STRONG DENSITY OF SPHERICAL CHARACTERS ATTACHED TO UNIPOTENT SUBGROUPS

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Abstract. We prove the following result in relative representation theory of a reductive p-adic group G:

Let U be the unipotent radical of a minimal parabolic subgroup of G, and let ψ be an arbitrary smooth character of U. Let $S \subset \mathrm{Irr}(G)$ be a Zariski dense collection of irreducible representations of G. Then the span of the Bessel distributions B_π attached to representations π from S is dense in the space $\mathcal{S}^*(G)^{U\times U,\psi\times\psi}$ of all $(U\times U, \psi\times\psi)$-equivariant distributions on G.

We base our proof on the following results.

(1) The category of smooth representations $\mathcal{M}(G)$ is Cohen-Macaulay.
(2) The module $ind_U^G(\psi)$ is a projective module.

Contents

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1. Introduction

Let $\mathbf{G}$ be a reductive (connected) algebraic group defined over a non-Archimedean local field F. Let $G = \mathbf{G}(F)$ be the corresponding ℓ-group.

In this paper we prove few results about the representation theory of G and about its relation with unipotent subgroups of G. Our main result concerns density of spherical characters with respect to unipotent subgroups. In order to formulate it we recall the construction of spherical character attached to an irreducible smooth representation. We denote by $\mathcal{H}(G)$ the Hecke algebra of G consisting of smooth compactly supported measures on G. We also let $C^{-\infty}(G)$ the space of generalized functions on G. These are linear functionals on $\mathcal{H}(G)$.

Definition 1.0.1. *Let (π, V) be a smooth (complex) representation of G. Let π^* be the dual representation of π and let $\tilde{\pi} := (\pi^*)^\infty$ be the contragredient (i.e. smooth dual) representation of π. For $l_1 \in \pi^*$ and $l_2 \in \tilde{\pi}^*$, we define the spherical character as the generalized function $\chi^\pi_{l_1,l_2} \in C^{-\infty}(G)$ by*

$$\langle \chi^\pi_{l_1,l_2}, f\rangle := \langle f \cdot l_1, l_2\rangle,$$

for any smooth compactly supported measure $f \in \mathcal{H}(G)$.

Note that if $H_1, H_2 \subset G$ are two (closed) subgroups, $\chi_i : H_i \to \mathbb{C}$ are (continuous) characters and $l_1 \in (\pi^*)^{H_1,\chi_1}, l_2 \in (\tilde{\pi}^*)^{H_2,\chi_2}$, then the spherical character $\chi^\pi_{l_1,l_2}$ lies in the space of $(H_1 \times H_2, \chi_1 \times \chi_2)$-equivariant generalized functions. We shall denote this space by $C^{-\infty}(G)^{H_1\times H_2,\psi\times\psi}$.

We also recall the infinitesimal character map. Let $\mathcal{M}(G)$ be the category of smooth (complex) representations of G. We will denote by $\mathrm{Irr}(G)$ the set of isomorphism classes of irreducible smooth representations of G. Let $\mathfrak{z}(G)$ be the Bernstein center of $\mathcal{M}(G)$. Given an irreducible representation $(\pi, V) \in \mathrm{Irr}(G)$ the action of each $z \in \mathfrak{z}(G)$ on V is, by Schur's lemma, given by a multiplication with a complex scalar $\chi_{(V,\pi)}(z) \in \mathbb{C}$. Notice that $\chi_{(V,\pi)} : \mathfrak{z}(G) \to \mathbb{C}$ is an algebra homomorphism. Denote by

$$\Theta(G) := Mor_{\mathbb{C}}(\mathfrak{z}(G), \mathbb{C}) \quad (1)$$

the set of algebra homomorphisms of the center.

We obtain the *infinitesimal character* map:

$$\inf : \mathrm{Irr}(G) \to \Theta(G) \quad (2)$$

defined by $\inf(V) := \chi_{(V,\pi)}$.

In words, $\inf(V)$ is the character by which $\mathfrak{z}(G)$, the Bernstein's center of G, acts on the irreducible representation V.

Remark 1.0.2. *Let A be a unital algebra over $\mathbb{C}$. It is natural to consider the set of maximal ideal in A or the set of prime ideals in A as its spectrum. Our choice to consider $\mathrm{Spec}_{\mathbb{C}}(A) := Mor_{\mathbb{C}}(A, \mathbb{C})$ is guided by the fact that when $B = \Pi_{\alpha \in I} A_\alpha$ we obtain $\mathrm{Spec}_{\mathbb{C}}(B) = \bigsqcup_{\alpha \in I} \mathrm{Spec}_{\mathbb{C}}(A_\alpha)$. Such equality does not hold for the prime (or maximal) spectrum. For more details one can consult* [LLS91].

To formulate our density result, we introduce the ad-hoc notion of **rich** collections of irreducible representations.

Definition 1.0.3. *We say that a set $\Pi \subset \mathrm{Irr}(G)$ (of smooth irreducible representations of G) is* **rich** *if $\inf(\Pi) \subset \Theta(G)$ is Zariski dense.*

Our density theorem for spherical characters is the following:

Theorem A (Density; see §5 below)**.** *Let $U < G$ be a closed subgroup which is exhausted by its open compact subgroups (i.e. any compact subset of U is contained in a compact subgroup of U). Let ψ be a character of U. Assume that (G, U) is of finite type (see Definition 2.3.1 below). Let $\Pi \subset \mathrm{Irr}(G)$ be* **rich**.

Then the space of spherical characters

$$\mathrm{span}(\{\chi^{\pi}_{l_1,l_2} | \pi \in \Pi, l_1 \in (\pi^*)^{U,\psi}, l_2 \in (\tilde{\pi}^*)^{U,\psi}\})$$

is dense in the space $C^{-\infty}(G)^{U\times U, \psi\times\psi}$

The following theorem provides examples of pairs satisfying the conditions of the theorem above.

Theorem 1.0.4. *Let $\mathbf{U}$ be a maximal unipotent subgroup of $\mathbf{G}$ and let $U = \mathbf{U}(F)$. Then (G, U) is of finite type. Namely, for any character $\psi : U \to \mathbb{C}^\times$ and any irreducible $\pi \in \mathrm{Irr}(G)$ we have:*

$$\dim(\pi^*)^{U,\psi} < \infty$$

This theorem can be easily deduced from the results of [BH03], see Appendix B for more details.

The proof of Theorem A is based on the following two results.

Theorem B (See §3 below)**.** *The category $\mathcal{M}(G)$ of smooth representations of G is Cohen-Macaulay, i.e. for any finitely generated projective module $P \in \mathcal{M}(G)$ the algebra $End(P)$ is a Cohen-Macaulay module of full dimension (see Definition 3.0.1 below) over its center.*

Theorem C (See §4 below)**.** *Let $\mathbf{G}, G, U, \psi$ and F be as in Theorem A.*

Then the module $ind_U^G(\psi)$ is a projective object in the category $\mathcal{M}(G)$ of smooth representations of G.

Using the Bernstein decomposition of the category $\mathcal{M}(G)$, the proof of Theorem C is easy. It is merely the observation that finitely presented flat modules are projective.

1.1. **Related Works.**

1.1.1. *Density of spherical characters.* Density of characters of irreducible representations inside the space of *Ad*-invariant distribution, is a classical result in representation theory (See [DKV84, Kaz86]). In [AGS15] a general result about density of spherical characters of *admissible* representations was proven. Theorem A give a very strong kind of density for the unipotent case. We do not expect this kind of density to hold in general.

1.1.2. *Cohen-Macaulay property of* $\mathcal{M}(G)$. A proof of Theorem B was recently published as Proposition 3.1 of [BBK18]. For completeness we include the proof here.

1.1.3. *Projectivity of* $ind_U^G(\psi)$. For non-degenerate ψ, the small Whittaker module $\mathcal{W} = ind_U^G(\psi)$ is studied in [BH03]. In particular it is shown to be finitely generated over any component of $\Theta(G)$, ([BH03, Theorem 4.2,(2)]).

In recent works, the small Whittaker module $\mathcal{W}$ was studied further and, in the quasi-split case, its projectivity was shown in [CS19, Corollary A.6]. Note that the proof in [CS19] can be easily generalized to cover Theorem C.

Our argument here is slightly more general, it is based on Bernstein's decomposition of $\mathcal{M}(G)$ (showing in particular that $\mathcal{M}(G)$ is locally Noetherian) and a general result from homological algebra connecting flatness to projectivity for finitely presented modules.

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2. The Bernstein decomposition of the category $\mathcal{M}(G)$ and the Bernstein center

In this section we summarize the parts of the theory of the Bernstein center and Bernstein decomposition that are used in this paper.

Recall that $\mathbf{G}$ is a reductive (connected) algebraic group defined over a non-Archimedean local field F and $G = \mathbf{G}(F)$.

Throughout the body of the paper we fix a minimal parabolic subgroup $\mathbf{B} \subset \mathbf{G}$ and its Levi subgroup $\mathbf{T}$. All the parabolic subgroups and Levi subgroups that we consider are assumed to contain $\mathbf{T}$ (even if we do not say that explicitly). Note that any parabolic subgroup that contains $\mathbf{T}$ has a unique Levi subgroup M that contains $\mathbf{T}$.

By parabolic and Levi subgroups of G we mean the groups of F-points of parabolic and Levi subgroups of $\mathbf{G}$. We now list few notations to formulate some results from the theory of Bernstein center.

Notation 2.0.1. *Let* $\mathbf{P}$ *be a parabolic subgroup of* $\mathbf{G}$ *and* $\mathbf{M}$ *be its Levi subgroup. Let* ρ *be a cuspidal representation of* $M = \mathbf{M}(F)$. *Denote*

- $i_M^G : \mathcal{M}(M) \to \mathcal{M}(G)$ *the (normalized) parabolic induction from M with respect to P.*
- $\bar{i}_M^G(\rho) : \mathcal{M}(M) \to \mathcal{M}(G)$ *the (normalized) parabolic induction from M with respect to the opposite parabolic subgroup $\bar{\mathbf{P}}$.*
- $r_M^G : \mathcal{M}(G) \to \mathcal{M}(M)$ *the (normalized) Jacquet functor.*
- $\bar{r}_M^G : \mathcal{M}(G) \to \mathcal{M}(M)$ *the (normalized) Jacquet functor with respect to the opposite parabolic subgroup $\bar{\mathbf{P}}$.*
- *We denote by G^0 the subgroup of G generared by compact subgroups.*
- *For a Levi subgroup $M \subset G$ we denote by $\mathfrak{X}_M$ the complex torus of unramified characters on M. Note that the ring $O(\mathfrak{X}_M)$ of regular functions on $\mathfrak{X}_M$ is isomorphic to the group algebra $\mathbb{C}[M/M^0]$.*
- *By a cuspidal data we mean a pair (M, ρ) consisting of a Levi subgroup $M \subset G$ and its irreducible cuspidal representation ρ.*

The following theorems summarizes the results proved in [Ber84, Ber87, BR, Bus01]. See [AS20, Theorem 2.5] and [AAG12, §§2.1] for detailed references for each item.

2.1. **Bernstein Decomposition.**

Theorem 2.1.1.

(1) *The functor r_M^G is right adjoint to $\bar{i}_M^G$, that is*

$$\mathrm{Hom}_G(\bar{i}_M^G(V), W) \cong \mathrm{Hom}_M(V, r_M^G(W)).$$

(2) *For a cuspidal data (M, ρ) let $\Psi_G(M, \rho) = \bar{i}_M^G(\rho \otimes O(\mathfrak{X}_M))$ be the normalized parabolic induction of $\rho \otimes O(\mathfrak{X}_M)$, where the action of M is diagonal. Then $\Psi_G(M, \rho) \in \mathcal{M}(G)$ is a projective generator of a direct summand of the category $\mathcal{M}(G)$.*

(3) *We call the category generated by $\Psi_G(M, \rho)$ a Bernstein block of $\mathcal{M}(G)$. The collection of those blocks is denoted by Ω. For $\omega \in \Omega$ and $V \in \mathcal{M}(G)$ we denote by $V_\omega \in \omega$ to be the corresponding direct summand. We have*

$$\mathcal{M}(G) = \prod_{\omega \in \Omega} \omega.$$

(4) *For a cuspidal data (ρ, M), the Bernstein center $\mathfrak{z}(G)$ acts through a character $\chi_{(M,\rho)}$ on $i_M^G(\rho)$. This gives a bijection between the set of conjugacy classes of cuspidal data and the variety $\Theta(G)$ (see (1)).*

The following gives a description of the algebra $\mathrm{End}_G(\Psi_G(M, \rho))$ and its center.

Theorem 2.1.2.

(1) *Let ρ be an irreducible cuspidal representation of G, and set $\mathfrak{I}_\rho := \{\psi \in \mathfrak{X}_G | \psi\rho \simeq \rho\}$ and embed $O(\mathfrak{X}_G)$ in $\mathcal{R}_{(G,\rho)} := \mathrm{End}(\Psi_G(G, \rho))$. Then there exists a decomposition*

$$\mathcal{R}_{(G,\rho)} = \bigoplus_{\psi \in \mathfrak{I}_\rho} O(\mathfrak{X}_G)\nu_\psi,$$

such that $\nu_\psi f = f_\psi \nu_\psi$ and $\nu_\psi \nu_{\psi'} = c_{\psi,\psi'} \nu_{\psi\psi'}$ where f_ψ is the translation of $f \in O(\mathfrak{X}_G)$ by ψ and $c_{\psi,\psi'}$ are scalars.

In particular $Z(\mathcal{R}_{(G,\rho)}) \cong O(\mathfrak{X}_G)^{\mathfrak{I}_\rho} \cong O(\mathfrak{X}_G/\mathfrak{I}_\rho)$

(2) For a cuspidal data (ρ, M), the natural embeding

$$End_M(\Psi_M(M,\rho))) \to End_G(\Psi_G(M,\rho))$$

gives an isomorphism

$$Z(End_M(\Psi_M(M,\rho)))^{W_{M,\rho,G}} \cong Z(End_G(\Psi_G(M,\rho)))$$

where $W_{M,\rho,G} = \{g \in G | gMg^{-1} = M; \ \rho|_{M^0} \circ Ad(g) \cong \rho|_{M^0}\}$. cf. [BR, pp 73-74].

2.2. Splitting subgroups.

Theorem 2.2.1.

(1) There exists a local base $\mathcal{B}$ of the topology at the unit $e \in G$, such that every $K \in \mathcal{B}$ is a compact (open) subgroup satisfying the following:
   (a) The category $\mathcal{M}(G,K)$ of representations generated by their K-fixed vectors is a direct summand of the category $\mathcal{M}(G)$.
   (b) The functor $V \to V^K$ is an equivalence of categories from $\mathcal{M}(G,K)$ to the category of modules over the Hecke algebra $\mathcal{H}(G,K)$.
   (c) The algebra $\mathcal{H}(G,K)$ is Noetherian.
   (d) For any (standard) Levi subgroup $M \subset G$ and for any $V \in \mathcal{M}(G)$ the map
   $$V^K \to r^G_M(V)^{K \cap M}$$
   is onto.

   We will call an open compact subgroup K satisfying these properties, a splitting subgroup.

(2) For any splitting subgroup $K \subset G$ we have
$$\mathcal{M}(G,K) = \bigoplus_{\omega \in \Omega_K} \omega$$
for some finite subset $\Omega_K \subset \Omega$.

(3) We have
$$\bigcup_{K \in \mathcal{B}} \Omega_K = \Omega$$

2.3. Relation to harmonic analysis.
We recall the following definition of pairs of finite types and some of their properties.

Definition 2.3.1 (cf. [AS20, §3])**.** *Let $H \subset G$ be a closed subgroup. We say that (G,H) is of finite type if for any character $\chi : H \to \mathbb{C}^\times$ and any $\pi \in \mathrm{Irr}(G)$ we have*

$$\dim((\pi^*)^{H,\chi}) < \infty$$

Theorem 2.3.2 ([AGS15, Theorem B.0.2])**.** *If (G,H) is of finite type then for any character $\chi : H \to \mathbb{C}^\times$ and any compact open $K < G$ the module $ind^G_H(\chi)^K$ is finitely generated $\mathcal{H}(G,K)$-module.*

Corollary 2.3.3. *If G, H, χ are as above, then for any $\omega \in \Omega(G)$ the representation $ind_H^G(\chi)_\omega$ is finitely generated.*

Proof. This follows immediately from the previous theorem (Theorem 2.3.2) and from Theorem 2.2.1 (2, 3). □

Theorem 1.0.4 provides examples of pairs of finite type.

3. Cohen-Macaulay property of the category of smooth representations of reductive p-adic groups

In this section we prove Theorem B. We first fix conventions regarding the concept of Cohen-Macaulay modules. We use the notion of Cohen-Macaulay modules over local rings, and their dimension. Since the dimension of a module over non-local ring might vary from point to point, there are several useful version of the notion of Cohen-Macaulay module in this more general situation. In the following we consider some of these versions.

Definition 3.0.1. *Let A be a commutative unital $\mathbb{C}$-algebra and let V be an A-module.*

- *We say that V is locally Cohen-Macaulay if for any $p \in \operatorname{Spec}(A)$, the module V_p is Cohen-Macaulay over A_p.*
- *We say that V is equi-dimensional Cohen-Macaulay module if there is an integer $d \geq 0$ such that for any $p \in \operatorname{Spec}(A)$, the module V_p is Cohen-Macaulay module of dimension d over A_p.*
- *We say that V is Cohen-Macaulay of full dimension if for any $p \in \operatorname{Spec}(A)$, the module V_p is Cohen-Macaulay module of dimension $\dim_p(\operatorname{Spec}(A))$.*
- *We say that V is component-wise Cohen-Macaulay if for any connected component $X \subset \operatorname{Spec}(A)$ the restriction $M|_X$ is equi-dimensional Cohen-Macaulay module over $O_X(X)$.*

We note that the following implications holds:

- equi-dimensional Cohen-Macaulay module is component-wise Cohen-Macaulay module.
- Component wise Cohen-Macaulay module is locally Cohen-Macaulay module.
- Full dimension Cohen-Macaulay module is component-wise Cohen-Macaulay.

Let us recall the formulation of Theorem B:

Theorem 3.0.2 (See also [BBK18, Proposition 3.1])**.** *The category $\mathcal{M}(G)$ of smooth representations of G is Cohen-Macaulay, that is, for any finitely generated projective module $P \in \mathcal{M}(G)$ the algebra $\operatorname{End}(P)$ is a Cohen-Macaulay module of full dimension over its center.*

Proof. Using Theorem 2.1.1(2,3), without loss of generality, we may assume that $P = \Psi_G(M, \rho) \cong i_M^G(ind_{M^0}^M(\rho|_{M^0}))$, where $M < G$ is a Levi subgroup and $\rho \in \mathrm{Irr}(M)$ is a cuspidal representation. We have

$$\mathrm{End}(P) = \mathrm{Hom}_G(i_M^G(ind_{M^0}^M(\rho|_{M^0})), i_M^G(ind_{M^0}^M(\rho|_{M^0}))) =$$
$$= \mathrm{Hom}_M(ind_{M^0}^M(\rho|_{M^0}), \bar{r}_M^G(i_M^G(ind_{M^0}^M(\rho|_{M^0}))))$$

Let $R = ind_{M^0}^M(\rho|_{M^0})$ and $Q := \bar{r}_M^G(i_M^G(ind_{M^0}^M(\rho|_{M^0})))$. Note that R is finitely generated and hence, by [BR, Proposition 33], Q is also finitely generated module. By Theorem 2.1.2(2) and [BBG97, Theorem 2.1], it is enough to show that $\mathrm{Hom}_M(R, Q)$ is finitely generated Cohen-Macaulay module over the center of $\mathrm{End}_M(R)$. For this it is enough to show that $\mathrm{Hom}_M(R, Q)$ is finitely generated Cohen-Macaulay over $\mathbb{C}[M/M^0]$. By Theorem 2.1.1(1, 2) Q is projective. Hence, by Theorem 2.1.1(2) we can decompose $Q = Q' \oplus Q''$ where Q' is a direct summand of R^n (for some n) and $Hom_M(R, Q'') = 0$. Thus, by [BBG97, Theorem 2.1], it is enough to show that $\mathrm{End}_M(R)$ is finitely generated Cohen-Macaulay module of full dimension over $\mathbb{C}[M/M^0]$. This follows from Theorem 2.1.2(1). □

4. Projectivity of $ind_U^G(\psi)$ – proof of Theorem C

In this section we prove Theorem C. For this we first prove flatness of the module $ind_U^G(\psi)$.

4.1. Flatness of $ind_U^G(\psi)$.

Proposition 4.1.1. *Let L be an ℓ-group and let $U < L$ be a closed subgroup that is exhausted by its open compact subgroups. Let ψ be a character of U. Then $\mathcal{W} := ind_U^L(\psi)$ is a flat $\mathcal{H}(L)$-module.*

Proof. Consider $\mathcal{W}$ as a right representation via the inversion on L. Consider also the functor

$$F : \mathcal{M}(\mathcal{H}(L)) \to Vect$$

given by

$$F(M) = ind_U^L(\psi) \otimes_{\mathcal{H}(L)} M.$$

We have to show that F is exact.

Let

$$W : \mathcal{M}(L) \to Vect$$

be the functor given by

$$W(\pi) = \pi_{U,\psi^{-1}}.$$

Since U is exhausted by its open compact subgroups, the Jacquet's lemma (see e.g. [BZ76, Lemma 2.35]) implies that W is an exact functor.

Let $M \in \mathcal{M}(\mathcal{H}(L))$, using

$$ind_U^L(\psi) \cong \psi \otimes_{\mathcal{H}(U)} \mathcal{H}(L),$$

we obtain

$$F(M) \cong (\psi \otimes_{\mathcal{H}(U)} \mathcal{H}(L)) \otimes_{\mathcal{H}(L)} M \cong M_{U,\psi} = W(M).$$

Hence

$$F \cong W.$$

The theorem follows. ∎

This theorem together with Theorem 2.2.1(1b) gives us the following:

Corollary 4.1.2. *Let $U < G$ be a closed subgroup that is exhausted by its open compact subgroups. Let ψ be a character of U and $\mathcal{W} := ind_U^L(\psi)$. Let $K < G$ be a splitting subgroup. Then the right $\mathcal{H}(G,K)$-module $\mathcal{W}^K$ is flat.*

4.2. Proof of projectivity of $ind_U^G(\psi)$ (Theorem C).

Proof of Theorem C. By assumption, the pair (G,U) is of finite type. Let $\mathcal{W} = ind_U^G(\psi)$. By Theorem 2.3.2 the module $\mathcal{W}^K$ is finitely generated over $\mathcal{H}(G,K)$. The algebra $\mathcal{H}(G,K)$ is Noetherian (by Theorem 2.2.1(1c)) so the module $\mathcal{W}^K$ is finitely presented. Combining with Theorem 4.1.2, we see that $\mathcal{W}^K$ is a finitely presented flat module. It is well known that a finitely presented flat module is projective (e.g. [Rot09, Theorem 3.56]). It follows that $\mathcal{W}^K$ is a projective $\mathcal{H}(G,K)$-module. Thus it is a projective object in $\mathcal{M}(\mathcal{H}(G,K))$.

For a splitting subgroup $K \subset G$ we let

$$I_K : \mathcal{M}(\mathcal{H}(G,K)) \to \mathcal{M}(G,K)$$

be the equivalence of categories as in Theorem 2.2.1. For a pair of splitting subgroups $K < L < G$ and $V \in \mathcal{M}(G)$ we can decompose $V^K = I_L(V^L)^K \oplus V_{K,L}$. By Theorem 2.2.1(1a) we can choose a descending sequence K_n of splitting subgroups that forms a basis for the topology at 1. We get

$$\mathcal{W} = \bigoplus_n I_{K_n}(\mathcal{W}_{K_n,K_{n-1}}).$$

Note that $\mathcal{W}_{K_n,K_{n-1}}$ is a direct summand of $\mathcal{W}^{K_n}$ and hence projective. We obtain that $I_{K_n}(\mathcal{W}_{K_n,K_{n-1}})$ is a projective object in $\mathcal{M}(G)$. Finally, being a direct sum of projective objects, $\mathcal{W}$ is also a projective object of $\mathcal{M}(G)$. ∎

5. Density of spherical characters – proof of Theorem A

5.1. Sketch of the proof. We consider $V := \mathcal{S}(G)_{U\times U,\psi\times\psi}$ as a module over $\mathfrak{z}(G)$ and thus as a sheaf over $\Theta(G)$. We have to show the vanishing of any section $v \in V$ that satisfy the identities

$$\langle \xi, v\rangle = 0$$

for any spherical character ξ of any $\pi \in \Pi$.

We do it by three steps:

(1) We show that for any such section v, there exist a Zariski dense subset $C \subset \Theta(G)$ such that for any $\chi \in C$ we have $v|_\chi = 0$.
(2) We show that V is (locally) finitely generated Cohen-Macaulay module of full dimension (over each component).

(3) We show that, given an element v of a Cohen-Macaulay module of full dimension M over a domain A, if there exist a Zariski dense subset $C \subset \text{Spec}(A)$ such that for any $x \in C$, then we have $v|_x = 0$ then $v = 0$.

The proof of Step (1) is in §§5.2. For this we identify an open dense subset of $\Theta(G)$ such that the category $\mathcal{M}$ have a very simple structure over points in this set (see Definition 5.2.1 and Proposition 5.2.3).

Step (2) is based on Theorems B and C. It is proven in Lemma 5.4.1.

The proof of Step (3) is in §§5.3.

5.2. **Good characters.** In this subsection we identify an open dense subset $\mathcal{U}_G$ of the set $\Theta(G)$ of characters of $\mathfrak{z}(G)$ such that the category $\mathcal{M}$ have very simple structure over these characters.

Definition 5.2.1. *We say that a cuspidal data (ρ, M) is good if the following holds*

- $i_M^G(\rho)$ *is irreducible.*
- $(W_{M,\rho,G})_\rho = M$. *Here $(W_{M,\rho,G})_\rho$ is the stabilizer of ρ under the action of the group $W_{M,\rho,G}$ defined in Theorem 2.1.2(2).*

If (M, ρ) is good, we say that $\chi_{(M,\rho)} \in \Theta(G)$ is good, where $\chi_{(M,\rho)}$ is defined in Theorem 2.1.1 (4).

We denote by $\mathcal{U}_G \subset \Theta(G)$ the set of good characters.

Notation 5.2.2. *Let $\mathcal{C}$ be an $\mathbb{C}$-abelian category and $\mathfrak{z}$ be its center. Given a character $\chi : \mathfrak{z} \to \mathbb{C}$ we define a full subcategory:*

$$\mathcal{C}|_\chi := \{c \in Ob(\mathcal{C}) | \forall \lambda \in \mathfrak{z} \text{ we have } \lambda|_c = \chi(\lambda) Id_c\}$$

Proposition 5.2.3.

(1) The set $\mathcal{U}_G \subset \Theta(G)$ is a Zariski open dense.

(2) $\forall \chi \in \mathcal{U}_G$ we have $\mathcal{M}(G)|_\chi \cong Vect$. In other words: there exist a unique (up to an isomorphism) irreducible representation $\rho \in \mathcal{M}(G)|_\chi$, and any representation in $\mathcal{M}(G)|_\chi$ is direct sum of several copies of ρ.

In order to prove this proposition we need some preparations.

Recall that any $\mathbb{C}$-abelian category $\mathcal{C}$ is a module category over the category of modules over its center $\mathfrak{z}$. In particular, for an object $M \in Ob(\mathcal{C})$ and a character $\chi : \mathfrak{z} \to \mathbb{C}$ one can define $M \otimes_{\mathfrak{z}} \chi$. Explicitly, it can be described as a quotient of M by the sum

$$\sum_{\lambda \in \mathfrak{z}} \text{Ker}(\lambda|_M - \chi(\lambda) id_M)$$

Lemma 5.2.4. *Let $\mathcal{C}, \mathfrak{z}$ and χ be as above. Let P be a projective generator of $\mathcal{C}$. Then $P \otimes_{\mathfrak{z}} \chi$ is a projective generator of $\mathcal{C}|_\chi$*

Proof. Let $X \in \mathcal{C}|_\chi$. We have

$$\mathrm{Hom}_{\mathcal{C}|_\chi}(P \otimes_{\mathfrak{z}} \chi, X) \cong \mathrm{Hom}_{\mathcal{C}}(P, X).$$

This implies the assertion. □

Lemma 5.2.5 (cf. Theorem 27 from [BR])**.** *Let (ρ, M) be a cuspidal data for G. Then for a generic unramified character χ of M the representation $i_M^G(\chi\rho)$ is irreducible.*

Lemma 5.2.6. *Let (M, ρ) be a cuspidal data. Let $w \in W_{M,\rho,G}$ then $i_M^G(\rho)$ and $i_M^G(Ad(w)\rho)$ have the same Jördan-Holder components.*

For completeness, we include the proof of this Lemma in §§5.5.

Proof of Proposition 5.2.3.

(1) By lemma 5.2.5 and the fact that $(W_{M,\rho,G})_\rho/M$ is finite we get that $\mathcal{U}_G \subset \Theta(G)$ is an open dense set.

(2) Let $\chi \in \mathcal{U}_G$ and (M, ρ) be a corresponding cuspidal data. To show that $\mathcal{M}(G)|_\chi \cong Vect$, we will exhibit a projective generator for $\mathcal{M}(G)|_\chi$ whose Endomorphism algebra is isomorphic to a matrix algebra.

By Theorem 2.1.1 (2), the representation $\Psi_G(M, \rho) = i_M^G(ind_{M^0}^M(\rho|_M))$ is a projective generator of $\mathcal{M}_{[M,\rho]}$. Thus by Lemma 5.2.4 the representation

$$Q := (i_M^G(ind_{M^0}^M(\rho|_{M^0}))) \otimes_{\mathfrak{z}(G)} \chi$$

is a projective generator of the category $\mathcal{M}(G)|_\chi$. In order to show that $\mathrm{End}_G(Q)$ is isomorphic to a matrix algebra we show that Q is isotypic (semi-simple) representation.

Theorem 2.1.2 (2) gives us the morphism $\nu : \mathfrak{z}(G) \to \mathfrak{z}(M)$. Thus, via ν, every representation $\tau \in \mathcal{M}(M)$ have an action of $\mathfrak{z}(G)$. We have

$$\begin{aligned} Q &\cong i_M^G(ind_{M^0}^M(\rho|_{M^0}) \otimes_{\mathfrak{z}(G)} \chi) \\ &\cong i_M^G(ind_{M^0}^M(\rho|_{M^0}) \otimes_{\mathbb{C}[M/M^0]} (\mathbb{C}[M/M^0] \otimes_{\mathfrak{z}(G)} \chi)) \end{aligned}$$

Our next step is to show that $\mathbb{C}[M/M^0] \otimes_{\mathfrak{z}(G)} \chi$ is a direct sum of characters.

Theorem 2.1.2 (1) gives us morphisms:

$$\mathfrak{z}(M) \to Z(\mathcal{R}_{(G,\rho)}) \cong O(\mathfrak{X}_G)^{\mathfrak{I}_\rho} \to O(\mathfrak{X}_G).$$

Denote this composition by μ.

Since $\Theta(G)$ is a countable disconnected union of algebraic varieties, we can use the classical language of algebraic geometry when operating with it locally. Since the action of $\mathfrak{I}_\rho$ on $\mathfrak{X}_G$ is free, we obtain that the corresponding map

$$\mu_* : \mathrm{Spec}(\mathbb{C}[M/M^0]) \to \Theta(M)$$

is étale.

Notice also that ν induces a map $\nu_* : \Theta(M) \to \Theta(G)$ that is étale over χ, by Theorem 2.1.2 (2) and the fact that $\chi \in \mathcal{U}_G$ is good.

Hence the map $\nu_* \circ \mu_* : \operatorname{Spec} \mathbb{C}[M/M^0] \to \Theta(G)$ is étale at χ. Let η be the character of $\mathfrak{z}(M)$ acting on ρ. We have that $\nu_*(\eta) = \chi$. By Theorem 2.1.2 (2), $\nu_*^{-1}(\chi) = W_{M,\rho,G} \cdot \eta$.

Therefore

$$\mathbb{C}[M/M^0] \otimes_{\mathfrak{z}(G)} \chi \cong \bigoplus_i \chi_i$$

with $\chi_i \in \mathfrak{X}_M$ and where

$$\chi_i \rho \simeq Ad(w_i)\rho,$$

for some $w_i \in W_{M,\rho,G}$. We get

$$Q \cong \bigoplus_i i_M^G(Ad(w_i)\rho).$$

By Lemma 5.2.6 all $i_M^G(Ad(w_i)\rho)$ are the same in the Grothendieck group of $\mathcal{M}(G)$. Since χ is good, these induced representations are all irreducible and hence isomorphic. This implies that Q is isotypic (semi-simple) representation and hence

$$\mathcal{M}(G)|_\chi \cong \mathcal{M}(End_G(Q)) \cong \mathcal{M}(Mat_{k\times k}(\mathbb{C})) \cong Vect.$$

□

Corollary 5.2.7. *Let $V = \mathcal{S}(G)_{U\times U,\psi\times\psi}$. Let $\mathcal{U}_G \subset \Theta(G)$ be as above. For any $\chi \in \mathcal{U}_G$, we let $\pi_\chi \in \inf^{-1}(\chi) \subset \operatorname{Irr}(G)$, be an irreducible representation with infinitesimal character χ. Then the space $(V^*)^{\mathfrak{z}(G),\chi^{-1}}$ is generated by the spherical characters of π_χ.*

Proof. By [AGS15, Proof of Proposition D], for any $\chi \in \Theta(G)$, the space $(V^*)^{\mathfrak{z}(G),\chi}$ is the space spanned by spherical characters of admissible representations of G on which $\mathfrak{z}(G)$ acts by χ^{-1}. The corollary follows now from the previous proposition (Proposition 5.2.3(2)). □

5.3. **Support of elements of Cohen-Macaulay modules.** In this subsection we show a strong restriction on the support of sections of Cohen-Macaulay modules of full dimension, see Corollary 5.3.3.

Definition 5.3.1. *Let A be a commutative unital finitely generated $\mathbb{C}$-algebra. We say that an A-module M is relatively torsion free if for any non-zero element $m \in M$, and for any $x \in \operatorname{Supp}(m)$ we have $\dim_x(\operatorname{Supp}(m)) = \dim_x(M)$.*

Lemma 5.3.2. *Let A be a commutative unital finitely generated $\mathbb{C}$-algebra. An equidimensional Cohen-Macaulay module M over A is relatively torsion free.*

Proof. By [BBG97, Criterion 2.5] there exists a polynomial algebra $B \subset A$ such that M is finitely generated (locally) free B-module. Let $\pi : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ be the projection. Then $\pi|_{Supp(M)}$ is a finite map. The support of m in $\mathrm{Spec}(B)$ is the projection $\pi(\mathrm{Supp}(m))$ and the support of M in $\mathrm{Spec}(B)$ is the projection $\pi(\mathrm{Supp}(M))$. Now

$$\dim(\mathrm{Supp}(m)) = \dim(\mathrm{Supp}_B(m)) = \dim(\mathrm{Supp}_B(M)) = \dim(M).$$

□

Corollary 5.3.3. *Let A be a commutative unital finitely generated $\mathbb{C}$-algebra without zero divisors and let M be a Cohen-Macaulay module of full dimension over A. Let $m \in M$ and assume that there exist a dense subset $S \subset \mathrm{Spec}(A)$ such that for each $x \in S$ we have $m|_x = 0$. Then $m = 0$.*

Proof. Consider M as a sheaf over $\mathrm{Spec}\,(A)$. There exist an open dense subset $U \subset \mathrm{Spec}(A)$ such that $M|_U$ is locally free. The assumption implies that $m|_U = 0$. By Lemma 5.3.2 we get $m = 0$. □

5.4. Cohen-Macaulay property of $\mathcal{S}(G)_{U\times U,\psi\times\psi}$ and the proof of Theorem A.

Lemma 5.4.1. *For any Bernstein block $\omega \in \Omega(G)$, the module $(\mathcal{S}(G)_{U\times U,\psi\times\psi})_\omega$ is a finitely generated Cohen-Macaulay module of full dimension over $(\mathfrak{z}(G))_\omega$.*

Proof. The inversion on G gives an anti-involution of $\mathcal{H}(G)$. It allows to make a right module V^R from a left module V of $\mathcal{H}(G)$. Note that

$$\mathcal{S}(G)_{U\times U,\psi\times\psi} \cong ind_U^G(\psi)^R \otimes_{\mathcal{H}(G)} ind_U^G(\psi).$$

By Theorem C and Corollary 2.3.3 the module $ind_U^G(\psi)_\omega$ is a direct summand of $\mathcal{H}(G)_\omega^{n_\omega}$ for some n_ω. So, it is enough to check that $\mathcal{H}(G)_\omega^{n_\omega} \otimes_{\mathcal{H}(G)_\omega} \mathcal{H}(G)_\omega^{n_\omega} \cong \mathcal{H}(G)_\omega^{n_\omega^2}$ is a Cohen-Macaulay module of full dimension over $\mathfrak{z}(G)_\omega$. This follows from Theorem B. □

Proof of Theorem A. Let C be the collection of characters of $\mathfrak{z}(G)$ corresponding to Π.

Without loss of generality $C \subset \mathcal{U}_G$ (see Definition 5.2.1). Let $V = \mathcal{S}(G)_{U\times U,\psi\times\psi}$ as above. Fix $f \in V$. We have to show that if for any spherical character ξ of an irreducible representation $\pi \in \Pi$ we have

$$\langle \xi, f \rangle = 0,$$

then $f = 0$. By Corollary 5.2.7 we obtain that $f|_\chi = 0$ for any $\chi \in C$. Note that $V \cong ind_U^G(\psi)^R \otimes_{\mathcal{H}(G)} ind_U^G(\psi)$. Thus, by Lemma 5.4.1, the module V is Cohen-Macaulay of full dimension over $\mathfrak{z}(G)$. Passing to a single component of $\Theta(G)$ and applying Corollary 5.3.3 we obtain $f = 0$. □

5.5. **Proof of Lemma 5.2.6.** Let $I(\chi) : i_M^G(\chi\rho) \to i_M^G(Ad(w)\chi\rho)$ be the intertwining operator defined for $Re(\chi) >> 0$ (see e.g. [Mui08]). By Lemma 5.2.5, for generic χ it is an isomorphism. Fix now $f \in \mathcal{H}(G)$, we have that

$$tr(f, i_M^G(\chi\rho)) = tr(f, i_M^G(Ad(w)\chi\rho))$$

for such χ. But both sides are algebraic functions (e.g. Lemma 5.13 of [Mui08]) of χ hence, by linear independence of characters we obtain the result.

Appendix A. Degenerate characters of unipotent groups

In this appendix we prove some statement (Proposition A.0.1 below) on characters of maximal unipotent subgroups of G. We use this result in the next appendix in order to prove Theorem 1.0.4

We will use the notations of [BH03, §1]. In particular we fix:

- $\mathbf{S}$ - a maximal split torus of $\mathbf{G}$.
- $\mathbf{T}$ - the centralizer of $\mathbf{S}$ in $\mathbf{G}$.
- A minimal parabolic $\mathbf{B}$ of $\mathbf{G}$, containing $\mathbf{T}$.
- $\mathbf{U}$ - the unipotent radical of $\mathbf{B}$.
- $U := \mathbf{U}(F)$.
- Φ - the set of relative roots corresponding to $(\mathbf{G}, \mathbf{S})$.
- Φ^+ - the set of positive roots corresponding to $\mathbf{B}$.
- Δ - the set of simple roots corresponding to $\mathbf{B}$.

Proposition A.0.1. *If ξ is degenerate character of U then there exist a proper parabolic $\mathbf{P} \subset \mathbf{G}$ containing $\mathbf{B}$ such that $\xi|_{\mathbf{V}(F)} = 1$, where $\mathbf{V} \subset \mathbf{P}$ is the unipotent radical of $\mathbf{P}$*

For the proof we will need some recollections from [Bor91] and results from [BH02]. For any root $\alpha \in \Phi^+$ one can define a subgroup $U_{(\alpha)} \subset U$ as in [Bor91, Proposition 21.9 (i)] or [BH02, section 1.1].

Lemma A.0.2. *If $\gamma \in \Phi^+$ is not colinear to a simple root, then $U_{(\gamma)} \subset [U, U]$*

This lemma is proven in [BH02]. As it is not formulated explicitly there, we indicate its proof.

Proof of Lemma A.0.2. Let $char(F)$ be the characteristic of F. Recall that by [BH02] we have $[U, U] = [\mathbf{U}, \mathbf{U}](F)$. We split the proof of the lemma into several cases

Case 1: $char(F) \neq 2$.
In this case the statement follows from [BH02, Theorem 4.1 (2)].

Case 2: $char(F) = 2$, and $\mathbf{G}$ is absolutely almost simple.
This case follows from [BH02, Theorem 2.1(2,3)].

Case 3: $char(F) = 2$, and $\mathbf{G}$ is F-almost simple, simply connected group.
By [Tit66, 3.1.2] and [BH02, Proposition 4.3], in this case $\mathbf{G}$ is a restriction of scalars of an absolutely almost simple group $\mathbf{G}'$. Now, by [BH02, Lemma 4.4, Lemma 4.5(2) and Proposition 4.5] the result follows from the previous case.

Case 4: $char(F) = 2$ and $\mathbf{G}$ simply connected.
This case follows from the previous one, since any simply connected group is a product of simply connected, almost simple groups.

Case 5: $char(F) = 2$, and G is semi-simple.
This case follows from the previous one, since by [BH02, Proposition 5.2] the group $\mathbf{U}$ does not change when we replace $\mathbf{G}$ by an isogenus group.

Case 6: $char(F) = 2$.
This case follows from the previus one, since $\mathbf{U} \subset [\mathbf{G}, \mathbf{G}]$.

□

One can assign to a set of simple roots $I \subset \Delta$ a maximal parabolic subgroup $\mathbf{P}_I$, see [Bor91, subsection 21.11, Proposition 21.12]. Let $\mathbf{U}_I$ be the unipotent radical of $\mathbf{P}_I$ and let $U_I = \mathbf{U}_I(F)$. Unfolding the concept of direct spanning (see [Bor91, subsection 14.3]) we have:

Lemma A.0.3 (cf. [Bor91, Proposition 21.9 (ii)])**.** *For a subset $I \subset \Delta$ denote by $[I]$ the set of elements in Φ that are non negative integral combinations of elements of I. We have*

$$U_I = \prod_{\beta \in \Phi^+ \setminus [I]} U_{(\beta)},$$

where the product can be taken in any order.

Proof of Proposition A.0.1. By the definition, a degenerate character of U is a character that is trivial on $U_{(\alpha)}$ for some some $\alpha \in \Delta$. Let α be such that $\xi|_{U_{(\alpha)}} = 1$ and let $\mathbf{P} := \mathbf{P}_{\Delta - \{\alpha\}}$. It is left to show that

$$\xi|_{U_{\Delta - \{\alpha\}}} = 1.$$

By Lemma A.0.3, it is enough to show that for any $\beta \in \Phi^+ \setminus [\Delta - \{\alpha\}]$ we have $\xi|_{U_{(\beta)}} = 1$. As $\beta \in \Phi^+ \setminus [\Delta - \{\alpha\}]$ we will do it by considering two cases:

Case 1: β is not co-linear to a simple root.
In this case the assertion follows from Lemma A.0.2

Case 2: β is co-linear to a simple root.
In this case β have to be co-linear to α and we have $U_{(\beta)} \subset U_{(\alpha)}$. By the assumption on α this implies that $\xi|_{U_{(\beta)}} = 1$.

□

Appendix B. Finite multiplicity result for degenerate Whittaker models

In this appendix we deduce Theorem 1.0.4 from the special case where the character ψ is non-degenerate, a result proven in [BH03, §4].

The reduction to the non-degenerate case is based on induction and Proposition A.0.1.

Proof of Theorem 1.0.4. The case when ξ is non-degenerate is proven in [BH03, §4]. We will prove the general case by induction on the dimension of $\mathbf{G}$. We can assume that ξ is degenerate. Using Proposition A.0.1 we can find a proper parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ such that ξ is trivial on $\mathbf{V}(F)$, where $\mathbf{V} \subset \mathbf{P}$ is the unipotent radical of $\mathbf{P}$. Let $\mathbf{M}$ be the (standard) Levi subgroup of $\mathbf{P}$.

Note that ξ is the pullback of a character ξ_0 of the group $\mathbf{U}(F)/\mathbf{V}(F) = (\mathbf{U}/\mathbf{V})(F) =: \mathbf{N}(F)$.

We can identify $\mathbf{N}$ with the unipotent radical of the minimal parabolic subgroup of $\mathbf{M}$.

We have

$$(\pi^*)^{(U,\xi)} \cong ((r_M^G(\pi))^*)^{(N,\xi_0)}.$$

Since π is irreducible it is finitely generated and admissible (see e.g. [BR, Theorem 12]). This implies that the representation $r_M^G(\pi)$ is finitely generated and admissible (see e.g. [BR, Proposition 19, Jacquet's lemma (page 64)]) Hence it is of finite length (See e.g. [Cas, Theorem 6.3.10]). The assertion follows now from the induction assumption. □

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