In a recent article [1], Godbillon and Vey constructed a certain element \( \Omega \in H^{p+1}(M, \mathbb{R}) \) for an orientable foliation \( \mathcal{F} \) of codimension \( p \) on a manifold \( M \). They showed also, that in the case \( p = 1 \) the class \( \Omega \) is related to the cohomologies of the Lie algebra of formal vector fields in \( p \) variables [3]. Denote by \( \mathcal{T}(\mathcal{F}) \) the subbundle of the tangent bundle \( \mathcal{T}(M) \) which corresponds to the foliation \( \mathcal{F} \), and let \( Q(\mathcal{F}) = \mathcal{T}(M)/\mathcal{T}(\mathcal{F}) \).

In this article, a homomorphism \( \iota_\mathcal{F}: H^*(\mathbb{W}_p; \mathbb{R}) \to H^*(\mathcal{S}(\mathcal{F}), \mathbb{R}) \) is constructed for an arbitrary foliation \( \mathcal{F} \), where \( \mathcal{S}(\mathcal{F}) \) is the space of the coordinate frame of the bundle \( Q(\mathcal{F}) \).

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1. We recall several definitions and results of [2].

**Definition 1.** Let \( a \) be a topological Lie algebra over the field \( \mathbb{R} \). A principal \( a \)-space is a manifold \( S \) (possibly infinite dimensional) together with a given homomorphism \( \varphi \) of the algebra \( a \) into the Lie algebra \( \mathfrak{a}(S) \) or vector fields on the manifold \( S \); moreover, \( \varphi \) must satisfy the following conditions:

1) Let \( s \in S \). Then the continuous map \( \varphi: a \to T(s) \), is an isomorphism of linear topological spaces, where \( T(s) \) is the tangent space of \( S \) at the point \( s \).

2) The 1-form \( \omega \) on the manifold \( S \) with values in \( a \), given by the formula \( \omega(s) = \varphi^{-1}(\xi) (\xi \in T(s)) \), is infinitely differentiable.

Denote by \( C(a) = (C^q(a), \varphi) \) the standard cochain complex of the Lie algebra \( a \) with coefficients in \( \mathbb{R} \) [3]. Recall that the elements of \( C^q(a) \) are \( q \)-linear skew-symmetric continuous functions \( \eta_1, \eta_2, ..., \eta_q \), where \( \eta_1, \eta_2, ..., \eta_q \in a \). If \( S \) is a principal \( a \)-space, then one can find a homomorphism \( \psi: C(a) \to \mathcal{Q}(S) \), where \( \mathcal{Q}(S) \) is the de Rham complex of the manifold \( S \). In particular, if \( c \in C^q(a) \), then we define the differential form \( \psi(c) \in \mathcal{Q}(S) \) by the formula \( \psi(c)(\xi_1, ..., \xi_q) = c(\omega(\xi_1), ..., \omega(\xi_q)) (\xi_1, ..., \xi_q \in T(s)) \).

It is easy to verify [2] that \( \psi \) is a homomorphism of complexes.

We introduce now the example of a principal \( \mathbb{W}_p \)-space from [2].

**Example.** Let \( N \) be a manifold of dimension \( p \). Consider the manifold \( S(N) \) of formal coordinate systems on \( N \), constructed as follows. Let \( S(k) \) denote the manifold of \( k \)-jets at the point \( 0 \) of regular maps of \( \mathbb{R}^p \) to \( N \). The manifold \( S(k) \) forms the projective system \( N \to S^1(N) \to S^2(N) \to ... \). Define \( S(N) \) as \( \lim S^k(N) \).

Now construct the homomorphism \( \varphi: \mathbb{W}_p \to S^1(N) \). Let \( s, s', s'' \) be regular maps of \( \mathbb{R}^p \) to \( N \). The ordered set of vectors \( \xi_1, ..., \xi_p \) is constructed as follows. Let \( \xi^s = (s; s'; s'') \in T^q(S_q(N)) \). The vectors \( \xi^s \) are constructed as follows. Let \( \xi^s \) be a diffeomorphism of some neighborhood of the point \( s \in \mathbb{R}^p \) onto a neighborhood of the point \( s \in \mathbb{R}^p \), the \( i \)-jet of which corresponds to the

*All manifolds, functions, vector fields, and so forth are taken to be of class \( C^\infty \).
point $s^i$. Further, let $(\eta)$ be a vector field on $\mathbb{R}^p$, the i-jet of which coincides with the i-jet of the formal vector field $\eta_i$, and $\tilde{\eta}(\eta)$ is the corresponding local vector field on $N$. Consider the one-parameter family of local diffeomorphisms of the manifold $N$, which correspond to the field $\tilde{\eta}(\eta)$. This family induces a one-parameter family of local diffeomorphisms of the manifold $S^k(N)$, to which family there corresponds a certain local vector field $\xi^k$ on $S^k(N)$. It is easy to show that, when $i > k$, the tangent vector of the field $\xi^k$ at the point $s^k$ is independent of the choice of $\eta$ and $\xi$ as well as being independent of $i$. The value of the vector field $\xi^k$ at the point $s^k$ is taken as $\xi^k$. We define the map $\tau$ by putting $\tau(\eta)(s) = \xi = (\xi^1, ..., \xi^r, ...)$. 

**THEOREM 2.** For any manifold $N$ of dimension $p$, the pair $(s^k, \eta)$ is a principal $W_2$-space.

2. Let $\mathcal{F}$ be a foliation of codimension $p$ in a manifold $M$ of dimension $n$. We now construct the homomorphism $I_\mathcal{F}: H^*(\mathcal{W}_p, \mathbb{R}) \to H^*(S^k(\mathcal{F}, \mathbb{R}))$.

**Definition 2.** Let $x \in M$ and let $u$ be a regular mapping of some neighborhood of the point $x$ into $\mathbb{R}^p$. The k-jet space at $x$ of such mappings is denoted by $J^k_x$. Let $S^k(M, \mathcal{F})$ denote the manifold $S^k(M, \mathcal{F}) = (x, [j]) x \in M, j \in J^k_x$ and let $S(M, \mathcal{F}) = \lim_{\leftarrow} S^k(M, \mathcal{F})$.

Now construct the homomorphism $\psi_\mathcal{F}: C(\mathcal{W}_p) \to \Omega(S(U, \mathcal{F}))$. Let $U$ be some neighborhood of $M$ and let $U \to N$ be a regular mapping which is a fibration whose fibers coincide with the leaves of $\mathcal{F}$. For each $k$, the mapping $J^k_x(U, \mathcal{F}) \to S^k(U, \mathcal{F})$ can be constructed. In particular, let $u$ be a map of a neighborhood of the point $x \in U$ into $\mathbb{R}^p$, which corresponds to the jet $J^k_x(U, \mathcal{F})$. Since $u$ is constant on the leaves of the foliation $\mathcal{F}$, it induces a regular mapping $\bar{u}$ of a neighborhood of the point $(s) \in N$ into $\mathbb{R}^p$. Put $\xi^k = j((\bar{u}^{-1}), j)$ is the k-jet of the mapping inverse to $\bar{u}$. The mappings $\xi^k$ define the mapping $I_\mathcal{F}: S(U, \mathcal{F}) \to S(M, \mathcal{F})$.

Now construct the global form $\psi_\mathcal{F}: (\mathcal{W}_p) \to \Omega(S(U, \mathcal{F}))$ by putting $\psi_\mathcal{F}(\eta)(s) = (\xi)$, where $\xi = (\xi^1, ..., \xi^r, ..., \xi^r)$. The value of the vector field $\xi^k$ at the point $s^k$ is taken as $\xi^k$. We define the map $\tau$ by putting $\tau(\eta)(s) = \eta^k = (\eta^1, ..., \eta^r, ..., \eta^r)$. 

The mapping $\psi_\mathcal{F}: C(\mathcal{W}_p) \to \Omega(S(M, \mathcal{F}))$ is a homomorphism of complexes. Therefore it induces a map of homologies $I_\mathcal{F}: H^*(\mathcal{W}_p, \mathbb{R}) \to H^*(S(M, \mathcal{F}), \mathbb{R})$.

We now note that the fibration $S_{\infty}: S(M, \mathcal{F}) \to S^1(M, \mathcal{F})$ has a cross section $h: S^1(M, \mathcal{F}) \to S(M, \mathcal{F})$ and any two such cross sections are homotopic.

Put $I_\mathcal{F} = h^* \circ \psi_\mathcal{F}: H^*(\mathcal{W}_p, \mathbb{R}) \to H^*(S^1(M, \mathcal{F}), \mathbb{R})$. From what has been said above, it follows that the homomorphism $I_\mathcal{F}$ does not depend on the choice of cross section $h$. It is also clear that the space $S^1(M, \mathcal{F})$ coincides with that described in the introduction.

3. For the case $p = 1$, we describe the relation of the mapping $I_\mathcal{F}$ to the Godbillon-Vey class $\alpha$.

Let $\alpha$ be a nondegenerate 1-form on M which generates an orientable foliation $\mathcal{F}$ of codimension 1. Let $\alpha_1$ be such a 1-form on $M$ such that $\alpha = \alpha_1 \wedge \alpha$. Then $\alpha_1 \wedge \alpha$ is a closed form which gives rise to the cohomology class $\alpha$ in $\mathcal{W}_1$.

The mapping $\tau: M \to S^1(M, \mathcal{F})$ is given by the formula $\tau(s) = (s, \alpha_1)$, where $s \in M$, while $\alpha$ is taken to be a 1-jet at the point $x$ of the image of $M$ in $\mathbb{R}$. Denote by $\tilde{\tau}$ the continuous mapping of complexes $C(\mathcal{W}_1) \to \Omega(S(M, \mathcal{F}))$ by $\psi_\mathcal{F}(\eta) \to \Omega(S^1(M, \mathcal{F}))$. Define the cochain $\gamma \in C^1(\mathcal{W}_1)$ by $\gamma(P, \frac{\partial}{\partial x}) = P(0)$. Then $d\gamma = \gamma_1 \wedge \gamma$, where $\gamma_1(\frac{\partial}{\partial x}) = -P'(0)$.

It follows from [3] that the natural nontrivial cohomology class $\delta$ of the algebra $\mathcal{W}_1$ is the cocycle $\gamma_1 \wedge \phi_1 \in C^1(\mathcal{W}_1)$. It is easy to verify that $\tau(\gamma) = \alpha$. Put $\tilde{\tau}(\eta_1) = \alpha_1$. Then $\alpha = \alpha_1 \wedge \alpha$ and $\tau(\eta_1 \wedge \phi_1) = \alpha_1 \wedge \alpha$. Thus, $I_\mathcal{F}(\delta) = \alpha$.

**LITERATURE CITED**


