Problem assignment 1

Advanced Algebra II - Class Field Theory

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Separability

Let us fix a field K and some imbedding of K into an algebraically closed field Ω (for example we can take $\Omega = \bar{K}$).

For every finite field extension L/K we consider the set $M(L) = \mathrm{M}or_{K-alg}(L, \Omega)$ and define the separable degree $[L : K]_s := |M(L))|$.

We have shown in class that $[L : K]_s \leq [L : K]$. The field extension L/K is called *separable* if this is an equality.

1. Let $K \subset L \subset M$ be a tower of finite field extensions. Show that M/K is separable iff M/L and L/K are separable.

An element $\lambda \in \Omega$ is called *separable* over K if the field $K < \lambda >$ is a finite separable extension of K.

2. Let $P = \sum a_n x^n \in K[x]$ be a monic polynomial. Show that P does not have multiple roots in Ω iff $gcd(P, DP) = 1$, where DP is the usual derivative of P, $DP = \sum n a_n x^{n-1}$.

Show that if P is irreducible then this is equivalent to $DP \neq 0$.

3. Show that an element $\lambda \in \Omega$ is separable over K iff it is algebraic and its minimal polynomial $P \in K[x]$ does not have multiple roots in Ω (i.e. $DP \neq 0$).

Show that if λ is separable over K then it is separable over any subfield $L \subset \Omega$ containing K.

4. Let L/K be a finite field extension. Show that L/K is separable iff all elements of L are separable over K iff L is generated over K by elements separable over K.

5. A field K is called *perfect* if any its finite extension is separable.

Show that any field of characteristic 0 is perfect

Show that a field K of characteristic p is perfect iff the Frobenius morphism $Fr : K \to K$, $x \mapsto x^p$, is bijective.

Show that all finite fields are perfect.

Let A be a commutative K-algebra (always with 1) of finite dimension n. Define the set $M(A) := \mathrm{M}or_{K-alg}(A, \Omega)$ and the number $[A : K]_s = |M(A)|$.

6. Show that elements of the set $M(A)$ are linearly independent in the Ω-linear space $H(A) = \mathrm{H}om_K(A, \Omega)$. Deduce from this that $[A : K]_s \leq n = dim_K(A)$.

Show that the algebra A is separable over K iff $[A : K]_s = n$ iff A is isomorphic to a direct sum of separable field extensions of K.

7. Let L/K be a finite field extension. Consider an L- algebra $A = L_L$ obtained from L by extension of scalars from K to L.

Show that L is separable over K iff the algebra A is semisimple (e.i. does not have nilpotent elements).

Suppose that L/K is a Galois extension. Show that then the L-algebra A splits.