ALL REDUCTIVE p-ADIC GROUPS ARE TAME

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In this paper we prove the following theorem.

THEOREM 1. Let G be a reductive group over the locally compact nondiscrete non-Archimedean field F (more precisely, a group of its F-points), considered as a locally compact group, and let K be an open compact subgroup in G. Then there exists an $N = N(G, K)$ such that for any irreducible unitary representation π of G on the Hilbert space V, the dimension of the space V^K of K-invariant vectors does not exceed N.

It follows from this theorem that G is a tame group (a group of type I). For the case $G = GL_n(F)$, this theorem was proved in a recently appearing preprint of R. Howe, which uses a subtler method (and obtains a stronger result). Our proof is completely elementary (modulo the results of [1]).

The assertion of Theorem 1 was stated as a hypothesis in [1]. There, it was shown that for its proof it is sufficient to consider only square-integrable representations π.

We denote by $\mathcal{H}_K$ the convolution algebra of finite functions on G which are two-sided invariant with respect to K. Theorem 1 is equivalent to the assertion that for the algebra $\mathcal{H}_K$, the dimensions of all unitary irreducible representations are finite and bounded (see [1]). Since for square-integrable representations the space $V^{K'}$ is finite-dimensional for all open subgroups K' (see [1]), it is sufficient to prove that there are arbitrarily small compact open subgroups K in G for which the following assertion is valid:

Assertion (A). All finite-dimensional irreducible representations of the algebra $\mathcal{H}_K$ have bounded dimension.

We will prove Assertion (A) under the following assumptions on G and K.

I. G is a locally compact group, and K an open compact subgroup in G.

II. There are given in G subgroups Z, K_0, Γ^+, and Γ^-, elements $a_1, a_2, \ldots, a_l$, and a finite set Ω such that:

a) Z lies in the center of G;

b) $a_1, \ldots, a_l$ commute among themselves; we denote by A^+ the semigroup with unit which they generate;

c) K_0 is a compact subgroup, and $G = K_0A^+\Omega Z K_0$ (the Cartan decomposition);

d) $K \subset K_0$ and K_0 normalizes K;

e) $\Gamma^- \subset K$, $\Gamma^+ \subset K$, and $K = \Gamma^-\Gamma^+$;

f) $a_i\Gamma^-a_i^{-1} \subset \Gamma^-$, $a_i^{-1}\Gamma^+a_i \subset \Gamma^+$ for all i.

Let G be a reductive group over F, Z its center, A a maximal split torus, P a minimal parabolic subgroup containing A, $\bar{U}$ a unipotent subgroup complementary to P and normalizing the group A. Let Δ be the set of roots of G with respect to A, and Δ_+ be the set of positive roots corresponding to P. We put $\tilde{A}^+ = \{a \in A \mid |\alpha(a)| \geq 1 \text{ for all } \alpha \in \Delta_+\}$ (we regard α as a homomorphism $\alpha: A \to F^*$). As shown by Brunat and Tit-s (see [1]), there exist in G an open compact subgroup K_0 and a finite set $\tilde{\Omega}$ such that $G = K_0\tilde{A}^+\tilde{\Omega}K_0$.

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It is easily verified that it is possible to find in $\widetilde{A}^+$ elements $a_1, \ldots, a_l$ and a compact set C such that $\widetilde{A}^+ = A^+C(Z \cap A)$, where A^+ is the semigroup generated by $a_1, \ldots, a_l$. Choosing Ω so that $C\widetilde{\Omega}K_0 \subset \Omega K_0$, we fulfill condition c). If we now choose in K_0 a sufficiently small open normal divisor K, then the groups K, $\Gamma^- = K \cap \overline{U}$, $\Gamma^+ = K \cap P$ satisfy conditions d), e), f). (This is a result of Jacquet; see [2].)

Example: $G = GL_{l+1}(F)$, $\Omega = \{1\}$, $K_0 = GL_{l+1}(\mathcal{O})$, Z is the center of G, $K = K_\nu = \{x \in G \mid \|1 - x\| \leqslant |\pi|^\nu\}$, where $\nu \geq 1$ (here $\mathcal{O}$ is the ring of integers of the field F, π is the generator of the maximal ideal in $\mathcal{O}$, and for the matrix $x = (x_{ij})$, $\|x\| = \max |x_{ij}|$), $\Gamma^- = K \cap \bar{U}$, $\Gamma^+ = K \cap B$, where B is the group of upper triangular matrices and $\overline{U}$ is the group of lower triangular matrices with units on the diagonal. Also, a_j (j = 1, ..., l) is a diagonal matrix, $(a_j)_{ii} = 1$ for $i \leq j$, $(a_j)_{ii} = \pi$ for $i > j$.

Our proof is based on the following facts from linear algebra.

PROPOSITION 1. Let $\mathcal{L}$ be an algebra,* $\mathcal{A}$, $\mathcal{Z}$ subalgebras in $\mathcal{L}$, $A_1, \ldots, A_l \in \mathcal{A}$, $X_1, \ldots, X_p$, $Y_1, \ldots, Y_q \in \mathcal{L}$. Let us assume that $\mathcal{Z}$ lies in the center of the algebra $\mathcal{L}$, $\mathcal{Z} \subset \mathcal{A}$, $\mathcal{A}$ is the commutative algebra generated by $A_1, \ldots, A_l$ and $\mathcal{Z}$, and that any element $X \in \mathcal{L}$ can be written in the form $X = \sum X_i P_{ij} Y_j$, where $P_{ij} \in \mathcal{A}$ ($i = 1, \ldots, p$; $j = 1, \ldots, q$). Then any irreducible finite-dimensional representation of the algebra $\mathcal{L}$ has dimension at most $(pq)^{2^{l-1}}$.

PROPOSITION 2. Let V be an n-dimensional space, and $\mathcal{R} \subset \operatorname{End} V$ the commutative subalgebra generated by the operators $A_1, \ldots, A_l$ (and the identity). Then $\dim \mathcal{R} \leqslant f_l(n)$, where $f_l(n) = n^{2-1/(2^{l-1})}$.

Proposition 1 follows from Proposition 2. In fact, if $\rho: \mathcal{L} \to \operatorname{End} V^n$ is an irreducible representation, then $\rho(\mathcal{Z}) = \mathbf{C}\cdot 1$ (by Schur's lemma), $\dim \rho(\mathcal{L}) = n^2$ (by Burnside's theorem), $\dim \rho(\mathcal{A}) \leqslant f_l(n)$ by Proposition 2, and $\dim \rho(\mathcal{L}) \leqslant pq \dim \rho(\mathcal{A})$ by virtue of the conditions, so that $n^2 \leq pqf_l(n)$, from which follows the assertion of Proposition 1. We prove Proposition 2 at the end of the paper.

We now establish that for a pair G, K satisfying conditions I and II, the algebra $\mathcal{H}_K$ satisfies the conditions of Proposition 1.† If $g \in G$, then by $\overline{KgK}$ we will denote the function in $\mathcal{H}_K$ with integral 1 concentrated on the double coset KgK.

LEMMA. a) If $g, h \in G$ and g or h normalizes K, then $\overline{KgK} \cdot \overline{KhK} = \overline{KghK}$. In particular, this is so if g or h lies in Z.

b) If $g, h \in A^+$, then $\overline{KgK} \cdot \overline{KhK} = \overline{KghK}$.

Proof. In both cases it is necessary to prove that KgKhK = KghK. In case a) this is obvious, and in case b), we have $KgKhK = Kg\Gamma^-\Gamma^+hK = K(g\Gamma^-g^{-1})gh(h^{-1}\Gamma^+h)K = KghK$ (since $g\Gamma^-g^{-1} \subset \Gamma^-$, $h^{-1}\Gamma^+h \subset \Gamma^+$). The lemma is proved.

We denote by $\mathcal{Z}$ and $\mathcal{A}$ the spaces of functions in $\mathcal{H}_K$ concentrated on KZK and KA^+ZK, respectively. Then $\mathcal{Z}$ is a central subalgebra in $\mathcal{H}_K$, $\mathcal{A}$ is a commutative subalgebra, $\mathcal{Z} \subset \mathcal{A}$ and $\mathcal{A}$ is generated by $\mathcal{Z}$ and the elements $A_i = \overline{Ka_iK}$.

Case 1. We assume that all elements in Ω normalize the group K (usually this is so, since usually $\Omega = \{1\}$). Let x_i, y_j be sets of representatives of right (and thus, also double) cosets with respect to K in K_0 and ΩK_0, respectively. We put $X_i = \overline{Kx_iK}$, $Y_j = \overline{Ky_jK}$. It follows from the lemma that $\mathcal{H}_K = \sum X_i \mathcal{A} Y_j$, so the conditions of Proposition 1 are satisfied.

General Case. Let us choose x_i and y_j so that $\cup x_iK = K_0$ and $\cup y_jK = K\Omega K_0$. Let M be the space of functions in $\mathcal{H}_K$ concentrated on $KA^+ZK\Omega K_0$. It is clear that M is an $\mathcal{A}$-module with respect to left multiplication by $\mathcal{A}$ and $\mathcal{H}_K = \sum X_i M$, where $X_i = \overline{Kx_iK}$. Therefore, it is enough to prove that M is a finitely generated $\mathcal{A}$-module.

For any $p \in A^+$, we put $\Gamma_p^+ = p^{-1}\Gamma^+p \subset \Gamma^+$. Just as in the lemma, we convince ourselves that

$$Kp_1K \cdot Kp_2y_jK = Kp_1p_2\Gamma_{p_2}^+ y_jK \text{ and } Kp_1p_2y_jK = Kp_1p_2\Gamma_{p_1p_2}^+ y_jK.$$

Therefore, if $\Gamma_{p_1p_2}^+ y_jK = \Gamma_{p_2}^+ y_jK$, then $\overline{Kp_1K} \cdot \overline{Kp_2y_jK} = \overline{Kp_1p_2y_jK}$.

*An algebra is everywhere understood to mean an associative algebra over **C** with unit.

† This, in essence, was established in [3].

For any subgroup $\Gamma \subset K$, we put

$$\|\Gamma\| = \sum_j \text{(the number of cosets with respect to } K \text{ in } \Gamma y_j K).$$

Clearly, if $\Gamma' \subset \Gamma$, then $\|\Gamma'\| \leq \|\Gamma\|$, while the equality $\|\Gamma'\| = \|\Gamma\|$ means that $\Gamma' y_j K = \Gamma y_j K$ for all j.

We consider the integral quadrant $D = \{z = (z_1, \ldots, z_l) \mid z_i \in \mathbf{Z}^+\}$, and for every $z = (z_1, \ldots, z_l) \in D$, we put $p_z = a_1^{z_1} \cdot \ldots \cdot a_l^{z_l} \in A^+$ and $f(z) = \|\Gamma_{p_z}\|$. We will say that $z' < z$ if $z' \neq z$ and $z - z' \in D$. If $z' < z$, then $f(z') \geq f(z)$; if, in addition, $f(z') = f(z)$, then $\Gamma_{p_z} y_j K = \Gamma_{p_{z'}} y_j K$ for every j, so that the classes $\overline{Kp_z y_j K}$ lie in the $\mathcal{A}$-module generated by the classes $\overline{Kp_{z'} y_j K}$. Since the classes $\overline{Kp_z y_j K}$ ($z \in D$, j arbitrary) generate M as a $\mathcal{Z}$-module, we can choose as generators of M as an $\mathcal{A}$-module the elements $\overline{Kp_z y_j K}$, which correspond to singular points z, i.e., z such that for all $z' < z$ we have the strict inequality $f(z') > f(z)$.

We have arrived at a combinatorial problem: we are given on D a function f with values in $\mathbf{Z}^+$ and need to show that the number of points singular for f is finite.

For the proof, we note that if z is a singular point, then $f(z) < f(0)$, and using induction on $f(0)$, we can compute that in the quadrant $z + D$ there are a finite number of singular points. Since the complement $D \setminus (z + D)$ can be covered by a finite number of quadrants of rank $(l-1)$, using induction on l, we can compute that because there are a finite number of singular points in each of these quadrants, there are also a finite number in D.

Thus, the algebra $\mathcal{H}_K$ satisfies the conditions of Proposition 1, whence follow Assertion (A) and Theorem 1.

Proof of Proposition 2. (The proof is that of D. A. Kazhdan.) Since the algebra $\mathcal{R}$ is commutative, we can decompose the space V into the direct sum of $\mathcal{R}$-invariant subspaces V_j such that for every $P \in \mathcal{R}$ and every j, all eigenvalues of the operator $P|_{V_j}$ coincide. Clearly, we can restrict ourselves to the case $V = V_j$, and subtracting suitable constants from the operators A_i, we may assume that all the A_i are nilpotent.

Let $\varphi_l(n)$ be the maximum possible dimension of $\mathcal{R}$ for given l and n (we assume that all the A_i are nilpotent). We prove that

$$\varphi_l(n) \leqslant \varphi_l\left(\left[n - \frac{\varphi_l(n)}{n}\right]\right) + \varphi_{l-1}(n). \qquad (*)$$

Since $f_l(n) \geqslant f_l([n - f_l(n)/n]) + f_{l-1}(n)$, Proposition 2 follows from induction on l and n in (*).

Let I be the ideal in $\mathcal{R}$ generated by the operators A_i, I^k a power of it, $V^k = I^k V$. Then $V = V^0 \supset V^1 \supset \ldots \supset V^n = 0$. Let L be a subspace in V complementary to V^1, and $m = \dim L$. It is clear that $I^k L$ generates V^k modulo V^{k+1}, so that $\mathcal{R}L = V$.

Hence it follows that any operator $P \in \mathcal{R}$ is determined by its values on L, since $P\left(\sum_i P_i l_i\right) = \sum_i P_i (P l_i)$, so that $\dim \mathcal{R} \leqslant nm$. We may assume that $\dim \mathcal{R} = \varphi_l(n)$, whence $m \geq \varphi_l(n)/n$.

Let $\mathcal{R}''$ be the subalgebra generated by $A_2, A_3, \ldots, A_l$, and let $\mathcal{R}' = A_1\mathcal{R}$. Then $\mathcal{R} = \mathcal{R}' + \mathcal{R}''$. Since A_1 carries V into V^1, $\dim \mathcal{R}'$ does not exceed the dimension of the algebra obtained by restricting $\mathcal{R}$ to V^1, i.e., $\dim \mathcal{R}' \leqslant \varphi_l(\dim V^1) = \varphi_l(n - m) \leqslant \varphi_l([n - \varphi_l(n)/n])$. On the other hand, $\dim \mathcal{R}'' \leqslant \varphi_{l-1}(n)$, which implies the formula (*).

LITERATURE CITED

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