Problem assignment 1

Introduction to Differential Geometry

Joseph Bernstein November 3, 2005.

Problems in linear algebra.

1. Let $A : V \to W$ be a morphism of vector spaces, $K = \ker A$ its kernel and $I = ImA$ its image.

Show that $K \subset V$ and $I \subset W$ are linear subspaces.

Show that A is mono iff $K = 0$. Show that if $K = 0$ and $I = W$ then A is an isomorphism, i.e. there exists an inverse morphism $B : W \to V$ such that compositions $A \circ B$ and $B \circ A$ are identity morphisms.

2. Let V be a vector space and $L \subset V$ a subspace. Show that there exists a vector space Q and an epimorphism $p : V \to Q$ such that $\ker p = L$.

Show that the pair (Q, p) is uniquely defined up to canonical isomorphism (i.e. any two such pairs are canonically isomorphic).

The space Q is called the **quotient space**; usually it is denoted by V/L.

[P] 3. Let V be vector space of dimension $n < \infty$ and $L \subset V$ be a subspace of dimension l. Show that there exists a basis $e_1, ..., e_n$ of the space V such that vectors $e_1, ..., e_l$ form a basis of L.

Show that in this case the vectors $e_{l+1}, ..., e_n$ (or more precisely their images) form a basis of the quotient space V/L.

[P] 4. Let V be a vector space of dimension n, $L, L' \subset V$ subspaces of V. Show that if $\dim L + \dim L' > n$ then L and L' have a non-zero intersection.

[P] 5. Let V be a vector space of dimension n and $L \subset V$ a subspace. Consider its orthogonal complement $L^\perp \subset V^*$ defined by $L^\perp := \{f \in V^* | f|L = 0\}$.

(i) What is the dimension of $L^\perp$?

(ii) Show that $(R \cap L)^\perp = R^\perp + L^\perp$ and $(R + L)^\perp = R^\perp \cap L^\perp$.

(iii) Show that $(L^\perp)^\perp = L$.

(iv) Show that $L^\perp$ is naturally isomorphic to $(V/L)^*$.

6. Let B be a symmetric bilinear form on V. Denote by Q the corresponding quadratic form on V defined by $Q(x) = B(x, x)$.

(i) Show that the form B could be recovered from Q.

(ii) Show that a function Q on V is a quadratic form iff in any coordinate system it could be written as $\sum a_{ij} x_i x_j$.

[P] (iii) Show that Q is a quadratic form iff it is homogeneous function of degree 2 which for any $a, b \in V$ satisfies the condition that the function $Q(x + a + b) - Q(x + a) - Q(x + b) + Q(x)$ is constant function.

7. Let V, Q be a finite dimensional Euclidean space. Show that it is isomorphic to $(\mathbf{R}^n, Q_0)$, where Q_0 is the standard quadratic form $Q_0(x_1, ..., x_n) = \sum x_i^2$.

[P] 8.. Let Q' be a quadratic form on an Euclidean space (V, Q). Show that there exists a constant $C > 0$ such that $|Q'(x)| \leq CQ(x)$ for all $x \in V$.