Problem assignment 1

Algebraic Geometry and Commutative Algebra II

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One of important results about the notion of dimension is

Principal Ideal Theorem. Let X be an irreducible algebraic variety of dimension n and f a regular function on X. Denote by $Z = V(f)$ the subvariety of zeroes of f.

Then either $f = 0$ and then $Z = X$ has dimension n or $f \neq 0$ and then every irreducible component of Z has dimension exactly $n - 1$ (in particular it might happen that Z is empty).

1. Let X be an irreducible algebraic variety of dimension n and $Y \subset X$ a closed irreducible subvariety of dimension d. Show that we can include Y in a chain of irreducible closed subvarieties $Y = X_d \subset X_{d+1} \subset ... \subset X_n = X$ where $\dim X_i = i$.

2. Let X be an algebraic variety. Suppose it is locally irreducible. Show that every connected component of X is irreducible.

Use this to show that any smooth connected variety is irreducible.

Definition. Let Y be an irreducible algebraic variety, P a property which holds for some points $y \in Y$. We say that the property P holds for **generic point** of Y if the set of points for which P holds contains an open dense subset of Y.

3. Let $\pi : X \to Y$ be a dominant morphism of irreducible algebraic varieties of relative dimension k (i.e. $k = \dim X - \dim Y$). For every point $y \in Y$ consider the fiber $F_y = \pi^{-1}(y)$.

(i) Show that for generic point $y \in Y$ $\dim F_y = k$.

(ii) Show that for every point $y \in Y$ dimension of every irreducible component of the fiber F_y is $\geq k$.

4. Let V be a finite dimensional vector space over k and $\mathbf{V}$ the corresponding affine variety.

(i) Fix a number l. Define the structure of an algebraic variety on the set G_l of all affine (i.e not necessarily passing through 0) linear subspaces $L \subset V$ of codimension l.

(ii) Prove the following

Proposition. Let Y be an algebraic subvariety of $\mathbf{V}$. Show that the following conditions are equivalent:

(a) $\dim Y \leq k$

(b) For generic point $L \in G_l$ with $l > k$ the space L does not intersect Y.

(c) For generic point $L \in G_k$ the intersection of L with Y is finite.

(Hint. Consider the incidence variety $Z \subset Y \times G_l$ consisting of points (y, L) such that $y \in L$ and compute its dimension using projections to Y and to G_l).

This proposition can be used as a definition of dimension, and as a powerful tool for computing dimensions of different varieties.

5. Let $\pi : X \to Y$ be a morphism of algebraic varieties, $x \in X$ and $y = \pi(X)$. Assume that the differential $d\pi : T_x X \to T_y Y$ is onto and that x is a smooth point of X.

(i) Show that x is a smooth point of the fiber $F = \pi^{-1}(y)$.

(ii) Show that y is a smooth point of Y.