where $\lambda \in \mathbb{C}$ is the parameter of the representation π_∞ . Such an integral could be regularized by the analytic continuation method (see [G1]). Hence we extended the functional to the whole space $\mathcal{K}^{\psi_\infty}(\pi_\infty)$, and from the uniqueness of Whittaker functional it follows that $\mathbf{i}(\psi_\infty^+, \bar{\psi}_\infty^-) \delta_- = j_{\pi_\infty, \psi_\infty}(a) \delta_1$. \square

Remark. One can use the asymptotic expansion for the Bessel function obtained in [JY], Proposition 2.3, to give a proof for the Lemma for a non-archimedean field arguing as in the case of reals.

4.4.5. *Proof of (4.3).* This is a simple computation following [S], [BM1]. The Bessel function of an induced representation $\pi_{\mathfrak{p}}(\chi_1, \chi_2)$ is given by

$$(4.20) \quad j_{\pi_{\mathfrak{p}}}(1) = \lim_{N \rightarrow \infty} \int_{p^{-N} \leq |t|_{\mathfrak{p}} \leq p^N} \chi_1^{-1} \chi_2(t) \psi_{\mathfrak{p}}(x - x^{-1}) |t|_{\mathfrak{p}}^{-1} dt .$$

The integral stabilizes as $N \rightarrow \infty$, and for unramified $\psi_{\mathfrak{p}}$ and χ_i , in fact, stabilizes at $N = 1$. We have $j_{\pi_{\mathfrak{p}}}(1) = \int_{p^{-1} \leq |t|_{\mathfrak{p}} \leq p} \chi_1^{-1} \chi_2(t) \psi_{\mathfrak{p}}(x - x^{-1}) |t|_{\mathfrak{p}}^{-1} dt = \int_{|t|_{\mathfrak{p}}=1} 1 dt + \chi_1^{-1} \chi_2(\varpi) \int_{|t|_{\mathfrak{p}}=p^{-1}} \psi_{\mathfrak{p}}(-x^{-1}) |t|_{\mathfrak{p}}^{-1} dt + \chi_1^{-1} \chi_2(\varpi^{-1}) \int_{|t|_{\mathfrak{p}}=p} \psi_{\mathfrak{p}}(x) |t|_{\mathfrak{p}}^{-1} dt$. We obtain correspondingly

$$(4.21) \quad j_{\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}}(1) = 1 - q_{\mathfrak{p}}^{-1} - \chi_1^{-1} \chi_2(\varpi) q_{\mathfrak{p}}^{-1} - \chi_1 \chi_2^{-1}(\varpi) q_{\mathfrak{p}}^{-1} .$$

4.4.6. *Proof of Proposition 4.2.* Let $\pi_{\mathfrak{p}} \simeq \pi_{\mathfrak{p}}(\chi_1, \chi_2)$ be an unramified representation. Denote by $\alpha_{\mathfrak{p}}^2 = \chi_1^{-1} \chi_2(\varpi)$. We have

$$L(1, \pi_{\mathfrak{p}}, Ad) = 1 / (1 - q_{\mathfrak{p}}^{-1})(1 - \alpha_{\mathfrak{p}}^2 q_{\mathfrak{p}}^{-1})(1 - \alpha_{\mathfrak{p}}^{-2} q_{\mathfrak{p}}^{-1}) .$$

Using (4.4) we write $\lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}) = 1 - q_{\mathfrak{p}}^{-1} - \alpha_{\mathfrak{p}}^2 q_{\mathfrak{p}}^{-1} - \alpha_{\mathfrak{p}}^{-2} q_{\mathfrak{p}}^{-1} = L(1, \pi_{\mathfrak{p}}, Ad)^{-1} - r_{\mathfrak{p}}(\alpha_{\mathfrak{p}}, q_{\mathfrak{p}})$, where $r_{\mathfrak{p}}(\alpha_{\mathfrak{p}}, q_{\mathfrak{p}}) = q_{\mathfrak{p}}^{-2} + \alpha_{\mathfrak{p}}^2 q_{\mathfrak{p}}^{-2} + \alpha_{\mathfrak{p}}^{-2} q_{\mathfrak{p}}^{-2} - q_{\mathfrak{p}}^{-3}$. From this we deduce that

$$(4.22) \quad \lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}) L(1, \pi_{\mathfrak{p}}, Ad) = (L(1, \pi_{\mathfrak{p}}, Ad)^{-1} - r_{\mathfrak{p}}(\alpha_{\mathfrak{p}}, q_{\mathfrak{p}})) L(1, \pi_{\mathfrak{p}}, Ad) = 1 - r_{\mathfrak{p}}(\alpha_{\mathfrak{p}}, q_{\mathfrak{p}}) L(1, \pi_{\mathfrak{p}}, Ad) = 1 - Q_{\mathfrak{p}}(\alpha_{\mathfrak{p}}, q_{\mathfrak{p}}) .$$

Note that the term $Q_{\mathfrak{p}}(\alpha_{\mathfrak{p}}, q_{\mathfrak{p}}) = q_{\mathfrak{p}}^{-2} \frac{(1 + \alpha_{\mathfrak{p}}^2 + \alpha_{\mathfrak{p}}^{-2} - q_{\mathfrak{p}}^{-1})}{(1 - q_{\mathfrak{p}}^{-1})(1 - \alpha_{\mathfrak{p}}^2 q_{\mathfrak{p}}^{-1})(1 - \alpha_{\mathfrak{p}}^{-2} q_{\mathfrak{p}}^{-1})}$ is expected to be bounded by $q_{\mathfrak{p}}^{-2+\varepsilon}$ according to the Ramanujan-Petersen conjecture, and hence this leads to an absolutely convergent Euler product. Namely, according to any non-trivial bound towards Ramanujan, there exists $\sigma > 0$ such that $|\alpha_{\mathfrak{p}}| \leq q_{\mathfrak{p}}^{\frac{1}{2}-\sigma}$. Hence $|Q_{\mathfrak{p}}(\alpha_{\mathfrak{p}}, q_{\mathfrak{p}})| \ll q_{\mathfrak{p}}^{-1-\sigma'}$ for some $\sigma' > 0$. This implies that $|\lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}) L(1, \pi_{\mathfrak{p}}, Ad)| \leq 1 - q_{\mathfrak{p}}^{-1-\sigma'}$ and hence the Euler product $\prod_{\mathfrak{p}} \lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}) L(1, \pi_{\mathfrak{p}}, Ad)$ is absolutely convergent.

APPENDIX A. COMPUTATION FOR THE RAMANUJAN CUSP FORM

A.1. **Numerical evaluation.** Let $\Delta(z) = \sum_{n \geq 1} \tau(n) q^n$ be the classical cusp form, studied by Ramanujan in [Ra], with $\tau(n)$ the Ramanujan tau function. The holomorphic cusp form Δ has weight 12 and level 1. In the adelic language, Δ corresponds to a cuspidal automorphic representation $\pi_{\Delta} = \otimes_{p \leq \infty} \pi_p$ of $GL(2)$ over \mathbb{Q} with trivial central character. The corresponding local components are unramified for all finite primes p , and π_{∞} is isomorphic to the discrete series representation of $GL_2(\mathbb{R})$ with the lowest weight vector of weight 12 (in [BM1] such a representation is denoted by π_6 ; see Appendix B.2 below). The

Satake parameters $(\alpha_p, \alpha_p^{-1})$ of a local representation π_p at $p < \infty$ are given by $\alpha_p + \alpha_p^{-1} = \tau(p)p^{-\frac{11}{2}}$ (since the Ramanujan conjecture is known for Δ , we have $|\alpha_p + \alpha_p^{-1}| \leq 2$). Below we attempt to calculate the constant $\lambda(\pi_\Delta, \psi)$ for the additive character $\psi(x) = e^{2\pi ix}$ of $\mathbb{Q} \setminus \mathbb{A}_\mathbb{Q}$. We have

$$\begin{aligned} \lambda_p(\pi_p, \psi_p) &= 1 - p^{-1} - \alpha_p^2 p^{-1} - \alpha_p^{-2} p^{-1} = 1 - (1 + \alpha_p^2 + \alpha_p^{-2})p^{-1} \\ &= 1 - ((\alpha_p + \alpha_p^{-1})^2 - 1)p^{-1} = 1 - (\tau^2(p)p^{-11} - 1)p^{-1} = 1 - \tau(p^2)p^{-12}. \end{aligned}$$

We also have

$$\begin{aligned} L(1, \pi_p, Ad) &= 1/(1 - p^{-1})(1 - \alpha_p^2 p^{-1})(1 - \alpha_p^{-2} p^{-1}) = \\ &= (1 - p^{-1} - \alpha_p^2 p^{-1} - \alpha_p^{-2} p^{-1} + \alpha_p^2 p^{-2} + \alpha_p^{-2} p^{-2} + p^{-2} - p^{-3})^{-1} = \\ &= (1 - (\tau^2(p)p^{-11} - 1)p^{-1} + (\tau^2(p)p^{-11} - 1)p^{-2} - p^{-3})^{-1}. \end{aligned}$$

We now consider a finite product over first N primes

$$(A.1) \quad \tilde{\lambda}_N^f(\pi_\Delta, \psi) = \prod_{i \leq N} \lambda(\pi_{p_i}, \psi_{p_i}) L(1, \pi_{p_i}, Ad)$$

for $N \geq 1$, and $\lambda_N^f(\pi_\Delta, \psi) = \tilde{\lambda}_N^f(\pi_\Delta, \psi)/L(1, \pi, Ad)$. We have $L(1, \pi, Ad) = 0.63179294573\dots$ (see [Za] and Rubinstein's Sage L -functions Calculator). We computed a numerical approximation $\lambda_{100}^f(\pi_\Delta, \psi) = 1.49154\dots$, and the archimedean counterpart $\lambda^\infty(\pi_6, \psi_\infty)$ given by the value of the classical Bessel function $j_{\pi_d, \psi_\infty}(1) = 2\pi J_{11}(4\pi) = 1.8305\dots$ (see (B.4)).

Hence we obtain the following numerical approximation for $\lambda(\Delta, e^{2\pi ix}) = 4.32\dots$. In particular, one can see that it is different from 1 (since it is easy to estimate the absolutely convergent remainder).

A.2. An infinite product. We do not understand the nature of the constant $\lambda(\pi, \psi)$. According to the local unramified computation (see (4.4)), the local constant $\lambda_p(\pi_p, \psi_p)$ and the Euler polynomial $L(1, \pi_p, Ad)$ have the same linear part. This allowed us to prove Proposition 4.2 by approximating $\lambda_p(\pi_p, \psi_p)$ with $L(1, \pi_p, Ad)^{-1}$. We can iterate this process.

We consider the polynomial $l(a, x) = 1 - (a + 1 + a^{-1})x$ in x (so that $\lambda_p(\pi_p(\alpha_p, \alpha_p^{-1})) = l(\alpha_p^2, q_p^{-1})$). Let us introduce a family of polynomials

$$(A.2) \quad p_l(a, x) = \prod_{i=-l}^l (1 - a^i x), \quad l \geq 0$$

(e.g., $p_0(a, x) = 1 - x$, $p_l(a, x) = (1 - a^l x)(1 - a^{-l} x)p_{l-1}(a, x)$). In particular, we have $L(s, \pi_p(\alpha_p, \alpha_p^{-1}), Sym^{2l}) = p_l(\alpha_p^2, q_p^{-s})^{-1}$ for the symmetric power L -function. From an easy inductive argument, it follows that there are integer

coefficients $m_{kl} \in \mathbb{Z}$, $k, l \in \mathbb{Z}_+$ such that

$$(A.3) \quad l(a, x) = p_1(a, x) \prod_{k=2}^{\infty} \left[\prod_{l=0}^{k-1} p_l(a, x^k)^{m_{kl}} \right]$$

as a formal power series identity in $\mathbb{Z}[a + a^{-1}][[x]]$.

We now introduce the constant $\lambda^f(\pi, \psi) = \prod_{\mathfrak{p} < \infty} \lambda_{\mathfrak{p}}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}})$. Assuming coefficients m_{kl} do not grow too fast (although they do grow exponentially), the formula (A.3) suggests the following representation (at least in the unramified case):

$$(A.4) \quad \lambda^f(\pi, \psi) = L(1, \pi, Sym^2) \prod_{k=2}^{\infty} \left[\prod_{l=0}^{k-1} L(k, \pi, Sym^{2l})^{-m_{kl}} \right].$$

We note that all L -functions appearing in the infinite product are in the region of the absolute convergence (assuming the Ramanujan conjecture).

The sequence m_{kl} could be interpreted as a virtual representation of $SL(2, \mathbb{C}) \times \mathbb{G}_m$. Even some basic properties of the sequence m_{kl} are not clear to us. In particular, we do not know if these coefficients are non-negative (i.e., is it true that $m_{kl} \geq 0$; this would mean that the corresponding virtual representation is a genuine representation). Also we would like to have an estimate for the growth rate of m_{kl} in order to justify convergence of the infinite product (A.4).

Here we list the first few coefficients m_{kl} (kindly computed by S. D. Miller):

$k \setminus l$	0	1	2	3	4	5	6	7	8	9	10
1		1									
2		1									
3		1	1								
4		2	1	1							
5	1	3	3	2	1						
6	1	7	6	5	2	1					
7	5	13	15	12	7	3	1				
8	9	31	33	31	18	10	3	1			
9	25	67	84	74	52	29	12	4	1		
10	55	163	198	192	137	85	39	16	4	1	
11	144	383	500	483	375	240	127	55	19	5	1

APPENDIX B. KIRILLOV MODEL

Here we collect various facts about the Kirillov model for representations of $GL(2)$ (for more detail, see [JL], [Be], [Ba], [BS], [BM1], [BM2], [S]).

B.1. Non-archimedean Kirillov model. Let $\pi_{\mathfrak{p}}$ be an irreducible infinite dimensional unitary representation of $GL(2)$ over a local field $k_{\mathfrak{p}}$. We fix a non-trivial character $\psi_{\mathfrak{p}}$ of $N_{\mathfrak{p}}$, and choose a non-zero Whittaker functional

$\mathcal{W}^{\psi_{\mathfrak{p}}}$ on $\pi_{\mathfrak{p}}$. Such a functional gives rise to the Kirillov model for $\pi_{\mathfrak{p}}$. Let $S^+(k_{\mathfrak{p}}^{\times})$ be the space of smooth (locally constant for $\mathfrak{p} < \infty$) functions on $k_{\mathfrak{p}}^{\times}$ of rapid decay at infinity (relative to the completion $k_{\mathfrak{p}}^{\times} \subset k_{\mathfrak{p}}$ at 0). Consider the map $k^{\mathcal{W}^{\psi_{\mathfrak{p}}}} : V_{\pi_{\mathfrak{p}}} \rightarrow S^+(k_{\mathfrak{p}}^{\times})$ given by $\left(k^{\mathcal{W}^{\psi_{\mathfrak{p}}}}(v)\right)(a) = \mathcal{W}^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}}(\bar{a}))$, $\bar{a} = \begin{pmatrix} a & \\ & 1 \end{pmatrix}$, $a \in k_{\mathfrak{p}}^{\times}$, for any vector $v \in V_{\pi_{\mathfrak{p}}}$ in the space of smooth vectors in $\pi_{\mathfrak{p}}$. The image $\mathcal{K}^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}})$ of this map is called the (smooth) Kirillov model of $\pi_{\mathfrak{p}}$.

We now describe the structure of $\mathcal{K}^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}})$ where $k_{\mathfrak{p}}$ is a non-archimedean local field. Let $\mathcal{S}(k_{\mathfrak{p}}^{\times})$ be the space of Schwartz functions on $k_{\mathfrak{p}}^{\times}$ (i.e., locally constant functions of compact support on $k_{\mathfrak{p}}^{\times}$). For a supercuspidal representation $\pi_{\mathfrak{p}}$, we have $\mathcal{K}^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}}) = \mathcal{S}(k_{\mathfrak{p}}^{\times})$. For induced representations $\pi_{\mathfrak{p}}(\chi_1, \chi_2)$, the space $\mathcal{K}^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}})$ is a linear span of $\mathcal{S}(k_{\mathfrak{p}}^{\times})$ and of a finite-dimensional space $V(\chi_1, \chi_2)$ with $\dim V(\chi_1, \chi_2) = 1$ or 2 . One can take as a basis of $V(\chi_1, \chi_2)$ functions on $k_{\mathfrak{p}}^{\times}$ with the support in $\mathcal{O}_{\mathfrak{p}} \cap k_{\mathfrak{p}}^{\times}$. More precisely, for an irreducible $\pi_{\mathfrak{p}}(\chi_1, \chi_2)$, $V(\chi_1, \chi_2) = \mathbb{C}\text{-span}(f_1, f_2)$ with $f_i = \chi_i(a)|a|_{\mathfrak{p}}^{\frac{1}{2}} \chi_{\mathcal{O}_{\mathfrak{p}}}$ and $\chi_{\mathcal{O}_{\mathfrak{p}}}$ which is the characteristic function of $\mathcal{O}_{\mathfrak{p}}$, for $\pi_{\mathfrak{p}}(\chi_1, \chi_2)$ with $\chi_1 = \chi_2$, $V(\chi_1, \chi_2) = \mathbb{C}\text{-span}(f_1)$ with $f_1 = \chi_1(a)|a|_{\mathfrak{p}}^{\frac{1}{2}} \chi_{\mathcal{O}_{\mathfrak{p}}}$, and for for $\pi_{\mathfrak{p}}(\chi_1, \chi_2)$ with $\chi_1 = \chi_2|\cdot|_{\mathfrak{p}}$, $V(\chi_1, \chi_2) = \mathbb{C}\text{-span}(f_1, f_2)$ with $f_1 = \chi_1(a)|a|_{\mathfrak{p}}^{\frac{1}{2}} \chi_{\mathcal{O}_{\mathfrak{p}}}$ and $f_2 = \chi_2(a)|a|_{\mathfrak{p}}^{\frac{1}{2}} \log|a|_{\mathfrak{p}} \chi_{\mathcal{O}_{\mathfrak{p}}}$.

The action of $\mathrm{GL}(2, k_{\mathfrak{p}})$ on $\mathcal{K}^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}})$ can be described as follows. The action of the Borel subgroup does not depend on the representation (however, the space of smooth vectors does!), but only on its central character, and is given by

$$(B.1) \quad \begin{aligned} \pi_{\mathfrak{p}}(n(x))f(a) &= \psi_{\mathfrak{p}}(ax)f(a), \\ \pi_{\mathfrak{p}}(\bar{t})f(a) &= f(ta), \\ \pi_{\mathfrak{p}}(\bar{z})f(a) &= \omega_{\pi_{\mathfrak{p}}}(z)f(a), \end{aligned}$$

where $a \in k^{\times}$, $\bar{t} = \begin{pmatrix} t & \\ & 1 \end{pmatrix}$, $\bar{z} = \begin{pmatrix} z & \\ & z \end{pmatrix}$ and $f \in \mathcal{K}^{\psi_{\mathfrak{p}}}(\pi_{\mathfrak{p}})$. Hence the Whittaker functional $\mathcal{W}^{\psi_{\mathfrak{p}}}$ we started with is given by the evaluation at $a = 1$ (i.e., is given by the delta function $\delta_1(f) = f(1)$).

The action of w defines the action of G (via the Bruhat decomposition), and it is known that $\pi_{\mathfrak{p}}(w)$ is given by the integral transform

$$(B.2) \quad \pi_{\mathfrak{p}}(w)f(a) = \int_{k_{\mathfrak{p}}^{\times}} \omega_{\pi_{\mathfrak{p}}}^{-1}(t) j_{\pi_{\mathfrak{p}}}(ta) f(t) d^{\times}t,$$

where the kernel $j_{\pi_{\mathfrak{p}}} = j_{\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}}$ is called the Bessel function of the representation $\pi_{\mathfrak{p}}$. The function $j_{\pi_{\mathfrak{p}}}$ is a smooth function (i.e., a locally constant for non-archimedean $k_{\mathfrak{p}}$ and smooth for $k_{\mathfrak{p}}$ archimedean). We will need a non-trivial

bound on $j_{\pi_{\mathfrak{p}}}$ near 0. We have

$$(B.3) \quad |j_{\pi_{\mathfrak{p}}}(a)| \leq C_{\pi_{\mathfrak{p}}} |\omega_{\pi_{\mathfrak{p}}}(a)|^{-1/2} |a|_{\mathfrak{p}}^{-1/4}$$

for $|a|_{\mathfrak{p}} \leq 1$. This is proved in [Ba], Corollary 4.2 (see also [JY] for the crucial computation of the germ of the corresponding orbital integral).

B.2. Kirillov model for $\mathrm{GL}_2(\mathbb{R})$. We recall here the structure of the Kirillov model for unitary representations of $\mathrm{GL}_2(\mathbb{R})$. The results we quote are discussed at length (and proved) in [BM2] from where we borrow notations as well. We will cover only representations with the *trivial* central character.

Let $\eta \in \{0, 1\}$ and $s \in \mathbb{C}$, $\mathrm{Re}(s) \geq 0$. Let $\Pi_{\eta, s}$ be the (induced) representation of G in the space of smooth functions $f : G \rightarrow \mathbb{C}$ satisfying $f(n(x)\bar{a}z(b)h) = \mathrm{sign}^{\eta}(a)|a|^{1/2+s}f(h)$. For $s \neq d - \frac{1}{2}$ where d is a positive integer, the representation $\Pi_{\eta, s}$ is irreducible and we denote it by $\pi_{\eta, s}$. In that case, it is a unitarizable representation for $\mathrm{Re}(s) = 0$ (the principal series representations) and for real s satisfying $0 < s < 1/2$ (the complementary series representations). For $s = d - \frac{1}{2}$ with $d \in \mathbb{N}$, the representation $\Pi_{\eta, s}$ has the unique irreducible subspace which we denote by π_d (suppressing η since $\pi_{1, d-1/2} \simeq \pi_{0, d-1/2}$). The quotient space $W_d = V_{\Pi_{1, d-1/2}}/V_{\pi_{1, d-1/2}}$ is finite-dimensional of dimension $2d - 1$ (e.g., $W_1 \simeq \mathbb{C}$). The lowest weight vector in the representation π_d has the weight $2d$. (Note that in notations of [BM2], Theorem 2.5.3, we have $k = 2d$.)

Let $\psi_{\infty}(x) = e^{2\pi i x}$. The Bessel function $j_{\pi_d, \psi_{\infty}}$ for representation of discrete series π_d , $d \in \mathbb{N}$, is given by

$$(B.4) \quad j_{\pi_d, \psi_{\infty}}(a) = (-1)^d 2\pi |a|^{\frac{1}{2}} J_{2d-1}(4\pi |a|^{\frac{1}{2}})$$

for $a > 0$ and $j_{\pi_d, \psi_{\infty}}(a) = 0$ for $a < 0$. Here J_n is the classical J -Bessel function (see [Ma]).

For a principal series representation $\pi_{0, ir}$, $ir \in i\mathbb{R}$, we have

$$j_{\pi_{0, ir}, \psi_{\infty}}(x) = -\pi |x|^{\frac{1}{2}} \sin(\pi ir)^{-1} (J_{2ir}(4\pi |x|^{\frac{1}{2}}) - J_{-2ir}(4\pi |x|^{\frac{1}{2}}))$$

for $x > 0$, and $j_{\pi_{0, ir}, \psi_{\infty}}(x) = -\pi |x|^{\frac{1}{2}} \sin(\pi ir)^{-1} (I_{2ir}(4\pi |x|^{\frac{1}{2}}) - I_{-2ir}(4\pi |x|^{\frac{1}{2}}))$ for $x < 0$. Analogous formulas are known for representations of complementary series (e.g., see [BM2]).

We note that for the classical Bessel function one has a well-developed asymptotic expansion (see [Ma]). In particular, the classical J -Bessel function satisfies for $0 < z \ll 1$,

$$(B.5) \quad J_{\alpha}(z) = \sum_{m=0}^N \frac{(-1)^m}{m! \Gamma(m + \alpha + 1)} \left(\frac{z}{2}\right)^{2m+\alpha} + O(|z|^{2N+\alpha+1}),$$

and $J_\alpha(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)$ for $|z| \rightarrow \infty$ and $Im(z)$ bounded. As a result, we have a similar asymptotic expansion at 0 and ∞ for Bessel functions of all representations.

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