Problem assignment 1

Functions of Complex Variable 2

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1. Define the integral $\int_0^\infty x^\lambda e^{ix} dx$ and compute its value as a function of λ.

2. Prove that $\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$.

3. Compute the logarithmic derivative Γ'/Γ at points $z = 1, 2, 3....$

4. (i) Let f be a non zero meromorphic function on $\mathbf{C}$. Show that its logarithmic derivative $g = f'/f$ is a meromorphic function with poles of first order and integral residues.

(ii) Conversely show that any meromorphic function g with poles of first order and integral residues is a logarithmic derivative of a meromorphic function.

5. Let f, g be entire functions.

(i) Show that they have gcd (greatest common divisor) h. This means that f and g are divided by h and h divides any other entire function u with this property.

(ii) Show that there exist entire functions A, B such that $h = Af + Bg$.

6. Let f, g be entire functions of order ρ.

(i) Show that $f + g$ and fg are entire functions of order ρ.

(ii) Show that if the function $h = f/g$ is entire then it is also of order ρ.

7. (i) Compute the function $\prod_{n=1}^\infty (1 + \frac{z^4}{n^4})$

(ii) Show that $e^z - 1 = ze^{z/2} \prod_{n=1}^\infty (1 + \frac{z^2}{4\pi^2 n^2})$

8. Let $f = f(z)$ be an entire function that has no more than exponential growth, i.e. $|f(z)| \leq C \exp(C|z|)$. Suppose we also know that it is periodic with period 1.

Show that f is a polynomial of the function $q = \exp(2\pi i z)$.