FEASIBILITY OF THE ANALYTIC CONTINUATION f_+^λ FOR CERTAIN POLYNOMIALS f

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In a lecture by I. M. Gel'fand at the Amsterdam Congress the following problem was posed.

Let $f(x_1, x_2, \ldots, x_k)$ be a polynomial with real coefficients. Consider the function $f_{+0}^\lambda = f^\lambda$, $f > 0$, $f_{+0}^\lambda = 0$, for $f \leq 0$ (λ is a complex number). For Re $\lambda > 0$ f_{+0}^λ is a locally integrable function. Denote by f_+^λ the corresponding generalized function. It depends analytically on λ. It is required to show that f_+^λ is analytically continuable, as a meromorphic function of λ, to the whole complex plane, and that its poles lie on a finite number of arithmetic progressions (see [1], p. 58, and [2] Chapter 3, Section 2, 4). We prove this assertion for a certain class of polynomials f.

Let M be a real linear space with coordinates $x_1, x_2, \ldots, x_k$; $f(x_1, \ldots, x_k)$, a polynomial with real coefficients; x, a point of M; A, the ring of rational functions a on M satisfying $a(x) \neq \infty$ (the local ring at the point x); $m = \{a \in A, a(\mathbf{x}) = 0\}$, a maximal ideal in A; $f_i = \frac{\partial f}{\partial x_i}$ the partial derivatives of the function f; $I = (f_1, \ldots, f_k)A$, an ideal in the ring A.

Definition. A point x is called simple with respect to the polynomial f if the following conditions are fulfilled:

1. $f(x) = 0$.

2. In some neighborhood of the point x in complex space the differerential of f is everywhere different from zero; moreover, this may be true at the point x.

It is well-known that this is equivalent to the condition

2a. $I \supset m^N$ for some N, or equivalently, A/I is a finite-dimensional space.

3. The function f admits the representation $f = \sum \alpha_i f_i$, where $\alpha_i \in A$, $\alpha_i(x) = 0$.

Examples. a) f is a non-singular homogeneous form, b) $f = \sum x_i^{n_i}$.

Remark 1. Condition 3 does not allow from conditions 1 and 2; e.g., $f(x_1, x_2) = x_1^5 + x_2^5 + x_1^2 \cdot x_2^2$.

THEOREM 1. If x is a simple point with respect to f, then in some neighborhood U of the point x, f_+^λ is a meromorphic function of λ.

Let R denote the ring of differential operators with coefficients from A.

LEMMA 1. There is a differential operator $D \in R$ and a nonzero polynomial H(n) with constant coefficients, such that $D \circ f^{n+1} = H(n) \cdot f^n$ for any natural number n.

Proof of the Theorem from the Lemma. Let U be a neighborhood of x in which all coefficient of D are regular. The equation $D \circ f_+^{n+1} = H(n) \cdot f_+^n$ follows easily from Lemma 1 for sufficiently large natural numbers n, since there are sufficiently smooth functions on both sides. Now we apply the following theorem due to Carlson: If $g(\lambda)$ is an analytic function for Re $\lambda > 0$, if $|g(\lambda)| < c_1 e^{c\,\mathrm{Re}\lambda}$, and g (n) = 0 for all sufficiently large natural numbers n, then $g(\lambda) \equiv 0$. For a proof see, e.g., [3], Chapter 9, Section 3. (In order

Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 2, No. 1, pp. 92-93, January-March, 1968. Original article submitted November 4, 1967.

to investigate this case we must multiply $g(\lambda)$ by $e^{-c\lambda}$ and transform the right half-plane into the unit disk). Consider the function $g(\lambda) = ((D \circ f_+^{\lambda+1} - H(\lambda) \cdot f_+^{\lambda}), \varphi)$, where φ is an infinitely differential function with compact support. It is easily seen that the conditions of Carlson's theorem are fulfilled, and thus that $g(\lambda) \equiv 0$. Whence it follows that $D \circ f_+^{\lambda+1} = H(\lambda) \cdot f_+^{\lambda}$ or $f_+^{\lambda} = f_+^{\lambda} = \frac{D \circ f_+^{\lambda+1}}{H(\lambda)}$. We show that f_+^{λ} is meromorphic for Re $\lambda > -n$. This is so far $n = 0$. If it is true for $n = l - 1$, then the function on the right is meromorphic for Re $\lambda > -l$, and so f_+^{λ} is meromorphic for Re $\lambda > -l$. By induction, the theorem is proved.

Remark 2. The poles of f_+^{λ} are concentrated on a finite number of arithmetic progressions λ_i, $\lambda_i - 1$, $\lambda_i - 2, \ldots$, where λ_i is a root of the polynomial $H(\lambda)$. If distinct progressions overlap, multiple poles may arise.

Proof of Lemma 1. Consider the following elements of the ring R: $D_i = \frac{\partial}{\partial x_i}$, $P = \sum \alpha_i D_i$ (see item 3 of the definition), $S_i = f_i P - f D_i = f_i (P + 1) - D_i f$. In this connection $P \circ f = \sum \alpha_i f_i = f$, $S_i \circ f = f_i f - f f_i = 0$. Since P and S_i give vector fields Leibnitz's formula holds for them, and thus, $P \circ f^n = n f^n$, $S_i \circ f^n = 0$.

LEMMA 2. There exists a nonzero polynomial M(P) with constant coefficients, such that M(P) is representable in the form $M(P) = \sum J_i f_i$, where $J_i \in R$ and the equation is considered on the ring R.

Proof. Let $I = (l_1, l_2, \ldots, l_k)$ denote a multiple-index $D^l = D_1^{l_1} \cdot D_2^{l_2} \cdot \ldots \cdot D_k^{l_k} \in R$, $|l| = l_1 + \ldots + l_k$. For any natural number L, we depict P^L in the form $P^L = \sum_{|l| \leqslant L} D^l \gamma_l^L$, where γ_{0l}^L is some function from A; i.e., we transfer all coefficients in differentiations to the right. It is easy to see that the γ_{0l}^L are polynomials in the α_i and their derivatives, where in each term γ_{0l}^L has at least an $|l|$-multiple zero, $\gamma_{0l}^L \in m^{|l|}$. If we look at M(P) in the form $M(P) = \sum_{L=0}^{q} b_L P^L = \sum D^l b_L \gamma_l^L$, then in order that the condition of the lemma be satisfied, it is necessary that $\sum b_L \gamma_l^L \in I$ for all l. If $|l| \geq N$ (N is such that $I \supset m^N$), then this condition is automatically satisfied. Thus, there remain a finite number of conditions of the form $\sum_{L=0}^{q} b_L \bar{\gamma}_l^L = 0$, $|l| < N$, where $\bar{\gamma}_l^L$ is the image of γ_{0l}^L in A/l. Since A/l is finite-dimensional, this results in a finite number of linear homogeneous equations in b_L (namely, there are K of these equations where K is equal to dim A/I times {the number of multiple-indices l with the condition $|l| < N$}), and by taking q = K, we can find a nontrivial solution $\{b_L\}$. The lemma is proven.

$$M(P)(P+1) = \sum J_i f_i (P+1) = \sum J_i S_i + \sum J_i D_i f.$$

Denoting H(P) by $H(P) = M(P)(P+1)$, $D = \sum J_i D_i$, we obtain

$$D \circ f^{n+1} = \left(\sum J_i D_i f\right) \circ f^n = H(P) \circ f^n = H(n) \cdot f^n,$$

since $S_i \circ f^n = 0$, $P^L \circ f^n = n^L \cdot f^n$. The proof is completed.

THEOREM 2. If all real solutions of the equation $f(x) = 0$ are simple points with respect to f, then f_+^λ is a meromorphic function of λ in the whole complex plane.

Proof. Each point may be covered by a neighborhood U (defined by the algebraic conditions $a_i \neq \infty$, where a_i are rational functions) in which f_+^λ is meromorphic. From each covering a finite subcovering U_j may be chosen (this follows easily from a theorem due to Hilbert concerning the zeros). Then by standard means a partition of unity φ_j is constructed in such a way that φ_j is infinitely differentiable, $\varphi_j \geq 0$, $\sum \varphi_j = 1$, $\varphi_j = 0$ on the complement of U_j. Then $f_+^\lambda = \sum (f_+^\lambda \varphi_j)$. On the right side is the sum of a finite number of meromorphic functions, which proves the theorem.

LITERATURE CITED

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