Problem assignment 2

Analysis on Manifolds

Joseph Bernstein November 19, 2003

1. Let $\nu : X \to Y$ be a submersive morphism of manifolds.

(i) Show that the image of ν is open in Y.

(ii) Show that if X is compact and non-empty and Y is connected then the morphism ν is epimorphic.

2. (i) Let $Z \subset X \subset Y$ be a system of manifolds. Show that locally it is diffeomorphic to a system of linear spaces.

(ii) Let Z, W be a system of two submanifolds in a manifold X. Show that if they are transversal then locally this system is diffeomorphic to a system of linear spaces.

3. Construct a morphism $\nu : \mathbf{R} \to \mathbf{R}$ which has infinite set of critical values.

Can you construct a morphism $\nu : S^1 \to S^1$ with the same property ?

4. Let $G \subset GL(n, \mathbf{R})$ be one of the following subgroups: $SL(n, \mathbf{R})$, $O(n)$, $SO(n)$, group T_n of upper triangular matrices.

(i) Show that each of this groups is a manifold. Compute for each of these groups the tangent space at identity element e .

(ii) Describe tangent spaces at all points of the groups in (i).

5. Let V be the space of matrices $Mat(m, n)$ of size $m \times n$. For every r consider the subset $M_r \subset V$ of matrices of rank r.

(i) Show that this is a submanifold. Compute its dimension.

(ii) For every point $m \in M_r$ describe the tangent space $T_m(M_r) \subset V$

6. Let $\nu : X \to X$ be an automorphism of a smooth manifold X, $x \in X$ a fixed point of ν. We say that the point x is a Lefschetz fixed point for ν if the differential $d\nu : T_x X \to T_x X$ does not have eigenvalue 1.

Suppose that X is compact and all fixed points of ν are Lefschetz.

(i) Show that this situation is stable.

(ii) Show that the morphism ν has finite number of fixed points.

7. Let $\nu(t) : X \to Y (0 \leq t \leq 1)$ be a smooth homotopy. Sow that there exists a smooth homotopy $\mu(t) : X \to Y (0 \leq t \leq 1)$ such that $\mu(t) = \nu(0)$ for $t < 1/4$ and $\mu(t) = \nu(1)$ for $t > 3/4$.

Show that homotopy is an equivalence relation on the set of morphisms $Mor(X, Y)$.

8. Let $\nu : X \to Y$ be a morphism of manifolds. Suppose that at all points $x \in X$ the rank of the (linear) tangent map $d\nu$ equals k.

Show that morphism ν is locally diffeomorphic to a linear morphism of linear spaces.