Problem assignment 1

Analysis on Manifolds

Joseph Bernstein November 2, 2003

Let M denote a manifold of dimension m (for example $\mathbf{R}^m$).

1. Let f be a (smooth) function on M, $Z = Z(f)$ the sets of its zeroes.

Suppose that at some point $a \in M$ we have $df \neq 0$, where $df \in T_a(M)^*$ be the differential of the function f at the point a.

(i) Show that near the point a (i.e. in some neighborhood of a in M) the subset $Z(f) \subset M$ is a submanifold of dimension $m - 1$.

(ii) Show that the tangent space $T_a(Z)$ as a subspace of $T_a(M)$ is given by equation $df = 0$.

2. (i) Show that if $r > 0$ then the sphere S_r of radius r in $\mathbf{R}^n$ is a submanifold. Describe its tangent space at every point.

(ii) Consider a subset $C \subset \mathbf{R}^n$ given by equation $x_n^3 = \sum_1^{n-1} x_i^2$.

Describe at what points it is a manifold and at what points it is not.

3. Let S^{n-1} be the unit sphere in $\mathbf{R}^n$. Consider the open set $U = S^{n-1} \backslash A$, where A is the point $(0, ..., 0, 1)$. Show that U is a manifold and describe a diffeomorphism of U with $\mathbf{R}^{n-1}$

4. Let $f_1, ..., f_k$ be a collection of smooth functions on M. Denote by $Z = Z(f_1, ...f_k)$ the set of common zeroes of functions f_i.

Suppose that at some point $a \in Z$ differentials df_i of functions f_i are linearly independent. Show that then Z near the point a is a manifold. Describe the tangent space $T_a(Z)$.

5. Cut off functions. Show that for every $\delta > 0$ there exists a smooth function p on $\mathbf{R}^n$ such that

$0 \leq p \leq 1$, $p \equiv 0$ outside the ball B_1 of radius 1 around 0 and $p \equiv 1$ inside the ball $B_{1-\delta}$ of radius $1 - \delta$ around 0.

(**Hints**. First check that the function $u(t)$ in one variable given by $u(t) = 0$ for $t \leq 0$ and $u(t) = exp(-1/t^2)$ for $t > 0$ is smooth.

Then construct a monotone smooth function $h(t)$ in one variable such that $h(t) \equiv 0$ for $t \leq -\delta$ and $h(t) \equiv 1$ for $t > 0$.)

6. Lagrange's lemma. Consider the Euclidean space $V \cong \mathbf{R}^n$ with coordinates x_i.

Show that any smooth function $f \in C^\infty(V)$ can be written in the form $f = f(0) + \sum x_i h_i$, where h_i are smooth functions on V.

Hint. In case of one variable x we can define $h(x) = \int_0^1 u(tx)dt$, where $u = \frac{df}{dx}$.