

Problem assignment 2

Introduction to Differential Geometry

Joseph Bernstein

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Problems about metric spaces.

1. Let X be a compact metric space.

[P] (i) Show that any closed subset $Z \subset X$ is compact (with respect to induced metric).

(ii) Show that for any continuous map $\nu : X \rightarrow Y$ from X to a metric space Y the subset $\nu(X) \subset Y$ is closed and compact.

[P] 2.. Let $X \subset \mathbf{R}^n$ be a compact subset. Show that X is bounded and closed subset of \mathbf{R}^n .

[P] 3. Let $A, B \subset X$ be two non-empty subsets of a metric space X . We define the distance $d(A, B)$ by $d(A, B) = \inf\{d(a, b) | a \in A, b \in B\}$.

(i) Show that if A is a set consisting of one point a then $d(A, B) = 0$ iff a lies in the closure of the set B .

(ii) Suppose $X = \mathbf{R}^n$, A is closed and B is compact. Show that there exist points $a \in A$ and $b \in B$ such that $d(a, b) = d(A, B)$.

(ii) Construct example of two closed subsets $A, B \subset \mathbf{R}^n$ which do not intersect but $d(A, B) = 0$.

4. Let $\nu : X \rightarrow Y$ be a map of two metric spaces.

(i) Consider a point $x \in X$ and its image $y = \nu(x) \in B$.

Show that ν is continuous at the point x iff it satisfies

(*) For any neighborhood $V \subset Y$ of the point y the subset $\nu^{-1}(V) \subset X$ is a neighborhood of x .

(ii) Show that ν is continuous (i.e. continuous at all points) iff it satisfies

(**) For any open subset $V \subset Y$ the subset $\nu^{-1}(V) \subset X$ is open.

[P] 5. Let X be a metric space. Show that it is compact iff it satisfies the following

Finite Covering Property. Any open covering $\{U_\alpha\}$ contains a finite subcovering $\{U_{\alpha_i}\}$.

[P] 6. Let X be a compact metric space and $\{F_\alpha\}$ a family of closed subsets of X .

Suppose we know that any finite collection of these subsets has non-empty intersection. Show that all these subsets have non-empty intersection.

7. Show that a map $f : X \rightarrow \mathbf{R}^n$ is continuous iff its coordinate functions f^i are continuous.

[P] 8. Let C be a compact subset of a metric space X and $U \subset X$ be an open subset which contains C . Show that there exists $\varepsilon > 0$ such that U contains ε -neighborhood of C .

[P] 9. (i) Find the maximal area of a triangle inscribed into a unit circle.

(ii) Can you describe how to evaluate the maximal area of a convex 10-gon inscribed into the unit circle.