CHARACTERISTIC CLASSES OF FOLIATIONS

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In a recent article [1], Godbillon and Vey constructed a certain element $\Omega \in H^{2p+1}(M, \mathbf{R})$ for an orientable foliation $\mathscr{F}$ of codimension p on a manifold M.* They showed also, that in the case p = 1 the class Ω is related to the cohomologies of the Lie algebra of formal vector fields in p variables [3]. Denote by $T(\mathscr{F})$ the subbundle of the tangent bundle T(M) which corresponds to the foliation $\mathscr{F}$, and let $Q(\mathscr{F}) = T(M)/T(\mathscr{F})$.

In this article, a homomorphism $f_{\mathscr{F}}: H^*(W_p; \mathbf{R}) \to H^*(S^1(\mathscr{F}), \mathbf{R})$ is constructed for an arbitrary foliation $\mathscr{F}$, where $S^1(\mathscr{F})$ is the space of the coordinate frame of the bundle $Q(\mathscr{F})$.

The basis for our construction is a differential geometry construction from [2]; see also [4]. The authors wish to thank I. M. Gel'fand and D. A. Kazhdan for the opportunity of seeing their manuscript [2], as without it the present work probably would not have been completed. The authors thank V. I. Arnol'd, S. M. Vishik, and D. B. Fuks for their helpful comments.

1. We recall several definitions and results of [2].

Definition 1. Let $\mathfrak{a}$ be a topological Lie algebra over the field R. A principal $\mathfrak{a}$-space is a manifold S (possibly infinite dimensional) together with a given homomorphism φ of the algebra $\mathfrak{a}$ into the Lie algebra $\mathfrak{a}(S)$ or vector fields on the manifold S; moreover, φ must satisfy the following conditions:

1) Let $s \in S$. Then the continuous map $\varphi_s: \mathfrak{a} \to T(S)_s$ is an isomorphism of linear topological spaces, where $T(S)_s$ is the tangent space of S at the point s.

2) The 1-form ω on the manifold S with values in $\mathfrak{a}$, given by the formula $\omega(\xi) = \varphi_s^{-1}(\xi)$ $(\xi \in T(S)_s)$, is infinitely differentiable.

Denote by $C(\mathfrak{a}) = \{C^q(\mathfrak{a}), d^q\}$ the standard cochain complex of the Lie algebra $\mathfrak{a}$ with coefficients in R [3]. Recall that the elements of $C^q(\mathfrak{a})$ are q-linear skew-symmetric continuous functions $(\eta_1, \eta_2, \ldots, \eta_q)$, where $\eta_1, \eta_2, \ldots, \eta_q \in \mathfrak{a}$. If S is a principal $\mathfrak{a}$-space, then one can find a homomorphism $\psi: C(\mathfrak{a}) \to \Omega(S)$, where $\Omega(S)$ is the de Rham complex of the manifold S. In particular, if $c \in C^q(\mathfrak{a})$, then we define the differential form $\psi(c) \in \Omega(S)$ by the formula $\psi(c)(\xi_1, \ldots, \xi_q) = c(\omega(\xi_1), \ldots, \omega(\xi_q))$ $(\xi_1, \ldots, \xi_q \in T(S)_s, s \in S)$.

It is easy to verify [2] that ψ is a homomorphism of complexes.

We introduce now the example of a principal W_p-space from [2].

Example. Let N be a manifold of dimension p. Consider the manifold S(N) of formal coordinate systems on N, constructed as follows. Let $S^k(N)$ denote the manifold of k-jets at the point 0 of regular maps of R^p to N. The manifold $S^k(N)$ forms the projective system $N = S^0(N) \leftarrow S^1(N) \leftarrow \ldots \leftarrow S^k(N) \leftarrow \ldots$ Define S(N) as $\varprojlim S^k(N)$.

Now construct the homomorphism $\varphi: W_p \to \mathfrak{a}(S(N))$. Let $\eta \in W_p$ and $s = (s^0, s^1, \ldots, s^k, \ldots) \in S$ $(s^k \in S^k(N))$. For the element η construct the vector $\xi_s \in T(S(N))_s$, i.e., the ordered set of vectors $\xi_s = (\xi^0, \ldots, \xi^k, \ldots)$, where $\xi^k \in T(S^k(N))_{s^k}$. The vectors ξ^k are constructed as follows. Let $\bar{s}^i$ be a diffeomorphism of some neighborhood of the point $0 \in \mathbf{R}^p$ onto a neighborhood of the point $x = s^0 \in N$, the i-jet of which corresponds to the

*All manifolds, functions, vector fields, and so forth are taken to be of class C^∞.

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point s^i. Further, let $(\bar\eta)$ be a vector field on R^p, the i-jet of which coincides with the i-jet of the formal vector field η, and $\bar{s}^i(\bar\eta)$ is the corresponding local vector field on N. Consider the one-parameter family of local diffeomorphisms of the manifold N, which correspond to the field $\bar{s}^i(\bar\eta)$. This family induces a one-parameter family of local diffeomorphisms of the manifold $S^k(N)$, to which family there corresponds a certain local vector field $\tilde\xi^k$ on $S^k(N)$. It is easy to show that, when $i > k$ the tangent vector of the field $\tilde\xi^k$ at the point s^k is independent of the choice of $\bar{s}^i$ and $\bar\eta$ as well as being independent of i. The value of the vector field $\tilde\xi^k$ at the point s^k is taken as ξ^k. We define the map φ by putting $\varphi(\eta)(s) = \xi_s = (\xi^0, \ldots, \xi^k, \ldots)$.

THEOREM [2]. For any manifold N of dimension p, the pair $(S(N), \varphi)$ is a principal W_p-space.

2. Let $\mathcal{F}$ be a foliation of codimension p in a manifold M of dimension n. We now construct the homomorphism $l_{\mathcal{F}}: H^*(W_p, R) \to H^*(S^1(\mathcal{F}), R)$.

Definition 2. Let $x \in M$ and let u be a regular mapping of some neighborhood of the point x into R^p, which satisfies the following conditions: (a) u is constant on the leaves of the foliation $\mathcal{F}$, and (b) u(x) = 0. The k-jet space at x of such mappings is denoted by J_x^k. Let $S^k(M, \mathcal{F})$ denote the manifold $S_k(M, \mathcal{F}) = \{x, j \mid x \in M, j \in J_x^k\}$ and let $S(M, \mathcal{F}) = \varprojlim S^k(M, \mathcal{F})$.

Now construct the homomorphism $\psi_{\mathcal{F}}: C(W_p) \to \Omega(S(M, \mathcal{F}))$. Let U be some neighborhood of M and $l: U \to N$ be a regular mapping which is a fibration whose fibers coincide with the leaves of $\mathcal{F}$. For each k, the mapping $l^k: S^k(U, \mathcal{F}) \to S^k(N)$ can be constructed. In particular, let u be a map of a neighborhood of the point $x \in U$ into R^p, which corresponds to the jet $s^k \in S^k(U, \mathcal{F})$. Since u is constant on the leaves of the foliation $\mathcal{F}$, it induces a regular mapping $\tilde{u}$ of a neighborhood of the point $l(x) \in N$ into R^p. Put $l^k(s^k) = j(\tilde{u}^{-1})$, where $j(\tilde{u}^{-1})$ is the k-jet of the mapping inverse to $\tilde{u}$. The mappings l^k $(k = 0, 1, \ldots)$ define the mapping $l^\infty: S(U, \mathcal{F}) \to S(N)$.

Now construct the homomorphism $\psi_{\mathcal{F}, U}: C(W_p) \to \Omega(S(U, \mathcal{F}))$. Since S(N) is a principal W_p-space, there is a differential form $\psi(c) \in \Omega^q(S(N))$ (see Sec. 1) corresponding to each cochain $c \in C^q(W_p)$. Put $\psi_{\mathcal{F}, U}(c) = (l^\infty)^* \psi(c)$. $S(U, \mathcal{F})$ is an open subset of $S(M, \mathcal{F})$. It is easy to verify that on the intersection of two such neighborhoods $S(U, \mathcal{F})$ and $S(U', \mathcal{F})$ the forms $\psi_{\mathcal{F}, U}(c)$ and $\psi_{\mathcal{F}, U'}(c)$ coincide $(c \in C^q(W_p))$. Denote by $S(M, \mathcal{F})$ the global form so obtained on $\psi_{\mathcal{F}}(c)$.

The mapping $\psi_{\mathcal{F}}: C(W_p) \to \Omega(S(M, \mathcal{F}))$ is a homomorphism of complexes. Therefore it induces a map of homologies $\psi_{\mathcal{F}}^*: H^*(W_p, R) \to H^*(S(M, \mathcal{F}), R)$.

We now note that the fibration $S_{\infty, 1}: S(M, \mathcal{F}) \to S^1(M, \mathcal{F})$ has a cross section $h: S^1(M, \mathcal{F}) \to S(M, \mathcal{F})$ and any two such cross sections are homotopic.

Put $l_{\mathcal{F}} = h^* \circ \psi_{\mathcal{F}}^*: H^*(W_p, R) \to H^*(S^1(M, \mathcal{F}), R)$. From what has been said above, it follows that the homomorphism $l_{\mathcal{F}}$ does not depend on the choice of cross section h. It is also clear that the space $S^1(M, \mathcal{F})$ coincides with that described in the introduction.

3. For the case p = 1, we describe the relation of the mapping $l_{\mathcal{F}}$ to the Godbillon-Vey class Ω. Let α be a nondegenerate 1-form on M which generates an orientable foliation $\mathcal{F}$ of codimension 1. Recall the construction of the class Ω [1]. Let α_1 be such a 1-form on M such that $d\alpha = \alpha_1 \wedge \alpha$. Then $\alpha_1 \wedge d\alpha_1$ is a closed form which gives rise to the cohomology class Ω.

The mapping $\tau: M \to S^1(M, \mathcal{F})$ is given by the formula $\tau(x) = (x, \alpha_x)$, where $x \in M$, while α is taken to be a 1-jet at the point x of the image of M in R. Denote by $\tilde\tau$ the continuous mapping of complexes $C(W_1) \xrightarrow{\psi_{\mathcal{F}}} \Omega(S(M, \mathcal{F})) \xrightarrow{h^*} \Omega(S^1(M, \mathcal{F})) \xrightarrow{\tau^*} \Omega(M)$. Define the cochain $\gamma \in C^1(W_1)$ by $\gamma\left(P \frac{\partial}{\partial x}\right) = P(0)$. Then $d\gamma = \gamma_1 \wedge \gamma$, where $\gamma_1\left(P \frac{\partial}{\partial x}\right) = -P'(0)$.

It follows from [3] that the natural nontrivial cohomology class δ of the algebra W_1 is the cocycle $\gamma_1 \wedge d\gamma_1 \in C^3(W_1)$. It is easy to verify that $\tilde\tau(\gamma) = \alpha$. Put $\tilde\tau(\gamma_1) = \alpha_1$. Then $d\alpha = \alpha_1 \wedge \alpha$ and $\tilde\tau(\gamma_1 \wedge d\gamma_1) = \alpha_1 \wedge d\alpha_1$. Thus, $l_{\mathcal{F}}(\delta) = \Omega$.

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