Problem assignment 3

Algebraic Geometry and Commutative Algebra II

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Some cohomological constructions.

1. Five lemma. Let L, M be two complexes and $\nu : L \to M$ a morphism of complexes, i.e. a collection of morphisms $\nu_i : L^i \to M^i$ commuting with differentials.

Let us assume that the complexes are exact, morphisms ν_1 and ν_{-1} are isomorphisms, ν_2 is epimorphic and ν_{-2} is mono.

Sow that the morphism ν_0 is an isomorphism.

Cone construction. Let $\nu : L \to M$ be a morphism of complexes. We construct a new complex $Cone(\nu)$ as follows. We extend ν to a complex of complexes placing $L^\cdot$ and $M^\cdot$ in places -1 and 0, consider the corresponding bicomplex B and set $Cone(\nu) := Tot(B)$.

2. (i) Write explicit formulas for the complex $Cone(\nu)$. Show that there exists a short exact sequence of complexes $0 \to M \to Cone(\nu) \to L[1] \to 0$.

Deduce from this a long exact sequence connecting cohomologies of L, M and $Cone(\nu)$.

(ii) Show that the morphism of complexes ν is a quasiisomorphism iff the complex $Cone(\nu)$ is acyclic.

(ii) Show that if ν is injective then $Cone(\nu)$ is quasiisomorphic to the quotient complex M/L.

3. Let $\nu : B \to B'$ be a morphism of bicomplexes. Suppose we know that for every row ν induces quasiisomorphism of the complexes $\nu : Row^j(B) \to Row^j(B')$. Show (under appropriate finiteness assumptions) that ν induces a quasiisomorphism of total complexes $Tot(B) \to Tot(B')$.

(**Hint.** Using problem 2 reduce this statement to Grothendieck's lemma).

Truncation. Let k be an integer. We define truncation functor $\tau_{\leq k}$ from category of complexes into itself as follows. For a complex $M^\cdot$ we consider subcomplex $L = \tau_{\leq j} M$, where $L^i = M^i$ for $i < k$, $L^i = 0$ for $i > k$ and $L^k = \ker(M^k \to M^{k+1})$.

4. Show that $H^i(L) = H^i(M)$ for $i \leq k$ and $H^i(L) = 0$ for $i > k$.

Compute cohomologies of the quotient complex M/L.

5. Let B^{ij} be a bicomplex. Suppose that every row is acyclic outside column 0. Prove (under appropriate finiteness assumptions) that the total complex $Tot(B)$ has the same cohomologies as the complex combined from objects $H^0(B^{ij}, d)$.

In fact these complexes are quasiisomorphic.