Second adjointness for representations of reductive $p$-adic groups

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§0. Introduction

0.1. In this paper, which was written in 1987, I continue the investigation of induced representations of reductive $p$-adic groups, started in [BZ]. The main tools of the investigation are induction functors $i_{GM}$ and Jacquet functors $r_{MG}$. More precisely, let $G$ be a reductive $p$-adic group and $\text{Alg} G$ the category of algebraic (in other terminology, smooth) representations of $G$. For any parabolic subgroup $P < G$ with Levi component $M$ we define the induction functor $i_{GM} : \text{Alg} M \to \text{Alg} G$ and Jacquet functor $r_{MG} : \text{Alg} G \to \text{Alg} M$ as in [BZ].

Frobenius reciprocity implies that functor $r_{MG}$ is left adjoint to $i_{GM}$. Recently, I have discovered to my great surprise, that these functors are also adjoint in the opposite direction. More precisely, let $\mathcal{P}$ be the parabolic subgroup opposite to $P$ with Levi component $M$. Then we can define functors $\tilde{i}_{GM} : \text{Alg} M \to \text{Alg} G$ and $\tilde{r}_{MG} : \text{Alg} G \to \text{Alg} M$ in the same way as $i$ and $r$, but using $\mathcal{P}$ instead of $P$.

Main theorem. Functor $i_{GM}$ is left adjoint to $r_{MG}$, and $\tilde{i}_{GM}$ is left adjoint to $\tilde{r}_{MG}$.

This innocent-looking statement is in fact very powerful. For instance, it implicitly contains the strong admissibility theorem (indeed, it implies that functors $r_{MG}$ commute with direct product and hence products of quasicuspidal representations are quasicuspidal. But this means that for a given open subgroup $K \subset G$ there exists a uniform bound on supports of all $K$-invariant matrix coefficients of all cuspidal representations of $G$, i.e. all these supports lie in some subset $S \subset G$, compact modulo center).

The aim of this paper is to prove the main theorem and to show how it implies many important results about induced representation: description of the center of category $\text{Alg} G$, matrix Paley-Wiener theorem, cohomological duality in $\text{Alg} G$.

More precise versions of the theorem are formulated in §. They allow to prove Zelevinsky’s conjecture, that duality, which he defined on the Grothendieck group of representations of $GL(n)$, actually carries irreducible representations into irreducible ones (see [Z]). I should add, that this way of proving Zelevinsky’s conjecture was suggested to me by V. Drinfeld many years ago. He explained to me that for the group $G = SL(2)$, $\text{Ext}^1$ (trivial representation) $= \text{Steinberg representation}$. 

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0.2. Contragredient properties of functor $r_{\mu_G}$.

Another, essentially equivalent, form of the main theorem describes how to compute contragredient representations of $r_{\mu_G} (\pi)$. For induction functor we have the Frobenius reciprocity $(i_{\mu_G} (\rho))^\sim = i_\mu (\tilde{\rho})$, where $^\sim$ denotes the contragredient representation (see [B2]).

**Theorem.** There is a functorial isomorphism

$$(r_{\mu_G} (\pi))^\sim \cong \pi_{\mu_G} (\bar{\pi}), \quad \pi \in \text{Alg} \ G.$$  

0.3. Matrix Paley-Wiener theorem.

Let $\rho$ be an irreducible cuspidal representation of $M$. Consider the family of induced representations $\pi_\chi = i_{\mu_M} (\chi \cdot \rho)$, parametrized by unramified characters $\chi$ of $M$, with underlying family of vector spaces $E_\chi$.

Let $H = H(G)$ be the algebra of compactly supported locally constant measures on $G$. Any element $h \in H(G)$ induces the family of operators $h_\chi = \pi_\chi (h) : E_\chi \to E_\chi$.

This family has the following properties:

(PW1) $h_\chi$ is a regular function of parameter $\chi$ (unramified characters of $M$ form a group isomorphic to $(\mathbb{C}^*)^d$ and function $h_\chi$ is algebraic on $(\mathbb{C}^*)^d$).

(PW2) There exists an open subgroup $K \subset G$ such that operators $h_\chi$ are left and right invariant with respect to $\pi(K)$.

(PW3) For any intertwining operator $A : E_\chi \to E_{\chi'}$ one has $h_{\chi'} \circ A = A \circ h_\chi$.

**Theorem.** Let $a_\chi : E_\chi \to E_\chi$ be a family of operators, satisfying (PW1)-(PW3). Then $a_\chi = h_\chi$ for some $h \in H(G)$.

**Remark.** It is clear, that it is sufficient to check property (PW3) only on Zariski dense subsets of parameters $\chi$ and $\chi'$.

This theorem follows easily from the following corollary of the main theorem: functor $i_{GM}$ carries projective generators into projective generators.

0.4. Cohomological duality theorem.

Let us denote by $\gamma_\pi^r$ and $\gamma_\pi$ left and right actions of $G$ on $H(G)$. For any $\pi \in \text{Alg} \ G$, we can consider spaces $\text{Ext}^i_\pi (\pi) = \text{Ext}^i_{A(G)} (\pi, (\gamma_\pi, H(G)))$ as $G$-modules, using right action $\gamma_\pi$.

**Theorem.** If $\pi$ is irreducible then for exactly one index $i$, $\text{Ext}^i_\pi (\pi) \neq 0$. Moreover, representation $\text{Ext}^i_\pi (\pi)$ is irreducible and $\pi \mapsto \text{Ext}^i_\pi (\pi)$ defines a duality on the set of equivalence classes of irreducible algebraic representations of $G$.

§1. Generalities from Algebra and Category Theory
1.1. Idempotented Algebras and Nondegenerate Modules.

We consider a class of rings slightly more general than rings with identity.

**Definition.** An associative ring $\mathcal{H}$ is called an *idempotented ring* if for each finite subset \( \{x_i\} \in \mathcal{H} \) there exists an idempotent $e \in \mathcal{H}$ such that $ex_i = x_i = xe$ for all $i$.

**Example.** Each ring with identity is an idempotented ring. More generally, let $\mathcal{H}_\alpha$, $\alpha \in I$, be a direct system of rings and $\mathcal{H} = \varinjlim_{\alpha \in I} \mathcal{H}_\alpha$. Suppose that the ordered set $I$ is filtered (i.e. for each $\alpha, \beta \in I$ there exists $\gamma \in I$ such that $\alpha < \gamma, \beta < \gamma$) and all $\mathcal{H}_\alpha$ are rings with identities (but ring homomorphisms $\mathcal{H}_\alpha \to \mathcal{H}_\beta$ for $\alpha < \beta$ are not supposed to map identities into identities). Then $\mathcal{H}$ is an idempotented ring.

In fact, any idempotented ring can be presented in such a way. Namely, consider the set $I = \text{Idem}\mathcal{H}$ of idempotents in $\mathcal{H}$ with partial order $e \leq f$ if $e\mathcal{H}e \subset f\mathcal{H}f$. Then $\mathcal{H} = \varprojlim_{e \in \text{Idem}\mathcal{H}} e\mathcal{H}e$, where $e\mathcal{H}e$ is the ring with identity $e$.

Usually we consider $\mathcal{H}$ to be an algebra over some field $k$ and call $\mathcal{H}$ an idempotented algebra.

A (left) module $M$ over an idempotented ring $\mathcal{H}$ is called *nondegenerate* if $\mathcal{H}M = M$ or equivalently $\varinjlim_{e \in \text{Idem}\mathcal{H}} eM = M$. If $\mathcal{H}$ is a ring with identity, this is just the usual condition that $1$ acts on $M$ as identity.

The category of nondegenerate $\mathcal{H}$-modules we denote $\mathcal{M}(\mathcal{H})$. Each $\mathcal{H}$-module $M$ contains the maximal nondegenerate submodule $\mathcal{H}M$, which we call the *nondegenerate part* of $M$. It is easy to see that $\mathcal{M}(\mathcal{H})$ is an abelian category with direct limits and filtered direct limits in $\mathcal{M}(\mathcal{H})$ are exact. Category $\mathcal{M}(\mathcal{H})$ also has arbitrary direct products (and, hence, inverse limits). Namely, for a family \( \{M_\alpha \in \mathcal{M}(\mathcal{H})\} \) the product $\prod_\alpha M_\alpha$ in $\mathcal{M}(\mathcal{H})$ is equal to the nondegenerate part of the set theoretic direct product,

$$
\prod_\alpha M_\alpha = \mathcal{H} \left( \prod_\alpha M_\alpha \right) = \varinjlim_{e \in \text{Idem}\mathcal{H}} \left( \prod_\alpha (eM_\alpha) \right).
$$

1.2. Projective and Injective $\mathcal{H}$-Modules.

For each idempotent $e \in \mathcal{H}$ the functor $M \to eM$ is exact on $\mathcal{M}(\mathcal{H})$. Since $e\mathcal{H} = \text{Hom}_\mathcal{H}(e\mathcal{H}, M)$, it shows, that $e\mathcal{H}$ is a finitely generated projective object in $\mathcal{M}(\mathcal{H})$. The family of modules $e\mathcal{H}$ for $e \in \text{Idem}\mathcal{H}$ form a system of projective generators for category $\mathcal{M}(\mathcal{H})$. In particular, $\mathcal{M}(\mathcal{H})$ has enough projective objects, i.e. each module $M \in \mathcal{M}(\mathcal{H})$ is a quotient of a projective one.

Similarly, one can see that $\mathcal{M}(\mathcal{H})$ has enough injective objects. Namely for each $e \in \text{Idem}\mathcal{H}$ and each injective $\mathbb{Z}$-module $U$ denote by $I(e,U)$ the nondegenerate part of $\mathcal{H}$-module $\text{Hom}_\mathcal{Z}(e\mathcal{H}, U)$. Then the functor $M \to \text{Hom}_\mathcal{H}(M, I(e,U)) = \text{Hom}_\mathcal{Z}(eM, U)$ is exact on $\mathcal{M}(\mathcal{H})$, i.e. $I(e,U)$ is an injective object, and $\{I(e,U)\}$ form a system of injective cogenerators in $\mathcal{M}(\mathcal{H})$.

We will denote by $\mathcal{M}^R(\mathcal{H})$ the category of nondegenerate right $\mathcal{H}$-modules, which we identify with category $\mathcal{M}(\mathcal{H}^o)$, where $\mathcal{H}^o$ is the opposite algebra. We define in a usual way the tensor product $M' \otimes_{\mathcal{H}} M$ of nondegenerate right and left $\mathcal{H}$-modules. It is easy to see that all the usual properties of $\otimes$ hold in this
case; we will use them freely. Note, that formula $M' \otimes_\mathcal{H}(\mathcal{H}e) = M'e$ shows, that $\mathcal{H}e$ is a flat $\mathcal{H}$-module, which implies that all projective $\mathcal{H}$-modules are flat.

Let $\mathcal{H}$ be an idempotent algebra over a field $k$. For each $\mathcal{H}$-module $M \in \mathcal{M}(\mathcal{H})$ we define the contragredient module $\bar{M} \in \mathcal{M}^R(\mathcal{H})$ as a nondegenerate part of the dual space $M^* = \text{Hom}_k(M, k)$, i.e. $\bar{M} = \lim_{\rightarrow} (M^*)^*$. Similarly we define the functor $\sim : \mathcal{M}^R(\mathcal{H}) \to \mathcal{M}(\mathcal{H})$. It is easy to check that $\sim$ is an exact contravariant functor, with duality property $\text{Hom}_K(M, N) = \text{Hom}_{K^R}(\bar{N}, \bar{M}), M \in \mathcal{M}(\mathcal{H}), N \in \mathcal{M}^R(\mathcal{H})$. In particular, $\sim$ maps projective objects into injective ones.

1.3. Hecke Algebras.

Let $G$ be an $\ell$-group, i.e. a Hausdorff topological group, which has a basis of neighborhoods of $e \in G$, consisting of open compact subgroups (see [BZ1]). Let $\mathcal{H} = \mathcal{H}(G)$ be the Hecke algebra of locally constant distributions (or complex valued measures) on $G$ with compact support. Then $\mathcal{H}$ is an idempotent algebra (over $\mathbb{C}$ and category $\mathcal{M}(\mathcal{H}(G))$ is naturally identified with category $\mathcal{M}(G)$ of $G$-modules (see... [BZ1]).

Let $K \subset G$ be an open compact subgroup, $e_K \subset \mathcal{H}(G)$ be the normalized Haar measure on $K$. Then $e_K \mathcal{H}(G)e_K$ is the subalgebra $\mathcal{H}_K(G)$ of $K$-biinvariant measures. The system of idempotents $\{e_K\}$ is cofinal in Idem $\mathcal{H}(G)$, i.e. $\mathcal{H}(G) = \lim_K \mathcal{H}_K(G)$.

The involution $\iota : g \mapsto g^{-1}$ on $G$ defines the natural antiisomorphism $\iota : \mathcal{H}(G) \rightarrow \mathcal{H}(G)$. Using this antiautomorphism we will usually identify $\mathcal{M}(\mathcal{H})$ with $\mathcal{M}^R(\mathcal{H})$, though sometimes it is more convenient to separate them.

1.4. Jordan-Hölder Content of a Module.

We want to describe some general properties of the category $\mathcal{M}(\mathcal{H})$. It is convenient to do it in a more general setting.

Let $\mathcal{M}$ be an abelian category with (arbitrary) direct sums (and, hence, direct limits). We will assume that $\mathcal{M}$ satisfies some axioms.

(A1) Filtered direct limits in $\mathcal{M}$ are exact.

In [Gr] this axiom is called AB... It is equivalent (see [...]) to

(A1') Let $M_\alpha \subset M$ be a filtered system of submodules, $N \subset M$. Then

$$N \cap \left( \sum_\alpha M_\alpha \right) = \sum_\alpha \left( N \cap M_\alpha \right) .$$

An object $M \in \mathcal{M}$ is called finitely generated if for any filtered system of proper subobjects $M_\alpha \subset M$ the subobject $\sum_\alpha M_\alpha \subset M$ is proper. For a finitely generated object $M$ the functor $\text{Hom}(M, \cdot) : \mathcal{M} \to Ab$ preserves direct sums.

An object $M \in \mathcal{M}$ is called noetherian, if every of its subobjects is finitely generated or, equivalently, if each ascending chain of subobjects $M_1 \subset M_2 \subset \ldots$ of $M$ is stable.

Category $\mathcal{M}$ is called locally noetherian if each finitely generated object of $\mathcal{M}$ is noetherian.
(42) Every object $M \in \mathcal{M}$ is a union of finitely generated subobjects.

In order to avoid set-theoretical troubles we also add

(43) Isomorphism classes of finitely generated objects in $\mathcal{M}$ form a set.

We denote by $Irr \mathcal{M}$ the set of isomorphism classes of irreducible (i.e. simple) objects in $\mathcal{M}$. For every $E \in \mathcal{M}$ we denote by $JH(E) \subset Irr \mathcal{M}$ the subset of irreducible subquotients of $E$.

For each idempotent ring $\mathcal{H}$ the category $\mathcal{M} = \mathcal{M}(\mathcal{H})$ satisfies axioms A1 - A3. We will denote $Irr(\mathcal{M}(\mathcal{H}))$ by $Irr \mathcal{H}$ and $Irr \mathcal{M}(G)$ by $Irr \mathcal{H}$ (see 1.3).

**Lemma.**  (i) Let $E' \subset E$. Then $JH(E) = JH(E') \cup JH(E/E')$.

(ii) $JH(E) = \emptyset$ iff $E = 0$

(iii) If $E_a \subset E$, then $JH(\bigcup_a E_a) = \bigcup_a JH(E_a)$.

**Proof:**

(i) is clear.

(ii) Let $E \not= 0$. By A2 $E$ has a nonzero finitely generated submodule $E'$. By Zorn's lemma $E'$ has an irreducible quotient, i.e. $JH(E) \not= \emptyset$.

(iii) Let $I = \{\alpha\}$ be the indexing set of $E_a$. If $I$ is finite, the statement follows from (i) by induction. Hence, replacing system $\{E_a\}$ by a system, consisting of finite sums of $E_a$ we can assume, that $\{E_a\}$ is a filtered direct system. Let $Q = E'/E''$ be a simple subquotient of $\bigcup_a E_a$, i.e. $E'' \not\subset E' \subset \bigcup_a E_a$. Suppose that for all $\alpha \notin JH(E_a)$. Then for every $\alpha E' \cap (E'' + E_a) = E''$. By A1 $E' \cap \bigcup_a (E'' + E_a) = \bigcup_a E' \cap (E'' + E_a) = E''$, which contradicts the inclusion $E' \subset \bigcup_a E_a$, since $E'' \not= E'$.

1.5. **Decomposition of Categories.** Suppose that the category $\mathcal{M}$ is split into a product of two subcategories $\mathcal{M} = \mathcal{M}' \times \mathcal{M}''$. This splitting induces a disjoint union decomposition $Irr \mathcal{M} = Irr \mathcal{M}' \cup Irr \mathcal{M}''$. We want to show that this decomposition completely describes the splitting.

For each subset $S \subset Irr \mathcal{M}$ denote by $\mathcal{M}(S)$ the full subcategory of $\mathcal{M}$ defined by $\mathcal{M}(S) = \{ E \in \mathcal{M} \mid JH(E) \subset S \}$. Lemma 1.4 shows that $\mathcal{M}(S)$ is an abelian subcategory, closed with respect to subquotients, extensions and direct limits. For every $E \in \mathcal{M}$ we denote by $E_S$ the union of all submodules $E' \subset E$, which lie in $\mathcal{M}(S)$. Then $E_S$ also lies in $\mathcal{M}(S)$. Let $S' \subset Irr \mathcal{M}$ be another subset, which does not intersect $S$. Then for each $E \in \mathcal{M}(S) \cap \mathcal{M}(S')$ we have $JH(E) = \emptyset$, i.e. $E = 0$. This implies that the categories $\mathcal{M}(S'), \mathcal{M}(S')$ are orthogonal, i.e. $\text{Hom}_\mathcal{M}(E,E') = 0$ for $E \in \mathcal{M}(S)$, $E' \in \mathcal{M}(S')$. Also for every $E \in \mathcal{M}$, $E_S \cap E_{S'} = 0$, i.e. $E \supset E_S \oplus E_{S'}$. 

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Definition. We say that a subset $S \in \text{Irr} \mathcal{M}$ splits an object $E \in \mathcal{M}$ if $E = E_S \oplus E_{\overline{S}}$, where $\overline{S} = \text{Irr} \mathcal{M} \setminus S$. We say that $S$ splits $\mathcal{M}$ if it splits all objects in $\mathcal{M}$.

More generally, suppose we have a disjoint union decomposition $\text{Irr} \mathcal{M} = \bigcup_{a \in A} S_a$. We say that this decomposition $\{S_a\}$ splits $E$ if $E = \bigoplus_{a \in A} E_{S_a}$. We say that the decomposition $\{S_a\}$ splits $\mathcal{M}$ if it splits all objects in $\mathcal{M}$. In this case $\mathcal{M}$ is equivalent to the category $\prod_{a \in A} \mathcal{M}(S_a)$.

Lemma. Let $\text{Irr} \mathcal{M} = \bigcup_{a \in A} S_a$ be a disjoint union decomposition. Suppose it splits an object $E \in \mathcal{M}$. Then it splits all quotients of $E$.

Proof: Let $E = \bigoplus_{a \in A} E_a$, $E_a \in \mathcal{M}(S_a)$. It is sufficient to check that for every subobject $L \subset E \mathcal{L} = \sum_{a} (L \cap E_a)$. Put $C = L / \sum_{a} (L \cap E_a)$. Then for $a$

$$JH(C) \subset \mathcal{JH}(L/L \cap E_a) \subset JH(E/E_a) \subset \bigcup_{\beta \neq a} JH(E_\beta) \subset \overline{S_a}.$$ .

This implies, that $JH(C) \subset \bigcap_{a} (\overline{S_a}) = \emptyset$, i.e. $C = 0$.

Remark. Let $\mathcal{H}$ be an idempotent ring. Suppose category $\mathcal{M} = \mathcal{M}(\mathcal{H})$ has a decomposition $\mathcal{M} = \prod_{a} \mathcal{M}_a$. Applying this decomposition to the $\mathcal{H}$-module $\mathcal{H}$ we see that $\mathcal{H} = \bigoplus_{a} \mathcal{H}_a$. Since right multiplications in $\mathcal{H}$ are morphisms in $\mathcal{M}(\mathcal{H})$ all $\mathcal{H}_a$ are two-sided ideals. It is easy to see that $\mathcal{M}_a = \mathcal{M}(\mathcal{H}_a)$.

Conversely, each decomposition $\mathcal{H} = \bigoplus_{a} \mathcal{H}_a$ of $\mathcal{H}$ into a direct sum of two-sided ideals leads to the decomposition $\mathcal{M}(\mathcal{H}) = \prod_{a} \mathcal{M}(\mathcal{H}_a)$.

1.6. Realization of an Abelian Category as a Category of Modules.

Let $\mathcal{M}$ be an abelian category, satisfying $A1 - A3$. Let $P \in \mathcal{M}$ be a finitely generated projective object, $\Lambda = \text{End}_{\mathcal{M}}(P)^o$ ($^o$ denotes the opposite algebra).

We define the functor $r = r_P : \mathcal{M} \to \mathcal{M}(\Lambda)$ by $r(E) = \text{Hom}_{\mathcal{M}}(P,E)$. It is exact and commutes with direct sums. Functor $r$ has a left adjoint functor $i = i_P : \mathcal{M}(\Lambda) \to \mathcal{M}$. Indeed, every $\Lambda$-module $M$ can be presented as a cokernel of a morphism $\nu_M$ of free $\Lambda$-modules $\nu_M : \bigoplus_{\alpha} \Lambda \to \bigoplus_{\beta} \Lambda$, where $\nu_M$ is given by a matrix $\{\nu_{\alpha, \beta} \in \Lambda\}$. We define $i(M)$ as a cokernel of a morphism $\nu' : \bigoplus_{\alpha} P \to \bigoplus_{\beta} P$, where $\nu'$ is given by the same matrix $\{\nu_{\alpha, \beta} \in \Lambda\}$. In case when $\mathcal{M} = \mathcal{M}(\mathcal{H})$ the functor $i$ can be described as $i(M) = P \bigotimes_{\Lambda} M$.

Lemma. Suppose that $P$ is a generator of the category $\mathcal{M}$, i.e. the functor $r$ is faithful, or, equivalently, $\text{Hom}_{\mathcal{M}}(P,Q) \neq 0$ for $Q \in \text{Irr} \mathcal{M}$. Then functor $r$ and $i$ are inverse and define an equivalence of categories

$$\mathcal{M} \xrightarrow{i} \mathcal{M}(\Lambda).$$

Proof: See [ ...].

This lemma allows us to realize $\mathcal{M}$ as a category of modules over some algebra with identity. This realization is not unique, it depends on the choice of $P$. Let us describe the relation between two such realizations.
Let $A$ be an algebra with identity, $P \in \mathcal{M}(A)$ a finitely generated projective generator, $\Lambda = (\text{End}_A P)^\circ$. Then $P$ is an $A - \Lambda$-bimodule. We define a dual $\Lambda - A$-bimodule $P^*$ by $P^* = \text{Hom}_A(P, A)$.

**Proposition.** $P^*$ is a finitely generated projective generator in $\mathcal{M}(\Lambda)$, $\text{End}_\Lambda(P^*) = A^\circ$ and the functors $i : \mathcal{M}(\Lambda) \to \mathcal{M}(A)$, $r : \mathcal{M}(A) \to \mathcal{M}(\Lambda)$ are canonically isomorphic to $r(E) = P^* \otimes_A E, E \in \mathcal{M}(A)$ and $i(M) = \text{Hom}_\Lambda(P^*, M), M \in \mathcal{M}(\Lambda)$.

**Proof:**

**Step 1.** For any $E \in \mathcal{M}(A)$ the natural morphism $P^* \otimes_A E \to \text{Hom}_A(P, E) = r(E)$ is an isomorphism.

Indeed, this is true for $P = A$, hence for $P = A^n$ and hence for $P$ which is a direct summand of $A^n$.

**Step 2.** Since $P$ is a generator of $\mathcal{M}(A)$, $A$ is a direct summand of $P^m$ for some natural $n$. Hence $r(A) = P^*$ is a direct summand of $r(P)^n = A^n$, i.e. $P^*$ is a finitely generated projective $\Lambda$-module.

**Step 3.** Since functors $r$ and $i$ are mutually inverse, we have

$$\text{Hom}_A (E, i(M)) = \text{Hom}_A(r(E), M) =$$

$$= \text{Hom}_A \left( P^* \otimes_A E, M \right) = \text{Hom}_A(E, \text{Hom}_A(P^*, M))$$

which implies that $i(M)$ is canonically isomorphic to $\text{Hom}_\Lambda (P^*, M)$. Since the functor $i$ is faithful, $P^*$ is a generator of $\mathcal{M}(\Lambda)$.

**Step 4.** We have $r(P) = \text{Hom}_A(P, P) = \Lambda \in \mathcal{M}(\Lambda)$, $r(A) = P^* \otimes_A A = P^*$ and hence $i(\Lambda) = \text{Hom}_A(P^*, \Lambda) = P$, $i(P^*) = \text{Hom}_A(P^*, P^*) = A \in \mathcal{M}(A)$. This implies that as an algebra $\text{End}_\Lambda(P^*) = A^\circ$.

**Corollary.** $P$ is a right projective $\Lambda$-module and $\text{End}_A(P) = A$.

Indeed, since $P = \text{Hom}_A(P^*, \Lambda)$, it is a right projective $\Lambda$-module, dual to $P^*$. Hence $\text{End}_A(P) = \text{End}_A(P^*)^\circ = A$.

**1.7. Realization of a Subcategory as a Category of Modules.**

Let $P \in \mathcal{M}$ be a finitely generated projective object, which we do not suppose to be a generator. Consider subset $S = S_P \subset \text{Irr} \mathcal{M}$ of irreducible quotients of $P$. We say that $P$ splits the category $\mathcal{M}$ if the subset $S$ splits $\mathcal{M}$, i.e. $\mathcal{M} = \mathcal{M}(S) \times \mathcal{M}(\bar{S})$. (see...).

**Corollary.** Suppose $P$ splits $\mathcal{M}$. Then functors $r, i$ give equivalence of categories $\mathcal{M}(S) \overset{r}{\cong} \mathcal{M}(\Lambda)$. Moreover, $\mathcal{M}(\bar{S}) = \{ E \in \mathcal{M}(S) \mid \text{Hom}(P, E) = 0 \}$ $\mathcal{M}(S) = \{ E \in \mathcal{M}(S) \mid E \text{ is a quotient of } \bigoplus_{i} P \}$.

This easily follows from 1.6.
Example. Let \( \mathcal{H} \) be an idempotent ring \( \mathcal{M} = \mathcal{M}(\mathcal{H}) \). Choose an idempotent \( e \in \mathcal{H} \) and put \( P = \mathcal{H}e \). Then \( P \) is a finitely generated projective object in \( \mathcal{M} \), \( \Lambda = (\text{End}_\mathcal{M} P)^\circ \) coincides with the subalgebra \( e\mathcal{H}e \subset \mathcal{H} \) and functors \( r : \mathcal{M}(\mathcal{H}) \rightarrow \mathcal{M}(\Lambda) \), \( i : \mathcal{M}(\Lambda) \rightarrow \mathcal{M}(\mathcal{H}) \) are given by \( r(E) = eE \), \( i(M) = P \otimes \Lambda M \).

We say that idempotent \( e \) splits \( \mathcal{M} \) if the subset \( S = S_e = \{ \omega \in \text{Irr} \mathcal{M} \mid e\omega \neq 0 \} \) splits \( \mathcal{M} \). In this case functors \( r \) and \( i \) give equivalence of categories \( \mathcal{M}(S) \xrightarrow{r} \mathcal{M}(\Lambda) \) and \( \mathcal{M}(S) = \{ E \in \mathcal{M}(\mathcal{H}) \mid E \text{ is generated by } eE \} \), \( \mathcal{M}(\mathcal{S}) = \{ E \in \mathcal{M}(\mathcal{H}) \mid eE = 0 \} \).

1.8. The Central Algebra of \( \mathcal{M} \).

Let \( \mathcal{M} \) be an abelian category.

Definition. The central algebra \( Z(\mathcal{M}) \) is defined as \( Z(\mathcal{M}) = \text{End}(\text{Id}_\mathcal{M}) \), where \( \text{Id}_\mathcal{M} : \mathcal{M} \rightarrow \mathcal{M} \) is the identity functor. In other words, an element \( z \in Z(\mathcal{M}) \) is a collection of morphisms \( z_M : M \rightarrow M \) for all \( M \in \text{Ob} \mathcal{M} \), such that for each morphism \( \alpha : M \rightarrow N \)

\[ z_N \circ \alpha = \alpha \circ z_M. \]

If \( \mathcal{M} = \mathcal{M}(\mathcal{H}) \) or \( \mathcal{M}(G) \) we will also use notations \( Z(\mathcal{H}) \) or \( Z(G) \) instead of \( Z(\mathcal{M}(\mathcal{H})) \) or \( Z(\mathcal{M}(G)) \).

Lemma. Let \( \mathcal{H} \) be an idempotent ring. Then the morphism \( z \mapsto z_H \) identifies \( Z(\mathcal{H}) \) with the algebra \( \text{End}_{\mathcal{H} \times \mathcal{H}}(\mathcal{H}) \) of endomorphisms of \( \mathcal{H} \) which commute with right and left multiplications. In particular, if \( \mathcal{H} \) has an identity, \( Z(\mathcal{H}) \) is isomorphic to the center of \( \mathcal{H} \).

Proof: is straightforward, see...

Corollary. Let \( P \) be a finitely generated projective generator in \( \mathcal{M} \), \( \Lambda = (\text{End}_\mathcal{M} P)^\circ \). Then the natural morphism \( z \mapsto z_P \in \Lambda \) gives an isomorphism of \( Z(\mathcal{M}) \) with the center of \( \Lambda \).

This follows from the lemma and 1.6.

§2. Decomposition theorem

2.0. Let \( G \) be a connected reductive \( p \)-adic group, \( \Theta(G) \) the set of infinitesimal characters of \( G \), \( \Theta(G) = \cup \Theta \) its decomposition into the union of connected components. For each \( \Theta \) consider the subset \( S_\Theta = \inf \cdot \text{ch}^{-1}(\Theta) \subset \text{Irr} G \) and denote by \( \mathcal{M}(\Theta) = \mathcal{M}(G, S_\Theta) \) the corresponding subcategory in \( \mathcal{M}(G) \), \( \mathcal{M}(\Theta) = \{ E \in \mathcal{M}(G), JH(E) \subset S_\Theta \} \) (see 1). In this section we prove the following

Decomposition theorem. \( \mathcal{M}(G) = \prod_{\Theta} \mathcal{M}(\Theta) \), where \( \Theta \) runs through all connected components of \( \Theta(G) \).

Our proof follows the proof in [ ] with slight modifications, which we will use later.
**Generalization.** Let $B$ be a commutative algebra with identity. Put $M(\Theta; B) = \{E \in M(G; B) | E \in M(\Theta) \text{ is } G\text{-module}\}$. Then decomposition theorem implies that $M(G; B) = \bigoplus M(\Theta; B)$.

2.1. Separation of compactly supported $G$-modules.

Let $G$ be an arbitrary $\ell$-group as in 1. A $G$-module $E$ is called *compactly supported* if for each open compact subgroup $K \subset G$ and each $\xi \in E$ the function $g \mapsto (e_K g e_K)\xi$ has a compact support of $G$. This implies that $E$ has compactly supported matrix coefficients. Using this fact and arguing exactly like in a case of compact groups, one can prove the following.

**Proposition.** (see [1]). Let $V$ be a finitely generated compactly supported $G$-module. Then $V$ is admissible and has finite length. The finite subset $S = JH(V) \subset \text{Irr} G$ splits the category $M(G)$ and each module $E \in M(G; S)$ is completely reducible.

2.2. Separation of cuspidal components.

Let $G$ be a reductive $p$-adic group. If the center $Z(G)$ of $G$ is compact, cuspidal $G$-modules are compactly supported and we can use 2.1 to separate them. In general they are compactly supported modulo center $Z(G)$. To study this case we will use the following property of $G$.

(*) $G$ has an open normal subgroup $G^0$ such that $Z(G) \cap G^0$ is compact, $Z(G) \cdot G^0$ has finite index in $G$ and the group $\Lambda = G/G^0$ is a lattice, i.e. is isomorphic to $\mathbb{Z}^d, d \in \mathbb{Z}^+.$

It is easy to see that such a subgroup $G^0$ is unique. By definition the group $\Psi(G)$ of unramified characters of $G$ coincides with

$$\text{Hom} \ (\Lambda, \mathbb{C}^*) = \{\psi : G \to \mathbb{C}^* : \psi|_{G^0} = 1\}.$$ 

**Lemma.** Let $(\rho, V)$ be a simple $G$-module. Then

(i) $\rho|_{G^0}$ is completely reducible of finite length. The subset $S_\rho = JH(\rho|_{G^0}) \subset \text{Irr} G^0$ is finite and is an $G$-orbit of the natural action of $G$ on $\text{Irr} G^0$.

(ii) The correspondence $\rho \mapsto S_\rho$ gives a bijection of the set of $\Psi(G)$ - orbits in $\text{Irr} G$ and $G$-orbits in $\text{Irr} G^0$, i.e. $S_\rho = S_{\rho'}$ iff $\rho' \approx \rho$ for some $\psi \in \Psi(G)$.

(iii) The stabilizer $\text{St}(\rho, \Psi)$ of $\rho$ in $\Psi(G)$ is finite. If we choose for each $\psi \in \text{St}(\rho, \Psi)$ a nonzero morphism $\alpha_\psi : (\rho, V) \to (\psi, V)$, then $\{\alpha_\psi\}$ is a $\mathbb{C}$-basis of $\text{End}_{G^0}(V)$.

**Proof:** (i), (ii) are proven in [1]. (iii) Put $A = \text{End}_{G^0}(V)$ and define the action of $G$ on $A$ by $g(a) = \rho(g)a\rho(g)^{-1}$. This action is trivial on $G^0$. Because of Schur’s lemma it is also trivial on $Z(G)$, so it is an action of the finite abelian group $G/G^0 \cdot Z(G)$. Using this we can decompose $A = \bigoplus A_\psi$, where $A_\psi$ are eigenspaces of the action. But $A_\psi = \text{Hom}_G(\rho, \psi) = \mathbb{C} \cdot a_\psi$ by Schur’s lemma, i.e. $A = \bigoplus \limits_\psi \mathbb{C} \cdot a_\psi$ with $\psi \in \text{St}(\rho, \Psi) \subset \text{Hom} \ (G/G^0 \cdot Z(G), \mathbb{C}^*)$. 

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Harish-Chandra theorem. (see [ ]) Let $\pi$ be a quasicuspidal $G$-module, i.e. $r_{MG}(\pi) = 0$ for all subgroups $M \leq G$. Then it is compactly supported modulo center, i.e. $\pi|_{\mathcal{O}}$ is compactly supported.

**Corollary.** Let $(\rho, V)$ be a cuspidal irreducible $G$-module. Then the cuspidal component $D = \Psi(G) \cdot \rho \in \text{Irr}G$ splits the category $\mathcal{M}(G)$.

**Proof:** Put $S = S_\rho = JH(\rho|_{\mathcal{O}}) \subset \text{Irr}G^0$. By 2.1 every $G$-module $E$ has a decomposition $E = E_S \oplus E_S$ with $E \in \mathcal{M}(G^0; S)$, $E_S \in \mathcal{M}(G^0; \overline{\mathcal{T}})$. Since this decomposition is canonical it is $G$-invariant, i.e. $E_S$ and $E_S$ are $G$-submodules. Lemma 2.2 implies that $E_S \in \mathcal{M}(G, D)$, $E_S^\perp \in \mathcal{M}(G, \overline{D})$.

2.3. Functors $i_{GM}$ and $r_{MG}$.

In order to deal with noncuspidal components we will use functors $i_{GM}$ and $r_{MG}$. Let us recall some elementary properties of these functors. For simplicity we consider only the case when $M$ is a standard Levi subgroup.

(i) Transitivity. Let $M < N < G$. Then $i_{GM} = i_{GN} \circ i_{NM}$, $r_{MG} = r_{MN} \circ r_{NG}$ (canonical isomorphisms).

(ii) Functor $r_{MG}$ is left adjoint to $i_{GM}$ (canonical adjointness). See [ ].

(iii) Functors $i_{GM}$ and $r_{MG}$ are exact and preserve direct sums. See [ ]

(iv) There exists a functorial isomorphism $i_{GM}(\overline{\sigma}) = (i_{GM}(\sigma))$, $\sigma \in \mathcal{M}(M)$ (canonical isomorphism). See [ ]

(v) Functor $r_{GM}$ maps finitely generated $G$-modules into finitely generated $M$-modules. See [ ].

(vi) Composition of functors $r$ and $i$.

We need some notations. For each $w \in W_G$ we fix a representative $w \in \text{Norm}(M_0, G)$. For each subgroup $H \subset G$ we put $w(H) = \tilde{w}H\tilde{w}^{-1}$ and denote by $w$ the corresponding functor $w: \mathcal{M}(H) \to \mathcal{M}(w(H))$.

Let $M, N < G$. Each double coset $W_N \backslash W_G / W_M$ has a unique representative of minimal length; we denote the set of these representatives by $W_G^{NM}$. For each $w \in W_G^{NM}$ we put $M_w = M \cap w^{-1}(N) < M$, $N_w = w(M_w) = w(M) \cap N < N$.

**Composition theorem.** Consider functors $F, F_w : \mathcal{M}(M) \to \mathcal{M}(N)$, for $w \in W_G^{NM}$, defined by $F = r_{NG} \circ i_{GM}$, $F_w = i_{NM} \circ w \circ r_{WM}$. More precisely, choose any ordering $\{w_1, \ldots, w_r\}$ of $W_G^{NM}$ such that $w_i < w_j$ implies $i \geq j$ (here $<$ is the standard partial order on $W$, see [ ]). Then $F$ has a canonical filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_V = F$ and $F_i / F_{i-1}$ is canonically isomorphic to $F_{w_i}$.

See the proof in [ ]. Canonicity of isomorphisms in ... and ... is discussed in appendix ...
(vii) Let $K \subset G$ be an open compact subgroup. We will use the following simple lemma, which describes $K$-invariant vectors in induced $G$-modules.

**Lemma.** Let $(P,M)$ be a standard parabolic pair. Fix a system $(g_1, \ldots, g_n)$ of representatives of double cosets $P \backslash G / K$ and consider open compact subgroups $\Gamma_1, \ldots, \Gamma_n \subset M$ defined by $\Gamma_i = \pi_{\rho^{-1}}(P \cap g_i K g_i^{-1})$. Also fix a Haar measure on the unipotent radical $U \subset P$. Then for every $V \in \mathcal{M}(M)$ and $E = \bigoplus_{i=1}^{n} \Gamma_i$, there exists a canonical functorial isomorphism $E_K \cong \bigoplus_{i=1}^{n} V_i$.

**Proof:** is straightforward.

2.4. Functors $i_{\alpha, \beta}$ and $r_{\alpha, \beta}$ and their properties.

Let $(M,D)$ be a standard cuspidal block (notation $(M,D) < (G, \Theta(G))$). It means that $M < G$ and $D$ is a cuspidal component of $\Theta(M)$. The subset $\Theta = i_{\alpha, \beta}(D) \subset \Theta(G)$ is a connected component. We say that the component $\Theta$ corresponds to the block $(M,D)$ and use the notation $(M,D) < (G, \Theta)$. Another standard cuspidal block $(N,D')$ corresponds to the same component $\Theta$ if and only if there exists $w \in W_G$ such that $w(M,D) = (N,D')$, i.e. $N = w(M)$, $D' = w(D)$. In this case we say that $(N,D')$ is associate to $(M,D)$ (notation $(N,D') \sim (M,D)$).

Standard cuspidal blocks will play a role similar to standard Levi subgroups. By 2.2, $\mathcal{M}(D)$ is a direct summand of $\mathcal{M}(M)$. We denote by $i_{\alpha, \beta} : \mathcal{M}(D) \rightarrow \mathcal{M}(M)$ and $p_{\alpha, \beta} : \mathcal{M}(M) \rightarrow \mathcal{M}(D)$ the corresponding inclusion and projection functors.

Consider the functors

$$i_{\alpha, \beta} = i_{\alpha, \beta} \circ i_{\alpha, D} : \mathcal{M}(D) \rightarrow \mathcal{M}(G)$$

$$r_{\alpha, \beta} = p_{\beta} \circ r_{\alpha, \beta} : \mathcal{M}(G) \rightarrow \mathcal{M}(D).$$

The following properties of these functors immediately follow from 2.3.

(i) $r_{\alpha, \beta}$ is left adjoint to $i_{\alpha, \beta}$.

(ii) $i_{\alpha, \beta}$ and $r_{\alpha, \beta}$ are exact and preserve direct sums.

(iii)

**Composition theorem.** Let $(M,D), (N,D')$ be standard cuspidal blocks $F : p_{\alpha, \beta} \circ i_{\alpha, \beta} : \mathcal{M}(D) \rightarrow \mathcal{M}(D')$. Then $F = 0$ unless $(M,D) \sim (N,D')$. If they are associate, $F$ is glued from the functors $w : \mathcal{M}(D) \rightarrow \mathcal{M}(D')$, where $w \in \{w \in W_G^{N,M} \mid w(M,D) = (N,D')\}.

**Proof:** By composition theorem $F$ is glued from $p_{\alpha, \beta} \circ i_{\alpha, \beta} \circ w \circ r_{\alpha, \beta} \circ i_{\alpha, \beta} \circ m_{\beta}$. If $M_w \neq M$, we have $r_{\alpha, \beta} \circ i_{\alpha, \beta} \circ m_{\beta} = 0$. If $N_w \neq N$, we have $p_{\alpha, \beta} \circ i_{\alpha, \beta} \circ m_{\beta} = 0$ (as right adjoint to $r_{\alpha, \beta} \circ i_{\alpha, \beta} \circ m_{\beta} = 0$). This proves the theorem.

**Proposition.** (i) The system of functors $r_{\alpha, \beta}$ for all $(M,D)_\sim < (G, \Theta(G))$ is faithful, i.e. $r_{DG}(E) = 0$ for all $(M,D)$ implies that $E = 0$.
(ii) Fix a connected component $\Theta \subset \Theta(G)$. Then the system of functors $r_{p,\rho}$ with $(M, D) < (G, \Theta)$ is faithful on $\mathcal{M}(\Theta)$.

(iii) Let $E$ be a $G$-module such that $r_{p,\rho'}(E) = 0$ for all standard cuspidal blocks $(N, D')$ which do not correspond to the component $\Theta$. Then $E \in \mathcal{M}(\Theta)$.

(iv) Conversely, if $E \in \mathcal{M}(\Theta)$, then $r_{p,\rho'}(E) = 0$ for $(N, D') \not\subset (G, \Theta)$.

(v) If $(M, D) < (G, \Theta)$, then $i_{\alpha_{\rho}}(\mathcal{M}(D)) \subset \mathcal{M}(\Theta)$.

**Lemma.** Let $w \in \text{Irr}G$, $\theta = \inf ch w \in \Theta(G)$ and $\Theta$ be a connected component of $\theta$. There exists a cuspidal block $(M, D) < (G, \Theta)$ such that $r_{p,\rho}(w) \neq 0$. For each cuspidal block $(N, D')$ which does not correspond to $\Theta$ $r_{p,\rho'}(w) = 0$.

**Proof:** We can find a cuspidal pair $(M, \rho)$ such that $M < G$ and $w \in i_{\alpha_{\rho}}(\rho)$. Let $D \subset \Theta(M)$ be a connected component of $\rho$. Then $\text{Hom}(w, i_{\alpha_{\rho}}(\rho)) = \text{Hom}(r_{p,\rho}(w), \rho) \neq 0$, i.e. $r_{p,\rho}(w) \neq 0$. If $(N, D') \not\subset (M, D)$, then $r_{p,\rho'}(w) \subset r_{p,\rho'} \circ i_{\alpha_{\rho}}(\rho) = 0$ by composition theorem 2.4 (iii).

**Proof of the proposition.** Since functors $r_{p,\rho}$ are exact the lemma implies (i), (ii) and (iii). Since for $(N, D') \not\subset (M, D)$ $r_{p,\rho'} \circ i_{\alpha_{\rho}} = 0$, (iii) implies (iv).

Let us prove (iv). Let $(M, D')$ be a standard Levi block such that the corresponding component $\Theta'$ differs from $\Theta$. Put $V = r_{p,\rho'}(E) \in \mathcal{M}(D')$. By (v) $i_{\alpha_{\rho}}(v) \in \mathcal{M}(\Theta)$ and hence $\text{Hom}_N(V, V) = \text{Hom}_N(r_{p,\rho'}(E), V) = \text{Hom}_G(E, i_{\alpha_{\rho'}}(V)) = 0$, i.e. $V = 0$.

**Corollary.** Let $N < G$, $\Theta \subset \Theta(G)$ be a connected component. Consider all components $\Theta_N \subset i_{\alpha_{\rho}}^{-1}N(\Theta) \subset \Theta(N)$ and the corresponding product category $\mathcal{M}' = \prod_{\Theta_N} \mathcal{M}(\Theta_N)$. Then

$$i_{\alpha_{\rho}}(\mathcal{M}') \subset \mathcal{M}(\Theta) \text{ and } r_{n,\rho}(\mathcal{M}(\Theta)) \subset \mathcal{M}' \text{.}$$

**Proof:** it easily follows from the composition theorem in 2.3 and the proposition.

### 2.5. Proof of decomposition theorem.

**Step 1.** For each standard cuspidal block $(M, D)$ define a functor $T_D = i_{\alpha_{\rho}} \circ r_{p,\rho} : \mathcal{M}(G) \rightarrow \mathcal{M}(G)$. Since the functor $r_{p,\rho}$ is left adjoint to $i_{\alpha_{\rho}}$, for each $G$-module $E$ we have a canonical functorial morphism $\alpha_{\rho,\rho} : E \rightarrow T_D(E)$. If $L \subset E$, then the restriction $\alpha_{\rho,\rho}|_L : L \rightarrow T_D(E)$ corresponds to the morphism $r_{p,\rho}(L) \rightarrow r_{p,\rho}(E)$. Since the functor $r_{p,\rho}$ is exact, this morphism is an inclusion. This proves that $\alpha_{\rho}(L) = 0$ if and only if $r_{p,\rho}(L) = 0$.

**Step 2.** Consider the product morphism

$$\alpha = \prod_{(M, D)} \alpha_{\rho} : E \rightarrow \prod_{(M, D)} T_D(E)$$

where the product is over all standard cuspidal blocks $(M, D)$. Then $r_{p,\rho}(\text{Ker } \alpha) = 0$ for all $(M, D)$, and, since $\{r\}$ is a faithful system of functors, $\text{Ker } \alpha = 0$. 

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**Step 3.** We want to show that the decomposition \( \text{Irr} G = \bigcup_{\Theta} S_\Theta \) splits a \( G \)-module \( E \). Since \( E \subseteq \prod_{(M,D)} T_D(E) \) it is sufficient to check that \( \{S_\Theta\} \) splits this product (see 1...). By proposition 2.4. (v) \( \{S_\Theta\} \) splits \( \bigoplus_{(M,D)} T_D(E) \), hence it would be sufficient to prove that \( \bigoplus_{(M,D)} T_D(E) \cong \prod_{(M,D)} T_D(E) \). This follows from the following general statement.

\(^(*)\) Let \( V_\Theta \in \mathcal{M}(\Theta) : \Theta \subset \Theta(G) \). Then \( \bigoplus_{\Theta} V_\Theta \approx \prod_{\Theta} V_\Theta \).

**Step 4.** As we saw in .. \( \prod_{\Theta} V_\Theta = \lim_{K} \left( \prod_{\Theta} V^K_\Theta \right) \). Hence \(^(*)\) follows from

\(^{(**)}\) Let \( K \subset G \) be an open compact subgroup. Then \( V^K_\Theta = 0 \) for all but a finite number of components \( \Theta \), so \( \bigoplus_{\Theta} V^K_\Theta = \prod_{\Theta} V^K_\Theta \).

Put \( S_K = \{L \in \text{Irr} G | L^K \neq 0\} \), \( \Theta_K(G) = \inf \cdot \text{ch} \cdot S_K \). If \( V^K_\Theta \neq 0 \), then \( V_\Theta \) has an irreducible subquotient in \( S_K \). Hence \(^{(**)}\) follows from

\(^{(***)}\) \( \Theta_K(G) \) is a union of a finite number of components.

Let \( \Theta \subset \Theta(G) \) be a connected component \( (M,D) < (G,\Theta) \), \( (\rho,V) \in D \). For every \( \psi \) put \( E_\psi = i_{\rho,M}(\psi \rho) \in \mathcal{M}(G) \). The lemma 2.3 ( ) shows that the space \( E_\psi \) does not depend on \( \psi \) and is equal to \( \bigoplus V^T \). For a given infinitesimal character \( \theta = (M,\psi \rho) \in \Theta \) the fiber \( \inf \cdot \text{ch}^{-1}(\theta) \subset \text{Irr} G \) coincides with \( JH(E_\psi) \). This implies, that \( \theta \in \Theta_K(G) \) iff \( \bigoplus V^T \neq 0 \).

Hence \( \Theta \) either lies in \( \Theta_K(G) \) or does not intersect it, i.e. \( \Theta_K(G) \) is a union of components. Moreover, \( \Theta \subset \Theta_K(G) \) iff \( D \subset \Theta_{\Gamma_i}(M) \) for some \( i \). So, using induction in \( \dim M \), we should estimate only the number of cuspidal components. In other words \(^{(***)}\) follows from

\(^{(***)}\) \( \Theta_K(G) \) contains a finite number of cuspidal connected components.

**Step 5.** Using 2.2 we see that \(^{(***)}\) is equivalent to

\(^{*****}\) \( \text{Irr}_K G^0 \) has a finite number of compactly supported \( G^0 \)-modules.

This statement is deduced in [ ] from the following

**Uniform admissibility theorem.** Let \( K \subset G \) be an open compact subgroup. There exists an effective constant \( C = C(G,K) \) such that for each simple \( G \)-module \( L \) \( \dim L^K \leq C(G,K) \).

**Remark.** The proof in [ ] does not give an effective estimate for the number and type of cuspidal components in \( \Theta(G) \). In .... we will give an effective estimate.
2.6. The faithfulness of the functor $r_{\rho_\sigma}$.

Fix a connected component $\Theta \subset \Theta(G)$. As we saw in 2.4, the system of functors $\{ r_{\rho_\sigma} (M, D) \in \mathcal{M}(\Theta) \}$ is faithful on $\mathcal{M}(\Theta)$. In fact, each of these functors is faithful. This fact allows us to simplify notations in many proofs.

**Proposition.** Let $(M, D) \in \mathcal{M}(\Theta)$. Then the functor $r_{\rho_\sigma}$ is faithful on $\mathcal{M}(\Theta)$. In particular, for every $G$-module $E \in \mathcal{M}(\Theta)$ the morphism $\alpha_D : E \rightarrow T_D E$, described in 2.5 is an inclusion.

The proof is based on the following lemma, due to Casseman

**Lemma.** Let $M < G$ be a maximal Levi subgroup, $D \subset \Theta(M)$ a cuspidal component, $\rho \in D$. Suppose that for some $w \in W_G$, $wM < G$ and $w(M, D) \neq (M, D)$. Then the $G$-module $\pi = i_{\rho_\sigma} (\rho)$ is irreducible.

**Proof:**

**Step 1.** Let $R(G)$ be the Grothendieck group of $G$-modules of finite length. By Langlands theory $R(G)$ is generated by $i_{\rho_\sigma}(\psi \sigma)$, where $N < G$, $\psi \in \Psi(N)$, $\sigma \in \text{Irr} N$ is a tempered $N$-module.

Consider the infinitesimal character $\theta$, corresponding to $(M, \rho)$ and a subgroup $R(\theta) \subset R(G)$, generated by $G$-modules with infinitesimal character $\theta$. Let $i_{\rho_\sigma}(\psi \sigma) \in R(\theta)$. If $N \neq G$ then, since $M$ is maximal, $(N, \psi \sigma)$ is conjugate to $(M, \rho)$ and hence $i_{\rho_\sigma}(\psi \sigma) \approx \pi$. Hence if exclude the possibility $N = G$, then $R(\theta) = \mathbb{Z} \cdot \pi$, i.e. $\pi$ is irreducible.

Suppose there exists a tempered $G$-module $\sigma \in \text{Irr} G$, and $\psi \in \Psi(G)$ such that $\psi \sigma \in R(\theta)$. Replacing $\rho$ by $\psi^{-1} \rho$ we can assume that $\psi = 1$, i.e. inf. ch. $\sigma = \theta$. Replacing the cuspidal pair $(M, \rho)$ by a conjugate one we can assume, that $\sigma \nmid \pi$.

**Step 2.** Since $M$ is a maximal Levi subgroup, there exist modulo $W_M$, only one nontrivial element $w \in W_G$ such that $wM < G$ (see [ ]).

Put $N = wM$, $D' = wD$, $\pi' = i_{\rho_\sigma}(\psi w)$. We have $r_{\rho_\sigma}(\pi) = \rho$, $r_{\rho_\sigma}(\pi) = \psi w$. Since the system of functors $r_{\rho_\sigma}$, $r_{\rho_\sigma}$ is faithful on $\mathcal{M}(\theta)$ and $r_{\rho_\sigma}(\sigma) \neq 0$, this implies that $r_{\rho_\sigma}(\sigma) = \rho$, $r_{\rho_\sigma}(\pi/\sigma) = \psi w$ and hence $r_{\rho_\sigma}(\sigma) = 0$. This shows that $\pi$ has length 2. Similarly, $\pi'$ has length 2.

Since $\sigma \in JH(\pi') = JH(\pi)$ and $\sigma \nmid \pi'$, there exists a nontrivial morphism $\pi' \rightarrow \sigma$.

**Step 3.** For every $G$-module $\tau$ denote by $\pi^+$ the Hermitian contragredient $G$-module. Then $\pi^+ \approx \sigma$, since $\sigma$ is tempered and hence unitary. Also $\rho^+$ lies on the same component $D$ as $\rho$, since $D$ contains some unitary $\text{M}$-modules. This implies that $\tau = (\pi')^+$ has a form $\tau = i_{\rho_\sigma}(\rho')$ with $\rho' \in D'$. Nontrivial morphism $\pi' \rightarrow \sigma$ gives a nontrivial morphism $\pi = \sigma^+ \rightarrow \tau$.

But $\text{Hom}_G(\sigma, \tau) = \text{Hom}_G(\pi, i_{\rho_\sigma}(\pi')) = \text{Hom}_G(\pi, i_{\rho_\sigma}(\rho'))$ i.e. $r_{\rho_\sigma}(\sigma) \neq 0$, which contradicts Step 2. This contradiction proves the lemma.

**Proof of the proposition.**

Let $E \in \mathcal{M}(\Theta), E \neq 0$. We have to prove that $r_{\rho_\sigma}(E) \neq 0$. By ..., we can find a standard cuspidal block $(N, D')$, associate to $(M, D)$ such that $r_{\rho_\sigma}(E) \neq 0$.
0. Let \((N, D') = w(M, D), w \in W_G\). We call the map \(w : M \to N\) elementary if there exists a Levi subgroup \(L < G\) such that \(M < L, N < L, w \in W_L\) and \(M\) is a maximal Levi subgroup in \(L\). It is shown in \([\text{____}]\) that any map \(w : M \to N\) can be obtained as a composition of elementary maps. Hence we can assume that \(w : M \to N\) is elementary.

Let \(\Theta' = i_{LM}(D) = i_{LN}(D') \subset \Theta(L), V = r_{\nu}(E) \in \mathcal{M}(L)\). Since \(r_{\nu}(V) = r_{\nu}(E) \neq 0\), \(V\) has a nontrivial \(D'\)-component. Hence replacing \(G\) by \(L\) and \(E\) by the \(\Theta'\)-component of \(V\) we can assume that \(M < G\) is a maximal Levi subgroup. We can also assume that \((M, D) \neq (N, D')\), otherwise \(r_{\nu}(E) = r_{\nu}(E) \neq 0\). Choose an irreducible quotient \(w \in E\). Then \(w \in JH\(i_{LM}(\rho)\)) for some \(\rho \in D\). By the lemma, \(i_{LM}(\rho)\) is irreducible, i.e. \(w = i_{LM}(\rho)\). This implies that \(r_{\nu}(w) \neq 0\) and hence \(r_{\nu}(E) \neq 0\).

Thus we have proved that \(r_{\nu}\) is faithful on \(\mathcal{M}(\Theta)\). The same arguments as in 2.5 show that \(\alpha : E \to TDE\) is an inclusion.

§3. Decomposition of category \(\mathcal{M}(G)\) with respect to a compact subgroup

3.1. Let \(K \subset G\) be an open compact subgroup \(H_K = H_K(G)\). Put \(S_K = \{L \subset \text{Irr}G | L^K \neq 0\}\). We say that the subgroup \(K\) splits \(\mathcal{M}(G)\) if the subset \(S_K\) splits \(\mathcal{M}(G)\), i.e. \(\mathcal{M}(G) = \mathcal{M}(S_K) \times \mathcal{M}(S_K)\). As shown in ... in this case we have

\[
\mathcal{M}(S_K) = \{E \in \mathcal{M}(G) | E\text{is generated by } E^K \},
\]
\[
\mathcal{M}(S_K) = \{E \in \mathcal{M}(G) | E^K = 0 \}
\]

and the functors

\[
r : \mathcal{M}(S_K) \to \mathcal{M}(H_K),
\]
\[
i : \mathcal{M}(H_K) \to \mathcal{M}(S_K)
\]

given by \(r(E) = E^K, i(M) = H \bigotimes_{H_K} M\) are mutually inverse equivalences of categories.

We want to show that there are a lot of subgroups \(K\) which split \(\mathcal{M}(G)\). In order to do this we describe some geometrical sufficient conditions on \(K\).

First of all, let us notice, that if \(S_K\) is a union of subsets \(S_{\Theta}\) for some components \(\Theta\), then \(K\) splits \(\mathcal{M}(G)\). In fact, one can prove that any splitting subset \(S \subset \text{Irr} G\) is a union of \(S_{\Theta}\) (it follows, for instance, from the description of \(Z(\mathcal{M}(G))\) below). So we want to find conditions which imply that \(S_K\) is a union of \(S_{\Theta}\).

3.2. Let \(P \subset G\) be a parabolic subgroup \(M = P/U\). For a compact open subgroup \(K \subset G\) put \(K_P = K \cap P, K_M = pr_{P \to M}(K_P)\). Let \(K \subset G, \Gamma \subset M\) be open compact subgroups. Consider the following conditions on \(K\) and \(\Gamma\).

(I) For each \(g \in G\) the subgroup \((gK)_M \subset M\) contains a subgroup, conjugate to \(\Gamma\).
(II) For any open subgroup \( N \subset G \) the subset \( (\tau_{r \rightarrow \eta})^{-1}(\Gamma) \cdot N \) contains a subgroup conjugate to \( K \).

Note that these conditions are invariant with respect to conjugation of \( P \), \( K \) or \( \Gamma \).

Lemma. (see.....).

(i) Suppose \( K, \Gamma \) satisfy I. Then for each \( M \)-module \( V \), \( V^{\Gamma} = 0 \) \( \Rightarrow i_{\sigma M}(V)^K = 0 \).

(ii) Suppose \( K, \Gamma \) satisfy II. Then for each \( G \)-module \( E \), \( E^K = 0 \) \( \Rightarrow r_{\sigma}(E)^\Gamma = 0 \).

Proof:

(i) Follows from Lemma ...

(ii) \( V \) is isomorphic to \( E_v \) as \( \Gamma \)-module (see ...). Denote by \( A : E \rightarrow E_V \) the natural projection. Suppose that \( E_v^\Gamma \neq 0 \) and choose \( \xi \in E \) such that \( v = A\xi \in E_v^\Gamma \setminus \{0\} \). Let \( N \) be the stabilizer of \( \xi \) in \( g \). Then for each \( g \in \tau_{r \rightarrow \eta}(r)^{-1} \cdot N \) we have

\[
A(g\xi) = A(\gamma m)\xi = \gamma A(\gamma n)\xi = \gamma A\xi = \gamma v = v.
\]

Choose a subgroup \( K' \subset \tau^{-1}(\Gamma) \cdot U \), conjugate to \( K \). Then \( A(e_n, \xi) = v \neq 0 \), i.e. \( E^{K'} \neq 0 \) and \( E^{K} \neq 0 \).

Corollary. Let \( K \subset G \) be an open compact subgroup such that for each parabolic subgroup \( P \) the pair \( K, \Gamma = K_M \subset M \) satisfy both conditions I and II. Then \( S_K \) is a union of \( S_\Theta \) and hence \( K \) splits \( M(G) \).

Proof: Let \( \Theta \subset \Theta(G) \) be a connected component \( (M, D) < (G, \Theta) \) corresponding standard cuspidal block. Let \( L \in S_\Theta \). Then by ... \( r_{\sigma}(L) \neq 0 \), so for some \( \psi \in \Psi(M) \) there exists an epimorphism \( r_{\sigma}(L) \rightarrow \psi \rho \) and an inclusion \( L \rightarrow i_{\sigma M}(\psi \rho) \). Hence

\[
L^K = 0 \Rightarrow r_{\sigma}(L)^\Gamma = 0 \Rightarrow V^{\Gamma} = 0 \text{ and }
\]

\[
V^{\Gamma} = 0 \Rightarrow i_{\sigma M}(\psi \rho) = 0 \Rightarrow L^K = 0.
\]

Thus the condition \( L^K = 0 \) does not depend on \( L \in S_\Theta \), i.e. either \( S_\Theta \subset S_K \) or \( S_\Theta \subset \overline{S_K} \).

Remarks.

(i) It is sufficient to check condition (I) for (finite number of) representatives \( \{g\} \) of double cosets \( P\backslash G/\text{Norm} K \). In particular, if \( K \) is a congruence subgroup, which is normalized by the maximal compact subgroup \( K_0 \), then Iwasawa decomposition \( G = PK_0 \) implies that \( I \) holds for \( \Gamma = K_M \).

(ii) Let \( (P, \overline{P}) \) be a parabolic pair. Suppose that \( K \subset U \Gamma \overline{U} \), where \( \Gamma \subset M = P \cap \overline{P} \). Then condition II holds. Indeed, put \( C = \overline{P}_{\Gamma \cap \Gamma \overline{P}}(K) \). Then we can find \( a \in Z(M) \) for which \( ^aC \) is arbitrarily small, and hence lie in \( N \), which implies \( ^aK \subset U \Gamma N \).

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Examples.

(1) A congruence subgroup $K$ of a nonzero level is normalized by $K_0$ and satisfies $KU \cup K_M \overline{M}$ for each standard parabolic pair $(P, \overline{P})$. Hence it splits $\mathcal{M}(G)$.

(2) Let $I$ be an Iwahori subgroup (see ...). Then it is easy to see that $I \subset U_{I,M}$. Choosing representatives $w \in W = K_0/I$ in $P\backslash G/I$ it is easy to check that $K, K_M$ satisfy condition I for each standard parabolic subgroup $\overline{P}$. Thus $I$ splits $\mathcal{M}(G)$. Another proof of the fact see in [ ]. In this case $S_I$ consists of one component $S_{01}$.

(3) The maximal compact subgroup $K_0$ does not split $\mathcal{M}(G)$ since trivial and Steinberg $G$-modules $\mathbb{C}$ and $St$ lie over the same component $\Theta \subset \Theta(G)$, but $\mathbb{C}^{K_0} \neq 0$ while $St^{K_0} = 0$.

§4. Noetherian properties of $\mathcal{M}(G)$

4.1. Structure of category $\mathcal{M}(D)$ for a cuspidal component $D$.

Let $D \subset \Theta(G)$ be a cuspidal component. Fix $(\rho, V) \in D$. Denote by $F$ the algebra of regular functions on algebraic variety $\Psi(G)$. It coincides with the group algebra of the lattice $L = G/G^0$ and hence has a natural structure of $G - F$-module. This module describes a universal $\psi(G)$-family of unramified characters of $G$ since its specialization at a point $\psi \in \Psi(G)$ is $\mathbb{C}_\psi$.

We denote by $\Pi(\rho)$ the $G - F$-modules $\Pi(\rho) = F \bigotimes_{\mathbb{C}} V$. As $G$-module $\Pi(\rho)$ does not depend on the choice of a point $\rho \in D$ (up to a noncanonical isomorphism). So we denote this $G$-module as $\Pi(D)$.

For every $\psi \in \text{Stab}(\rho, \Psi(G))$ we choose an isomorphism $\alpha_\psi : (\rho, V) \rightarrow (\psi \rho, V)$ and extend it to the automorphism of $\Pi(D)$ by $\alpha_\psi (v, f) = \alpha_\psi (v) \otimes \psi(f)$, where $\psi(f)$ is defined as $\psi(f)(\psi_1) = f(\psi^{-1} \psi_1)$.

Proposition. Let $D \subset \Theta(G)$ be a cuspidal component, $(\rho, V) \in D$.

(i) $\Pi(D)$ is a finitely generated projective generator in the category $\mathcal{M}(D)$.

(ii) $\text{End}_G \Pi(D) = \bigoplus_{\psi} F \cdot a_\psi$ where $\psi \in \text{Stab}(\rho, \Psi(G))$.

Proof:

(i) Since $F = \text{ind}_{G^0}^G (\mathbb{C})$, where $\mathbb{C}$ is the trivial $G^0$-module, $\Pi(D) = \text{ind}_{G^0}^D (\rho|_{G^0})$.

Hence for every $G$-module $E$ we have $\text{Hom}_G (\Pi(D), E) = \text{Hom}_{G^0}(V, E)$.

If $E \in \mathcal{M}(D)$ its restriction to $G^0$ is completely reducible (see 2.1), i.e. the functor $E \mapsto \text{Hom}_G (\Pi(D), E) = \text{Hom}_{G^0}(V, E)$ is exact and faithful. Hence $\Pi(D)$ is a projective generator of $\mathcal{M}(D)$. Since $G^0$ is open in $G$, $\Pi(D)$ is finitely generated.

(ii) $\text{Hom}_G (\Pi(D), \Pi(D)) = \text{Hom}_{G^0}(V, F \otimes V) = F \bigotimes_{\mathbb{C}} \text{Hom}_{G^0}(V, V)$, so the statement follows from 2.2.
Using ... we see that the category $\mathcal{M}(D)$ has a fairly simple description. Namely, put $\Lambda = \operatorname{End}_G(\Pi(D))^0$. Then $\mathcal{M}(D)$ is equivalent to the category $\mathcal{M}(\Lambda)$. The algebra $\Lambda$ is a free module over the subalgebra $F$ with generators $a_\psi$, i.e. $\Lambda = \bigoplus F : a_\psi$ with $\psi \in \operatorname{Stab}(\rho, \Psi(G))$, and following relations

(a) $a_\psi f a_\psi^{-1} = \psi(f), f \in F$.  
(b) $a_\psi a_\chi = c(\psi, \chi)a_{\psi \chi}$, where $c(\psi, \chi) \in \mathbb{C}$ are some constants, defining a projective representation of $\operatorname{Stab}(\rho, \Psi(G))$ in $V$.

**Corollary.**  
(i) The center $Z(\mathcal{M}(D))$ of the category $\mathcal{M}(D)$ is isomorphic to the algebra $Z(D) \subset F \subset \operatorname{End}(\Pi(D))$ of regular functions on $D$.  
(ii) Category $\mathcal{M}(D)$ is locally noetherian.  
(iii) Every finitely generated $G$-module $E \in \mathcal{M}(D)$ is $Z(D)$ admissible.

**Proof:**  
(i) Relations (a) - (b) show that $Z(D)$ coincides with the center of $\Lambda$. Using ... we see that it coincides with $Z(\mathcal{M}(D))$.  
(ii) Since $D \approx \Psi(G)/\operatorname{Stab}(\rho, \Psi(G))$, $F$, and hence $\Lambda$, is a finitely generated $Z(D)$-module. Since $Z(D)$ is a noetherian algebra, the category $\mathcal{M}(\Lambda) \approx \mathcal{M}(D)$ is locally noetherian.  
(iii) Since $\rho|_{\Lambda}$ is admissible (see 2.1), $\Pi(D)$ is $F$-admissible and hence $Z(D)$-admissible. Since any finitely generated $G$-module $E \in \mathcal{M}(D)$ is a quotient of $\Pi(D)^n, n \in \mathbb{Z}^+$, it is also $Z(D)$ admissible.

4.2. Noetherian properties of $\mathcal{M}(G)$.

**Theorem.** Category $\mathcal{M}(G)$ is locally noetherian. Functors $r$ and $i$ map finitely generated modules into finitely generated ones.

**Proof:**  
Step 1. Functor $r$ maps finitely generated modules into finitely generated ones. This easily follows from Iwasawa decomposition (see [ ]).  
Step 2. Let $(M, D)$ be a standard cuspidal block, $V \in \mathcal{M}(D)$ be a finitely generated $M$-module. Then $G$-module $E = i_{\alpha_0}(V)$ is noetherian. Let $\Theta = i_{\alpha_0}(D) \subset \Theta(G)$. Then $E \in \mathcal{M}(\Theta)$ (see ...). Since the functor $r_{\alpha_0}$ is faithful and exact on $\mathcal{M}(\Theta)$ it is sufficient to check, that $r_{\alpha_0}(E)$ is noetherian. But by 2.4 $r_{\alpha_0}(E) = r_{\alpha_0} \circ i_{\alpha_0}(V)$ is glued from $M$-modules $wV, w \in W(D)$, each of which is noetherian by Proposition 4.1.  
Step 3. Let $E$ be a finitely generated $G$-module. Then it is noetherian. Indeed, by 2. E imbeds into $\bigoplus_{(M, D)} T_D E$. Since it is finitely generated, its image lies in a finite sum. Using Steps 1,2 we see that each $G$-module $T_D E = r_{\alpha_0} \circ r_{\alpha_0}(E)$ is noetherian, and hence $E$ is noetherian.
Step 4. Let $N < G$, $V \in \mathcal{M}(N)$ be a noetherian $M$-module. Then $i_{\alpha N}(V)$ is noetherian $G$-module.

Repeating arguments in Step 3 we see that $V$ is contained in a finite sum $\bigoplus_{(M,D)} i_{\alpha N} \circ r_{D N}(V)$. Hence $i_{\alpha N}(V)$ is contained in a finite sum $\bigoplus i_{\alpha D} \circ r_{D N}(V)$, which is noetherian by Steps 1,2.

Generalization. Let $B$ be a commutative noetherian $\mathbb{C}$-algebra with identity. Then category $\mathcal{M}(G; B)$ is locally noetherian, and functors $i, r$ map noetherian $G - B$-modules into noetherian ones.

Generalization. Let $B$ be a commutative algebra with identity. Then $\mathcal{M}(D; B) \approx \mathcal{M}(\Lambda \otimes B)$, $Z(\mathcal{M}(D, B)) = Z(D) \otimes \mathbb{C}$. If $B$ is noetherian, then $Z(D) \otimes \mathbb{C}$ is noetherian, since $Z(D)$ is a finitely generated $\mathbb{C}$-algebra. This implies that $\mathcal{M}(D, B)$ is noetherian.

§5. Stabilization Theorem

5.1. Let $K \subset G$ be an open compact subgroup. For each $g \in G$ we put $h(g) = e_{Kg} \in H_K$, where $g$ stands for $\delta$-distribution at $g$. In other words, $h(g)$ is the unique normalized bi-$K$-invariant measure, supported on $KgK$.

In some cases we have equalities $h(a^i) = h(a)^i$ for $i \geq 0$ or $h(ab) = h(a)h(b)$.

(geometrically it means that $K^i gK = (KgK)^i$ and $KabK = KaKbK$ respectively.) We want to describe some sufficient conditions for these equalities. Essentially these conditions mean that $a, b$ are dominant with respect to some parabolic pair.

Definition. Let $(P, \overline{P})$ be a parabolic pair. We say that subgroup $K$ is in a good position with respect to $(P, \overline{P})$ if

(*) $K = K_+ \Gamma K_-$, where $K_+ = K \cap \overline{U}, \Gamma = K \cap M, K_+ = K \cap U$.

Suppose $(P, \overline{P})$ and $K$ are in a good position. We call element $a \in M$ dominant with respect to $(P, \overline{P}, K)$ if

(**) $a^{-1} K_- a \subset K_-$, $a \Gamma a^{-1} = \Gamma$, $aK_+a^{-1} \subset K_+$.

For each compact subgroup $C \subset G$ we denote by $e_c$ the distribution on $G$, which is the image of the normalized Haar measure on $c$. If $K$ is in a good position with respect to $(P, \overline{P})$, we have

$$e_K = e_{K_+} e \Gamma e_{K_-} = e_{K_+} e \Gamma e_{K_-}.$$ 

If $a, b$ are dominant with respect to $(P, \overline{P}, K)$ we have $h(ab) = h(a)h(b)$. Indeed,

$$KaKbK = KaK_+ \Gamma K_- bK = K(aK_+ a^{-1})(a \Gamma a^{-1}) ab(b^{-1} K_- b) K = KabK.$$
Example. Let $A \subset Z(M_0)$ be the maximal split torus, $\Lambda = \text{Hom}_{alg. gr.}(A, F^*)$ its character lattice, $\Sigma \subset \Lambda$ the root system of $G$ and $\Sigma^+ \subset \Sigma$ the system of positive roots, corresponding to $P_0$. Put $A^+ = \{a \in A \mid |\alpha(a)| \leq 1 \text{ for all } \alpha \in \Sigma^+\}$. Then there exist arbitrary small open compact subgroups $K \subset G$ (congruence subgroups) such that $(P_0, \overline{P}_0)$ and $K$ are in a good position, and all elements $a \in A^+$ are dominant with respect to $(P_0, \overline{P}, K)$. In particular, $\mathcal{H}_K$ contains a very big commutative subalgebra $\mathcal{A} = \text{span} \{h(a) | a \in A^+\}$.

In fact these congruence subgroups are in a good position with respect to each standard parabolic pair $(P, \overline{P})$ and all elements in $A^+ \cap Z(M)$ are dominant with respect to $(P, \overline{P}, K)$ (see [ ]).

5.2. To each element $g \in G$ naturally corresponds a parabolic pair. Namely, put $P_g = \{x \in G \mid$ the sequence $g^i x g^{-i}, \; i = 1, 2, \ldots, \text{ is bounded in } G\}.

Statement. $P_g$ is a parabolic subgroup of $G$, $(P_g, P_{g^{-1}})$ is a parabolic pair.

For regular semisimple $g$ the statement is proved in [c]. It is enough for our purposes.

Definition. Let $(P, \overline{P})$ be a parabolic pair. We say that an element $a \in M$ is strictly dominant with respect to $(P, \overline{P})$ if $(P, \overline{P}) = (P_a, P_{a^{-1}})$. Geometrically it means that operators $Ad a^i$ and $Ad a^{-1} | \overline{P}$ are strictly contractable and the family of operators $\{Ad a_i^i \mid i \in Z\}$ is uniformly bounded on $M$.

Let $(P, \overline{P})$ and $K$ be in a good position. We say that an element $a \in M$ is strictly dominant with respect to $(P, \overline{P}, K)$ if it is dominant and strictly dominant with respect to $(P, \overline{P})$.

Lemma. (i) Let $g \in G, \; (P, \overline{P}) = (P_g, P_{g^{-1}})$. There exist arbitrary small open subgroups $K \subset G$ in a good position with respect to $(P, \overline{P})$ such that $g$ is strictly dominant with respect to $(P, \overline{P}, K)$.

(ii) Let $K$ be in a good position with respect to $(P, \overline{P})$. There exist an element $a \in Z(M)$ strictly dominant with respect to $(P, \overline{P}, K)$.

Proof: Statement (i) is proved in [ ], (ii) is straightforward.

Fix an element strictly dominant with respect to $(P, \overline{P}, K)$ and consider increasing sequences of subgroups

$$U_n = a^{-n}K + a^n \subset U, \quad \overline{U}_n = a^nK - a^{-n} \subset \overline{U}.$$ When $n \to \infty$ these subgroups become arbitrary large, when $n \to -\infty$ they become arbitrary small.

Put $h = h(a)$. Using formulae in 5.1, we get for $n \geq 0$

$$h^n = e_K a^n e_K$$
$$e_K a^n = a^n e_{U_n} e_{\overline{U}_n}$$
$$h^n = e_K a^n e_K = a^n e_{U_n} e_K$$ and similarly

$$h^n = e_K e_{\overline{U}_n} a^n.$$
Proposition. Let $E$ be a $G$-module, $E_U$ the space of $U$-coinvariants of $E$ (see...) and $A : E \to E_U$ the natural $M$-equivariant projection. Denote by $A_k$ the corresponding morphism $A_K : E^K \to E^K_U = (E_U)^\Gamma$. Then

(i) $A_K h^n = a^n A_K$.

(ii) For $\xi \in e^K h^n \xi = 0$ iff $e_U \xi = 0$

In particular

\[ \text{Ker } A_K = \bigcup_n \text{Ker } e_U |_{E^K} = \{ \xi \in E^K | h^n \xi = 0 \text{ for larger } \} . \]

(iii) If $\xi \in E$ is $U_{-n}$-invariant, then $a^n e_{\Gamma} A \xi = A e_K a^n \xi$. In particular, for each $\eta \in E^\Gamma_U a^n \eta \in \text{Im } A_K$ for large $n$, i.e. $\bigcup a^{-n} \text{Im } A_K = E^\Gamma_U$.

Proof: Formula $h^n = a^n e_U e_K$ implies (i). Since the operator $a$ on $E^\Gamma_U$ is invertible, it also implies (ii). Using formula $a^n e_U e_{\Gamma} e_{U_{-n}} = e_K a^n$ we see that $a^n e_\Gamma \left( e_{U_{-n}} \xi \right) = a^n A e_U e_{\Gamma} e_{U_{-n}} \xi = A a^n e_U e_{\Gamma} e_{U_{-n}} \xi = A e_K a^n \xi$ which proves (iii).

This proposition means that space $E^\Gamma_U$ together with operator $g$ is naturally isomorphic to the localization of $E^K$ with respect to operator $h$.

5.3. Stabilization Theorem. Let $(P, \overline{P})$ be a parabolic pair, $K \subset G$ an open compact subgroup, in a good position with respect to $(P, \overline{P})$. Denote by $C = C_K$ a constant in uniform admissibility theorem (see....), i.e. a bound for $\dim E^K$ for $L \in \text{Irr } G$.

Let $a \in M$ be an element strictly dominant with respect to $(P, \overline{P}, K)$. Put $h = h(a) \in H_K$. For each $G$-module $E$ consider $h$ as an endomorphism of $E^K$.

Stabilization theorem. (i) For each $G$-module $E$ there exists a unique decomposition $E^K = E^K_0 \oplus E^K_\ast$ into $h$-invariant subspaces such that $h^* E^K_0 = 0$ and $h$ is invertible on $E^K_\ast$. Namely, $E^K_0 = \text{Ker } h^n$, $E^K_\ast = \text{Im } h^n$ for any $n \geq C$.

(ii) Let $C \subset U$, $\overline{C} \subset \overline{U}$ be sufficiently large open compact subgroups. Then for each $G$-module $E$

\[ E^K_0 = E^K \cap \text{Ker } e_C , \quad E^K_\ast = e_K e_{\overline{C}} E . \]

In particular, $E^K_0$, $E^K_\ast$ do not depend on the choice of $a$.

(iii) Consider the natural morphism $A_K : E^K \to E^K_U$. Then $E^K_0 = \text{Ker } A_K$ , $A_K : E^K_\ast \to E^K_U$ is an isomorphism.

Proof: Using formulas $h^n = a^n e_U e_K = e_K e_{U_{-n}} a^n$, we see that (i) implies (ii) for subgroups $C \supset U_n = a^{-n} K a^n$, $\overline{C} \supset \overline{U}_n = a^n K a^{-n}$. Using proposition 5.2 we see that (i) implies (iii). Hence it is enough to prove (i).
Step 1. Let $L$ be a $\mathbb{C}[x]$-module, i.e. a vector space with an endomorphism $x$. We say that $L$ is $x$-stable if $L$ has an $x$-invariant decomposition $L = L_0 \oplus L_x$ such that $xL_0 = 0$ and $x$ is invertible on $L_x$. Clearly, $L$ is $x$-stable $\iff L = \text{Ker } x \oplus \text{Im } x \iff \text{Ker } x^2 = \text{Ker } x$, $\text{Im } x^2 = \text{Im } x$ is invertible on $L / \text{Ker } x \approx \text{Im } x$.

It is easy to check that the direct sum of $x$-stable modules is $x$-stable and for each morphism $\alpha : L \to L'$ of $x$-stable $\mathbb{C}[x]$-modules $\text{Ker } \alpha$ and $\text{Coker } \alpha$ are $x$-stable $\mathbb{C}[x]$-modules.

Step 2. Denote by $\mathcal{M}' \subset \mathcal{M} (G)$ the subcategory of $G$-modules $E$ such that $E^k$ is $h^C$-stable. We have to show that $\mathcal{M}' = \mathcal{M} (G)$.

As follows from Step 1 direct sums of modules in $\mathcal{M}'$ and kernels and cokernels of morphisms of modules in $\mathcal{M}'$ lie in $\mathcal{M}'$.

Also, $\mathcal{M}'$ contains all irreducible $G$-modules. Indeed, for each irreducible $G$-module $L \dim L^K \leq C$, and hence the sequence of subspaces $\text{Im } h^i$ is constant for $i \geq C$, i.e. $h$ is invertible on $\text{Im } h^C$.

Step 3. Let $B$ be a commutative noetherian $\mathbb{C}$-algebra, $E$ $B$-admissible $\sigma - B$-module. Suppose that $r_{MG} (E)$ is $B$-admissible $M - B$-module. Then for some $n > 0 E^K$ is $h^n$-stable.

Indeed, since $E^K$ is noetherian $B$-module, the sequence of submodules $\text{Ker } h^n$ is stable. By proposition... $\text{Ker } A^K = \bigcup A^n_k$, and hence $\text{Ker } A^K = \text{Ker } h^n$ for some $n > 0$.

By proposition... $E_U^n$ is a union of $B$-submodules $a^{-n} \text{Im } A_k$. Since $E_U^n$ is finitely generated $B$-module it is equal to $a^{-n} \text{Im } A_k$ for some $\Gamma > 0$. Since $a$ is invertible on $E_U^n$ we see that $E_U^n = \text{Im } A_k = E^k / \text{Ker } A_k$.

Thus the operator $h$ is invertible on $E^K / \text{Ker } A_k = E^K / \text{Ker } h^n$, which implies that $E^K$ is $h^n$ stable.

Step 4. Let $(N, D)$ be a standard cuspidal block, $(\rho, V) \in D, \Pi (D) = F \otimes V$ be $G - F$-module described in... Put $(\Pi, E) = i_{GM} (\Pi (D))$. Then for some $n > 0 E^K$ is $h^n$-stable.

It is sufficient to check that $E$ and $r_{MG} (E)$ are $F$-admissible modules. By composition theorem $r_{MG} (E)$ is glued from $M$-modules $i_{M,M_w} \omega (\Pi (D))$. Hence $F$-admissibility of $E$ and $r_{MG} (E)$ follows from the following.

Lemma. The functor $i_{GM} : \mathcal{M} (A, B) \to \mathcal{M} (G, B)$ maps $B$-admissible modules into $B$-admissible ones.

This lemma is an immediate consequence of lemma...

Step 5. Module $(\Pi, E)$ is step 4 which lies in $\mathcal{M}'$, i.e. $E^K$ is $h^C$-stable. Indeed, it is sufficient to check that $\text{Ker } h^n \subset \text{Ker } h^C$. Let $\xi \in \text{ker } h^n$, $\xi = h^C \xi$. For each $\psi \in \Psi (M)$ consider specialization morphism $\Pi (D) \to \psi \rho$ and the corresponding morphism $\alpha_\psi : E \to E_\psi = i_{GM} (\psi \rho)$.

Lemma. (see [ ] ) For generic $\psi$ $G$-module $E_\psi$ is irreducible.

This lemma implies that for generic $\psi E_\psi \in \mathcal{M}'$. Since $h^n \alpha_\psi (\xi) = 0$, this implies that $\alpha_\psi (\xi') = h^C \alpha_\psi (\xi) = 0$ and hence $\xi' = 0$.
Step 6. Let \( (N, D) \) be a standard cuspidal block. Then \( i_{GN}(\mathcal{M}(D)) \subset \mathcal{M}' \).

Let \( \sigma \in \mathcal{M}(D) \). Since \( \Pi(D) \) is a projective generator in \( \mathcal{M}(D) \) we can represent \( \sigma \) as a cokernel of some morphism \( \gamma : \oplus_n \Pi(D) \to \oplus_n \Pi(D) \). Then \( i_{GN}(\sigma) = \text{Coker} (\oplus_n \Pi \to \oplus_n \Pi) \) (since functor \( i_{GN} \) is exact and preserves direct sums). Since \( \Pi \in \mathcal{M}' \) Step 2 implies that \( \sigma \in \mathcal{M}' \).

Step 7. Each \( G \)-module \( E \) lies in \( \mathcal{M}' \). Indeed, we can embed \( E \) into module \( E' = \oplus_{(N, D)} i_{GD} \circ r_{DG} \) as in..... By Step 6 \( E' \in \mathcal{M}' \). Similarly we embed \( E' / E \) into \( E'' \in \mathcal{M}' \). Then \( E = \ker (E' \to E'') \) lies in \( \mathcal{M}' \) by step 2.

5.4. Corollaries and Remarks to the Stabilization Theorem.

Generalized Jacquet Lemma. Let \( K \) be in a good position with respect to \( (P, \overline{P}) \). Then for each \( G \)-module \( E \) the morphism \( \Lambda_K : E^K \to E^U \) is an epimorphism. Moreover, it has a right inverse morphism \( B \), functorial in \( E \), i.e. \( E^U \) can be realized in a natural way as a direct summand of \( E^K \).

Corollary. Functor \( r^P_{MG} \) maps \( B \)-admissible \( G \)-\( B \)-modules into \( B \)-admissible \( M \)-\( B \)-modules.

We will prove more a general result.

Let \( B \) be a commutative \( \mathbb{C} \)-algebra with identity. Fix a class of objects \( C \subset \mathcal{M}(B) \) closed with respect to isomorphisms, finite direct sums and taking of direct summands (i.e. for \( x \oplus y \cong Z \), \( Z \in C \) iff \( X, Y \in C \)). Examples: \( C \) is the class of finitely generated \( B \)-modules, or the class of projective \( B \)-modules, or the class of flat \( B \)-modules and so on. We say that \( G \)-\( B \)-module \( E \) is of \( C \)-type if for each open compact subgroup \( K \subset G \) \( B \)-module \( E^K \) lies in \( C \).

Proposition. Fix a class \( C \subset \mathcal{M}(B) \) as above. Then functors \( i^P_{GM} : \mathcal{M}(M, B) \to \mathcal{M}(G, B) \), \( r^P_{MG} : \mathcal{M}(G, B) \to \mathcal{M}(M, B) \) map \( C \)-type modules into \( C \)-type modules.

Proof: For functor \( i^P_{GM} \) this follows from lemma..... Let \( E \) be a \( G \)-\( B \)-module of type \( C \) and \( \Gamma_0 \subset M \) an open compact subgroup. Choose an open compact subgroup \( K \subset G \) in a good position with respect to \( (P, \overline{P}) \) such that \( \Gamma = K \cap M \subset \Gamma_0 \). Then \( E^U_{\Gamma_0} \) is a direct summand of \( E^U \), which is a direct summand of \( E^K \). Hence \( B \)-module \( E^U_{\Gamma_0} \) lies in \( C \), which proves the proposition for functor \( r^P_{MG} \).

Remark. 1. Consider the decreasing sequence of right ideals \( J_n = h^n \mathcal{H}_K \subset \mathcal{H}_K \). Applying stabilization theorem to \( G \)-module \( \mathcal{H}(G)e_K \) we see that it is stable, namely

\[(*) \quad J_n = J_C \quad \text{for} \quad n \geq C \,.
\]

In fact this statement is equivalent to the theorem. Indeed, it implies that \( \text{Im} h^n = \text{Im} h^C \) for each \( G \)-module \( E \). Using the natural anti-involution of \( \mathcal{H}(G) \), given by the antiautomorphism \( g \mapsto g^{-1} \) on \( G \), we can deduce from \( (*) \) that \( \mathcal{H}_K h^n = h_K h^C \) for \( n \geq C \), which implies that \( \text{Ker} h^n = \text{Ker} h^C \).

Note, that \( (*) \) is purely geometrical statement, which has nothing to do with the representation theory. It would be very interesting to find a direct geometrical proof of \( (*) \). Such proof would probably give a reasonably precise
estimate for constant $C$ in (**). I was able to find such proof for congruence subgroups in $GL(Z)$, but not for higher rank. Another form of the statement (**), which does not involve the choice of $a$, is (**) For sufficiently large open compact subgroups $C \subset U$ the ideal $J_C = e_K \mathcal{H}(G) e_K$ does not depend on $C$. Namely, this is true for $C \supset a^{-C} K \cdot a^C$.

5.5. An Effective Bound of the Number of Cuspidal Components With a Given Conductor.

Fix an open compact subgroup $K \subset G$. We want to give an effective bound of the number of cuspidal components $D \subset \Theta_K(G)$.

Let $E$ be a $G$-module, $\xi \in E^K, \xi \in E^K$. We denote by $\varphi_{\xi, \xi}$ the matrix coefficient $\varphi_{\xi, \xi}(g) = \langle \xi, g\xi \rangle$.

Proposition. There exists a compact subset $S \subset G^0$, which can be effectively described in terms of $G$ and $K$, such that for each quasicuspidal $G$-module $E$, $\xi \in E^K, \xi \in E^K$ the matrix coefficient $\varphi_{\xi, \xi}$ vanishes on $G^0 \setminus S$.

This proposition gives a desired bound. Indeed, let $D_1, \ldots, D_r$ be different cuspidal components in $\Theta_K(G)$, $V_i \subset V_i, 0 \neq \xi_i \in V_i^K, 0 \neq \xi_i \in V_i^K, \varphi_i = \varphi_{\xi_i, \xi_i}$ for $i = 1, \ldots, r$. By 2. matrix coefficients $\varphi_i$ are linearly independent on $G^0$. Since they vanish on $G^0 \setminus S$ and are $K$-biinvariant, their number $r$ is less or equal to $\# \langle K \setminus S \rangle / K$.

Proof of Proposition. Let $A \subset Z(M_0)$ be the maximal split torus, $L$ the lattice of coweights of $A$, which we will identify with the quotient $L = A/A^0$ of $A$ by its maximal compact subgroup. Let $L^0 = L \cap G^0$ be the semisimple part of $L$, $L^0 = L^0 \cap A^+$, where $A^+$ is defined in example 5.1. In other words, $L^0 = \{ a \in L \mid (a, a) \leq 0 \}$. By 2. matrix coefficients $\varphi_i$ are biinvariant on $G^0$. Since they vanish on $G^0 \setminus S$ and are $K$-biinvariant, their number $r$ is less or equal to $\# \langle K \setminus S \rangle / K$.

Choose a congruence subgroup $K'$, which lies in the open subset $\bigcap_{x \in \Omega} xKx^{-1}$ and denote by $C = C_{K'}$ the constant in uniform admissibility theorem for $K'$. Put $S^0 = L^0 \setminus \left[ L^0 + c(L^0 \setminus \emptyset) \right], S = \Omega^{-1} S^0 \Omega$. We claim that $S$ is a desired subset. First of all, since $L^0$ is a strictly convex cone, set $S^0$ is finite, i.e., $S$ is compact. Let $E$ be a quasicuspidal $G$-module, $\xi \in E^K, \xi \in E^K, g \in G^0 \setminus S$. We want to show that $\varphi_{\xi, \xi}(g) = 0$. By definition $g = x^{-1} a' y$, where $x, y \in \Omega, a' \in L^0$ is of the form $a' = b + ca, b \in L^0, a \in L^0 \setminus \emptyset$. Put $h(a) = e_K ca_K$, and similarly for $a', b$. Since $a \in L^0 \setminus \emptyset$ the corresponding parabolic subgroup $P_a$ differs from $G$, i.e. $r_{MG}(E) = 0$. Hence for each vector $\xi \in E h(a)^n \eta = 0$ for large $n$ and by the stabilization theorem, $h(a)^{C_T} \eta = 0$. Hence

$\varphi_{\xi, \xi}(g) = \varphi_{x^\xi, y \xi}(a') = \left( x^\xi, a' y \xi \right) = \left( x^\xi, h(a') y \xi \right) = \left( x^\xi, h(b) h(a)^C y \xi \right) = 0$

Here we used that vectors $x^\xi$ and $y \xi$ are $K'$-invariant. Formula $h(a') = h(b) h(a)^C$ follows from 5.1. Note, that addition in $L$ becomes multiplication, when $L$ is considered as a subgroup of $G$. 

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**Remark.** All bounds we described are effective, but quite excessive. The most excessive is the estimate for the constant $C = C_K$ in the proof of uniform admissibility theorem. It would be interesting to find more precise bounds.

§6. Main Theorems About Functors Randi

6.1. Pairing Between $\tilde{E}_{\Gamma}$ and $E_U$.

Let $(\mathcal{P}, \mathcal{T})$ be a parabolic pair. For each $G$-module $E$ denote by $\tilde{E}$ the contragredient $G$-module and consider $M$-modules $\tilde{E}_{\Gamma} = (\tilde{E})_{\Gamma}$ and $E_U$.

**Theorem.** There exists a unique pairing $\{ \}$ : $\tilde{E}_{\Gamma} \times E_U \rightarrow \mathbb{C}$ satisfying the following condition on the asymptotic of matrix coefficients.

(ASS) Let $K \subset G$ be an open compact subgroup, $a \in M$ be an element strictly dominant with respect to $(\mathcal{P}, \mathcal{T})$. Then there exists $n_0$, depending only on $a$ and $K$, such that for each $\xi \in \tilde{E}$, $\xi \in E$, $i > n_0$ $\overline{(\xi, a^i \xi)} = \{ \overline{A \xi}, A^i \xi \}$ (here $A : E \rightarrow \tilde{E}_{\Gamma}$, $A : E \rightarrow E_U$ are natural projections).

The pairing $\{ \}$ is $M$-equivariant, functorial in $E$ and it gives an isomorphism of $M$-modules $\tilde{E}_{\Gamma} \rightarrow (E_U)^\sim$.

**Corollary.** There exists a canonical functorial isomorphism $r_{MG}^\mathcal{P}(E) \approx (r_{MG}(E))^\sim$.

In particular, for a standard Levi subgroup $M < G$ $\tilde{E}_{\Gamma} = r_{MG}(E)$.

**Proof:** Indeed, by definition $r_{MG}^\mathcal{P}(E) = \tilde{E}_{\Gamma} \otimes \Delta_{\mathcal{P}}^{1/2}, r_{MG}(E) = E_U \otimes \Delta_U^{1/2}$. Since $\Delta_{\mathcal{P}}$ and $\Delta_U$ are canonically dual (see appendix...), the theorem implies the corollary.


**Step 1.** Let $K \subset G$ be an open compact subgroup in a good position with respect to $(\mathcal{P}, \mathcal{T})$, $\Gamma = K \cap M$. First let us define the pairing $\{ \}$ : $\tilde{E}_{\Gamma} \times E_U \rightarrow \mathbb{C}$. By the stabilization theorem $A_K : E_U^K \rightarrow E_U^\Gamma$ is an isomorphism, so we can identify $E_U^\Gamma$ with a subspace $E_U^K \subset E^K$. Applying the stabilization theorem to the parabolic pair $(\mathcal{P}, \mathcal{T})$, subgroup $K$ and $G$-module $E$ we can identify $E_{\Gamma}^\mathcal{P}$ with the subspace $E^K \subset E^K$. Then the restriction of the pairing $\{ \} : \tilde{E}_{\Gamma} \times E^K \rightarrow \mathbb{C}$ defines a pairing $\{ \}$ : $E_U^\Gamma \times E_U^K \rightarrow \mathbb{C}$.

**Step 2.** Choose an element $a \in M$ strictly dominant with respect to $\mathcal{P}, K$ (see 5.2) and put $h = h(a)$, $h^* = h(a^{-1})$. For each $\xi \in E^K$, $\xi \in E^K$ we have

$$(\bar{\xi}, a^n \xi) = (\xi, h^n \xi) = (\bar{\xi}, h^n \xi) = \left( (h^*)^n \bar{\xi}, \xi \right).$$

Using stability theorem, we see that for $n > C_K (\xi, a^s \xi)$ depends only on projections of $\xi$ on $E^K$, and of $\xi$ and $E^K$. This shows that the pairing $\{ \}$ satisfies condition (ASS) for $a$ and $K$. Since $h$ is invertible on $E^K$, $\{ \}$ is uniquely determined by condition (ASS).
Step 3. Let $K' \subset K$ be a smaller subgroup, such that $a$ is strictly dominant with respect to $P, \mathcal{P}, K'$. Consider the corresponding pairing \( \{ \} : E_{U}^{L} \times E_{U}^{R} \to \mathbb{C} \). It satisfies (ASS) and by uniqueness property of \( \{ \} \) the restriction of \( \{ \} \) to $E_{U}^{L} \times E_{U}^{R}$ coincides with \( \{ \} \). Hence, choosing smaller and smaller subgroups $K$, we can define a pairing \( \{ \} : E_{U}^{L} \times E_{U}^{R} \to \mathbb{C} \) satisfying (ASS), and this pairing is unique. By construction the pairing \( \{ \} \) does not depend on $a$. This implies that it is $M$-equivariant.

Step 4. For each subgroup $K$ the space $E^{K}$ is dual to $E^{K}$ and the operator $h^{*}$ in $E^{K}$ is dual to the operator $h$ in $E^{K}$. Hence $E^{K}$ is dual to $E^{K}$. By definition of \( \{ \} E_{U}^{L} \approx E_{K}^{L} \approx E_{K}^{R} \), which implies that \( \{ \} \) gives an isomorphism of $E_{U}^{L}$ with module $(E_{U})^{-}$ contragredient to $E_{U}$.

6.3. Completion of $\sigma$-Modules. We want to describe the pairing \( \{ \} \) in a more direct and visual way, using the notion of completion of $G$-modules.

Definition. Let $E$ be a $G$-module. We define its completion $E^\wedge$ in any of three equivalent ways

(i) $E^\wedge = \text{Hom}_G(\mathcal{H}(G), E)$.

(ii) $E^\wedge = \lim_{\overrightarrow{K}} E^{K}$, where the inverse limit is over all open compact subgroups $K \subset G$ and for $K' \subset K$ the connecting morphism $E^{K'} : E^{K} \to E^{K}$ is given by $\xi \mapsto e_{K} \xi$.

(iii) $E^\wedge$ is the completion of $E$ in the topology, generated by open subset $\text{Ker} e_{K}$ for open compact subgroups $K \subset G$.

The algebra $D_{C}(G)$ of compactly supported distributions on $G$ acts on the completion $E^\wedge$ by $d\xi^\wedge(h) = \xi^\wedge(h \ast d)$. This action is continuous in the topology, described in (iii) and its restriction to $E \subset E^\wedge$ coincides with the natural action of $D_{C}(G)$ on $E$. In particular, $G$ acts on $E^\wedge$, but this representation usually is not smooth. The smooth part of $E^\wedge$ coincides with $E = \mathcal{H}(G)E^\wedge$.

It is easy to check that the functor $E \mapsto E^\wedge$ is exact and faithful. Moreover, if $E' \subset E$, then $(E')^\wedge = \text{Closure}_{E} E'$ in $E^\wedge = \{ \xi^\wedge \in E^\wedge \mid \mathcal{H}(G)\xi^\wedge \subset E' \subset E \}$.

It is easy to check that $(\mathcal{L}^\wedge)^* \approx L^*$ (the dual space). This gives the following realization of $E^\wedge$, convenient for computations:

Let us realize $E$ as a submodule of $\mathcal{L}$ for some $G$-module $L$ and then $E^\wedge$ can be described as

$$E^\wedge = \{ \xi^\wedge \in L^* \mid \mathcal{H}(G)\xi^\wedge \subset E \subset \mathcal{L} \}.$$

6.4.

Theorem. Let $(P, \mathcal{P})$ be a parabolic pair, $E$ a $G$-module. Then there exists a canonical isomorphism

$$ \mathcal{T} : (E^\wedge)^{U} \cong (E_{\mathcal{P}})^{\wedge} $$

where $(E^\wedge)^{U}$ is the space of $U$-invariants in $E^\wedge$. For each $\xi^\wedge \subset (E^\wedge)^{U}$ the vector $\eta^\wedge = \mathcal{T} \xi^\wedge$ is uniquely characterized by the following property.

(*) For each subgroup $K \subset G$ in a good position with respect to $(P, \mathcal{P})$ $\mathcal{T} e_{K} \xi^\wedge = e_{K} \eta^\wedge$.
This theorem allows us to give another description of the pairing \( \{ \cdot, \cdot \} \) in theorem 6.1. Namely, applying it to \( G \)-module \( E \) we see that \( \langle (E^\wedge)^U, (E^\wedge)^U = (E_U)^* \rangle^\wedge \) is canonically isomorphic to \( (E_U)^\wedge \). Hence \( E_U^\wedge \) is smooth part of \( (E_U)^\wedge \), which is the statement of theorem 6.1.

Proof of the Theorem.

Step 1. Let \( K' \subset K \subset G \) be open compact subgroups in a good position with respect to \( (P, \bar{P}) \). Then for each \( \xi \in E_K^* \), \( e_K \xi \in E_K^* \) and \( A e_K \xi = e_K A \xi \). Indeed, let \( C \subset U \) be a very large open compact subgroup, \( L = e_C E \). By stabilization theorem (applied to \( \bar{P}, P, K \)) \( E_K^* = e_K L \) and \( E_K^* = e_K L \), which implies that \( E_K^* = e_K E_K^* \). Moreover, for each \( \eta \in L \) \( A(e_K \eta) = A(e_K, e_K \eta) = e_K A(e_K \eta) = e_K A(\eta) \) and similarly for \( K' \). Hence if \( \xi = e_K' \eta \), we have \( A(e_K \xi) = e_K A(\eta) \). Clearly \( E^\wedge = \{ \xi \in E^\wedge \mid e_K \xi \in E_K^* \text{ for all good } K \} \).

Step 2. Consider the inverse system \( \{ e_K \} \) where \( K \) runs through all good subgroups (i.e., open compact subgroups in a good position with respect to \( (P, \bar{P}) \)). Step 1 shows that \( \{ E_K^* \} \) form a subsystem in \( \{ E^*_k \} \) and \( \bar{P} : E_K^* \to E_U^* \) gives an isomorphism of this subsystem with the system \( \{ E_U^* \} \). This allows us to identify \( (E_U^* )^\wedge = \lim_{\longrightarrow \bar{K}} E_U^\wedge \), with the subspace \( E_U^\wedge = \lim_{\longrightarrow \bar{K}} (E_k^*) \subset \lim_{\longrightarrow \bar{K}} (E_k^*) = E^\wedge \). Clearly \( E^\wedge = \{ \xi \in E^\wedge \mid e_K \xi \in E_K^* \text{ for all good } K \} \).

Step 3. Let us prove that \( E^\wedge = \langle E^\wedge \rangle^U \). Indeed \( P \xi \in E^\wedge \iff \text{ for all good } K \), \( e_K \xi \in E_K^* \text{ for all good } K \) and all open compact subgroups \( C \subset U \), \( e_K \xi \in e_K e_C E \) \iff \text{ for all } \). Choose a small subgroup \( K \subset G \) normalized by \( C \). Then the vector \( \xi = e_K \xi \) is \( C \)-invariant which implies that \( e_C \xi = \xi \). Hence \( e_K e_C \xi = e_K e_C e_K \xi = e_C e_K \xi = e_C \xi = \xi = e_K \xi \). Since this is true for arbitrary small \( K \), \( e_C e_K \xi = \xi \).

6.5. Second Adjointness of Functors \( i \) and \( r \).

Theorem. Let \( (P, \bar{P}) \) be a parabolic pair, \( M = P \cap \bar{P} \). Then the functor \( i^G_M : M(M) \to M(G) \) is canonically left adjoint to the functor \( r^G_M : M(G) \to M(M) \). In particular, for a standard Levi subgroup \( M < G \) the functor \( i^G_M \) is left adjoint to \( r^G_M \).

This theorem follows from Theorem 6.4 and the following form of Frobenius reciprocity.

Proposition. Let \( G \) be an \( \ell \)-group (see...), \( H \subset G \) a closed subgroup. Define the induction functor \( \text{ind} : M(H) \to M(G) \) as in \( (\text{....}) \), i.e., for \( V \in M(H) \) we define \( G \)-module \( E = \text{ind}(G, H, V) \) as \( E = \{ f : G \to V \mid f(hg) = hf(g) \text{ for } h \in H \text{; support of } f \text{ is compact modulo } H \text{ and } f \text{ is locally constant} \} \).

Define the twisted induction functor \( \text{ind}^\wedge(V) = \text{ind}(V \otimes \Delta_G \Delta_H^{-1}) \). Then for each \( V \in M(H) \), \( E \in M(G) \) there is a canonical functorial isomorphism

\[
\text{Hom}_G \left( \text{ind}^\wedge(V), E \right) = \text{Hom}_G (V, E^\wedge).
\]
In other words, the functor \( \text{ind}^\Delta \) is left adjoint to the functor \( S \), given by \( S(E) = H \)-smooth part of \( E^\wedge \).

**Proof of Proposition.** Let \( S(G) \) be the space of locally constant compactly supported functions on \( G \) with left action of \( G \). We have a canonical isomorphism \( \mathcal{H}(G) = S(G) \otimes \Delta_G(f \otimes \mathcal{M} \to f \cdot \mathcal{M}) \). We will identify \( S(G) \) with \( \text{ind}(G,1,\mathbb{C}) \) (since \( G \) acts on \( \text{ind}(\mathbb{C}) \) from the right, this identification involves change \( g \mapsto g^{-1} \)). By transitivity of induction we have \( \text{ind}(G, H, S(H)) = S(G) \).

This implies, that

\[
\text{ind}^\Delta(\mathcal{H}(H)) = \text{ind}(S(H) \cdot \Delta_G \cdot \Delta_H^{-1} \cdot \Delta_H) = \Delta_G \cdot \text{ind}(S(H)) = \Delta_G \cdot S(G) = \mathcal{H}(G) .
\]

Since \( \text{ind}^\Delta \) is an exact functor, preserving direct sums and \( \mathcal{H}(H) \) is a projective generator of \( \mathcal{M}(H) \), \( \text{ind}^\Delta(V) = \mathcal{H}(G) \otimes_{\mathcal{H}(H)} V \). This implies, that

\[
\text{Hom}_G(\text{ind}^\Delta(V), E) = \text{Hom}_G \left( \mathcal{H}(G) \otimes_{\mathcal{H}(H)} V, E \right) = \\
= \text{Hom}_H \left( V, \text{Hom}_G(\mathcal{H}(G), E) \right) = \text{Hom}_H(V, E^\wedge) = \\
= \text{Hom}_H(V, S(E)) .
\]

All isomorphisms above are canonical.

**Remark.** Let us describe explicitly morphism \( \alpha : V \to \text{ind}^\Delta(V)^\wedge \), corresponding to identity morphism of \( \text{ind}^\Delta(V) \). For \( v \in V \) we define \( \alpha(v) \in \text{ind}^\Delta(V)^\wedge \) by condition, that for each open compact subgroup \( K \subset G \) the function \( f_K = e_K^* \alpha(v) \in \text{ind}^\Delta(V) \) has the following form and vanishes outside of \( HK \) and

\[
f(hK) = h \in H \mapsto \kappa(v) \in \mathcal{M}_G \otimes \mathcal{M}_H^{-1} \left( \mathcal{M}_G(K)^{-1}, \mathcal{M}_H(H \cap K) \right) .
\]

where \( \mathcal{M}_G \in \Delta_G, \mathcal{M}_H \in \Delta_H. \)

**Proof of the Theorem.** Let \( V \subset \mathcal{M}(M), E \in \mathcal{M}(G) \). Using canonical isomorphisms \( \Delta_G \Delta_P^{-1} = \Delta_U^{-1} \) and \( \Delta_U^{-1} = \Delta_P^{-1} \) we have

\[
\text{Hom}_G \left( \text{ind}^\Delta(G,P,V \otimes \Delta_U^{1/2})E \right) = \\
= \text{Hom}_P \left( V \otimes \Delta_U^{1/2}, E^\vee \right) = \text{Hom}_M \left( V \otimes \Delta_U^{1/2}, (E^\vee)^U \right) = \\
= \text{Hom}_M \left( V, \text{Hom}_P(\mathcal{M}_G \otimes \Delta_U^{1/2}) \right) = \text{Hom}_M \left( V, \text{Hom}_P(\mathcal{M}_G \otimes \Delta_U^{1/2}) \right) .
\]

**Remark.** Let us write explicitly morphism \( \alpha : V \to \mathcal{M}_G \otimes \mathcal{M}_P \). Let \( v \in V \). Choose a subgroup \( K \) in a good position with respect to \( (P, \mathcal{P}) \), such that \( e_P v = v \). Then \( \alpha(v) \) is represented by \( \mathcal{M}_P^{1/2} \), where \( f : G \to V \otimes \Delta U^{-1/2} \) is supported on \( PK \) and for \( k \in K \) \( f(K) = v \mathcal{M}_U^{-1/2} \mathcal{M}_G^{-1}(K) \mathcal{M}_P(K \cap P) \).

Here \( \mathcal{M}_U \in D_U, \mathcal{M}_P \in \Delta_P \) are dual and \( \mathcal{M}_G = \mathcal{M}_P \mathcal{M}_P \). In particular, \( \mathcal{M}_G^{-1}(K) \mathcal{M}_P(K \cap P) = \mathcal{M}_P^{-1}(K) \). Identifying \( \mathcal{M}_U^{-1/2} \) with \( \mathcal{M}_P^{1/2} \) we can write
\[ \int \alpha(v) = \left( \int_{K_\pi \times \mathcal{M}_\pi} \right) \mathcal{M}_\pi^{-1}(K_\pi) = v. \]

This shows that $\alpha$ coincides with the morphism in the composition theorem, corresponding to the big cell $P\mathcal{F}$ and the point $w = 1 \in P\mathcal{F}$ (see....).