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The Cohen-Macaulay property of the category of $(\mathfrak{g}, K)$-modules

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Abstract. Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair. In this paper we prove that if P and P' are two projective $(\mathfrak{g}, K)$-modules, then $\mathrm{Hom}(P, P')$ is a Cohen-Macaulay module over the algebra $\mathcal{Z}(\mathfrak{g}, K)$ of K-invariant elements in the center of $U(\mathfrak{g})$. This fact implies that the category of $(\mathfrak{g}, K)$-modules is locally equivalent to the category of modules over a Cohen-Macaulay algebra, where by a Cohen-Macaulay algebra we mean an associative algebra that is a free finitely generated module over a polynomial subalgebra of its center.

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1. Introduction

1.1. Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair, i.e.

1) $\mathfrak{g}$ is a reductive Lie algebra over $\mathbb{C}$ endowed with an involution θ;
2) K is a complex reductive algebraic group acting by automorphisms of the pair $(\mathfrak{g}, \theta)$;
3) the Lie algebra $\mathfrak{k} = Lie(K)$ is identified with $\mathfrak{g}^\theta$ in such a way that the adjoint action of $\mathfrak{k}$ on $\mathfrak{g}$ coincides with the differential of the K-action on $\mathfrak{g}$.

Let $U(\mathfrak{g})$ (resp. $U(\mathfrak{k})$) be the universal enveloping algebra of $\mathfrak{g}$ (resp. of $\mathfrak{k}$). We shall denote by $\mathcal{Z}(\mathfrak{g})$ the center of $U(\mathfrak{g})$ and let us set $\mathcal{Z}(\mathfrak{g}, K) = \mathcal{Z}(\mathfrak{g})^K$. Note that $\mathcal{Z}(\mathfrak{g}, K)$ may be different from $\mathcal{Z}(\mathfrak{g})$ in the case when K is disconnected.

Our main object of study is the category $\mathcal{M}(\mathfrak{g}, K)$ of $(\mathfrak{g}, K)$-modules. By definition, a $(\mathfrak{g}, K)$-module is a vector space V, endowed with an algebraic action of the group K and an action of the Lie algebra $\mathfrak{g}$; these two actions are compatible in the following way:

1) the two natural actions of the Lie algebra $\mathfrak{k}$ on V coincide;

2) the action map $\mathfrak{g} \otimes V \to V$ is a morphism of K-modules (the second condition follows from the first one when K is connected).

1.2. The main result

Let P, P' be two $(\mathfrak{g}, K)$-modules. Then $\mathrm{Hom}_{\mathfrak{g},K}(P, P')$ is naturally a $\mathcal{Z}(\mathfrak{g}, K)$-module. Our main result is the following

Theorem. *Let P and P' be two projective finitely generated $(\mathfrak{g}, K)$-modules. Then the space $\mathrm{Hom}_{\mathfrak{g},K}(P, P')$ is a Cohen-Macaulay module over $\mathcal{Z}(\mathfrak{g}, K)$ of dimension ℓ (here ℓ is the split rank of the pair $(\mathfrak{g}, K)$ — cf.* 1.5*).*

We will review the theory of Cohen-Macaulay modules in section 2.

The geometric meaning of this theorem is that it allows us to give a relatively simple local description of the category $\mathcal{M}(\mathfrak{g}, K)$. Namely, in section 4 we will show that this category is locally equivalent to the category of modules over a Cohen-Macaulay algebra (cf. 4.2).

1.3. Harish-Chandra modules

There is a local description of the category $\mathcal{H}(\mathfrak{g}, K)$ of Harish-Chandra modules over the pair $(\mathfrak{g}, K)$, which is probably more transparent.

By a Harish-Chandra module we mean a finitely generated $(\mathfrak{g}, K)$-module V whose annihilator in $\mathcal{Z}(\mathfrak{g}, K)$ has finite codimension (this is equivalent to the fact that $\mathrm{Hom}_K(\sigma, V)$ is finite-dimensional for every finite-dimensional representation σ of K).

Now let U be any subset of $\mathrm{Specmax}(\mathcal{Z}(\mathfrak{g}, K))$. Denote by $\mathcal{H}(\mathfrak{g}, K, U)$ the full subcategory of $\mathcal{H}(\mathfrak{g}, K)$ consisting of Harish-Chandra modules supported inside U when viewed as sheaves over $\mathrm{Specmax}(\mathcal{Z}(\mathfrak{g}, K))$.

Theorem. *Every $\chi \in \mathrm{Specmax}(\mathcal{Z}(\mathfrak{g}, K))$ has a neighborhood U in the analytic topology such that the category $\mathcal{H}(\mathfrak{g}, K, U)$ is equivalent to the category of finite-dimensional modules over an algebra B that has the following properties:*

1) *B is an algebra over the ring $\mathcal{O}(D)$ of holomorphic functions on the unit ball $D \subset \mathbb{C}^\ell$;*
2) *as an $\mathcal{O}(D)$-module, B is free and finitely generated.*

Intuitively, the algebra B of the above theorem can be thought of as a holomorphic family of algebras of the same dimension, parameterized by points of D. Thus the theorem shows that a Harish-Chandra module can be thought of as a family of modules over this family of algebras, which is concentrated at a finite number of points of D.

The question of describing the freeness properties of the category $\mathcal{M}(\mathfrak{g}, K)$ over the center was raised in [BGG1] and [BGG2]. We consider Theorem 1.3 as answer to some questions posed in [BGG2].

1.4. Remark. It seems that the appearence of categories of Cohen-Macaulay type is a common phenomenon in representation theory. For example, Theorem 1.2 has the following analogue for p-adic groups, which in fact was one of our main motivations. Namely, let G be a reductive p-adic group and let P and P' be two smooth finitely generated projective G-modules. Then it can be shown that $\mathrm{Hom}_G(P, P')$ is a Cohen-Macaulay module over Bernstein's center (cf. [Ber]). This result is due to J. Bernstein (unpublished; see the proof in [Bez]). In the case of p-adic groups this theorem enables us to define some duality on the derived category of smooth G-modules which has a very interesting interplay with other dualities on the same category (cf. [Bez]). It is not clear whether an analogous theory can be developed for real groups as well.

1.5. Notations. Let $\mathfrak{p}$ denote the space $\{x \in \mathfrak{g} | \theta(x) = -x\}$. We have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The action of K on $\mathfrak{g}$ endows $\mathfrak{p}$ with the structure of a K-module.

Let ℓ be the split rank of the pair $(\mathfrak{g}, K)$, i.e the dimension of a maximal abelian reductive subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, contained in $\mathfrak{p}$.

For any vector space V, $S(V)$ will denote its symmetric algebra.

A filtered vector space is a space N equipped with an increasing filtration $F_i N$. We always assume that $F_i N = 0$ for $i << 0$ and $\bigcup F_i N = N$. We denote by $\mathrm{gr}(N)$ the associated graded space.

1.6. The Kostant-Rallis theorem

The proof of Theorem 1.2 is based on the following result of Kostant and Rallis.

Let K_0 be the identity component of K. Consider the natural action of the group K on symmetric algebra $S(\mathfrak{p})$ and denote by I the subalgebra of all elements invariant with respect to the subgroup K_0; in other words, $I = S(\mathfrak{p})^{\mathfrak{k}}$.

Theorem. *$S(\mathfrak{p})$ is a free module over subalgebra $I = S(\mathfrak{p})^{\mathfrak{k}}$.*

This statement is proven in [KR] (cf. also [BL]). Formally the statement in [KR] is slightly different (see [KR], Theorem 15). Namely they work with a slightly bigger group K_θ defined as the group of θ-invariant elements of the adjoint group G_{ad}. They consider the subalgebra J of K_θ invariants and show that $S(\mathfrak{p})$ is a free module over J.

However, we claim that the subalgebra J in fact coincides with I. Indeed, let us identify the algebra $S(\mathfrak{p})$ with the algebra of polynomial functions on $\mathfrak{p}$ using the Killing form. Then any K_0-invariant polynomial function $f \in I$ is completely determined by its restriction to the subalgebra $\mathfrak{a} \subset \mathfrak{p}$ — maximal abelian reductive subalgebra in $\mathfrak{p}$. As shown in [KR], Proposition 1, we have $K_\theta = FK_0$, where F is a finite subgroup which normalizes K_0 and acts trivially on $\mathfrak{a}$. This implies that the function f is K_θ-invariant, i.e. $I = J$.

1.7. Let ρ be a finite dimensional representation of $\mathfrak{k}$. Consider the natural $S(\mathfrak{p})^{\mathfrak{k}}$

action on $(\rho \otimes S(\mathfrak{p}))^{\mathfrak{k}}$.

Corollary. *The space $R = (\rho \otimes S(\mathfrak{p}))^{\mathfrak{k}}$ is a finitely generated free $I = S(\mathfrak{p})^{\mathfrak{k}}$-module.*

The fact that I-module R is finitely generated is standard (see e.g. [KR], Theorems 18, 19). The module R is a direct summand of an I-module $\rho \otimes S(\mathfrak{p})$ which is free by Theorem 1.6. Hence R is projective; since it is graded, this implies that it is a free I-module.

1.8. We would like to thank the referee for several useful remarks.

2. Cohen-Macaulay modules

2.0. Let A be a commutative finitely generated algebra over a field F. Let M be a finitely generated A-module.

According to the traditional definition (cf. [AK], [Ha], [Se]), the module M is said to be Cohen-Macaulay over A of dimension k if $\operatorname{depth}(M) = \dim \operatorname{supp} M = k$.

In this section we will review basic properties of Cohen-Macaulay modules and, in particular, describe some criteria for a module to be Cohen-Macaulay which have clear intuitive meaning.

2.1. We will be using the following property of Cohen-Macaulay modules

Theorem. *Let M be a module over two commutative F-algebras A and B as above, whose actions on M commute. Assume that M is finitely generated over each of them. Then M is Cohen-Macaulay of dimension k over one of them if and only if it is Cohen-Macaulay of dimension k over the other.*

In some sense the meaning of the theorem is that the module itself knows that it is Cohen-Macaulay of dimension k and the choice of a particular algebra acting on it does not change this fact.

2.2. We will deduce the theorem from the following

Lemma. *Let $\nu : B \to A$ be a morphism of commutative finitely generated algebras over the field F. Suppose A is finitely generated as a B-module. Then a finitely generated A-module M is Cohen-Macaulay of dimension k over algebra A if and only if it is Cohen-Macaulay of dimension k over B.*

See [Se], Chapter 5, Proposition 11.

2.3. *Proof of Theorem* 2.1**.** Let C be the F-algebra of endomorphisms of M generated by A and B. Clearly C is a finitely generated commutative F-algebra. Since C lies in a finitely generated A-module $\operatorname{End}_A(M)$ it is finitely generated as an A-module.

According to Lemma 2.2 M is Cohen-Macaulay of dimension k over A iff it is Cohen-Macaulay of dimension k over C. The same argument, with A replaced by B, shows that this holds iff M is Cohen-Macaulay of dimension k over B. □

2.4. Using the theorem we can formulate a convenient criterion which allows us to check whether a given finitely generated A-module M is Cohen-Macaulay. Namely, we can always write A as a quotient of a regular algebra C (we can take C to be a polynomial algebra) and thus reduce the problem to the case of a regular algebra.

Moreover, using the Noether normalization lemma, we can find a polynomial subalgebra $B \subset C$ of dimension $k = \dim \operatorname{supp} M$ such that M is a finitely generated B-module. Thus we reduced the question to the case when A is a regular algebra of pure dimension k. Now we can use the following

Lemma. *Suppose that A is a regular algebra of pure dimension k and M is a finitely generated A-module. Then M is Cohen-Macaulay of dimension k if and only if it is locally free over A.*

See [Se], Chapter 4, Corollary 2.

2.5. Results of 2.1–2.4 can be summarized as the following convenient and very intuitive criterion for a module to be Cohen-Macaulay.

Criterion. *Let M be a nonzero finitely generated module over an algebra A as above.*

(1) *The following conditions are equivalent:*
  (i) *M is Cohen-Macaulay of dimension k.*
  (ii) *There exists a regular subalgebra $B \subset A$ of pure dimension k such that M is a locally free B-module of finite rank.*
(2) *Suppose these conditions hold. Choose any commutative finitely generated F-algebra of endomorphisms B of the A-module M such that M is finitely generated as a B-module and B is a regular algebra of pure dimension $k =$ dim supp M. Then M will be locally free as a B-module.*

2.6. Sometimes it is convenient to use the following generalization of 2.4

Lemma. *Suppose that B is a regular algebra of pure dimension n and M a finitely generated B-module. Then M is Cohen-Macaulay of dimension k if and only if $\operatorname{Ext}^i_B(M, B) = 0$ for $i \neq n - k$.*

This lemma is proven in [AK], Chapter 3, Corollary 5.22. For some reason it is formulated in [AK] in a slightly weaker form, though the proof proves exactly this statement. We will discuss the meaning of this condition in 2.7–2.8.

2.7. Grothendieck duality and Cohen-Macaulay modules

The most understandable definition of Cohen-Macaulay modules can be obtained using Grothendieck duality for coherent sheaves (see [Ha]).

We will need just basic properties of this duality in the simplest case of finitely generated modules over algebras, which corresponds to affine schemes.

Let A be a commutative finitely generated F-algebra. For any finitely generated A-module M, we denote by $\mathbb{D}(M)$ the dual complex which is an object of the derived category $D(A)$ of A-modules. It is defined as $\mathbb{D}(M) := \mathrm{RHom}(M, \mathbb{D}(A))$, where $\mathbb{D}(A)$ is the dualizing complex of A.

The dualizing complex $\mathbb{D}(A)$ we normalize as follows: $\mathbb{D}(A) = p^!(\mathcal{O})$, where p is the natural morphism of F-schemes $p : \mathrm{Specmax}(A) \to \mathrm{Specmax}(F)$ and the functor $p^!$ is defined in [Ha] (see Chapter 3, section 8 or Appendix by P. Deligne).

Theorem. *Functor $\mathbb{D}$ commutes with direct images for finite morphisms.*

This theorem is proven in [Ha], Chapter 7, Corollary 3.4 (c). In our case it just means that if A is a B-algebra which is finitely generated as a B-module, then the restriction functor $R : D(A) \to D(B)$ commutes with duality.

Proposition. *Let A be a finitely generated F-algebra and M a finitely generated A-module. Then the following conditions are equivalent:*

(1) *Module M is Cohen-Macaulay of dimension k.*
(2) *The dual complex $\mathbb{D}(M)$ has nonzero cohomologies only in dimension $-k$.*

In fact the property (2) should be considered as a definition of Cohen-Macaulay modules. With this definition all the properties 2.1–2.6 become transparent in view of Theorem 2.7.

2.8. *Proof of Proposition.* Using the same arguments as in 2.4, we can reduce the proof to the case when A is a regular algebra of pure dimension n. In this case the dualizing complex $\mathbb{D}(A)$ is a locally free A module shifted to degree $-n$, so the condition (2) of the Proposition is equivalent to $\mathrm{Ext}^i_A(M, A) = 0$ for $i \neq n - k$.

Using the same arguments as in 2.3 and 2.4 we can reduce the proof to the case when $n = \dim \mathrm{supp}\, M$. In this case we have $\mathrm{Ext}^0_A(M, A) = \mathrm{Hom}_A(M, A) \neq 0$, so the condition (2) may hold only when $n = k$.

Now the proof follows from the following standard lemma (see [AK], chapter 3, Proposition 5.21)

Lemma. *Let M be a finitely generated module over an algebra A as in* 2.0. *Suppose that A is a regular algebra of pure dimension n. Then M is locally free over A iff* $\mathrm{Ext}^i_A(M, A) = 0$ *for $i \neq 0$.*

Remark. This proof also gives the proof of Lemma 2.6.

2.9. Duality on the category of Cohen-Macaulay modules

Let $\mathcal{CM}_k(A)$ denote the category of Cohen-Macaulay modules of dimension k over the algebra A. This is a full additive (but not abelian) subcategory of the category $\mathcal{M}(A)$ of all A-modules.

The definition 2.7 shows that on this category there is a natural duality functor $\mathcal{D}$ which is defined by $\mathcal{D}(M) = \mathbb{D}(M)[-k]$. The following results follow easily from proofs above:

Theorem.

(1) *Functor $\mathcal{D}$ defines a perfect duality on the category $\mathcal{CM}_k$, i.e. it is a contravariant functor and we have functorial isomorphisms* $\mathrm{Hom}(M, \mathcal{D}(N)) \simeq \mathrm{Hom}(N, \mathcal{D}(M))$ *and* $\mathcal{D}\mathcal{D}(M) \simeq M$.

(2) *Suppose that M is a Cohen-Macaulay module of dimension k over two commuting algebras A and B like in* 2.1. *Then the dual modules of M constructed over A and over B are canonically isomorphic.*

(3) *Let $\mu : M \to N$ be a morphism of Cohen-Macaulay modules of dimension k. Suppose that it is an isomorphism in dimension k, i.e. its kernel and cokernel have smaller dimension.*

Then $\mathrm{Ker}\, \mu = 0$ *and* $\mathrm{Coker}\, \mu$ *is a Cohen-Macaulay module of dimension $k-1$. Moreover, the dual morphism $\mathcal{D}(\mu) : \mathcal{D}(N) \to \mathcal{D}(M)$ has the same properties and there exists a canonical isomorphism* $\mathcal{D}(\mathrm{Coker}\, \mu) \simeq \mathrm{Coker}(\mathcal{D}(\mu))$.

(4) *When $k = 0$ the dual module $\mathcal{D}(M)$ is canonically isomorphic to the dual space M^* with the natural action of A.*

2.10. The next assertion is used in the proof of Theorem 1.2.

Proposition. *Let A be a filtered commutative algebra over a field F and let M be a filtered module over A. Suppose that the associated graded algebra* $\mathrm{gr}(A)$ *is a finitely generated F-algebra and* $\mathrm{gr}(M)$ *is a Cohen-Macaulay module of dimension k over the algebra* $\mathrm{gr}(A)$. *Then M is a Cohen-Macaulay module of dimension k over the algebra A.*

Proof of the Proposition. Standard arguments show that M is a finitely generated A-module.

We can easily construct a polynomial algebra B with filtration and a morphism of filtered algebras $\nu : B \to A$ such that the algebra $\mathrm{gr}(B)$ is also a polynomial algebra and the morphisms $\mathrm{gr}(\nu) : \mathrm{gr}(B) \to \mathrm{gr}(A)$ and $\nu : B \to A$ are epimorphic. Using Lemma 2.2 we can replace everywhere A by B and thus assume that algebras A and $\mathrm{gr}(A)$ are polynomial algebras of the same dimension (say n).

According to 2.6, we must check that $\mathrm{Ext}^i(M, A) = 0$ if $i \neq n - k$.

However, by a standard spectral sequence argument, $\mathrm{Ext}^i_A(M,A) = 0$ if the same is true for $\mathrm{Ext}^i_{\mathrm{gr}(A)}(\mathrm{gr}(M), \mathrm{gr}(A))$ and the latter holds by our assumptions and Lemma 2.6. □

Remark. It is possible to prove this Proposition directly, without using Lemma 2.6. Namely, using a graded version of Noether's normalization lemma we can find a graded polynomial subalgebra $C' \subset \mathrm{gr}(A)$ of dimension k such that the module $\mathrm{gr}(M)$ is finitely generated over C'. Then we can lift this subalgebra to A, i.e. find a polynomial subalgebra $C \subset A$ such that $\mathrm{gr}(C) = C'$.

By Criterion 2.5, the C'-module $\mathrm{gr}(M)$ is locally free, and hence free since it is a graded module. This implies that M is a free C-module. By Criterion 2.5, M is a Cohen-Macaulay module over A of dimension k.

3. Proof of Theorem 1.2

3.1. The modules P_σ

Let σ be a finite dimensional representation of K. Let $\mathfrak{Vec}$ denote the category of complex vector spaces. Consider the functor $F_\sigma : \mathcal{M}(\mathfrak{g}, K) \to \mathfrak{Vec}$, defined by $F_\sigma(V) = \mathrm{Hom}_K(\sigma, V)$ for any $V \in \mathcal{M}(\mathfrak{g}, K)$. This functor is representable by a module $P_\sigma \in \mathcal{M}(\mathfrak{g}, K)$, where P_σ can be constructed as $U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \sigma$. One can easily see that each P_σ is projective and that any finitely generated $(\mathfrak{g}, K)$-module is a quotient of P_σ for some σ. Hence the modules P_σ form a system of projective generators for the category $\mathcal{M}(\mathfrak{g}, K)$. Therefore, any projective finitely generated $(\mathfrak{g}, K)$-module is a direct summand of some P_σ, and since the property of being Cohen-Macaulay is stable under taking direct summands, it is sufficient to prove our assertion for $P = P_\sigma$ and $P' = P_{\sigma'}$.

3.2. By definition of P_σ's we have $\mathrm{Hom}_{\mathfrak{g},K}(P_\sigma, P_{\sigma'}) = \mathrm{Hom}_K(\sigma, P_{\sigma'})$ as $\mathcal{Z}(\mathfrak{g}, K)$-modules (here $\mathcal{Z}(\mathfrak{g}, K)$ acts on the right hand side via its action on $P_{\sigma'}$). Hence by 3.1, Theorem 1.2 is equivalent to the following proposition.

Proposition. *The space $M = \mathrm{Hom}_K(\sigma, P_{\sigma'})$ is a Cohen-Macaulay module of dimension ℓ over the algebra $\mathcal{Z}(\mathfrak{g}, K)$.*

3.3. Consider a $\mathcal{Z}(\mathfrak{g})$-module $M' = \mathrm{Hom}_{\mathfrak{k}}(\sigma, P_{\sigma'})$. As a $\mathcal{Z}(\mathfrak{g}, K)$-module M identifies with the space of invariants of a finite group K/K^0 of components of the group K acting on M'. Thus M is a direct summand of M' and hence it is enough to check that M' is Cohen-Macaulay over $\mathcal{Z}(\mathfrak{g}, K)$. However, by Lemma 2.2 this is equivalent to the fact that M' is Cohen-Macaulay over $\mathcal{Z}(\mathfrak{g})$, and this is what we are going to prove.

3.4. We will apply Lemma 2.10 to M' as above and $A = \mathcal{Z}(\mathfrak{g})$. The algebra $\mathcal{Z}(\mathfrak{g})$

carries the natural filtration induced from that of $U(\mathfrak{g})$ and $\mathrm{gr}(\mathcal{Z}(\mathfrak{g}))$ is isomorphic to $S(\mathfrak{g})^{\mathfrak{g}}$. The latter is isomorphic to a polynomial algebra by Chevalley's theorem.

The module $P_{\sigma'} = U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \sigma'$ is also endowed with a filtration coming from $U(\mathfrak{g})$ and this filtration is preserved by the action of K. Therefore $M' = \mathrm{Hom}_{\mathfrak{k}}(\sigma, P_{\sigma'})$ is a filtered $\mathcal{Z}(\mathfrak{g})$-module and we must ensure that $\mathrm{gr}(M')$ is Cohen-Macaulay of dimension l over $S(\mathfrak{g})^{\mathfrak{g}}$.

3.5. Using the splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ we can identify $S(\mathfrak{p})$ with the quotient $S(\mathfrak{g})/\mathfrak{k}S(\mathfrak{g})$. This defines a K-equivariant morphism $S(\mathfrak{g}) \to S(\mathfrak{p})$ and by restriction a morphism $\nu : S(\mathfrak{g})^{\mathfrak{g}} \to S(p)^{\mathfrak{k}}$. The following assertion is well-known (cf. [KR], [Wa]):

Lemma.

(1) $I = S(\mathfrak{p})^{\mathfrak{k}}$ *is a polynomial algebra in* ℓ *variables.*
(2) $S(\mathfrak{p})^{\mathfrak{k}}$ *is finitely generated as a module over* $S(\mathfrak{g})^{\mathfrak{g}}$.

It is easy to see that $\mathrm{gr}(P_{\sigma'})$ is isomorphic to $S(\mathfrak{p}) \otimes \sigma'$. This implies that the module $\mathrm{gr}(M')$ is isomorphic to $\mathrm{Hom}_{\mathfrak{k}}(\sigma, S(\mathfrak{p}) \otimes \sigma') = (\mathrm{Hom}(\sigma, \sigma') \otimes S(\mathfrak{p}))^{\mathfrak{k}}$. It is easily checked that the algebra $S(\mathfrak{g})^{\mathfrak{g}} = \mathrm{gr}(\mathcal{Z}(\mathfrak{g}))$ is acting on this module via the above homomorphism $\nu : S(\mathfrak{g})^{\mathfrak{g}} \to S(\mathfrak{p})^{\mathfrak{k}}$.

3.6. We must show that $\mathrm{gr}(M') \simeq (\mathrm{Hom}(\sigma, \sigma') \otimes S(\mathfrak{p}))^{\mathfrak{k}}$ is a Cohen-Macaulay module of dimension ℓ over $S(\mathfrak{g})^{\mathfrak{g}}$. By Lemma 3.5(2) and Theorem 2.1 this is equivalent to showing that $(\mathrm{Hom}(\sigma, \sigma') \otimes S(p))^{\mathfrak{k}}$ is Cohen-Macaulay of dimension ℓ over $S(\mathfrak{p})^{\mathfrak{k}}$.

However, by Corollary 1.7 the latter module is free over $S(\mathfrak{p})^{\mathfrak{k}}$. This concludes the proof in light of Lemmas 2.4 and 3.5(1). □

4. Cohen-Macaulay categories

4.0. In this section we will try to explain the categorical meaning of Theorem 1.2.

4.1. Localization of categories

Let C be an abelian category closed under inductive limits. Let Z be a commutative algebra mapping to the center of C. In other words, we are given a homomorphism $\phi : Z \to \mathrm{End}\,(\mathrm{Id}_C)$, where Id_C denotes the identity functor on C. For any object $X \in C$ we denote by ϕ_X the corresponding morphism $Z \to \mathrm{End}_C(X)$.

Let $\mathfrak{f}$ be a multiplicatively closed subset in Z and let $Z_{\mathfrak{f}}$ denote the localization of Z with respect to $\mathfrak{f}$. We would like to describe the localization $C_{\mathfrak{f}}$ of the category C with respect to $\mathfrak{f}$.

We say that a functor $L : C \to C'$ from C to another abelian category C' is $\mathfrak{f}$-inverting if for every $X \in C$ and $z \in \mathfrak{f}$, $L(\phi_X(z))$ is invertible in $\mathrm{End}_{C'}(L(X))$.

Lemma-Definition. *A localization of C with respect to $\mathfrak{f}$ is a category $C_{\mathfrak{f}}$ endowed with an $\mathfrak{f}$-inverting functor $L_{can} : C \to C_{\mathfrak{f}}$, which is universal in the following sense:*

For each pair (C', L), where C' is an abelian category and L is an $\mathfrak{f}$-inverting functor $C \to C'$, there exists a functor $G : C_{\mathfrak{f}} \to C'$ together with an isomorphism of functors $L \simeq G \circ L_{can}$. Such $C_{\mathfrak{f}}$ exists and is unique up to a canonical equivalence.

Proof-Construction. To construct such $C_{\mathfrak{f}}$, put the objects of $C_{\mathfrak{f}}$ to be the objects of C and for $X, Y \in C$, set $\mathrm{Hom}_{C_{\mathfrak{f}}}(X,Y) := \mathrm{Hom}_C(X,Y) \otimes_Z Z_{\mathfrak{f}}$.

The functor L_{can} is now the obvious one: it sends $X \in C$ to X viewed as an object of $C_{\mathfrak{f}}$ and $L_{can} : \mathrm{Hom}_C(X,Y) \to \mathrm{Hom}_{C_{\mathfrak{f}}}(X,Y)$ is the natural embedding. □

The next assertion is straightforward.

Proposition. *Consider the full subcategory $C'_{\mathfrak{f}}$ of C consisting of objects $X \in C$ such that for every $z \in \mathfrak{f}$, $\phi_X(z)$ is invertible in $\mathrm{End}_C(X)$. Then the natural functor $C'_{\mathfrak{f}} \to C_{\mathfrak{f}}$ is an equivalence of categories.*

Example. Let C be the category of modules over an associative algebra E and let Z be a central subalgebra of E. Then for every multiplicatively closed subset $\mathfrak{f} \subset Z$ the category $C_{\mathfrak{f}}$ is equivalent to the category of modules over $E \otimes_Z Z_{\mathfrak{f}}$.

Definition. Let $L : C \to C_1$ be a functor between abelian categories. The category C_1 is said to be a central localization of C if there exists a pair $(Z, \mathfrak{f})$ as above, such that the pair (C_1, L) is equivalent to the pair $(C_{\mathfrak{f}}, L_{can})$.

4.2. Definition. Let F be a field. A Cohen-Macaulay algebra over F is an associative F-algebra E such that there exists a subalgebra $\mathcal{O}$ of the center of E with the following properties:

(1) $\mathcal{O}$ is a finitely generated algebra over F and is a regular domain;
(2) As an $\mathcal{O}$-module, E is finitely generated and locally free.

Now we can give a definition of a Cohen-Macaulay category.

Definition. Let C be an abelian category over a field F closed under inductive limits. We say that C is of a Cohen-Macaulay type if there exists an algebra Z mapping to the center of C (as in 4.1) with the following property:

For every maximal ideal $\mathfrak{m}$ of Z, the localization $C_{\mathfrak{f}}$ of C with respect to $\mathfrak{f} := Z \backslash \mathfrak{m}$ is equivalent to a central localization in the sense of 4.1 of the category of modules over a Cohen-Macaulay algebra.

4.3. Theorem. *The category $\mathcal{M}(\mathfrak{g}, K)$ is a Cohen-Macaulay category over $\mathbb{C}$.*

Proof. Step 1. Let Z be equal to $\mathcal{Z}(\mathfrak{g}, K)$. Let also $\mathfrak{m}$ be a maximal ideal of $\mathcal{Z}(\mathfrak{g}, K)$. By a theorem of Harish-Chandra (cf. [Vo, 5.4]) there are finitely many isomorphism classes of simple $(\mathfrak{g}, K)$-modules with the central character corresponding to $\mathfrak{m}$. Therefore, there exists a finite-dimensional K-module σ such that $\mathrm{Hom}_K(\sigma, V) \neq 0$ for every non zero $(\mathfrak{g}, K)$-module V annihilated by $\mathfrak{m}$.

Claim. *Let* $\mathfrak{f} = \mathcal{Z}(\mathfrak{g}, K) \setminus \mathfrak{m}$. *Then the category* $\mathcal{M}(\mathfrak{g}, K)_{\mathfrak{f}}$ *is equivalent to the category of right modules over the algebra* $\mathrm{End}_{\mathcal{M}(\mathfrak{g},K)}(P_\sigma) \underset{\mathcal{Z}(\mathfrak{g},K)}{\otimes} \mathcal{Z}(\mathfrak{g}, K)_{\mathfrak{f}}$.

Proof of the claim. To prove the assertion, it is enough to show that $L_{can}(P_\sigma)$ is a projective generator of the category $\mathcal{M}(\mathfrak{g}, K)_{\mathfrak{f}}$.

It is clear that the functor L_{can} of 4.1 sends projective objects to projective ones. This implies that $L_{can}(P_\sigma)$ is a projective object of $\mathcal{M}(\mathfrak{g}, K)_{\mathfrak{f}}$. Let us show that it is a generator of the category. In other words, we have to show that for any $(\mathfrak{g}, K)$-module V, the equality $\mathrm{Hom}_{\mathcal{M}(\mathfrak{g},K)_{\mathfrak{f}}}(L_{can}(P_\sigma), L_{can}(V)) = 0$ implies that $L_{can}(V) = 0$.

We can assume that V is finitely generated. For any finite dimensional K-module τ consider $\mathcal{Z}(\mathfrak{g}, K)$-module $F_\tau(V) := \mathrm{Hom}_K(\tau, V)$ defined in 3.1. This module is finitely generated and it is easy to see that

(i) $F_\tau(V/\mathfrak{m}V) = F_\tau(V)/\mathfrak{m}F_\tau(V)$,
(ii) $\mathrm{Hom}_{\mathcal{M}(\mathfrak{g},K)_{\mathfrak{f}}}(L_{can}(P_\tau), L_{can}(V)) = (F_\tau(V))_{\mathfrak{f}}$.

According to the Nakayama lemma, for every finitely generated $\mathcal{Z}(\mathfrak{g}, K)$-module M the equality $M/\mathfrak{m}M = 0$ is equivalent to $M_{\mathfrak{f}} = 0$. From this we see that if $\mathrm{Hom}_{\mathcal{M}(\mathfrak{g},K)_{\mathfrak{f}}}(L_{can}(P_\sigma), L_{can}(V)) = 0$ then $F_\sigma(V/\mathfrak{m}V) = 0$.

But σ was chosen in such a way that this implies that $V/\mathfrak{m}V = 0$. Now reversing the argument we see that for any τ $\mathrm{Hom}_{\mathcal{M}(\mathfrak{g},K)_{\mathfrak{f}}}(L_{can}(P_\tau), L_{can}(V)) = 0$, which implies that $L_{can}(V) = 0$. □

Step 2. By Theorem 1.2, the algebra $\mathrm{End}_{\mathcal{M}(\mathfrak{g},K)}(P_\sigma)$ is a Cohen-Macaulay module over $\mathcal{Z}(\mathfrak{g}, K)$ of dimension ℓ. Criterion 2.5 then implies that it is a locally free finitely generated module over some regular subalgebra $\mathcal{O}$ in $\mathcal{Z}(\mathfrak{g}, K)$, i.e. that $\mathrm{End}_{\mathcal{M}(\mathfrak{g},K)}(P_\sigma)$ is a Cohen-Macaulay algebra. Thus, we have shown that for every maximal ideal $\mathfrak{m} \in Spec(\mathcal{Z}(\mathfrak{g}, K))$, the localization of the category $\mathcal{M}(\mathfrak{g}, K)$ with respect to multiplicative subset $\mathfrak{f} = \mathcal{Z}(\mathfrak{g}, K) \setminus \mathfrak{m}$ is equivalent to a localization of the category of modules over a Cohen-Macaulay algebra. □

4.4. Theorem 4.3 proved above has an analytic analog (Theorem 1.3), when the localization of $\mathcal{Z}(\mathfrak{g}, K)$ with respect to a subset $\mathcal{Z}(\mathfrak{g}, K) \setminus \mathfrak{m}$ is replaced by the algebra of holomorphic functions on a small ball in $\mathrm{Specmax}(\mathcal{Z}(\mathfrak{g}, K))$ around the point corresponding to $\mathfrak{m}$.

Its proof goes along the same lines as the one of Theorem 4.3 after adopting the following strengthening of Harish-Chandra's theorem (cf. [Vo, 5.4]):

Theorem. *Let* U *be a compact subset of* $\mathrm{Specmax}(\mathcal{Z}(\mathfrak{g}, K))$. *Then there exists a finite-dimensional* K*-representation* σ *such that for every irreducible* $(\mathfrak{g}, K)$*-module* V *with a central character belonging to* U, *one has* $\mathrm{Hom}_K(\sigma, V) \neq 0$.

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